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APPROXIMATE AMENABILITY AND CONTRACTIBILITY OF HYPERGROUP ALGEBRAS

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Abstract. Let K be a hypergroup. The purpose of this article is to study the notions of amenability of the hypergroup algebras $L(K)$, $M(K)$, and $L(K)^{**}$. Among other results, we obtain a characterization of approximate amenability of $L(K)^{**}$. Moreover, we introduce the Banach space $L_\infty(K, L(K))$ and prove that the dual of a Banach hypergroup algebra $L(K)$ can be identified with $L_\infty(K, L(K))$. In particular, $L(K)$ is an F-algebra. By using this fact, we give necessary and sufficient conditions for K to be left-amenable.

1. Introduction and preliminaries

For a locally compact Hausdorff space K, let $M(K)$ be the Banach space of all bounded complex regular Borel measures on K. For $x \in K$, the unit point mass at x will be denoted by δ_x. Let $M_1(K)$ be the set of all probability measures on K, and let $C_b(K)$ be the Banach space of all continuous bounded complex-valued functions on K. We denote by $C_0(K)$ the space of all continuous functions on K vanishing at infinity, and by $C_c(K)$ the space of all continuous functions on K with compact support.

The space K is called a *hypergroup* if there is a map $\lambda : K \times K \longrightarrow M_1(K)$ with the following properties.

(i) For every $x, y \in K$, the measure $\lambda_{(x,y)}$ (the value of λ at (x, y)) has a compact support.

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(ii) For each $\psi \in C_c(K)$, the map $(x, y) \longmapsto \psi(x * y) = \int_K \psi(t)\, d\lambda_{(x,y)}(t)$ is in $C_b(K \times K)$ and $x \longmapsto \psi(x * y)$ is in $C_c(K)$, for every $y \in K$.

(iii) The convolution $(\mu, \nu) \longmapsto \mu * \nu$ of measures defined by

$$\int_K \psi(t)\, d(\mu * \nu)(t) = \int_K \int_K \psi(x * y)\, d\mu(x)\, d\nu(y)$$

is associative, where $\mu, \nu \in M(K), \psi \in C_0(K)$ (note that $\lambda_{(x,y)} = \delta_x * \delta_y$).

(iv) There is a unique point $e \in K$ such that $\lambda_{(x,e)} = \delta_x$ for all $x \in K$.

When $\lambda_{(x,y)} = \lambda_{(y,x)}$, we say that K is a *commutative hypergroup* (for more details, see [4], [8], [16]). Let K be a foundation; that is, $K = \mathrm{cl}(\bigcup_{\mu \in L(K)} \operatorname{supp} \mu)$. We define

$$L(K) = \big\{\mu \;\big|\; \mu \in M(K), x \mapsto |\mu| * \delta_x, x \mapsto \delta_x * |\mu| \text{ are norm-continuous}\big\}.$$

It is easy to see that $M(K)$ is a Banach algebra and that $L(K)$ is an ideal in $M(K)$. An invariant measure (Haar measure) m on K is a positive nonzero regular Borel measure on K such that $m * \delta_x = m$, for all $x \in K$. If K admits a Haar measure m, then $L(K) = L^1(K, m)$ (see [8]).

An involution on a hypergroup K is a homeomorphism $x \mapsto \tilde{x}$ in K such that $\tilde{\tilde{x}} = x$ and $e \in \operatorname{supp} \lambda_{(x,\tilde{x})}$ for all $x \in K$. For each $\mu \in M(K)$, define $\widetilde{\mu} \in M(K)$ by $\widetilde{\mu}(A) = \overline{\mu(\widetilde{A})}$; that is, $\int_K f(x)\, d\widetilde{\mu}(x) = \int_K f(\tilde{x})\, d\overline{\mu}(x)$, for each $f \in C_c(K)$. Then $\mu \longrightarrow \tilde{\mu}$ is an involution on $M(K)$ such that $M(K)$ and $L(K)$ are Banach $*$-algebras and $\widetilde{\lambda_{(x,y)}} = \lambda_{(\tilde{y},\tilde{x})}$, whenever $x, y \in K$ (see [4]).

Let K be a foundation hypergroup without a Haar measure. With these conditions $L(K)$ is a general hypergroup algebra which includes not only group algebras but also most of the semigroup algebras. We recall (see [16, Proposition 1]) that the algebra $L(K)$ possesses a bounded approximate identity. Also, in this article the Banach space $L(K)^* \cdot L(K)$ is denoted by B. Medghalchi [16] showed that B^* (dual of B) is a Banach algebra by an Arens-type product and that $L(K) \subseteq B^*$. For $f \in B$, if K admits an invariant measure (Haar measure m), then by Proposition 2.4 of [17], $B = LUC(K)$ where

$$LUC(K) = \big\{f \;\big|\; f \in C_b(K), x \to l_x f \text{ from } K \text{ into } C_b(K) \text{ is continuous}\big\},$$

and $l_x f(y) = f(x * y)$ for any $y \in K$.

Let A be a Banach algebra, and let X be a Banach A-bimodule. A continuous derivation $D : A \longrightarrow X$ is said to be *approximately inner* if there exists a net $\{\zeta_i\}$ in X such that $D(a) = \lim_i(a.\zeta_i - \zeta_i.a)$ for all $a \in A$, in the norm topology. The Banach algebra A is called *approximately amenable* if every derivation from A into the dual A-bimodule X^* is approximately inner for all Banach A-bimodules X. Similarly, a complex Banach algebra A is called an *F-algebra* if it is the (unique) predual of a W^*-algebra M and the identity element u of M is a multiplicative linear functional on A.

Ghahramani, Loy, Willis, and Zhang introduced and studied concepts of approximate amenability (contractibility) and uniform approximate amenability (contractibility) for Banach algebras (for more details, see [5]–[7]). Medghalchi [17] introduced cohomology on hypergroup algebras. He showed that the amenability of $L(K)$ implies the left amenability of K; however, the converse is not valid any longer even if K is commutative and discrete. Moreover, Skantharajah [22] initiated and studied the notion of amenability for hypergroup algebras in the sense of Jewett [2]. The concept of φ-amenability of Banach algebras was introduced by Kaniuth, Lau, and Pym [12]. Similarly, character-amenable Banach algebras were introduced and investigated in [10]. These concepts generalize the concept of left amenability for F-algebras introduced by Lau [14].

This article is organized as follows. In Section 2, we investigate the concepts of approximate amenability and contractibility for Banach algebras $M(K)$, $L(K)$, and $L(K)^{**}$. As one of the interesting results, in Theorems 2.2 and 2.4 we show that K is left-amenable if the hypergroup algebra $L(K)$ is approximately amenable as a Banach algebra, but the converse is not true. Ghahramani and Loy [5, Theorem 3.2] showed that a necessary and sufficient condition for $M(G)$ to be approximately amenable is that G be discrete and amenable (see [5, Theorem 3.1]); we prove that for a hypergroup, this is not true. Moreover, in Theorem 2.6, for a hypergroup with an involution, we prove that the finiteness of K is equivalent to the contractibility of $L(K)$. Also, in Theorem 2.8, we show that K is discrete and amenable if $L(K)^{**}$ is approximately amenable; the converse is not necessarily true. But, for a hypergroup with an involution, in Theorem 2.9, we obtain that a necessary and sufficient condition for $L(K)^{**}$ to be approximately amenable is that K be finite.

