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DERIVATIONS ON GENERALIZED SEMIDIRECT PRODUCTS OF BANACH ALGEBRAS

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Abstract. Let A and B be Banach algebras, let $\theta : A \to B$ be a continuous Banach algebra homomorphism, and let I be a closed ideal in B. Then the l^1-direct sum of A and I with a special product becomes a Banach algebra, denoted by $A \bowtie^{\theta} I$, which we call the *generalized semidirect product* of A and I. In this article, among other things, we first characterize derivations on $A \bowtie^{\theta} I$ and then we compute the first cohomology group of $A \bowtie^{\theta} I$ when the first cohomology groups of A with coefficients in A and I are trivial. As an application we characterize the first cohomology group of second duals of dual Banach algebras. Then we provide a nice characterization of the first cohomology group of $A \bowtie^{\mathrm{id}} A$. Furthermore, we calculate the first cohomology group of some concrete Banach algebras related to locally compact groups.

1. Introduction and preliminaries

Let A be a Banach algebra, and let X be a Banach A-bimodule. A *derivation* from A into X is a bounded linear map satisfying

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

For each $x \in X$, we denote by ad_x the derivation $D(a) = a \cdot x - x \cdot a$ for all $a \in A$, which is called an *inner derivation*. We denote by $\mathcal{Z}^1(A, X)$ the space of all derivations from A into X and by $\mathcal{B}^1(A, X)$ the space of all inner derivations from A into X. The *first (Hochschild) cohomology group* of A with coefficients in X, denoted by $\mathcal{H}^1(A, X)$, is the quotient space $\mathcal{Z}^1(A, X)/\mathcal{B}^1(A, X)$. This group

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gives a considerable information about the structure of A. Let $n \in \mathbb{N} \cup \{0\}$. A Banach algebra A is called *n-weakly amenable* (*weakly amenable* in case $n = 1$) if $\mathcal{H}^1(A, A^{(n)}) = 0$, where $A^{(n)}$ is the nth dual space of A ($A^{(0)} = A$) which is canonically an A-bimodule (see [5]).

Although the first cohomology group for Banach algebras has been studied extensively for many years, there are still few techniques available for explicitly calculating the various first cohomology groups. Generally, the best that one might hope for is to be able to determine whether or not a particular cohomology group is trivial. The triviality of the first cohomology group of A with coefficients in A for some classes of Banach algebras are already known. For example, if A is a von Neumann algebra, a commutative C^*-algebra, a W^*-algebra, or a simple unital C^*-algebra (i.e., A has no proper closed two-sided ideal), then $\mathcal{H}^1(A, A) = 0$ (see [14]). I. M. Singer and J. Wermer [15] have shown that if A is a semisimple commutative Banach algebra, then $\mathcal{H}^1(A, A) = 0$. For a locally compact group G, V. Losert [12] showed that $\mathcal{H}^1(M(G), M(G)) = 0$, where $M(G)$ denotes the measure algebra of G. It is known that if A is commutative and weakly amenable and X is a symmetric Banach A-bimodule ($a \cdot x = x \cdot a$ for each $a \in A$ and $x \in X$), then $\mathcal{H}^1(A, X) = 0$.

In [8], B. E. Forrest and L. W. Marcoux calculated the first cohomology group of triangular Banach algebras. Analogously, in a joint work with A. R. Medghalchi [13], we calculated the first cohomology group of module extension Banach algebras. Motivated by these earlier investigations, in this article, we study the first cohomology group of those Banach algebras that have a semidirect product-like structure, which we call the *generalized semidirect product construction* and denote by $A \bowtie^\theta I$.

The organization of this article is as follows. In Section 2, we introduce the Banach algebra $A \ltimes^\theta I$ and give some examples of this construction. In Section 3, we first characterize derivations on $A \bowtie^\theta I$ and then we compute the first cohomology group of $A \bowtie^\theta I$ when the first cohomology group of A with coefficients in A and I vanish. As an application, we provide a characterization of the first cohomology group of second duals of dual Banach algebras and that of ϕ-Lau products of Banach algebras. We also show that $\mathcal{H}^1(M(G), L^1(G)) = 0$ for each locally compact group G. Section 4 is devoted to the characterization of the first cohomology group of the special case $A \bowtie^{\mathrm{id}} A$.

