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BEREZIN TRANSFORM OF THE ABSOLUTE VALUE OF AN OPERATOR

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Abstract. In this article, we concentrate on the Berezin transform of the absolute value of a bounded linear operator T defined on the Bergman space $L^2_a(\mathbb{D})$ of the open unit disk. We establish some sufficient conditions on T which guarantee that the Berezin transform of $|T|$ majorizes the Berezin transform of $|T^*|$. We have shown that T is self-adjoint and $T^2 = T^3$ if and only if there exists a normal idempotent operator S on $L^2_a(\mathbb{D})$ such that $\rho(T) = \rho(|S|^2) = \rho(|S^*|^2)$, where $\rho(T)$ is the Berezin transform of T. We also establish that if T is compact and $|T^n| = |T|^n$ for some $n \in \mathbb{N}$, $n \neq 1$, then $\rho(|T^n|) = \rho(|T|^n)$ for all $n \in \mathbb{N}$. Further, if $T = U|T|$ is the polar decomposition of T, then we present necessary and sufficient conditions on T such that $|T|^{1/2}$ intertwines with U and a contraction X belonging to $\mathcal{L}(L^2_a(\mathbb{D}))$.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $dA(z) = \frac{1}{\pi}\, dx\, dy$ denote the normalized Lebesgue area measure on $\mathbb{D}$ in the complex plane $\mathbb{C}$. For $1 \leq p < \infty$ and $f : \mathbb{D} \longrightarrow \mathbb{C}$ Lebesgue measurable, let $\|f\|_p = (\int_{\mathbb{D}} |f|^p\, dA)^{1/p}$. The Bergman space $L^p_a(\mathbb{D})$ is the Banach space of analytic functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ such that $\|f\|_p < \infty$. The Bergman space $L^2_a(\mathbb{D})$ is a Hilbert space; it is a closed subspace (see [4]) of the Hilbert space $L^2(\mathbb{D}, dA)$, with the inner product given by $\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z)$, $f, g \in L^2(\mathbb{D}, dA)$. Let P denote the orthogonal projection of

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$\overline{L^2(\mathbb{D}, dA)}$ onto $L_a^2(\mathbb{D})$. Let $K(z, \bar{w})$ be the function on $\mathbb{D}\times\mathbb{D}$ defined by $K(z, \bar{w}) = \overline{K_z(w)} = \frac{1}{(1-z\bar{w})^2}$. The function $K(z, \bar{w})$ is called the *reproducing kernel* of $L_a^2(\mathbb{D})$. For any $n \geq 0, n \in \mathbb{Z}$, let $e_n(z) = \sqrt{n+1}z^n$; then $\{e_n\}$ forms an orthonormal basis for $L_a^2(\mathbb{D})$. Let $k_a(z) = \frac{K(z,\bar{a})}{\sqrt{K(a,\bar{a})}} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$. These functions k_a are called the *normalized reproducing kernels* of $L_a^2(\mathbb{D})$; it is clear that they are unit vectors in $L_a^2(\mathbb{D})$. Let $L^\infty(\mathbb{D}, dA)$ be the Banach space of all essentially bounded measurable functions f on $\mathbb{D}$ with $\|f\|_\infty = \operatorname{ess\,sup}\{|f(z)| : z \in \mathbb{D}\}$, and let $H^\infty(\mathbb{D})$ be the space of bounded analytic functions on $\mathbb{D}$. Let $\mathcal{L}(H)$ be the space of all bounded linear operators from the separable Hilbert space H into itself, and let $\mathcal{LC}(H)$ be the space of all compact operators in $\mathcal{L}(H)$. An operator $A \in \mathcal{L}(H)$ is called *positive* if $\langle Ax, x\rangle \geq 0$ holds for every $x \in H$, in which case we write $A \geq 0$. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| = (A^*A)^{1/2}$. If H is infinite-dimensional, then the map $|\cdot|$ on $\mathcal{L}(H)$ is not Lipschitz-continuous. We define $\rho : \mathcal{L}(L_a^2(\mathbb{D})) \longrightarrow L^\infty(\mathbb{D})$ by $\rho(T)(z) = \widetilde{T}(z) = \langle Tk_z, k_z\rangle, z \in \mathbb{D}$. A function $g(x, \bar{y})$ on $\mathbb{D}\times\mathbb{D}$ is called *of positive type* (or *positive definite*), written $g \gg 0$, if

$$\sum_{j,k=1}^{n} c_j\overline{c_k}g(x_j, \overline{x_k}) \geq 0 \tag{1.1}$$

for any n-tuple of complex numbers $c_1, \dots, c_n$ and points $x_1, \dots, x_n \in \mathbb{D}$. We write $g \gg h$ if $g - h \gg 0$. We say that $\Upsilon \in \mathcal{A}$ if $\Upsilon \in L^\infty(\mathbb{D})$, and it is such that

$$\Upsilon(z) = \Theta(z, \bar{z}), \tag{1.2}$$

where $\Theta(x, \bar{y})$ is a function on $\mathbb{D}\times\mathbb{D}$, meromorphic in x and conjugate-meromorphic in y and there exists a constant $c > 0$ such that

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0 \quad \text{for all } x, y \in \mathbb{D}.$$

The function Θ given in (1.2), if it exists, is uniquely determined by Υ. (For more details, see [8] and [10].)

2. Majorization of Berezin transform

In this section, we present certain sufficient conditions on $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ which guarantee that the Berezin transform of $|T|$ majorizes the Berezin transform of $|T^*|$.

Theorem 2.1. *If $\phi \in \mathcal{A}$ and $0 \leq \phi$, then there exists a positive operator $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\phi(z) = \widetilde{S}(z)$ for all $z \in \mathbb{D}$.*

Proof. To prove the theorem, it suffices to show that $0 \leq \phi \in \mathcal{A}$ if and only if there exists a positive operator $S \in \mathcal{L}(L_a^2)$ such that $\phi(z) = \langle Sk_z, k_z\rangle$ for all $z \in \mathbb{D}$. So let $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ be a positive operator. Let $\Theta(x, \bar{y}) = \frac{\langle SK_y, K_x\rangle}{\langle K_y, K_x\rangle}$, where $K_x = K(\cdot, \bar{x})$ is the unnormalized reproducing kernel at x. Then $\Theta(x, \bar{y})$ is a function on $\mathbb{D}\times\mathbb{D}$, meromorphic in x, and conjugate-meromorphic in y. Let $\phi(z) = \Theta(z, \bar{z})$.

Then $\phi(z) = \langle Sk_z, k_z\rangle$ for all $z \in \mathbb{D}$ and $\phi \in L^\infty(\mathbb{D})$, as S is bounded. Now let $f = \sum_{j=1}^n c_j K_{x_j}$ where c_j are constants and $x_j \in \mathbb{D}$ for $j = 1, 2, \ldots, n$. Since S is bounded and positive, there exists a constant $c > 0$ such that $0 \leq \langle Sf, f\rangle \leq c\|f\|^2$. But

$$\begin{aligned}\langle Sf, f\rangle &= \Big\langle S\Big(\sum_{j=1}^n c_j K_{x_j}\Big), \sum_{j=1}^n c_j K_{x_j}\Big\rangle \\ &= \sum_{j,k=1}^n c_j\overline{c_k}\langle SK_{x_j}, K_{x_k}\rangle \\ &= \sum_{j,k=1}^n c_j\overline{c_k}\Theta(x_k, \overline{x_j})K(x_k, \overline{x_j})\end{aligned}$$

and $c\|f\|^2 = c\langle f, f\rangle = c\sum_{j,k=1}^n c_j\overline{c_k}K(x_k, \overline{x_j})$. Hence we get

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0.$$

Thus $\phi \in \mathcal{A}$.