Let G be a locally compact group. By Theorem 3.2 of [5] and Johnson's classical result, the approximate amenability of $L^1(G)$ is equivalent to the amenability of $L^1(G)$. Therefore, it is natural to ask the following question on hypergroups: Is the approximate amenability of $L(K)$ equivalent to the amenability of $L(K)$? We have yet to find an answer to this question.

In Section 3, we first introduce the Banach space $L_\infty(K, L(K))$. In Theorem 3.2, we prove that the dual of the Banach hypergroup algebra $L(K)$ can be identified with $L_\infty(K, L(K))$ and hence $L(K)$ is an F-algebra. This allows us to give an alternative theorem similar to Theorem 3.2 of [5] (see Theorem 3.5).

2. Approximate amenability of $L(K)$ and $L(K)^{**}$

Throughout this paper, K is a foundation hypergroup without a Haar measure. For $f \in B$ and $x \in K$, we will denote $l_x f$ by $\langle l_x f, \nu\rangle = \langle f, \delta_x * \nu\rangle$ whenever $\nu \in L(K)$. Since $B = L(K)^* \cdot L(K)$, $f = g \cdot \mu$ $(g \in L^*(K), \mu \in L(K))$. Therefore,

$$\langle l_x f, \nu\rangle = \langle g \cdot \mu, \delta_x * \nu\rangle = \langle g, \mu * \delta_x * \nu\rangle = \big\langle g \cdot (\mu * \delta_x), \nu\big\rangle.$$

Hence, $l_x f = g \cdot (\mu * \delta_x)$. It follows that $l_x f \in B$. Also, by Proposition 2 of [16], $1 \in B$, where 1 is the constant function.

Definition 2.1. Let K be a hypergroup. A linear functional $m : B \longrightarrow \mathbb{C}$ is called a *mean* if $m(1) = \|m\| = 1$. A mean on B is called a *left-invariant mean* if $m(l_x f) = m(f)$, for $f \in B$ and $x \in K$. A hypergroup K is called *left-amenable* if there exists a left-invariant mean on B.

Now we are in a position to prove a theorem that generalizes one side of Theorem 3.2 of [5] to hypergroups.

Theorem 2.2. *If $L(K)$ or $M(K)$ is approximately amenable, then K is left-amenable.*

Proof. Let $L(K)$ be approximately amenable, and let $X = \frac{B}{\mathbb{C}1}$, where 1 is the constant function. With left action $f \cdot \mu$ and right action $\mu \cdot f$, B is a Banach $M(K)$-bimodule, where

$$\langle f.\mu, \nu\rangle = \langle f, \mu * \nu\rangle, \qquad \mu \cdot f = \mu(K)f,$$

for $f \in B$, $\mu \in M(K)$, and $\nu \in L(K)$. Since the space $\mathbb{C}1$ is a closed sub-bimodule of B, X is a Banach $M(K)$-bimodule. We know that $\delta_e \in B^*$ and $\delta_e \notin X^*$, since $X^* = \{F \in B^* | F(1) = 0\}$. Let $\nu_0 = \delta_e$ and $D : \mu \longmapsto \mu \cdot \nu_0 - \mu(K)\nu_0$ (the action $\mu \cdot \nu_0$ is dual action), where $\mu \in M(K)$. In particular, $D(\delta_x) = \delta_x \cdot \nu_0 - \nu_0$, for $x \in K$. It is clear that D is a derivation on $M(K)$ into X^*. We can consider D as a derivation on $L(K)$ into X^*. On the other hand, $L(K)$ is approximately amenable. So, there is a net (m_α) in X^* such that

$$D(\mu) = \lim_\alpha(\mu \cdot m_\alpha - m_\alpha \cdot \mu) = \mu \cdot \nu_0 - \mu(K)\nu_0,$$

for $\mu \in L(K)$. Therefore,

$$\lim_\alpha\big(\mu \cdot (\nu_0 - m_\alpha) - \mu(K)(\nu_0 - m_\alpha)\big) = 0. \tag{2.1}$$

Taking $\mu \in L(K), \mu \geq 0, \|\mu\| = 1$, and $x \in K$. Therefore, we have

$$D(\delta_x) = \delta_x \cdot \nu_0 - \nu_0 = (\delta_x \cdot \nu_0) \cdot \mu - \nu_0 \cdot \mu = D(\delta_x) \cdot \mu = D(\delta_x * \mu) - \delta_x \cdot D(\mu).$$

Since $\delta_x * \mu \in M_1(K)$, $m_\alpha \cdot (\delta_x * \mu) = m_\alpha$. Then

$$\begin{aligned}
\delta_x \cdot \nu_0 - \nu_0 = D(\delta_x) &= D(\delta_x * \mu) - \delta_x \cdot D(\mu)\\
&= \lim_\alpha\big[(\delta_x * \mu) \cdot m_\alpha - m_\alpha \cdot (\delta_x * \mu) - \delta_x \cdot (\mu \cdot m_\alpha - m_\alpha \cdot \mu)\big]\\
&= \lim_\alpha\big[(\delta_x * \mu) \cdot m_\alpha - m_\alpha - \delta_x \cdot (\mu \cdot m_\alpha - m_\alpha)\big]\\
&= \lim_\alpha[\delta_x \cdot m_\alpha - m_\alpha].
\end{aligned}$$

It follows that

$$\lim_\alpha\big[\delta_x \cdot (\nu_0 - m_\alpha) - (\nu_0 - m_\alpha)\big] = 0.$$

For each α, $(\nu_0 - m_\alpha)(1) = 1$. Thus, $\|\nu_0 - m_\alpha\| \neq 0$. Now, taking $n_\alpha = \frac{\nu_0 - m_\alpha}{\|\nu_0 - m_\alpha\|}$, we have $\|n_\alpha\| = 1$, and by (2.1), $\lim_\alpha(\delta_x \cdot n_\alpha - n_\alpha) = 0$ in norm, where $x \in K$. Take $n \in X^*$ as n is a *weak** cluster point of (n_α). Then $\delta_x \cdot n = n$, and thus, n is a left-invariant mean on B because

$$\langle l_x f, n\rangle = \langle f \cdot \delta_x, n\rangle = \langle f, \delta_x \cdot n\rangle = \langle f, n\rangle,$$

for all $f \in B$ and $x \in K$. So, K is left-amenable.

Now suppose that $M(K)$ is approximately amenable. Since $L(K)$ is a closed ideal of $M(K)$ with a bounded approximate identity, $L(K)$ is approximately amenable (see [5, Corollary 2.3]). Thus, K is left-amenable. $\square$

For a locally compact group G, it was shown (see [5, Theorem 3.2]) that $L^1(G)$ is approximately amenable if and only if G is amenable. The following example indicates that the converse of the above theorem is not true for hypergroups.

Example 2.3. Let $(R_n)_{n\in\mathbb{N}_0}$ be a polynomial sequence defined by a recurrence relation

$$R_1(x)R_n(x) = a_nR_{n+1}(x) + b_nR_n(x) + c_nR_{n-1}(x),$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, and $R_0(x) = 1, R_1(x) = \frac{1}{a_0}(x - b_0)$, $a_n > 0, b_n \geq 0$, for all $n \in \mathbb{N}$. We assume that $a_n + b_n + c_n = 1$ for $n \in \mathbb{N}$. Define a convolution on $l^1(\mathbb{N}_o)$ such that

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} g(n, m, k)\delta_k,$$

where $g(n, m, k) > 0$. Then $(\mathbb{N}_0, *)$ is a discrete commutative hypergroup with the unit element 0 which is called the *polynomial hypergroup* on $\mathbb{N}_0$ induced by $(R_n)_{n\in\mathbb{N}_0}$. Since $(\mathbb{N}_0, *)$ is a commutative hypergroup, $(\mathbb{N}_0, *)$ is amenable (see [22, Example 3.3(a)]). Consider the class of polynomial hypergroups induced by the ultraspherical polynomials $(R_n^{(\alpha)})_{n\in\mathbb{N}_0}$, $\alpha \geq \frac{-1}{2}$ (see [13]). The Banach algebra $\ell^1(\mathbb{N}_0)$ of the polynomial hypergroup is induced by the ultraspherical polynomials $(R_n^{(\alpha)})_{n\in\mathbb{N}_0}$.