2. The Banach algebra $A \bowtie^\theta I$

Let A and B be Banach algebras; let $\theta : A \to B$ be a continuous Banach algebra homomorphism, where, without loss of generality, we can assume that $\|\theta\| \leq 1$; and let I be a closed ideal in B. Then the subset

$$C = \{(a, \theta(a) + i) : a \in A, i \in I\}$$

of $A \times B$ is a Banach algebra under the pointwise product and the l^1-norm. Algebraists call C the *amalgamated algebra of A with B along I with respect to θ* (see, e.g., [7] and references therein). It can be easily seen that the algebra C is also a Banach algebra under the norm $\|(a, \theta(a) + i)\| = \|a\| + \|i\|$. For all $a \in A$ and $i \in I$, we have

$$\begin{aligned}\big\|\big(a,\theta(a)+i\big)\big\|_1 &= \|a\|+\big\|\theta(a)+i\big\| \\ &\le \|a\|+\|\theta\|\|a\|+\|i\| \\ &\le \big(\|\theta\|+1\big)\big(\|a\|+\|i\|\big)=\big(\|\theta\|+1\big)\big\|\big(a,\theta(a)+i\big)\big\|.\end{aligned}$$

So, by the open mapping theorem, two norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent on C.

We also consider the Banach algebra $D=\{(a,i):a\in A, i\in I\}$, the l^1-direct sum of A and I, with the following product formula:

$$(a,i)\cdot(a',i')=\big(aa',\theta(a)i'+i\theta(a')+ii'\big).$$

Now the mapping $\varphi:(C,\|\cdot\|)\to D$ by $\varphi((a,\theta(a)+i))=(a,i)$ is an isometric isomorphism of Banach algebras. Therefore, D and C are the same objects in the category of Banach algebras. We are going to work on the Banach algebra D rather than C and denote it by $A\bowtie^\theta I$. Since the semidirect product of Banach algebras has the same structure (see Example 2.1 below), we call $A\bowtie^\theta I$ the *generalized semidirect product of A and B along I with respect to θ*. It can be easily seen that $A\cong A\times\{0\}$ is a closed subalgebra of $A\bowtie^\theta I$, that $I\cong\{0\}\times I$ is a closed ideal of it, and that $\frac{A\bowtie^\theta I}{I}\cong A$. Also, $A\bowtie^\theta I$ is commutative if and only if A and $\theta(A)+I$ are commutative.

The Banach algebra $A\bowtie^\theta B$ has been considered and studied by some authors (see, e.g., [1], [3], [10]). This class does not contain any concrete Banach algebra to our knowledge. So it seems that the Banach algebra $A\bowtie^\theta B$ is not so interesting. On the other hand, there are many classes of concrete Banach algebras having a generalized semidirect product structure, where I is a proper ideal of B.

Example 2.1.

(i) If $\theta=0$, then $A\bowtie^0 I$ is nothing other than the Cartesian product of A and I.

(ii) Let A be a nonunital Banach algebra. Then $A^\#=\mathbb{C}\oplus A$, the unitization of A, is the generalized semidirect product of $\mathbb{C}$ with $A^\#$ along A with respect to the homomorphism $\theta:\mathbb{C}\to A^\#$ defined by $\theta(\lambda)=(\lambda,0)$.

(iii) Let A be a Banach algebra, and let X be a Banach A-bimodule. The module extension Banach algebra corresponding to A and X, denoted by $\mathcal{S}=A\oplus X$, is the l^1-direct sum of A and X equipped with the product formula $(a,x)\cdot(a',x')=(aa',ax'+xa')$ for $a,a'\in A$ and $x,x'\in X$. Then $\mathcal{S}$ is the generalized semidirect product of A with $\mathcal{S}$ along X with respect to the injection $\theta:A\to\mathcal{S}$ defined by $\theta(a)=(a,0)$. We remark that the class of module extension Banach algebras includes the class of triangular Banach algebras.

(iv) Let A be a Banach algebra, and let ϕ be a nonzero character on A. Then $A\bowtie^\phi\mathbb{C}$ is the Banach algebra with the underlying Banach space $A\oplus_1\mathbb{C}$ and the product

$$(a,\lambda)\cdot(a',\lambda')=\big(aa',\phi(a)\lambda'+\phi(a')\lambda+\lambda\lambda'\big).$$

Note that here we have assumed that $I=B=\mathbb{C}$ and $\theta=\phi$.