Now let $\phi \in \mathcal{A}$ and $\phi(z) = \Theta(z, \bar{z})$, where $\Theta(x, \bar{y})$ is a function on $\mathbb{D} \times \mathbb{D}$, meromorphic in x, and conjugate-meromorphic in y. We will prove the existence of a positive, bounded operator $S \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\phi(z) = \langle Sk_z, k_z\rangle$. Let

$$Sf(x) = \int_{\mathbb{D}} f(z)\Theta(x, \bar{z})K(x, \bar{z})\, dA(z). \tag{2.1}$$

Indeed,

$$\begin{aligned}Sf(x) &= \langle Sf, K_x\rangle \\ &= \langle f, S^*K_x\rangle \\ &= \int_{\mathbb{D}} f(z)\overline{\langle S^*K_x, K_z\rangle}\, dA(z) \\ &= \int_{\mathbb{D}} f(z)\langle SK_z, K_x\rangle\, dA(z) \\ &= \int_{\mathbb{D}} f(z)\Theta(x, \bar{z})K(x, \bar{z})\, dA(z).\end{aligned}$$

Then

$$\begin{aligned}\langle SK_y, K_x\rangle &= \int_{\mathbb{D}} K_y(z)\Theta(x, \bar{z})K(x, \bar{z})\, dA(z) \\ &= \int_{\mathbb{D}} K_y(z)\Theta(x, \bar{z})\overline{K_x(z)}\, dA(z) \\ &= \overline{\langle\overline{\Theta(x, \bar{z})}K_x, K_y\rangle} \\ &= \overline{\overline{\Theta(x, \bar{y})}\langle K_x, K_y\rangle} \\ &= \Theta(x, \bar{y})\langle K_y, K_x\rangle.\end{aligned}$$

Hence $\Theta(x, \bar{y}) = \frac{\langle SK_y, K_x\rangle}{\langle K_y, K_x\rangle}$ and $\phi(z) = \Theta(z, \bar{z}) = \langle Sk_z, k_z\rangle$. We will now prove that S is positive and bounded. That is, there exists a constant $c > 0$ such that $0 \leq \langle Sf, f\rangle \leq c\|f\|^2$ for all $f \in L^2_a(\mathbb{D})$. Since $\phi \in \mathcal{A}$, there exists a constant $c > 0$ such that for all $x, y \in \mathbb{D}$,

$$cK(x, \bar{y}) \gg \Theta(x, \bar{y})K(x, \bar{y}) \gg 0. \tag{2.2}$$

Let $f = \sum_{j=1}^{n} c_j K_{x_j}$, where c_j are constants, $x_j \in \mathbb{D}$ for $j = 1, 2, \ldots, n$. Then from (2.2) it follows that $\langle Sf, f\rangle = \sum_{j,k=1}^{n} c_j\overline{c_k}\Theta(x_k, \overline{x_j})K(x_k, \overline{x_j}) \geq 0$ and that

$$\begin{aligned}\langle Sf, f\rangle &= \sum_{j,k=1}^{n} c_j\overline{c_k}\Theta(x_k, \overline{x_j})K(x_k, \overline{x_j})\\ &\leq c\sum_{j,k=1}^{n} c_j\overline{c_k}K(x_k, \overline{x_j})\\ &= c\|f\|^2.\end{aligned}$$

Since the set of vectors $\{\sum_{j=1}^{n} c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, 2, \ldots, n\}$ is dense in $L^2_a(\mathbb{D})$, we have $0 \leq \langle Sf, f\rangle \leq c\|f\|^2$ for all $f \in L^2_a(\mathbb{D})$ and thus S is bounded and positive. □

Theorem 2.2. *Let $T \in \mathcal{L}(L^2_a(\mathbb{D}))$. Then*

$$\left|\langle Tf, g\rangle\right|^2 \leq \big\langle |T|f, f\big\rangle\big\langle |T|g, g\big\rangle, \tag{2.3}$$

where $f, g \in \mathbb{B} = \{\sum_{j=1}^{n} c_j K_{y_j}, c_j, j = 1, 2, \ldots, n$ are constants, $y_j \in \mathbb{D}, j = 1, \ldots, n\}$ if and only if

$$\Theta_{|T|}(x, \bar{y})K(x, \bar{y}) \gg \Theta_{|T^*|}(x, \bar{y})K(x, \bar{y}) \tag{2.4}$$

holds for all $x, y \in \mathbb{D}$. If either (2.3) or (2.4) holds, then $\rho(|T^*|) \leq \rho(|T|)$.

Proof. Let $T \in \mathcal{L}(L^2_a(\mathbb{D}))$. Suppose that (2.3) holds for all $f, g \in \mathbb{B}$. Let $f = \sum_{j=1}^{n} c_j K_{y_j}$, where c_j are constants, $y_j \in \mathbb{D}$, for $j = 1, 2, \ldots, n$ and $g = \sum_{i=1}^{m} d_i K_{x_i}$, where d_i are constants, $x_i \in \mathbb{D}$ for $i = 1, 2, \ldots, m$. Then by (2.3),

$$\left|\langle Tf, g\rangle\right| \leq \big\langle |T|f, f\big\rangle^{1/2}\big\langle |T|g, g\big\rangle^{1/2}.$$

Since the set of vectors $\{\sum c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, 2, \ldots, n\}$ is dense in $L^2_a(\mathbb{D})$, we have

$$\left|\langle Tf, g\rangle\right|^2 \leq \big\langle |T|f, f\big\rangle\big\langle |T|g, g\big\rangle \tag{2.5}$$

for all $f, g \in L^2_a(\mathbb{D})$. It is straightforward to see that (2.5) implies (2.3). Thus (2.3) holds if and only if (2.5) holds. Now suppose that, for all $x, y \in \mathbb{D}$,

$$\Theta_{|T|}(x, \bar{y})K(x, \bar{y}) \gg \Theta_{|T^*|}(x, \bar{y})K(x, \bar{y}).$$

This then implies that $\langle |T|K_y, K_x\rangle \geq \langle |T^*|K_y, K_x\rangle$ for all $x, y \in \mathbb{D}$. Thus

$$\sum_{i,j=1}^{n} c_j\overline{c_i}\big\langle |T|K_{x_j}, K_{x_i}\big\rangle \geq \sum_{i,j=1}^{n} c_j\overline{c_i}\big\langle |T^*|K_{x_j}, K_{x_i}\big\rangle.$$