Theorem 2.4. *Let $\mathbb{N}_0$ be the class of polynomial hypergroups generated by the ultraspherical polynomials $(R_n^\alpha)_{n\in\mathbb{N}_0}$, $\alpha > 0$. Then $\ell^1(\mathbb{N}_0)$ is not approximately amenable.*

Proof. Assume toward a contradiction that $\ell^1(\mathbb{N}_0)$ is approximately amenable. Since $\ell^1(\mathbb{N}_0)$ is Abelian, it is pseudoamenable (see [9, Corollary 3.4]). Therefore, $\ell^1(\mathbb{N}_0)$ is weakly amenable (see [9, Corollary 3.7]). This is impossible (see [13]). $\square$

Remark 2.5. Ghahramani and Loy [5] showed that the group G is amenable and discrete if and only if $M(G)$ is approximately amenable. By Theorem 2.2, the hypergroup K is left-amenable if $M(K)$ is approximately amenable. But we do not know, if $M(K)$ is approximately amenable, whether K is discrete. Let $\mathbb{N}_0$ be the class of polynomial hypergroups generated by the ultraspherical polynomials $(R_n^\alpha)_{n\in\mathbb{N}_0}$, $\alpha > 0$. Then $(\mathbb{N}_0, *)$ is a discrete, commutative, and amenable hypergroup (see [22, Example 3.3(a)]). By Theorem 2.4, $M(K) = \ell^1(\mathbb{N}_0)$ is not approximately amenable. It follows that it is not necessarily true that $M(K)$ is approximately amenable if K is amenable and discrete.

We now state and prove another interesting theorem.

Theorem 2.6. *Let K be a hypergroup with an involution. Then $L(K)$ is contractible if and only if K is finite.*

Proof. Let $L(K)$ be contractible. By Theorem 2.8.48 of [3], $L(K)$ is biprojective and unital. Therefore, K is discrete. Since K is discrete and has an involution, Jewett's and Dunkl's definitions of hypergroup coincide. It follows that K has a Haar measure and $\ell^1(K) = L(K)$. Now, since $L(K)$ is biprojective and $\mathbb{C}$ is an essential module over $L(K)$, $\mathbb{C}$ is projective. On the other hand, the map $\varphi_K : L(K) \longrightarrow \mathbb{C}$ defined by $\varphi_K(\mu) = \mu(K)$ is admissible. Therefore, φ_K has a right inverse morphism ρ. Take $P_0 := \rho(1) \in L(K)$, so

$$f * P_0 = f * \rho(1) = \rho(f \cdot 1) = \rho\big(\varphi_K(f)\big),$$

for any $f \in L(K)$. Now, suppose that $f \in C_c^+(K)$ and $\|f\|_1 = 1$. Then, $\|l_x f\|_1 = 1$, where $l_x f(y) = f(x * y)$ for all $x, y \in K$. We have

$$\begin{aligned}\|l_x f * P_0 - l_x P_0\|_1 &= \big\|l_x(f * P_0) - l_x P_0\big\|_1 = \big\|l_x(f * P_0 - P_0)\big\|_1 \\ &\leq \|f * P_0 - P_0\|_1 = 0.\end{aligned}$$

Hence, $P_0 = l_x f * P_0 = l_x P_0$ almost everywhere. Since $\varphi_K(P_0) = \varphi_K(\rho(1)) = 1$, P_0 is equal to a nonzero constant almost everywhere. It follows that the characteristic function $1_K \in L(K)$, since $P_0 \in L(K)$. On the other hand, $c1_K = 1_K * 1_K \in C_0(K)$ where $c > 0$ (see [2, Proposition 1.4.11]). Thus, K is compact. From this it follows that K is finite.

Conversely, assume that K is finite. So, $\ell^1(K)$ is amenable (see [1, Theorem 3.3]). Therefore, there exists $M \in (\ell^1(K) \hat{\otimes} \ell^1(K))^{**}$ such that M is a virtual diagonal for $\ell^1(K)$. On the other hand, $(\ell^1(K) \hat{\otimes} \ell^1(K))^{**} = \ell^1(K) \hat{\otimes} \ell^1(K)$. It follows that M is a diagonal for $\ell^1(K)$. Thus, $\ell^1(K)$ is contractible (see [3, Theorem 1.9.21]). $\square$

In this article, the second dual $L(K)^{**}$ with the first Arens product is denoted by $(L(K)^{**}, \square)$. Also, $\pi : L(K)^{**} \longrightarrow B^*$ is the adjoint of the embedding of B in $L(K)^*$. By a well-known result of Ghahramani, Loy, and Willis [6, Theorem 2.1], if $L^1(G)^{**}$ is weakly amenable, then $M(G)$ is weakly amenable. The following theorem extends this result to hypergroups.

Theorem 2.7. *Let K be a hypergroup. Then we have the following.*

(i) *If B^* is weakly amenable, then $M(K)$ is weakly amenable.*
(ii) *If $(L(K)^{**}, \square)$ is approximately amenable (weakly amenable), then $M(K)$ is approximately amenable (weakly amenable).*

Proof. (i) For $f \in M(K)^*$, define $T_f \in B^{**}$ by $\langle T_f, \mu + m\rangle = f(\mu)$, where $\mu \in M(K)$ and $m \in C_0(K)^\perp$ ($B^* = M(K) \oplus C_0(K)^\perp$). Assume that $M(K)$ is not weakly amenable. So, there is a noninner derivation $D : M(K) \longrightarrow M(K)^*$. Define $\triangle : B^* \longrightarrow B^{**}$ by $\triangle(\mu + m) = T_{D(\mu)}$, for each $\mu \in M(K), m \in C_0(K)^\perp$. For each $\mu_1, \mu_2, \nu \in M(K)$ and $m_1, m_1, n \in C_0(K)^\perp$, we have

$$\begin{aligned}\big\langle(\mu_1 + m_1) \triangle (\mu_2 + m_2), \nu + n\big\rangle &= \big\langle \triangle(\mu_2 + m_2), (\nu + n)(\mu_1 + m_1)\big\rangle \\ &= \big\langle \triangle(\mu_2 + m_2), \nu * \mu_1 + n\mu_1 + \nu m_1 + n \,\square\, m_1\big\rangle \\ &= \langle T_{D(\mu_2)}, \nu * \mu_1 + n\mu_1 + \nu m_1 + n \,\square\, m_1\rangle \\ &= \big\langle D(\mu_2), \nu * \mu_1\big\rangle = \big\langle \mu_1 D(\mu_2), \nu\big\rangle \\ &= \langle T_{(\mu_1 D(\mu_2))}, \nu + n\rangle,\end{aligned}$$

since $C_0(K)^\perp$ is a closed ideal of B^*. It follows that $(\mu_1+m_1)\bigtriangleup(\mu_2+m_2) = T_{(\mu_1 D(\mu_2))}$. By a similar argument, $\bigtriangleup(\mu_2+m_2)(\mu_1+m_1) = T_{(D(\mu_2)\mu_1)}$. Therefore,

$$\begin{aligned}\bigtriangleup\big[(\mu_2+m_2)(\mu_1+m_1)\big] &= T_{D(\mu_2*\mu_1)} = T_{[D(\mu_2)\mu_1+\mu_2 D(\mu_1)]}\\ &= \bigtriangleup(\mu_2+m_2)(\mu_1+m_1) + (\mu_1+m_1)\bigtriangleup(\mu_2+m_2).\end{aligned}$$