(v) Let A and B be Banach algebras, and let ϕ be a nonzero character on A. Then $A\bowtie^\theta B$, the generalized semidirect product of A with $B^\#$ along

B with respect to the homomorphism $\theta : A \to B^{\#}$ defined by $\theta(a) = (\phi(a), 0)$, is a Banach algebra with the underlying Banach space $A \oplus_1 B$ and the following product formula:

$$(a, b) \cdot (a', b') = \big(aa', \phi(a)b' + \phi(a')b + bb'\big).$$

This is the so-called *ϕ-Lau product* of Banach algebras A and B and is denoted by $A \oplus_\phi B$ (see [16]). This class includes the class of Lau algebras introduced in [11].

(vi) Another interesting example is the semidirect product of two Banach algebras. Indeed, let B be a Banach algebra, let A be a closed subalgebra of B, and let I be a closed ideal in B. If $\iota : A \to B$ is the inclusion map, then the Banach algebra $C = A \bowtie^\iota I$ is $A \ltimes I$, the semidirect product of A and I (see [6, p. 8]). The special semidirect product $C = B \ltimes I$ has been interesting for algebraists and is studied by many authors. Here we give some important examples of Banach algebras that can be recognized as semidirect products.

   (a) Let A be a dual Banach algebra with predual A_*, and consider A^{**}, the second dual of A equipped with the first (or second) Arens product. It is shown in [6, Theorem 2.15] that $A^{**} = A \ltimes A_*^\perp$, where $A_*^\perp = \{F \in A^{**} : F = 0 \text{ on } A_*\}$. We remark that every von Neumann algebra, the measure algebra $M(G)$, and the Fourier–Stieltjes algebra $B(G)$ of a locally compact group G, as well as the second dual of an Arens regular Banach algebra, are examples of dual Banach algebras.
   (b) The measure algebra of a locally compact group G, $M(G)$, has a semidirect product structure. In fact, we have $M(G) = l^1(G) \ltimes M_c(G)$, where $l^1(G)$ and $M_c(G)$ denote the space of discrete measures and continuous measures in $M(G)$, respectively.
   (c) The Banach algebra $A \bowtie^{\mathrm{id}} A$ is nothing other than $A \ltimes A$.

3. Derivations on $A \bowtie^\theta I$

In this section, we first characterize derivations on $A \bowtie^\theta I$ and then give a characterization of $\mathcal{H}^1(A \bowtie^\theta I, A \bowtie^\theta I)$.

If A is a Banach algebra and X and Y are Banach A-bimodules, then we denote by $\mathrm{Hom}_A(X, Y)$ the space of continuous A-bimodule homomorphisms from X into Y.

Proposition 3.1. *Let the only A-bimodule homomorphism $T : I \to A$ vanishing on I^2 be $T = 0$, where I is an A-bimodule via θ. Then $D \in \mathcal{Z}^1(A \bowtie^\theta I, A \bowtie^\theta I)$ if and only if*

$$D(a, i) = \big(D_A(a), D_{A,I}(a) + D_I(i)\big) \quad (a \in A, i \in I),$$

where

(i) $D_A \in \mathcal{Z}^1(A, A)$,
(ii) $D_I \in \mathcal{Z}^1(I, I)$,
(iii) $D_{A,I} \in \mathcal{Z}^1(A, I)$,

(iv) $D_I(i\theta(a)) - D_I(i)\theta(a) = i\theta(D_A(a)) + iD_{A,I}(a)$ *for all* $a \in A$ *and* $i \in I$,

(v) $D_I(\theta(a)i) - \theta(a)D_I(i) = \theta(D_A(a))i + D_{A,I}(a)i$ *for all* $a \in A$ *and* $i \in I$.

Moreover, $D = \mathrm{ad}_{(a,i)}$ *if and only if* $D_A = \mathrm{ad}_a$, $D_{A,I} = \mathrm{ad}_i$, *and* $D_I = \mathrm{ad}_{i+\theta(a)}$.