Hence

$$\Big\langle |T|\Big(\sum_{j=1}^{n} c_j K_{x_j}\Big), \Big(\sum_{i=1}^{n} c_i K_{x_i}\Big)\Big\rangle \geq \Big\langle |T^*|\Big(\sum_{j=1}^{n} c_j K_{x_j}\Big), \Big(\sum_{i=1}^{n} c_i K_{x_i}\Big)\Big\rangle,$$

where $x_1, x_2, \ldots, x_n \in \mathbb{D}$ and $c_j, j = 1, \ldots, n$ are constants. Since

$$\Big\{\sum_{j=1}^{n} c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, \ldots, n\Big\}$$

is dense in $L_a^2(\mathbb{D})$, we have

$$\big\langle |T|g, g\big\rangle \geq \big\langle |T^*|g, g\big\rangle \tag{2.6}$$

for all $g \in L_a^2(\mathbb{D})$. Thus (2.4) implies (2.6). Now suppose that (2.6) holds. Then

$$\Big\langle |T|\Big(\sum_{j=1}^{n} c_j K_{x_j}\Big), \Big(\sum_{i=1}^{n} c_i K_{x_i}\Big)\Big\rangle \geq \Big\langle |T^*|\Big(\sum_{j=1}^{n} c_j K_{x_j}\Big), \sum_{i=1}^{n} c_i K_{x_i}\Big\rangle,$$

where $x_1, \ldots, x_n \in \mathbb{D}$ and $c_j, j = 1, \ldots, n$ are constants. This implies that

$$\sum_{i,j=1}^{n} c_j \overline{c_i} \big\langle |T| K_{x_j}, K_{x_i}\big\rangle \geq \sum_{i,j=1}^{n} c_j \overline{c_i} \big\langle |T^*| K_{x_j}, K_{x_i}\big\rangle.$$

Thus $\langle |T|K_y, K_x\rangle \geq \langle |T^*|K_y, K_x\rangle$ for all $x, y \in \mathbb{D}$. Hence (2.6) implies (2.4). Now we will show that (2.5) holds if and only if (2.6) holds. Let $T = U|T|$ be the polar decomposition of T. Then, since $|T^*| = U|T|U^*$, we obtain

$$\begin{aligned}
\big|\langle Tf, g\rangle\big|^2 &= \big|\big\langle U|T|^{1/2}|T|^{1/2}f, g\big\rangle\big|^2 \\
&= \big|\big\langle |T|^{1/2}f, |T|^{1/2}U^*g\big\rangle\big|^2 \\
&\leq \big\||T|^{1/2}f\big\|^2 \big\||T|^{1/2}U^*g\big\|^2 \\
&= \big\langle |T|f, f\big\rangle \big\langle |T^*|g, g\big\rangle;
\end{aligned}$$

for all $f, g \in L_a^2(\mathbb{D})$. Now if (2.6) holds, then $|\langle Tf, g\rangle|^2 \leq \langle |T|f, f\rangle\langle |T|g, g\rangle$ for all $f, g \in L_a^2(\mathbb{D})$. Thus (2.6) implies (2.5). If (2.5) holds, then we have

$$\begin{aligned}
\big|\big\langle |T^*|f, f\big\rangle\big|^2 &= \big|\big\langle U|T|U^*f, f\big\rangle\big|^2 \\
&= \big|\langle TU^*f, f\rangle\big|^2 \leq \big\langle |T|U^*f, U^*f\big\rangle \big\langle |T|f, f\big\rangle \\
&= \big\langle U|T|U^*f, f\big\rangle \big\langle |T|f, f\big\rangle = \big\langle |T^*|f, f\big\rangle \big\langle |T|f, f\big\rangle.
\end{aligned}$$

Hence $\langle |T^*|f, f\rangle \leq \langle |T|f, f\rangle$ for all $f \in L_a^2(\mathbb{D})$. This also implies that $\widetilde{|T^*|}(z) \leq \widetilde{|T|}(z)$ for all $z \in \mathbb{D}$. That is, $\rho(|T^*|) \leq \rho(|T|)$. $\square$

Lemma 2.3. *If $S, T \in \mathcal{L}(L_a^2(\mathbb{D}))$ are normal and $S^*S = T^*T$, then $\rho(|S|) = \rho(|T|)$.*

Proof. Let $S^*S = T^*T$. Let $S = U|S|$ and $T = V|T|$ be the polar decompositions of S and T. Since S and T are normal, it holds that U and V are unitary operators. Now $S^*S = T^*T$ implies that $|S|U^*U|S| = |T|V^*V|T|$. Thus $|S|^2 = |T|^2$. Since $|S|^2$ and $|T|^2$ are positive and they have unique positive square roots $|S|$ and $|T|$, we have $|S| = |T|$, and therefore $\rho(|S|) = \rho(|T|)$. □

If T is normal, then $T^*T = TT^*$. That is, $\langle |T^*|f, f\rangle \leq \langle |T|f, f\rangle$ for all $f \in L_a^2(\mathbb{D})$. Hence (2.3) holds. But (2.3) does not imply that T is normal. But when $T \in \mathcal{LC}(L_a^2(\mathbb{D}))$, that means that (2.3) implies that T is normal.

Theorem 2.4. *Let $T \in \mathcal{LC}(L_a^2(\mathbb{D}))$. Then (2.3) holds for all $x, y \in \mathbb{D}$ if and only if T is normal. In this case, $\rho(|T|) = \rho(|T^*|)$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. We will show that if $\langle |T^*|f, f\rangle \leq \langle |T|f, f\rangle$ for all $f \in L_a^2(\mathbb{D})$ then T is normal. Let $S = U|T|^{1/2}$. Then

$$SS^* = U|T|U^* = |T^*| \leq |T| = |T|^{1/2}U^*U|T|^{1/2} = S^*S.$$

Thus S is hyponormal. Now S is compact as T is compact. It follows from [3] that a compact hyponormal operator in $\mathcal{L}(L_a^2(\mathbb{D}))$ is normal. Thus S is normal and $UU^* = U^*U$ and $U|T|^{1/2} = |T|^{1/2}U$. Thus $U|T| = |T|U$, and hence T is normal. From Lemma 2.3, it follows that $\rho(|T|) = \rho(|T^*|)$. □

Theorem 2.5. *Let $A, B \in \mathcal{L}(L_a^2(\mathbb{D}))$, and assume that* Range$(A)$ *and* Range(B) *are closed. Then*

$$\big|\langle Cf, g\rangle\big|^2 \leq \langle Af, f\rangle\langle Bg, g\rangle, \tag{2.7}$$

where $f, g \in \mathbb{B}$ if and only if $A \geq 0, B \geq 0$ and there exists a contraction $K \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $\rho(C) = \rho(B^{1/2}KA^{1/2})$.

Proof. Suppose that (2.7) holds for all $f, g \in \mathbb{B}$. Let $f = \sum_{j=1}^n c_j K_{y_j}$ and $g = \sum_{i=1}^m d_i K_{x_i}$, where c_j and d_i are constants and $x_i, y_j \in \mathbb{D}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Then

$$\big|\langle Cf, g\rangle\big| \leq \langle Af, f\rangle^{1/2}\langle Bg, g\rangle^{1/2}.$$

Since the set of vectors $\{\sum c_j K_{x_j}, x_j \in \mathbb{D}, j = 1, \ldots, n\}$ is dense in $L_a^2(\mathbb{D})$, we have $|\langle Cf, g\rangle|^2 \leq \langle Af, f\rangle\langle Bg, g\rangle$ for all $f, g \in L_a^2(\mathbb{D})$. Now for $f, g \in L_a^2(\mathbb{D})$, we have

$$\begin{aligned}
\left\langle \begin{pmatrix} A & C^* \\ C & B \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix}\right\rangle &= \langle Af, f\rangle + \langle C^*g, f\rangle + \langle Cf, g\rangle + \langle Bg, g\rangle \\
&= \langle Af, f\rangle + \langle Bg, g\rangle + 2\operatorname{Re}\langle Cf, g\rangle \\
&\geq 2\langle Af, f\rangle^{1/2}\langle Bg, g\rangle^{1/2} + 2\operatorname{Re}\langle Cf, g\rangle \\
&\geq 2\big|\langle Cf, g\rangle\big| + 2\operatorname{Re}\langle Cf, g\rangle \\
&\geq 2\big|\langle Cf, g\rangle\big| - 2\big|\langle Cf, g\rangle\big| = 0.
\end{aligned}$$