It follows that $\bigtriangleup$ is a derivation and that $\bigtriangleup|_{M(K)} = D$. We prove that $\bigtriangleup$ cannot be inner. If $\bigtriangleup$ is inner, then there is an $F \in B^{**}$ such that $\bigtriangleup(G) = GF - FG$, for all $G \in B^*$. If $\Psi := G|_{M(K)}$, then Ψ is an element of $M(K)^*$. Now, for all $\mu \in M(K)$, we have

$$D(\mu) = \bigtriangleup(\mu) = \mu\Psi - \Psi\mu.$$

Hence, D is an inner derivation, and thus it is a contradiction. It follows that B^* is not weakly amenable.

(ii) Here, $L(K)$ has a bounded approximate identity $(e_\alpha)_\alpha$ with $\|e_\alpha\| = 1$ (see [8, Lemma 1]). Let E be a *weak** cluster point of (e_α) in $L(K)^{**}$. It is clear that E is a right identity for $L(K)^{**}$ and $\|E\| = 1$ (see [16, Lemma 5]). The map

$$\varphi : L(K)^{**} \longrightarrow E \,\Box\, L(K)^{**}, \qquad F \longmapsto E \,\Box\, F$$

is an epimorphism. On the other hand, $L(K)^{**}$ is approximately amenable, and therefore $E \,\Box\, L(K)^{**}$ is approximately amenable (see [5, Proposition 2.2]). By Theorems 7 and 4 of [16], $E \,\Box\, L(K)^{**}$ is isometrically isomorphic to $B^* = M(K) \oplus C_0(K)^\perp$, where $C_0(K)^\perp$ is a closed ideal in B^* and $C_0(K)^\perp = \{m \in B^* \mid \text{for all } f \in C_0(K), \langle m, f\rangle = 0\}$. Thus, $M(K)$ is approximately amenable (see [5, Corollary 2.1]).

Now, let $(L(K)^{**}, \Box)$ be weakly amenable, and let $M(K)$ be not weakly amenable. Then, by an argument similar to that of (i), the derivation $\bigtriangleup : B^* \to B^{**}$ is not inner. Now, let E be a right identity of $L(K)^{**}$. We have that $E \,\Box\, L(K)^{**}$ is isometrically isomorphic to B^*; therefore, we may consider $\bigtriangleup$ to be defined on $E \,\Box\, L(K)^{**}$. Now, define $\Lambda : L(K)^{**} \longrightarrow L(K)^{***}$ by $\Lambda(G) = \bigtriangleup(E \,\Box\, G)$, for all $G \in L(K)^{**}$. Since $L(K)^{**} = E \,\Box\, L(K)^{**} + (1-E) \,\Box\, L(K)^{**}$, Λ is a noninner derivation (see [6]). It follows that $L(K)^{**}$ is not weakly amenable, which is a contradiction of the hypothesis. Therefore, $M(K)$ is weakly amenable. $\Box$

Following [16, Definition 8], a compact set $Z \subseteq K$ is called a *compact carrier* for $m \in L(K)^{**}$ if for all $f \in L(K)^*$, $\langle m, f\rangle = \langle m, f\chi_{\rm z}\rangle$, where $f\chi_{\rm z}$ is defined by $\langle f\chi_{\rm z}, \mu\rangle = \langle f, \chi_{\rm z}\mu\rangle$, for all $\mu \in L(K)$. Now let

$$L_c(K)^{**} = \mathrm{cl}_{L(K)^{**}}\big\{m \;\big|\; m \in L(K)^{**}, m \text{ has a compact carrier}\big\}.$$

We now state and prove another interesting theorem.

Theorem 2.8. *Let K be a hypergroup, and let $(L(K)^{**}, \Box)$ be approximately amenable. Then K is discrete and left-amenable. The converse statement is not necessarily true.*

Proof. Let $(e_\alpha)_\alpha$, $\|e_\alpha\| = 1$, be a bounded approximate identity for $L(K)$ (see [16]), and let E be a *weak** cluster point of $(e_\alpha)_\alpha$ in $L(K)^{**}$ (E is also a right identity for $L(K)^{**}$). By hypothesis, $L(K)^{**}$ is approximately amenable. Then $L(K)^{**}$ has a left approximate identity $(F_\alpha)_\alpha$ (see [5, Lemma 2.2]). For each

$m \in L(K)^{**}$, $E \,\square\, m = \lim_\alpha (F_\alpha \,\square\, E) \,\square\, m = \lim_\alpha F_\alpha \,\square\, m = m$. Hence, E is an identity for $L(K)^{**}$ and so $L(K)^* = L(K)^* L(K) = B$ (see [15, Proposition 2.2]). This means that the natural embedding of B into $L(K)^*$ is the identity map and π is also. By Proposition 13(a) and Theorem 14(b) of [16], $M(K) = E \,\square\, L_c(K)^{**}$. Therefore, by Theorem 14(c) of [16], we have

$$M(K) = \bigcap_{E \in \varepsilon_1(K)} E \,\square\, L_c(K)^{**} = L(K).$$

So $M(K) = L(K)$. Thus, K is discrete. Also, by combining Theorems 2.7 and 2.2, K is left-amenable.

To show that the converse is not true, let $\mathbb{N}_0$ be a class of polynomial hypergroups generated by the ultraspherical polynomials $(R_n^\alpha)_{n \in \mathbb{N}_0}$, $\alpha > 0$. Then $\ell^1(\mathbb{N}_0)^{**}$ is not approximately amenable. This is because if $\ell^1(\mathbb{N}_0)^{**}$ is approximately amenable, then $\ell^1(\mathbb{N}_0)$ is approximately amenable (see [5, Theorem 2.3]). But, by Theorem 2.4, this is impossible. □

In [5, Theorem 3.3], it is shown that $L^1(G)^{**}$ is approximately amenable if and only if G is finite. For a hypergroup with an involution, the following theorem shows that this result remains true for approximate amenability.

Theorem 2.9. *Let K be a hypergroup with an involution $\sim\colon K \to K$, and endow $L(K)^{**}$ with the first Arens product. Then the following assertions are equivalent.*

(i) *$(L(K)^{**}, \square)$ is approximately amenable.*
(ii) *K is finite.*
(iii) *$(L(K)^{**}, \square)$ is amenable.*

Proof. (i)⇒(ii) By Theorem 2.8, K is discrete and left-amenable. By hypothesis, K is discrete with an involution. Then Jewett's and Dunkl's definitions of a hypergroup coincide. Therefore, K has a Haar measure and $L^1(K) = L(K)$ (see [11, Theorem 7.1.A]). Also, $TIM(L_\infty(K)) \neq \emptyset$ (topological two-sided invariant mean on $L_\infty(K)$; see [22, Theorem 3.2]). If m is a topological two-sided invariant mean on $L_\infty(K)$, then m is a two-sided invariant mean on $L_\infty(K)$ (see [22, Lemma 3.1]). An argument similar to [5, Theorem 3.3] shows that $|LIM(L_\infty(K))| = |IM(L_\infty(K))| = 1$. Now if K is infinite, this contradicts [22, Corollary 5.6]. Thus K is finite.