Proof. Let $\iota_A : A \to A \bowtie^\theta I$ and $\iota_I : I \to A \bowtie^\theta I$ be canonical injections, let $p_A : A \bowtie^\theta I \to A$ and $p_I : A \bowtie^\theta I \to I$ be canonical projections, and set

$$D_A = p_A \circ D \circ \iota_A, \qquad D_I = p_I \circ D \circ \iota_I,$$
$$D_{A,I} = p_A \circ D \circ \iota_I, \qquad \text{and} \qquad D_{A,I} = p_I \circ D \circ \iota_A.$$

Then

$$D(a,i) = \big(D_A(a) + D_{I,A}(i), D_{A,I}(a) + D_I(i)\big) \quad (a \in A, i \in I).$$

Let $(a,i), (a',i') \in A \bowtie^\theta I$. Then

$$\begin{aligned} D\big((a,i)\cdot(a',i')\big) &= D\big(aa', \theta(a)i' + i\theta(a') + ii'\big) \\ &= \big(D_A(aa') + D_{I,A}\big(\theta(a)i' + i\theta(a') + ii'\big), \\ &\qquad D_{A,I}(aa') + D_I\big(\theta(a)i' + i\theta(a') + ii'\big)\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} &(a,i)\cdot D(a',i') + D(a,i)\cdot(a',i') \\ &\quad = (a,i)\cdot\big(D_A(a') + D_{I,A}(i'), D_{A,I}(a') + D_I(i')\big) \\ &\qquad + \big(D_A(a) + D_{I,A}(i), D_{A,I}(a) + D_I(i)\big)\cdot(a',i') \\ &\quad = \big(aD_A(a') + aD_{I,A}(i'), i\theta\big(D_A(a')\big) + i\theta\big(D_{I,A}(i')\big) \\ &\qquad + \theta(a)D_{A,I}(a') + \theta(a)D_I(i') + iD_{A,I}(a') + iD_I(i')\big) \\ &\qquad + \big(D_A(a)a' + D_{I,A}(i)a', \theta\big(D_A(a)\big)i' + \theta\big(D_{I,A}(i)\big)i' \\ &\qquad + D_{A,I}(a)\theta(a') + D_I(i)\theta(a') + D_{A,I}(a)i' + D_I(i)i'\big). \end{aligned}$$

Putting $i = i' = 0$, one obtains

$$D_A(aa') = aD_A(a') + D_A(a)a'$$

and

$$D_{A,I}(aa') = \theta(a)D_{A,I}(a') + D_{A,I}(a)\theta(a') = a\cdot D_{A,I}(a') + D_{A,I}(a)\cdot a'.$$

Therefore, $D_A \in \mathcal{Z}^1(A,A)$ and $D_{A,I} \in \mathcal{Z}^1(A,I)$.

By letting $a = 0$ and $i' = 0$ ($a' = 0$ and $i = 0$), we find that $D_{I,A} \in \mathrm{Hom}_A(I,A)$.

By setting $a = a' = 0$ we get $D_{I,A}(ii') = 0$, and so, by the assumption, $D_{I,A} = 0$. This result, along with $a = a' = 0$, gives

$$D_I(ii') = iD_I(i') + D_I(i)i';$$

that is, $D_I \in \mathcal{Z}^1(I,I)$. Now put $a = 0$ and $i' = 0$. Then

$$D_I(i\cdot a') = D_I\big(i\theta(a')\big) = D_I(i)\theta(a') + i\theta\big(D_A(a')\big) + iD_{A,I}(a').$$

Similarly, by substituting $a' = 0$ and $i = 0$, we obtain

$$D_I(a\cdot i') = D_I\big(\theta(a)i'\big) = \theta(a)D_I(i') + \theta\big(D_A(a)\big)i' + D_{A,I}(a)i'.$$

If $D = \mathrm{ad}_{(a,i)}$, for some $a \in A$ and $i \in I$, then

$$\begin{aligned} D(a',i') &= \mathrm{ad}_{(a,i)}(a',i') = (a',i')\cdot(a,i) - (a,i)\cdot(a',i') \\ &= \big(a'a, \theta(a')i + i'\theta(a) + i'i\big) - \big(aa', \theta(a)i' + i\theta(a') + ii'\big) \\ &= \Big(a'a - aa', \big(\theta(a')i - i\theta(a')\big) + \big(i'\big(i+\theta(a)\big) - \big(i+\theta(a)\big)i'\big)\Big) \\ &= \Big(\mathrm{ad}_a(a'), \mathrm{ad}_i\big(\theta(a')\big) + \mathrm{ad}_{i+\theta(a)}(i')\Big). \end{aligned}$$