Thus $D = \left(\begin{smallmatrix} A & C^* \\ C & B \end{smallmatrix}\right)$ is a positive operator in $B(L_a^2 \oplus L_a^2)$. This impllies that $D = E^*E$ for some $E \in \mathcal{L}(L_a^2 \oplus L_a^2)$. Let $E = R \oplus S$, where $R, S \in \mathcal{L}(L_a^2, L_a^2 \oplus L_a^2)$. That is, if $f, g \in L_a^2(\mathbb{D})$, then

$$(R \oplus S)(f \oplus g) = Rf \oplus Sg = E(f \oplus 0) + E(0 \oplus g) = E(f \oplus g).$$

Then

$$\begin{aligned} D = \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} &= E^*E \\ &= \begin{pmatrix} R^* \\ S^* \end{pmatrix} \begin{pmatrix} R & S \end{pmatrix} \\ &= \begin{pmatrix} R^*R & R^*S \\ S^*R & S^*S \end{pmatrix}. \end{aligned}$$

Thus it follows that $A = R^*R \geq 0, B = S^*S \geq 0$, and $C^* = R^*S$. Since $\operatorname{Range}(A)$ is closed, $\operatorname{Range}(B)$ is closed; since $\operatorname{Range}(A) = \operatorname{Range} A^{1/2}$ and $\operatorname{Range}(B) = \operatorname{Range} B^{1/2}$, it holds that $\operatorname{Range} R$ and $\operatorname{Range} S$ are closed. Since $A = R^*R$ and $B = S^*S$, there exist partial isometries U_1 and U_2 in $\mathcal{L}(L_a^2(\mathbb{D}))$ such that $R = U_1 A^{1/2}$ and $S = U_2 B^{1/2}$ and $U_1 U_1^* = P_{\operatorname{Range}(R)}, U_2^* U_2 = P_{\mathcal{M}}$, where $P_{\operatorname{Range}(R)}$ denotes the projection onto $\operatorname{Range}(R)$ and $P_{\mathcal{M}}$ denotes an orthogonal projection onto a closed subspace $\mathcal{M}$ of $L_a^2(\mathbb{D})$. Thus $C^* = R^*S = A^{1/2} U_1^* U_2 B^{1/2}$. Let $K^* = U_1^* U_2$. Then

$$\begin{aligned} KK^* &= U_2^* U_1 U_1^* U_2 = U_2^* P_{\operatorname{Range}(R)} U_2 \\ &\leq U_2^* I_{\mathcal{L}(L_a^2 \oplus L_a^2)} U_2 = U_2^* U_2 = P_{\mathcal{M}} \\ &\leq I_{\mathcal{L}(L_a^2)}. \end{aligned}$$

Hence $C^* = A^{1/2} K^* B^{1/2}$. That is, $C = B^{1/2} K A^{1/2}$ and therefore, $\rho(C) = \rho(B^{1/2} K A^{1/2})$ for some contraction $K \in \mathcal{L}(L_a^2(\mathbb{D}))$. Let $A, B \in \mathcal{L}(L_a^2(\mathbb{D}))$, where $A \geq 0, B \geq 0$, and $\rho(C) = \rho(B^{1/2} K A^{1/2})$ for some contraction $K \in \mathcal{L}(L_a^2(\mathbb{D}))$. This implies that $C = B^{1/2} K A^{1/2}$ and $\|K^*\| = \|K\| \leq 1$. That is, $KK^* \geq 0$ and $\langle KK^* f, f \rangle \leq \|f\|^2$ for all $f \in L_a^2(\mathbb{D})$. Hence

$$\left| \langle K^* f, g \rangle \right|^2 \leq \|K^* f\|^2 \|g\|^2 \leq \|f\|^2 \|g\|^2.$$

Now

$$\begin{aligned} \left\langle \begin{pmatrix} I_{\mathcal{L}(L_a^2)} & K \\ K^* & I_{\mathcal{L}(L_a^2)} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle &= \langle f, f \rangle + \langle Kg, f \rangle + \langle K^* f, g \rangle + \langle g, g \rangle \\ &= \langle f, f \rangle + \langle g, g \rangle + 2 \operatorname{Re} \langle K^* f, g \rangle \\ &\geq 2 \langle f, f \rangle^{1/2} \langle g, g \rangle^{1/2} + 2 \operatorname{Re} \langle K^* f, g \rangle \\ &\geq 2 \left| \langle K^* f, g \rangle \right| + 2 \operatorname{Re} \langle K^* f, g \rangle \\ &\geq 2 \left| \langle K^* f, g \rangle \right| - 2 \left| \langle K^* f, g \rangle \right| = 0. \end{aligned}$$

Thus $\left(\begin{smallmatrix} I_{\mathcal{L}(L_a^2)} & K \\ K^* & I_{\mathcal{L}(L_a^2)} \end{smallmatrix} \right) \geq 0$. It then follows from

$$\begin{pmatrix} A & C^* \\ C & B \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{pmatrix} \begin{pmatrix} I_{\mathcal{L}(L_a^2)} & K^* \\ K & I_{\mathcal{L}(L_a^2)} \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 \\ 0 & B^{1/2} \end{pmatrix}$$

that $\left(\begin{smallmatrix} A & C^* \\ C & B \end{smallmatrix}\right) \geq 0$ in $\mathcal{L}(L^2_a \oplus L^2_a)$. Now from [1], it follows that

$$\left|\left\langle \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g \end{pmatrix} \right\rangle\right|^2 \\ \leq \left\langle \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} f \\ 0 \end{pmatrix}, \begin{pmatrix} f \\ 0 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} 0 \\ g \end{pmatrix}, \begin{pmatrix} 0 \\ g \end{pmatrix} \right\rangle$$

for all $f, g \in L^2_a(\mathbb{D})$. A direct computation of these inner products now yields $|\langle Cf, g\rangle|^2 \leq \langle Af, f\rangle\langle Bg, g\rangle$ for all $f, g \in L^2_a(\mathbb{D})$, and therefore (2.7) holds for all $f, g \in \mathbb{B} \subset L^2_a(\mathbb{D})$. □

Theorem 2.6. *Let A, B, and C be operators in $\mathcal{L}(L^2_a(\mathbb{D}))$ such that A and B are positive and $BC = CA$. If*

$$\left|\langle Cu, v\rangle\right|^2 \leq \langle Au, u\rangle\langle Bv, v\rangle \tag{2.8}$$

for all $u, v \in \mathbb{B}$, then

$$\left|\langle Cu, v\rangle\right|^2 \leq \left\langle f(A)^2 u, u\right\rangle\left\langle g(B)^2 v, v\right\rangle \tag{2.9}$$

for all $u, v \in \mathbb{B}$, where f and g are nonnegative continuous functions on $[0, \infty)$ that satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$.