(ii)⇒(iii) Since K is a finite hypergroup, $L(K)^{**} = M(K) = L(K)$ and K has a Haar measure. Now the mapping $T : M(K) \longrightarrow B(L^2(K))$ with $\mu \longmapsto T_\mu$ is defined in [11, Theorem 6.2I], where for all $f \in L^2(K)$

$$T_\mu(f) = \mu * f$$

is a faithful norm-decreasing unital $*$-representation of $M(K)$. We have that $L(K)^{**} = M(K)$ is $*$-semisimple and so it is semisimple (see [3, Theorem 3.1.17, p. 347]). Now, by the Wedderburn structure theorem (see [3, Theorem 1.5.9]), $L(K)^{**}$ is amenable.

(iii)⇒(i) This implication is trivial. □

3. A characterization of left amenability of a hypergroup

In this section, we first show that $L(K)$ is an F-algebra. Consider the product linear space $\prod_{\mu\in L(K)} L_\infty(|\mu|)$. Denote by $L_\infty(K, L(K))$ the linear subspace of all $f = (f_\mu)_\mu \in \prod_{\mu\in L(K)} L_\infty(|\mu|)$ such that

(i) $\|f\|_\infty := \sup_{\mu\in L(K)} \|f_\mu\|_{\infty,|\mu|} < \infty$,

(ii) if $\mu, \nu \in L(K)$ and $\mu \ll \nu$, then $f_\nu = f_\mu$, $|\mu|$-almost everywhere,

where $\|g\|_{\infty,|\mu|}$ denotes the essential supremum norm with respect to $|\mu|$.

Theorem 3.1. *For each $F \in L(K)^*$, there is a unique $f = (f_\mu)_\mu \in L_\infty(K, L(K))$ such that*

$$F(\mu) = \int f_\mu \, d\mu.$$

Moreover, $\|F\| = \|f\|_\infty$.

Proof. For each $\mu \in L(K)$, $F_\mu := F|_{L^1(|\mu|)}$ is a bounded linear functional F_μ on $L^1(|\mu|)$. Hence, by the Radon–Nikodym theorem, there is a function $f_\mu \in L_\infty(|\mu|) = L^1(|\mu|)^*$ such that for any $\nu \in L^1(|\mu|)$, we have

$$F(\nu) = F_\mu(\nu) = \int f_\mu \, d\nu.$$

In particular, $F(\mu) = \int f_\mu \, d\mu$. We claim that $f = (f_\mu)_{\mu\in L(K)} \in L_\infty(K, L(K))$. Let $\mu, \nu \in L(K)$ and $\mu \ll \nu$. We have

$$\int f_\mu \, d\mu = F_\mu(\mu) = F_\nu(\mu) = \int f_\nu \, d\mu.$$

Therefore, $f_\mu = f_\nu$ $|\mu|$-almost everywhere.

On the other hand, for each $\mu \in L(K)$,

$$\begin{aligned} \|f_\mu\|_{\infty,\mu} &= \|F_\mu\| = \sup\bigl\{\bigl|F_\mu(\nu)\bigr| : \nu \in L^1\bigl(|\mu|\bigr), \|\nu\| \le 1\bigr\} \\ &= \sup\bigl\{\bigl|F(\nu)\bigr| : \nu \in L^1\bigl(|\mu|\bigr), \|\nu\| \le 1\bigr\} \\ &\le \|F\|. \end{aligned}$$

Hence, $\|f\|_\infty \le \|F\|$. It follows that $f = (f_\mu)_{\mu\in L(K)} \in L_\infty(K, L(K))$. Also

$$\begin{aligned} \|F\| &= \sup\bigl\{|F(\mu)| : \mu \in L(K)), \|\mu\| \le 1\bigr\} \\ &= \sup\Bigl\{\Bigl|\int f_\mu \, d\mu\Bigr| : \mu \in L(K)), \|\mu\| \le 1\Bigr\} \\ &\le \sup_{\|\mu\|\le 1} \|f_\mu\|_{\infty,|\mu|}\|\mu\| \le \sup_{\mu\in L(K)} \|f_\mu\|_{\infty,|\mu|} = \|f\|_\infty. \end{aligned}$$

Thus, $\|F\| = \|f\|_\infty$.

To show uniqueness, let $f, g \in L_\infty(K, L(K))$ be such that for each $\mu \in L(K)$,

$$F(\mu) = \int f_\mu \, d\mu = \int g_\mu \, d\mu.$$

For each $\nu \ll \mu$, we have

$$\int f_\mu \, d\nu = \int f_\nu \, d\nu = \int g_\nu \, d\nu = \int g_\mu \, d\nu.$$

Therefore, $f_\mu = g_\mu$ in $L_\infty(|\mu|)$. This means that $f = g$. $\square$

We now state and prove another interesting theorem.

Theorem 3.2. *Let K be a hypergroup. Then $L(K)$ is an F-algebra.*

Proof. Let $T : L_\infty(K, L(K)) \longrightarrow L(K)^*$ be defined by

$$T(f)(\mu) = \int f_\mu \, d\mu, \quad \big(f \in L_\infty\big(K, L(K)\big), \mu \in L(K)\big).$$

First, we show that T is an isometric isomorphism of $L_\infty(K, L(K))$ onto $L(K)^*$. Let $\mu, \nu \in L(K)$. Without loss of generality, we can suppose that $\mu, \nu \geq 0$. Then $\mu \ll \mu + \nu$, $\nu \ll \mu + \nu$, and $\mu \ll \alpha\mu$, for all $\alpha \geq 0$. Therefore, by Theorem 3.1, for $f \in L_\infty(K, L(K)), \alpha \geq 0$,

$$\begin{aligned} T(f)(\mu + \nu) &= \int f_{\mu+\nu} \, d(\mu + \nu) \\ &= \int f_{\mu+\nu} \, d\mu + \int f_{\mu+\nu} \, d\nu \\ &= \int f_\mu \, d\mu + \int f_\nu \, d\nu = T(f)(\mu) + T(f)(\nu) \end{aligned}$$

and

$$T(f)(\alpha\mu) = \int f_{\alpha\mu} \, d(\alpha\mu) = \alpha \int f_{\alpha\mu} \, d\mu = \alpha \int f_\mu \, d\mu = \alpha T(f)(\mu).$$

Thus, $T(f)$ is a linear functional and then $T(f) \in L(K)^*$. Also, for any $\mu \in L(K)$

$$\big|T(f)(\mu)\big| = \left|\int f_\mu \, d\mu\right| \leq \|f_\mu\|_{\infty,|\mu|} \|\mu\|,$$

and hence, $\|T(f)\| \leq \|f\|_\infty$. Theorem 3.1 shows that T is onto and hence it is an isometry. On the other hand, by Exercise 1.1 and Example 2.1.4 of [19], $L_\infty(K, L(K))$ with the complex conjugation as an involution, the pointwise multiplications, and the norm $\|\cdot\|_\infty$, is a commutative C^*-algebra. Also, the constant function 1 is as in the identity. It follows that $L(K)^*$ is a W^*-algebra. Therefore, $L(K)$ is an F-algebra. $\square$

In Section 2, Theorem 2.4 indicates that, unlike the group case, the converse of Theorem 2.2 is not true for hypergroups. We restrict our discussion to φ-approximate amenability of $L(K)$ and character amenability of $L(K)^{**}$. In Theorem 3.5, however, by using Theorem 3.2, we will provide a characterization of left amenability of the hypergroup K.

Let $\triangle(L(K))$ be the set of all nonzero multiplicative linear functionals on $L(K)$. If $\varphi \in \triangle(L(K))$ and X is an arbitrary Banach space, then X can be viewed as a Banach left $L(K)$-module by the following actions. For $\mu \in L(K), x \in X$,

$\mu \bullet x = \varphi(\mu)x$. Throughout, by a $(\varphi, L(K))$-bimodule X, we mean that X is a Banach $L(K)$-bimodule for which the left module action is given by $\mu \bullet x = \varphi(\mu)x$.

We recall the definitions of φ-amenability and φ-approximate amenability (see [12]).

Definition 3.3. Let K be a hypergroup and $\varphi \in \triangle(L(K))$. Then $L(K)$ is called *φ-amenable* (resp., *approximately φ-amenable*) if every derivation D from $L(K)$ into the dual $L(K)$-bimodule X^* is inner (resp., approximately inner) for all $(\varphi, L(K))$-bimodules X.

Lemma 3.4. *Let K be a hypergroup, and let $F \in L(K)^*$ and $\mu, \nu \in L(K)$. Then*
 (i) $\langle F, \mu * \nu\rangle = \int \langle F, \delta_x * \nu\rangle \, d\mu$,
 (ii) $\langle F, \mu * \nu\rangle = \int \langle F, \mu * \delta_x\rangle \, d\nu$.

Proof. (i) Let $\nu \geq 0$, and we may assume that $C := \operatorname{supp} \mu$ is compact. Then $\phi : C \longrightarrow L(K)$ is defined by $\phi(x) = \delta_x * \nu$ and it is continuous. Thus, by [20, Theorems 3.20, 3.27], we can write $\int_C \varphi(x)\, d\mu \in L(K)$, that is, $\int_C \delta_x * \nu \, d\mu(x) \in L(K)$. On the other hand, for each $\psi \in C_0(K)$

$$\begin{aligned}\mu * \nu(\psi) &= \int_K \int_K \psi(x * y)\, d\mu(x)\, d\nu(y) = \int_K \int_K \psi(x * y)\, d\nu(y)\, d\mu(x) \\ &= \int_C \delta_x * \nu(\psi)\, d\mu(x).\end{aligned}$$

Hence,

$$\mu * \nu = \int_C \delta_x * \nu \, d\mu(x).$$

If $F \in L(K)^*$, then (see [20, Theorem 3.26])

$$\langle F, \mu * \nu\rangle = \left\langle F, \int_C \delta_x * \nu \, d\mu(x)\right\rangle = \int_C \langle F, \delta_x * \nu\rangle \, d\mu(x).$$

Finally, if (e_α) is a positive approximate identity of norm 1, then

$$\langle \nu F, \mu * e_\alpha\rangle = \int_C \langle \nu F, \delta_x * e_\alpha\rangle \, d\mu(x).$$

Hence, we have $\langle F, \mu * \nu\rangle = \int_C \langle F, \delta_x * \nu\rangle \, d\mu(x)$. We can now release the condition on ν.

(ii) Let $\mu \geq 0$, and we may assume that $C := \operatorname{supp} \nu$ is compact. Then $\phi : C \longrightarrow L(K)$ is defined by $\phi(x) = \mu * \delta_x$ and it is continuous. Now, proceeding exactly as above, we have

$$\langle F, \mu * \nu\rangle = \int \langle F, \mu * \delta_x\rangle \, d\nu. \qquad \square$$

We now give a characterization of left amenability of a hypergroup.

Theorem 3.5. *Let K be a hypergroup, and let $\varphi \in \triangle(L(K)$. Then the following assertions are equivalent.*

 (i) *$L(K)$ is approximately φ-amenable.*

(ii) K *is left-amenable.*
(iii) $L(K)$ *is φ-amenable.*

Proof. (i)⇒(ii) Let $L(K)$ be approximately φ-amenable. Then $X := L(K)$ is a Banach $(\varphi, L(K))$-bimodule with the right module action $\nu \cdot \mu := \nu * \mu$, for $\nu \in X, \mu \in L(K)$. Hence X^*, with dual module action, is a Banach $L(K)$-bimodule. Now, since $\varphi \in X^*$,

$$\langle \varphi \cdot \mu, \nu \rangle = \langle \varphi, \mu \bullet \nu \rangle = \big\langle \varphi, \varphi(\mu)\nu \big\rangle = \varphi(\mu)\langle \varphi, \nu \rangle$$

and

$$\langle \mu \cdot \varphi, \nu \rangle = \langle \varphi, \nu \cdot \mu \rangle = \langle \varphi, \nu * \mu \rangle = \varphi(\nu * \mu) = \varphi(\nu)\varphi(\mu) = \varphi(\mu)\langle \varphi, \nu \rangle,$$

for $\mu \in L(K), \nu \in X$. Thus, $\mu \cdot \varphi = \varphi(\mu)\varphi = \varphi \cdot \mu$. On the other hand, the space $\mathbb{C}$ is a Banach $(\varphi, L(K))$-sub-bimodule of X^*. So, $Y := \frac{X^*}{\mathbb{C}}$ is a Banach $L(K)$-bimodule. Let $\theta : X^* \longrightarrow Y$ be the canonical mapping, and let $n \in X^{**}$ with $n(\varphi) = 1$. Then, for $\mu \in L(K)$,

$$\langle \mu \cdot n - n \cdot \mu, \varphi \rangle = \langle \mu \cdot n, \varphi \rangle - \langle n \cdot \mu, \varphi \rangle = \langle n, \varphi \cdot \mu \rangle - \langle n, \mu \cdot \varphi \rangle = 0.$$

It follows that $\mu \cdot n - n \cdot \mu$ can be considered as an element of $\theta^*(Y^*)$, where θ^* is the adjoint of θ. Since θ^* is injective, we can define $D : L(K) \longrightarrow Y^*$ such that $\theta^* \circ D(\mu) = \mu \cdot n - n \cdot \mu$. It is easy to see D is a bounded derivation on $L(K)$. By the assumption, there exists a net $(\phi_\alpha) \subseteq Y^*$ such that

$$D(\mu) = \lim_\alpha (\phi_\alpha \cdot \mu - \mu \cdot \phi_\alpha), \quad \big(\mu \in L(K)\big).$$

Therefore,

$$\lim_\alpha \big(\big(\theta^*(\phi_\alpha) \cdot \mu - \mu \cdot \theta^*(\phi_\alpha)\big)\big) = \lim_\alpha \big(\big(\theta^*(\phi_\alpha \cdot \mu - \mu \cdot \phi_\alpha)\big) = \theta^*\big(D(\mu)\big) = \mu \cdot n - n \cdot \mu.$$

So, we have

$$\mu \cdot \big(\theta^*(\phi_\alpha) - n\big) = \big(\theta^*(\phi_\alpha) - n\big) \cdot \mu.$$

Define $n_\alpha := (n - \theta^*(\phi_\alpha)) \in L(K)^{**}$, for all α. Therefore,

$$\langle n_\alpha, \varphi \rangle = \langle n, \varphi \rangle - \big\langle \theta^*(\phi_\alpha), \varphi \big\rangle = \langle n, \varphi \rangle - \big\langle \phi_\alpha, \theta(\varphi) \big\rangle = 1 - 0 = 1.$$