Proof. From the proof of Theorem 2.5, it follows that the conditions (2.8) and (2.9) are equivalent to the fact that $\left(\begin{smallmatrix} A & C^* \\ C & B \end{smallmatrix}\right) \geq 0$ and $\left(\begin{smallmatrix} f(A)^2 & C^* \\ C & g(B)^2 \end{smallmatrix}\right) \geq 0$ in $\mathcal{L}(L^2_a \oplus L^2_a)$. Suppose that A and B are invertible. The proof follows from the following observations:

(i)
$$\begin{pmatrix} f(A)^2 & C^* \\ C & g(B)^2 \end{pmatrix} = \begin{pmatrix} f(A)A^{-1/2} & 0 \\ 0 & g(B)B^{-1/2} \end{pmatrix} \\ \times \begin{pmatrix} A & C^* \\ C & B \end{pmatrix} \begin{pmatrix} f(A)A^{-1/2} & 0 \\ 0 & g(B)B^{-1/2} \end{pmatrix}.$$

(ii) Since $BC = CA$, it follows that $h(B)C = Ch(A)$ for all continuous functions h on $[0, \infty)$.

(iii) Since $f(t)g(t) = t$ for all $t \in [0, \infty)$, we obtain $f(D)g(D) = D$ for any positive operator $D \in \mathcal{L}(L^2_a(\mathbb{D}))$. Thus $g(B)B^{-1/2}Cf(A)A^{-1/2} = C$. This last statement can be verified as follows. From (ii) it follows that

$$g(B)CA^{1/2} = Cg(A)A^{1/2} = CA^{1/2}g(A).$$

Now

$$\begin{aligned} CA^{1/2}g(A) &= g(B)CA^{1/2} \\ &= g(B)B^{1/2}C = g(B)B^{-1/2}BC \\ &= g(B)B^{-1/2}CA. \end{aligned}$$

Thus $g(B)B^{-1/2}Cf(A) = CA^{1/2}$. Hence $g(B)B^{-1/2}Cf(A)A^{-1/2} = C$.

We therefore have $\left(\begin{smallmatrix} A & C^* \\ C & B \end{smallmatrix}\right) \geq 0$ if and only if $\left(\begin{smallmatrix} f(A)^2 & C^* \\ C & g(B)^2 \end{smallmatrix}\right) \geq 0$. For the general case, apply the argument above to the invertible operators $A_\epsilon = A + \epsilon$ and $B_\epsilon = B + \epsilon$ for $\epsilon > 0$ and then let $\epsilon \longrightarrow 0$. □

For a self-adjoint operator $T \in \mathcal{L}(H)$, it follows from the spectral theorem that $-|T| \leq T \leq |T|$ or, equivalently, that $|\langle Tx, x\rangle| \leq \langle |T|x, x\rangle$ for all $x \in H$. But this is not true for arbitrary operators. For example, let $T = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $x = \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right)$. Then $|\langle Tx, x\rangle| = 2$ and $\langle |T|x, x\rangle = 1$.

Lemma 2.7. *If T is an operator in $\mathcal{L}(H)$, then $\left(\begin{smallmatrix} |T| & T^* \\ T & |T^*| \end{smallmatrix}\right)$ is a positive operator in $\mathcal{L}(H \oplus H)$, where $|T| = (T^*T)^{1/2}$ and $|T^*| = (TT^*)^{1/2}$.*

Proof. On $H \oplus H$, let $S = \left(\begin{smallmatrix} 0 & T^* \\ T & 0 \end{smallmatrix}\right)$. Then S is self-adjoint and $S^*S = \left(\begin{smallmatrix} T^*T & 0 \\ 0 & TT^* \end{smallmatrix}\right)$. By the uniqueness of the square root of a positive operator, it follows that $|S| = \left(\begin{smallmatrix} |T| & 0 \\ 0 & |T^*| \end{smallmatrix}\right)$. Since S is self-adjoint, it follows by the spectral theorem that $S + |S|$ is positive. Therefore, $\left(\begin{smallmatrix} |T| & T^* \\ T & |T^*| \end{smallmatrix}\right)$ is positive. □

Corollary 2.8. *Let $T \in \mathcal{L}(L^2_a(\mathbb{D}))$, and let f and g be as in the preceding theorem. Then*

$$\left|\langle Tu, v\rangle\right|^2 \leq \left\langle f\big(|T|\big)^2 u, u\right\rangle \left\langle f\big(|T^*|\big)^2 v, v\right\rangle$$

for all $u, v \in \mathbb{B}$. In this case, $\rho(f(|T^*|)) \leq \rho(f(|T|))$.

Proof. Since $T|T|^2 = |T^*|^2 T$, it follows that $T|T| = |T^*|T$. From Lemma 2.7 it follows that $\left(\begin{smallmatrix} |T| & T^* \\ T & |T^*| \end{smallmatrix}\right) \geq 0$ in $\mathcal{L}(L^2_a \oplus L^2_a)$. This is true if and only if

$$\left|\langle Tu, v\rangle\right|^2 \leq \left\langle |T|u, u\right\rangle \left\langle |T^*|v, v\right\rangle$$

for all $u, v \in \mathbb{B}$. From Theorem 2.6 it follows that

$$\left|\langle Tu, v\rangle\right|^2 \leq \left\langle f\big(|T|\big)^2 u, u\right\rangle \left\langle f\big(|T^*|\big)^2 v, v\right\rangle$$

for all $u, v \in \mathbb{B}$. Proceeding similarly as in Theorem 2.2, one can show that $\rho(f(|T^*|)) \leq \rho(f(|T|))$. □

3. Absolute value of an operator in $L^2_a(\mathbb{D})$

In this section, we again concentrate on the Berezin transform of the absolute value of a bounded linear operator defined on $L^2_a(\mathbb{D})$. We have established that T is self-adjoint and $T^2 = T^3$ if and only if there exists a normal idempotent operator S on $L^2_a(\mathbb{D})$ such that $\rho(T) = \rho(|S|^2) = \rho(|S^*|^2)$, where $\rho(T)$ is the Berezin transform of T. We also establish that if T is compact and $|T^n| = |T|^n$ for some $n \in \mathbb{N}$, $n \neq 1$, then $\rho(|T^n|) = \rho(|T|^n)$ for all $n \in \mathbb{N}$. Further, if $T = U|T|$ is the polar decomposition of T then we present necessary and sufficient conditions on T such that $|T|^{1/2}$ intertwines with U and a contraction X belonging to $\mathcal{L}(L^2_a(\mathbb{D}))$.