Also, if n_α and $\mu \in L(K)$, then $n_\alpha \cdot \mu = \mu \cdot n_\alpha$. Hence, we have

$$\langle f, n_\alpha \cdot \mu \rangle = \langle f, \mu \cdot n_\alpha \rangle = \langle f \cdot \mu, n_\alpha \rangle = \varphi(\mu)\langle f, n_\alpha \rangle,$$

for $f \in L(K)^*$. On the other hand, by Theorem 3.2, $L(K)^*$ is a W^*- algebra. So, if $\varphi(\mu) = 1$, then $\mu \cdot n_\alpha = n_\alpha \cdot \mu = n_\alpha$ and $\mu \cdot n_\alpha^* = n_\alpha^* \cdot \mu = n_\alpha^*$ (see [14, Theorem 4.1]). Thus, we can assume that n_α is self-adjoint. Let $n_\alpha = n_\alpha^+ - n_\alpha^-$ be the orthogonal decomposition of n_α. Then $n_\alpha \cdot \mu = n_\alpha^+ \cdot \mu - n_\alpha^- \cdot \mu$ and

$$\begin{aligned} \|n_\alpha^+ \cdot \mu\| + \|n_\alpha^- \cdot \mu\| &= \langle n_\alpha^+ \cdot \mu, \varphi \rangle + \langle n_\alpha^- \cdot \mu, \varphi \rangle = \langle n_\alpha^+, \mu \cdot \varphi \rangle + \langle n_\alpha^-, \mu \cdot \varphi \rangle \\ &= \varphi(\mu)\langle n_\alpha^+, \varphi \rangle + \varphi(\mu)\langle n_\alpha^-, \varphi, \rangle = \langle n_\alpha^+, \varphi \rangle + \langle n_\alpha^-, \varphi \rangle \\ &= \|n_\alpha^+\| + \|n_\alpha^-\|. \end{aligned}$$

Hence, $\mu \cdot n_\alpha^+ = n_\alpha^+ \cdot \mu = n_\alpha^+$, $\mu \cdot n_\alpha^- = n_\alpha^- \cdot \mu = n_\alpha^-$ (see [21, Theorem 1.14.3], the Jordan decomposition theorem), and n_α^+, n_α^- cannot both be zero . Without loss of generality, we assume that $n_\alpha^+ \neq 0$, for all α. Now, let $m_\alpha = \frac{1}{\|n_\alpha^+\|} n_\alpha^+$ ($\|m_\alpha\| = 1$), and let m be a *weak** cluster point of $(m_\alpha)_\alpha$. It is clear that m is a mean on X^*. Take $P_1(L(K)) := \{\mu \in L(K) | \mu \geq 0, \|\mu\| = 1\}$. For $f \in X^*$ and $\mu \in P_1(L(K))$, we have

$$\begin{aligned}\langle m, f\mu\rangle &= \lim_\alpha \langle m_\alpha, f\mu\rangle = \lim_\alpha \langle \mu \cdot m_\alpha, f\rangle = \lim_\alpha \frac{1}{\|n_\alpha^+\|}\langle \mu \cdot n_\alpha^+, f\rangle \\ &= \lim_\alpha \frac{1}{\|n_\alpha^+\|}\langle n_\alpha^+, f\rangle = \lim_\alpha \langle m_\alpha, f\rangle = \langle m, f\rangle.\end{aligned}$$

It follows that m is a topologically left-invariant mean on $L(K)^*$ because the linear span of $P_1(L(K))$ is $L(K)$. Now, let $\widetilde{m} = m|_B$ ($B = L(K)^* L(K)$). Then we have

$$\langle m, f \cdot \delta_x\rangle = \big\langle m, (g\mu)\cdot \delta_x\big\rangle = \big\langle m, g(\mu * \delta_x)\big\rangle = \langle m, g\rangle = \langle m, g\mu\rangle = \langle m, f\rangle,$$

for $f = g\mu \in B$ ($g \in L(K)^*, \mu \in L(K)$) and $x \in K$. Hence, $\widetilde{m}$ is a left-invariant mean on B. Thus, K is left-amenable.

(ii)⇒(iii) Suppose that m is a left-invariant mean on B. Let $\nu \in P_1(L(K))$ and $f = F\mu \in B$ ($F \in L(K)^*, \mu \in L(K)$). By Lemma 3.4(ii), we have

$$\begin{aligned}\langle m, f\cdot\nu\rangle &= \big\langle m, (F\mu)\cdot\nu\big\rangle = \big\langle m, F(\mu * \nu)\big\rangle \\ &= \langle m\cdot F, \mu * \nu\rangle = \int \langle m \cdot F, \mu * \delta_x\rangle \, d\nu \\ &= \int \big\langle m, F\cdot(\mu * \delta_x)\big\rangle \, d\nu = \int \big\langle m, (F\mu)\cdot \delta_x)\big\rangle \, d\nu \\ &= \int \langle m, f\cdot\delta_x)\rangle \, d\nu = \int \langle m, f\rangle \, d\nu = \langle m, f\rangle \nu(K) = \langle m, f\rangle.\end{aligned}$$

Now, let $\mu_0 \in P_1(L(K))$ be fixed. For $F \in L(K)^*$, define $f(x) := \langle F, \delta_x * \mu_0\rangle$. It is clear that $f \in C_b(K)$ and $f\cdot\mu \in C_b(K)$, for $\mu \in L(K)$ (see [16]). So, for all $\nu \in \mathrm{Ball}(L(K))$ and $x \in K$, we have

$$\begin{aligned}f_\nu(x) &= f(\nu * \delta_x) = \int f(t)\, d(\nu * \delta_x)(t) = \int \langle F, \delta_t * \mu_0\rangle \, d(\nu * \delta_x)(t) \\ &= \langle F, \nu * \delta_x * \mu_0\rangle.\end{aligned}$$

Hence, if $\nu \in \mathrm{Ball}(L(K))$ and $x, y \in K$, then

$$\begin{aligned}\big|f_\nu(x) - f_\nu(y)\big| &= \big|\langle F\cdot\nu, \delta_x * \mu_0 - \delta_y * \mu_0\rangle\big| \\ &= \big|\langle F, \nu * \delta_x * \mu_0 - \nu * \delta_y * \mu_0\rangle\big| \leq \|F\|\|\delta_x * \mu_0 - \delta_y * \mu_0\|.\end{aligned}$$

It follows that $\{f_\nu | \nu \in \mathrm{Ball}(L(K))\}$ is equicontinuous, and consequently, $f \in B$ (see [16, Proposition 2]). Now, define $M \in L(K)^{**}$ with $M(F) = m(f)$, for any $F \in L(K)^*$. By Lemma 3.4, if $\nu \in P_1(L(K))$, then

$$\begin{aligned}f\cdot\nu(x) &= f(\nu * \delta_x) = \int f(t)\, d(\nu * \delta_x)(t) = \int \langle F, \delta_t * \mu_0\rangle\, d(\nu * \delta_x)(t) \\ &= \langle F, \nu * \delta_x * \mu_0\rangle = \langle F\cdot\nu, \delta_x * \mu_0\rangle.\end{aligned}$$

Therefore, $M(F\cdot\nu)=m(f\cdot\nu)=m(f)=M(F)$, for $\nu\in L(K)$ and $F\in L(K)^*$. It follows that $L(K)$ is φ-amenable (see [12, Theorem 1.1]).

(iii)⇒(i) This implication is trivial. □

Definition 3.6. Let A be a Banach algebra. Then A is *right-* (resp., *left-*) *character-amenable* if for every $\varphi\in\triangle(A)\cup\{0\}$ and every (φ,A)-bimodule (resp., (A,φ)-bimodule) E, every derivation $D:A\longrightarrow E^*$ is inner. Also, A is character-amenable if it is both left- and right-character-amenable.