Theorem 3.1. *Let $A \in \mathcal{L}(L^2_a(\mathbb{D}))$. The operator A is self-adjoint and $A^2 = A^3$ if and only if there exists a normal idempotent operator $B \in \mathcal{L}(L^2_a(\mathbb{D}))$ such that $|B|^2 = |B^*|^2 = A$. In this case, $\rho(A) = \rho(|B|^2) = \rho(|B^*|^2)$.*

Proof. If $BB^* = A = B^*B$, then

$$A^3 = B^*BBB^*B^*B = B^*B^2B^{*^2}B = B^*BB^*B = A^2$$

and A is self-adjoint. Now assume that $A = A^*$ and that $A^3 = A^2$. Let $\widetilde{P}$ and $\widetilde{Q}$ be projections onto $\overline{\text{Range}\,A}$ and $\ker A = (\overline{\text{Range}\,A})^\perp$, respectively. Then $A\widetilde{P}f = \widetilde{P}Af = Af$ and $A\widetilde{Q}f = \widetilde{Q}Af = 0$. Since $A^3f = A^2f$, it holds that $A(A^2 - A) \times f = 0$. Thus the vector $(A^2 - A)f \in \ker A$. This impllies that $\widetilde{P}(A^2 - A)f = 0$ or $\widetilde{P}A^2 - A = 0$. Hence by taking adjoints, we obtain $A^2\widetilde{P} - A = 0$. Therefore $A(A\widetilde{P}f - f) = 0$ for any $f \in L^2_a(\mathbb{D})$. Consequently, $A\widetilde{P}f - f \in \ker A$. That is, $\widetilde{P}A\widetilde{P}f - \widetilde{P}f = 0$. Hence $\widetilde{P}A\widetilde{P} = \widetilde{P}$. Similarly, one can show that $(I-\widetilde{Q})A(I-\widetilde{Q}) = I - \widetilde{Q}$. Now $A^3 = A^2$ implies that $A^3 \geq 0$. Since $A = A^*$, and $A \in \mathcal{L}(L^2_a(\mathbb{D}))$, we obtain

$$0 \leq \langle A^2f, f\rangle = \langle A^3f, f\rangle = \langle A^2f, Af\rangle = \big\langle A(Af), Af\big\rangle.$$

Thus, if $h \in \overline{\text{Range}\,A}$, $\langle Ah, h\rangle \geq 0$. Now let $f \in L^2_a(\mathbb{D})$. Then $f = g + h$, where $g \in \ker A$ and $h \in (\ker A)^\perp = \overline{\text{Range}\,A}$. Now

$$\begin{aligned}\langle Af, f\rangle &= \big\langle A(g+h), g+h\big\rangle \\ &= \langle Ag, g\rangle + \langle Ag, h\rangle + \langle Ah, g\rangle + \langle Ah, h\rangle \\ &= \langle Ah, h\rangle \geq 0.\end{aligned}$$

Thus $A \geq 0$. Now $\widetilde{P}A^2\widetilde{P} = \widetilde{P}A^3\widetilde{P} = \widetilde{P}A\widetilde{P}A\widetilde{P}A\widetilde{P} = \widetilde{P}A\widetilde{P} = A$. Similarly, one can establish that $(I - \widetilde{Q})A^2(I - \widetilde{Q}) = A$. Since $\widetilde{P}$ is positive, we have $A\widetilde{P}A \geq 0$ and $(A\widetilde{P}A)^2 = A\widetilde{P}A^2\widetilde{P}A = AAA = A^2$. Since each positive operator has a unique positive square root, it holds that $A\widetilde{P}A = A$. Similarly, it is not difficult to see that $A(I - \widetilde{Q})A = A$. Now define the operator B by $B = \widetilde{P}A$. This implies that $B^* = A\widetilde{P}$. Since $B^2 = \widetilde{P}A\widetilde{P}A = \widetilde{P}A = B$, the operator B is an idempotent. Also $BB^* = \widetilde{P}A^2\widetilde{P} = A$ and $B^*B = A\widetilde{P}A = A$. Thus B is normal. Since $|B|^2 = B^*B = BB^* = |B^*|^2 = A$, the result follows. □

An operator $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ is said to be *hyponormal* if $T^*T \geq TT^*$. It is *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$ for a positive number p and log-*hyponormal* if T is invertible and $\log T^*T \geq \log TT^*$. The operator T is *paranormal* if $\|T^2f\| \geq \|Tf\|^2$ for all $f \in L^2_a(\mathbb{D})$. Let $T \in \mathcal{L}(L^2_a(\mathbb{D}))$. The operator T is said to be *quasinormal* if T commutes with $|T|^2$. Let $T = V|T|$ be the polar decomposition of T. If T is quasinormal, then it is not difficult to see that $V|T| = |T|V$. Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a continuous function. The function f is called *operator-monotone* if $A, B \in \mathcal{L}(L^2_a(\mathbb{D}))$, $0 \leq A \leq B$, implies that $f(A) \leq f(B)$.

Lemma 3.2. *Let $T \in \mathcal{L}(L^2_a(\mathbb{D}))$ be invertible. Then the following hold.*

(i) *If T is* log-*hyponormal, then $|T^2| \geq |T|^2$.*
(ii) *If $|T^2| \geq |T|^2$, then T is paranormal.*

Proof.

(i) Suppose that T is log-hyponormal. Then $\log|T|^2 \geq \log|T^*|^2$. From [6] and [7], it follows that $|T|^{2p} \geq (|T|^p|T^*|^{2p}|T|^p)^{1/2}$ for all $p \geq 0$. Let $p = 1$. Then we get

$$|T|^2 \geq \big(|T||T^*|^2|T|\big)^{1/2}. \tag{3.1}$$

Again from [6] and [7], it follows that (3.1) holds if and only if

$$|T|^2 \geq |T|T\big(T^*|T|^2T\big)^{-1/2}T^*|T|.$$

That is, if and only if $(T^*|T|^2T)^{1/2} \geq T^*T$. This is equivalent to say that $|T^2| \geq |T|^2$.

(ii) Suppose that $|T^2| \geq |T|^2$. Then for $f \in L_a^2(\mathbb{D})$ and $\|f\| = 1$, we obtain from [12] that

$$\begin{aligned}\|T^2f\|^2 &= \big\langle (T^2)^*T^2f, f\big\rangle\\ &= \big\langle |T^2|^2f, f\big\rangle\\ &\geq \big\langle |T^2|f, f\big\rangle^2\\ &\geq \big\langle |T|^2f, f\big\rangle^2 = \|Tf\|^4.\end{aligned}$$

Thus T is paranormal. $\square$

Theorem 3.3. *Let $T \in \mathcal{LC}(L_a^2(\mathbb{D}))$ and $|T^n| = |T|^n$ for some $n \in \mathbb{N}, n \neq 1$. Then (2.3) holds and $\rho(|T^n|) = \rho(|T|^n)$ for all $n \in \mathbb{N}$.*

Proof. Let $T \in \mathcal{LC}(L_a^2(\mathbb{D}))$, and let the spectral representation of T be as follows: $T = \sum_{i=1}^{\infty} \lambda_i(\psi_i \otimes \phi_i)$, where $\{\psi_i\}_{i=1}^{\infty}$ and $\{\phi_i\}_{i=1}^{\infty}$ are two orthonormal bases for $L_a^2(\mathbb{D})$, where $|\lambda_i| \longrightarrow 0$ as $i \longrightarrow \infty$. Then

$$|T| = \sum_{i=1}^{\infty} |\lambda_i|\phi_i \otimes \phi_i$$

and

$$|T^*| = \sum_{i=1}^{\infty} |\lambda_i|\psi_i \otimes \psi_i.$$

Since the eigenspace corresponding to $|\lambda_1|$ is finite-dimensional, there is a $k \in \mathbb{N}$ such that $|\lambda_1| = \cdots = |\lambda_k| > |\lambda_{k+1}|$. Thus

$$\begin{aligned}|\lambda|^{2m} &= \big\langle (T^*T)^m\phi_1, \phi_1\big\rangle = \langle T^{*m}T^m\phi_1, \phi_1\rangle\\ &= |\lambda|^2\big\langle (T^*)^{m-2}T^{m-2}T\psi_1, T\psi_1\big\rangle\\ &\leq |\lambda_1|^{2m-2}\langle T\psi_1, T\psi_1\rangle \leq |\lambda_1|^{2k}\end{aligned}$$

and therefore $\langle T^*T\psi_1, \psi_1\rangle = |\lambda_1|^2$. Hence $(|\lambda_1|^2 - T^*T)\psi_1 = 0$ as $|\lambda_1|^2 - T^*T \geq 0$. Proceeding similarly, one can show that $\{\psi_1, \ldots, \psi_j\} \subset \ker(T^*T - |\lambda_1|^2)$. Thus $\ker(TT^* - |\lambda_1|^2) = \ker(T^*T - |\lambda_1|^2) = M$ (let) and M reduces T to the normal operator. That is, $T^*T\psi_i = TT^*\psi_i$ for $1 \leq i \leq m$. Repeating this procedure to the other restrictions of T, one can derive that $T^*T\psi_i = TT^*\psi_i$ for all $i \in \mathbb{N}$.