Theorem 3.7. *Let K be a hypergroup, and let $(L(K)^{**},\Box)$ be character-amenable. Then K is finite.*

Proof. Since $L(K)^{**}$ is left-character-amenable, $L(K)^{**}$ has a left bounded approximate identity (see [10, Corollary 2.5]). By an argument similar to that in the proof of Theorem 2.8, K is discrete. Thus, $L_c(K)^{**}=M(K)$ and $L(K)^{**}=M(K)^{**}=B^*=M(K)\bigoplus C_0(K)^{\perp}$ (see [16, Theorem 14]). On the other hand, the map

$$\theta:M(K)\oplus C_0(K)^{\perp}\longrightarrow C_0(K)^{\perp},\qquad \mu\oplus m\longmapsto m\qquad \big(m\in C_0(K)^{\perp},\mu\in M(K)\big)$$

is an epimorphism. So, by Lemma 2.12 of [10], $C_0(K)^{\perp}$ is left- and right-character-amenable. Thus, $C_0(K)^{\perp}$ has a left bounded approximate identity $(e_\alpha)_\alpha$ and a right bounded approximate identity $(g_\alpha)_\alpha$ (see [10, Corollary 2.5]). Let $e_\alpha\longrightarrow e$ and $g_\alpha\longrightarrow g$ in *weak**-topology $\sigma(L(K)^{**},L(K)^*)$. Now, since (e_α) is a left bounded approximate identity, for each $m\in C_0(K)^{\perp}$, $e_\alpha\,\Box\,m\longrightarrow e\,\Box\,m$ in *weak**-topology $\sigma(L(K)^{**},L(K)^*)$ and $e_\alpha\,\Box\,m\longrightarrow m$ in the norm topology. Thus, e is a left identity for $C_0(K)^{\perp}$. Since (g_α) is a right bounded approximate identity, $eg_\alpha\to e$ in norm. But $eg_\alpha=g_\alpha$, so that $eg_\alpha\to e$ in norm. Therefore, $e=g$ is an identity for $C_0(K)^{\perp}$. By an argument similar to that in the proof of Theorem 2.5 of [18], K is compact. It follows that K is finite. □

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References

1. M. Amini and A. R. Medghalchi, *Amenability of compact hypergroup algebras*, Math. Nachr **287** (2014), no. 14–15, 1609–1617. Zbl 1304.43003. MR3266126. DOI 10.1002/mana.201200284. 556
2. W. R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, De Gruyter Stud. Math. **20**, De Gruyter, Berlin, 1995. Zbl 0828.43005. MR1312826. DOI 10.1515/9783110877595. 553, 556
3. H. G. Dales, *Banach Algebras and Automatic Continuity*, London Math. Soc. Monogr. (N.S.) **24**, Oxford Univ. Press, New York, 2000. Zbl 0981.46043. MR1816726. 555, 556, 558
4. C. F. Dunkl, *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. **179** (1973), 331–348. Zbl 0241.43003. MR0320635. DOI 10.2307/1996507. 552
5. F. Ghahramani and R. J. Loy, *Generalized notions of amenability*, J. Funct. Anal. **208** (2004), no. 1, 229–260. Zbl 1045.46029. MR2034298. DOI 10.1016/S0022-1236(03)00214-3. 552, 553, 554, 555, 557, 558

6. F. Ghahramani, R. J. Loy, and G. A. Willis, *Amenability and weak amenability of second conjugate Banach algebras*, Proc. Amer. Math. Soc. **124** (1996), no. 5, 1489–1497. Zbl 0851.46035. MR1307520. DOI 10.1090/S0002-9939-96-03177-2. 552, 556, 557
7. F. Ghahramani, R. J. Loy, and Y. Zhang, *Generalized notions of amenability, II*, J. Funct. Anal. **254** (2008), no. 7, 1776–1810. Zbl 1146.46023. MR2397875. DOI 10.1016/j.jfa.2007.12.011. 552
8. F. Ghahramani and A. R. Medghalchi, *Compact multipliers on weighted hypergroup algebras*, Math. Proc. Cambridge Philos. Soc. **98** (1985), no. 3, 493–500. Zbl 0584.43004. MR0803608. DOI 10.1017/S0305004100063696. 552, 557
9. F. Ghahramani and Y. Zhang, *Pseudo-amenable and pseudo-contractible Banach algebras*, Math. Proc. Cambridge Philos. Soc. **142** (2007), no. 1, 111–123. Zbl 1118.46046. MR2296395. DOI 10.1017/S0305004106009649. 555
10. Z. Hu, M. S. Monfared, and T. Traynor, *On character amenable Banach algebras*, Studia Math. **193** (2009), no. 1, 53–78. Zbl 1175.22005. MR2506414. DOI 10.4064/sm193-1-3. 553, 564
11. R. I. Jewett, *Spaces with an abstract convolution of measures*, Adv. Math. **18** (1975), no. 1, 1–101. Zbl 0325.42017. MR0394034. DOI 10.1016/0001-8708(75)90002-X. 558
12. E. Kaniuth, A. T. Lau, and J. Pym, *On φ-amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc. **144** (2008), no. 1, 85–96. Zbl 1145.46027. MR2388235. DOI 10.1017/S0305004107000874. 553, 561, 564
13. R. Lasser, *Various amenability properties of the L^1-algebra of polynomial hypergroups and applications*, J. Comput. Appl. Math. **233** (2009), no. 3, 786–792. Zbl 1182.43008. MR2583017. DOI 10.1016/j.cam.2009.02.046. 555
14. A. T. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups*, Fund. Math. **118** (1983), no. 3, 161–175. Zbl 0545.46051. MR0736276. DOI 10.4064/fm-118-3-161-175. 553, 562
15. A. T. Lau and A. Ülger, *Topological centers of certain dual algebras*, Trans Amer. Math. Soc. **394**, no. 3 (1996), 1191–1212. Zbl 0859.43001. MR1322952. DOI 10.1090/S0002-9947-96-01499-7. 558
16. A. R. Medghalchi, *The second dual algebra of a hypergroup*, Math. Z. **210** (1992), no. 4, 615–624. Zbl 0755.43001. MR1175726. DOI 10.1007/BF02571818. 552, 553, 557, 558, 563, 564
17. A. R. Medghalchi, *Cohomology on hypergroup algebras*, Studia Sci. Math. Hungar. **39** (2002), no. 3-4, 297–307. Zbl 1026.43001. MR1956941. DOI 10.1556/SScMath.39.2002.3-4.4. 552
18. A. R. Medghalchi and S. M. S. Modarres, *Amenability of the second dual of hypergroup algebras*, Acta Math. Hungar. **86** (2000), no. 4, 335–342. Zbl 0970.46030. MR1756256. DOI 10.1023/A:1006775726657. 564
19. G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, Boston, 1990. Zbl 0714.46041. MR1074574. 560
20. W. Rudin, *Functional Analysis*, 2nd ed., Internat. Ser. Pure Appl. Math., McGraw-Hill, New York, 1991. Zbl 0867.46001. MR1157815. 561
21. S. Sakai, C^*-algebras and W^*-algebras, Ergeb. Math. Grenzgeb. (3) **60**, Springer, New York, 1971. Zbl 0219.46042. MR0442701. 562
22. M. Skantharajah, *Amenable hypergroups*, Illinois J. Math. **36** (1992), no. 1, 15–46. Zbl 0755.43003. MR1133768. 553, 555, 558

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