From Theorem 2.4 it follows that T satisfies (2.3). Further, since T is normal, it holds that $\rho(|T^n|) = \rho(|T|^n)$ for all $n \in \mathbb{N}$. $\square$

Theorem 3.4. *Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Then the following are equivalent:*

(i) $\rho(T|T|^2) = \rho(|T|^2T)$;
(ii) $\rho(|T^n|) = \rho(|T|^n)$ *for all* $n \in \mathbb{N}$;
(iii) *there are integers k and m such that $\rho(|T^n|) = \rho(|T|^n)$ for $n = k, k+1, m$ and $m+1$, where $1 \leq k < m$.*

Proof. From Theorem 3.3 it is not difficult to verify that (i) implies (ii) and that (ii) implies (iii). We will only verify that (iii) implies (i). From (iii), it follows that $|T^n| = |T|^n$ for $n = k, k+1, m$, and $m+1$ for $1 \leq k \leq m$. Hence using mathematical induction, one can show that

$$T^*(T^*T)^mT = T^*(T^{*m}T^m)T = (T^*T)^{m+1} = T^*(TT^*)^mT.$$

Thus it follows that

$$\widetilde{Q}(T^*T)^m\widetilde{Q} = \widetilde{Q}(TT^*)^m\widetilde{Q} = (TT^*)^m,$$

where $\widetilde{Q}$ is the projection map from $L_a^2(\mathbb{D})$ onto $\overline{(\mathrm{Range}\, T)}$. Similarly, it follows that $\widetilde{Q}(T^*T)^k\widetilde{Q} = (TT^*)^k$, and hence

$$\big(\widetilde{Q}(T^*T)^m\widetilde{Q}\big)^{\frac{k}{m}} = \widetilde{Q}(T^*T)^k\widetilde{Q} = \widetilde{Q}\big((T^*T)^m\big)^{\frac{k}{m}}\widetilde{Q}.$$

Since $f(t) = t^{\frac{k}{m}}$ is an operator-monotone function, it follows from Theorem 3.3 and [1] that $\widetilde{Q}$ commutes with $(T^*T)^m$ and hence with T^*T. Hence $(\widetilde{Q}T^*T\widetilde{Q})^m = (TT^*)^m$ and therefore $\widetilde{Q}T^*T\widetilde{Q} = TT^*$. Thus

$$T^*TT = T^*T\widetilde{Q}T = \widetilde{Q}T^*T\widetilde{Q}T = TT^*T$$

and therefore $|T|^2T = T|T|^2$ and $\rho(|T|^2T) = \rho(T|T|^2)$. $\square$

Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$. Suppose that

$$\Theta_{|T|}(x,\bar{y})K(x,\bar{y}) \gg \big|\Theta_T(x,\bar{y})K(x,\bar{y})\big| \tag{3.2}$$

for all $x, y \in \mathbb{D}$. It is not difficult to see that (2.3) implies that (3.2). Thus if T is normal then (3.2) holds. Let $A, B \in \mathcal{L}(L_a^2(\mathbb{D}))$. The operator $X \in \mathcal{L}(L_a^2(\mathbb{D}))$ intertwines A and B if $AX = XB$. Let

$$\begin{aligned}\mathcal{B} = \Big\{X \in \mathcal{L}\big(L_a^2(\mathbb{D})\big) &: X = 2(I_{\mathcal{L}(L_a^2)} - C^*C)^{1/2}C,\\ &\text{where } C \in \mathcal{L}(L_a^2) \text{ and } \|C\| \leq 1\Big\}.\end{aligned}$$

For $X \in \mathcal{L}(L_a^2(\mathbb{D}))$, let

$$w(X) = \sup\big\{\big|\langle Xf, f\rangle\big| : f \in L_a^2(\mathbb{D}), \|f\| = 1\big\},$$

the numerical radius of X. It is well known (see [11]) that $w(|X|) = \|X\|$.

Theorem 3.5. *Let $T \in \mathcal{L}(L_a^2(\mathbb{D}))$ and $T = U|T|$ be the polar decompositions of T. Then (3.2) holds for all $x, y \in \mathbb{D}$ if and only if $|T|^{1/2}$ intertwines with U and an operator $X \in \mathcal{B}$. In this case, there is a sequence of operators $T_n \in \mathcal{L}(L_a^2(\mathbb{D}))$ which converges to U strongly, and there is a sequence $S_n \in \mathcal{L}(L_a^2(\mathbb{D}))$ which converges to X weakly and $S_n \in \mathcal{B}$ for all n.*

Proof. Suppose that (3.2) holds. Then

$$\Theta_{|T|}(x, \bar{y})K(x, \bar{y}) \gg \left|\Theta_T(x, \bar{y})K(x, \bar{y})\right|$$

for all $x, y \in \mathbb{D}$. This implies that

$$\Big\langle |T|\Big(\sum_{j=1}^{n} c_j K_{x_j}\Big), \sum_{i=1}^{n} c_i K_{x_i}\Big\rangle \geq \Big|\Big\langle T\Big(\sum_{j=1}^{n} c_j K_{x_j}\Big), \sum_{i=1}^{n} c_i K_{x_i}\Big\rangle\Big|,$$

where $x_1, x_2, \ldots, x_n \in \mathbb{D}$ and $c_j, j = 1, \ldots, n$ are constants. Since $\{\sum_{j=1}^{n} c_j K_{x_j}\}$ is dense in $L_a^2(\mathbb{D})$, it holds that $|\langle Tf, f\rangle| \leq \langle |T|f, f\rangle$ for all $f \in L_a^2(\mathbb{D})$. For $n \in \mathbb{N}$, define $S_n \in \mathcal{L}(L_a^2(\mathbb{D}))$ by

$$S_n = \Big(|T| + \frac{1}{n}\Big)^{-1/2} U \Big(|T| + \frac{1}{n}\Big)^{1/2}$$

and $T_n = T(|T| + \frac{1}{n})^{-1}$. Let $\{E_\lambda\}$ be the spectral family for $|T|$. Then T_n strongly converges to $I - E_0$ as $n \longrightarrow \infty$. The reason is as follows.

Notice that $|T| = \int_0^\infty \lambda \, dE_\lambda$ is the spectral decomposition of $|T|$. Let

$$V_n = |T|\Big(|T| + \frac{1}{n}\Big)^{-1}.$$

Then $V_n E_0 f = (|T| + \frac{1}{n})^{-1}|T|E_0 f = 0$ for $f \in L_a^2(\mathbb{D})$ and

$$\begin{aligned}
\big\|V_n f - (I - E_0)f\big\|^2 &= \big\|(V_n - I)(I - E_0)f\big\|^2 \\
&= \int_0^\infty \Big|\frac{\lambda}{\lambda + \frac{1}{n}} - 1\Big|^2 d\big\|E_\lambda(I - E_0)f\big\|^2 \\
&= \int_0^\infty \Big|\frac{\frac{1}{n}}{\lambda + \frac{1}{n}}\Big|^2 d\big\|E_\lambda(I - E_0)f\big\|^2.
\end{aligned}$$

From Lebesgue's dominated convergence theorem, it follows that V_n strongly converges to $I - E_0$ as $n \longrightarrow \infty$. Thus we have $T_n \longrightarrow U(I - E_0)$ strongly as $n \longrightarrow \infty$. Since E_0 is the projection onto the eigenspace $\{f \in L_a^2(\mathbb{D}) : Tf = 0\}$, we get $UE_0 = 0$. Consequently, $T_n \longrightarrow U$ strongly as $n \longrightarrow \infty$. Further, for all $f \in L_a^2(\mathbb{D})$,

$$\begin{aligned}
\langle S_n f, f\rangle &= \Big\langle U\Big(|T| + \frac{1}{n}\Big)^{1/2} f, \Big(|T| + \frac{1}{n}\Big)^{-1/2} f\Big\rangle \\
&= \Big\langle U\Big(|T| + \frac{1}{n}\Big)\Big(|T| + \frac{1}{n}\Big)^{-1/2} f, \Big(|T| + \frac{1}{n}\Big)^{-1/2} f\Big\rangle \\
&= \Big\langle T\Big(|T| + \frac{1}{n}\Big)^{-1/2} f, \Big(|T| + \frac{1}{n}\Big)^{-1/2} f\Big\rangle \\
&\quad + \frac{1}{n}\Big\langle U\Big(|T| + \frac{1}{n}\Big)^{-1/2} f, \Big(|T| + \frac{1}{n}\Big)^{-1/2} f\Big\rangle.
\end{aligned}$$

From (3.2), it follows that

$$\begin{aligned}\langle S_n f, f\rangle &\leq \Big\langle |T|\Big(|T|+\frac{1}{n}\Big)^{-1/2} f, \Big(|T|+\frac{1}{n}\Big)^{-1/2} f\Big\rangle + \frac{1}{n}\Big\|\Big(|T|+\frac{1}{n}\Big)^{-1/2} f\Big\|^2 \\ &= \Big\langle \Big(|T|+\frac{1}{n}\Big)\Big(|T|+\frac{1}{n}\Big)^{-1/2} f, \Big(|T|+\frac{1}{n}\Big)^{-1/2} f\Big\rangle \\ &= \langle f, f\rangle = \|f\|^2.\end{aligned}$$

If $S \in \mathcal{L}(L_a^2(\mathbb{D}))$, then it is known (see [9]) that $\frac{1}{2}\|S\| \leq w(S) \leq \|S\|$, where $w(S)$ is the numerical radius of S. Thus, we get $w(S_n) \leq 1$ and $\|S_n\| \leq 2$. By the Banach Alaoglu theorem (see [5, Theorem 1.23]), one can construct a subnet $\{S_j\}_{j\in J}$ converging weakly to some $X \in \mathcal{L}(L_a^2(\mathbb{D}))$ with $\|X\| \leq 2$ from the sequence $\{S_n\}_{n\in\mathbb{N}}$. Thus, we have $w(X) \leq 1$ since

$$\langle Xf, f\rangle = \lim_j \langle S_j f, f\rangle \leq \langle f, f\rangle.$$

From [2], it follows that $X \in \mathcal{B}$ and $S_n \in \mathcal{B}$ for all n. Now, from the definition of $\{S_j\}_{j\in J}$, we get

$$U\Big(|T|+\frac{1}{F(j)}\Big)^{1/2} = \Big(|T|+\frac{1}{F(j)}\Big)^{1/2} S_{F(j)} \tag{3.3}$$

for some mapping $F : J \longrightarrow \mathbb{N}$ (in fact, $S_j = S_{F(j)}$). Hence by taking weak limits of both sides of (3.3), we obtain $U|T|^{1/2} = |T|^{1/2}X$. Conversely, assume that $U|T|^{1/2} = |T|^{1/2}X$ for some $X \in \mathcal{L}(L_a^2(\mathbb{D}))$ with $w(X) \leq 1$. From [2], it follows that there exists a contraction $C \in \mathcal{L}(L_a^2(\mathbb{D}))$ such that $X = 2(I_{\mathcal{L}(L_a^2)} - C^*C)^{1/2}C$. Then for all $f \in L_a^2(\mathbb{D})$, we get

$$\begin{aligned}\big|\langle Tf, f\rangle\big| &= \big|\big\langle U|T|^{1/2}|T|^{1/2} f, f\big\rangle\big| \\ &= \big|\big\langle |T|^{1/2}X|T|^{1/2} f, f\big\rangle\big| \\ &= \big|\big\langle X|T|^{1/2} f, |T|^{1/2} f\big\rangle\big| \\ &\leq \big\langle |T|^{1/2} f, |T|^{1/2} f\big\rangle \\ &= \big\langle |T| f, f\big\rangle.\end{aligned}$$

The result follows. □

References

1. N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space, Vol. 1*, Pitman, Boston, 1981. Zbl 0467.47001. MR0615736. 158, 162
2. T. Ando, *Structure of operators with numerical radius one*, Acta Sci. Math. (Szeged) **34** (1973), 11–15. Zbl 0258.47001. MR0318920. 164
3. S. K. Berberian, *A note on hyponormal operators*, Pacific J. Math. **12** (1962), 1171–1175. Zbl 0129.08605. MR0149281. DOI 10.2140/pjm.1962.12.1171. 156
4. J. B. Conway, *A Course in Functional Analysis*, 2nd ed., Grad. Texts in Math. **96**, Springer, New York, 1990. Zbl 0706.46003. MR1070713. 151
5. R. G. Douglas, *Banach Algebra Techniques in Operator Theory*, Pure Appl. Math. **49**, Academic Press, New York, 1972. Zbl 0247.47001. MR0361893. 164

6. M. Fujii, T. Furuta, and E. Kamei, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl. **179** (1993), 161–169. Zbl 0788.47012. MR1200149. DOI 10.1016/0024-3795(93)90327-K. 161
7. T. Furuta, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl. **59** (1992), 180–190. Zbl 0792.47015. MR1246815. 161
8. G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, 2nd ed., Nauka, Moscow, 1966. Zbl 0148.30603. MR0219714. 152
9. P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Grad Texts in Math. **19**, Springer, New York, 1982. Zbl 0496.47001. MR0675952. 164
10. S. G. Krantz, *Function Theory of Several Complex Variables*, Wiley, New York, 1982. Zbl 0471.32008. MR0635928. 152
11. M. E. Omidvar, M. S. Moslehian, and A. Niknam, *Some numerical radius inequalities for Hilbert space operators*, Involve **2** (2009), no. 4, 469–476. Zbl 1185.47004. MR2579564. DOI 10.2140/involve.2009.2.471. 162
12. K. Zhu, *Operator Theory in Function Spaces*, Pure Appl. Math. **139**, Dekker, New York, 1990. Zbl 0706.47019. MR1074007. 161

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