Therefore, $D_A = \mathrm{ad}_a$, $D_{A,I} = \mathrm{ad}_i$, and $D_I = \mathrm{ad}_{i+\theta(a)}$. □

Proposition 3.2. *Let $T : I \to I$ be a derivation that is also an A-bimodule homomorphism. Then $D : A \bowtie^\theta I \to A \bowtie^\theta I$ defined by $D(a,i) = (0, T(i))$ is a derivation. Moreover, D is inner if and only if there exist $a \in Z(A)$ and $i \in Z_{\theta(A)}(I)$ such that $T = \mathrm{ad}_{i+\theta(a)}$, where $Z(A)$ is the algebraic center of A and $Z_{\theta(A)}(I) = \{i \in I : \theta(a)i = i\theta(a) \text{ for all } a \in A\}$.*

Proof. This is a straightforward verification. □

We write $C_A(I,I)$ for the subspace $\{T : I \to I : T = \mathrm{ad}_{i+\theta(a)}, a \in Z(A), i \in Z_{\theta(A)}(I)\}$ of $\mathcal{Z}^1(I,I) \cap \mathrm{Hom}_A(I,I)$. Note that if $\theta(A) + I$ is commutative, then $C_A(I,I) = \{0\}$.

Theorem 3.3. *Let $\mathcal{H}^1(A,A) = \mathcal{H}^1(A,I) = 0$, and let the only A-bimodule homomorphism $T : I \to A$ vanishing on I^2 be $T = 0$. Then, as vector spaces, we have*

$$\mathcal{H}^1(A \bowtie^\theta I, A \bowtie^\theta I) \cong \frac{\mathcal{Z}^1(I,I) \cap \mathrm{Hom}_A(I,I)}{C_A(I,I)}.$$

Proof. Define the mapping $\eta : \mathcal{Z}^1(I,I) \cap \mathrm{Hom}_A(I,I) \to \mathcal{H}^1(A \bowtie^\theta I, A \bowtie^\theta I)$ by $\eta(T) = [D_T]$, where $D_T \in \mathcal{Z}^1(A \bowtie^\theta I, A \bowtie^\theta I)$ is defined by $D_T(a,i) = (0, T(i))$ and $[D_T]$ denotes the equivalence class of D_T in $\mathcal{H}^1(A \bowtie^\theta I, A \bowtie^\theta I)$. By Proposition 3.2, η is well defined and clearly it is linear. To show that η is surjective, let D be a derivation on $A \bowtie^\theta I$. Then, by Proposition 3.1, D is of the form

$$D(a,i) = \big(D_A(a), D_{A,I}(a) + D_I(i)\big) \quad (a \in A, i \in I).$$

Since $\mathcal{H}^1(A,A) = \mathcal{H}^1(A,I) = 0$, there exist $b \in A$ and $j \in I$ such that $D_A = \mathrm{ad}_b$ and $D_{A,I} = \mathrm{ad}_j$. Set $T = D_I - \mathrm{ad}_{\theta(b)} - \mathrm{ad}_j$, which is a derivation on I. Let us check that T is an A-bimodule homomorphism by noting that I is an A-bimodule via θ. Let $a \in A$ and $i \in I$. Then by condition (v) in Proposition 3.1,

$$\begin{aligned} T(a\cdot i) &= T\big(\theta(a)i\big) = D_I\big(\theta(a)i\big) - \mathrm{ad}_{\theta(b)}\big(\theta(a)i\big) - \mathrm{ad}_j\big(\theta(a)i\big) \\ &= \theta(a)D_I(i) + \theta\big(\mathrm{ad}_b(a)\big)i + \mathrm{ad}_j(a)i - \mathrm{ad}_{\theta(b)}\big(\theta(a)i\big) - \mathrm{ad}_j\big(\theta(a)i\big) \\ &= \theta(a)D_I(i) + \theta(a)\theta(b)i - \theta(b)\theta(a)i + \theta(a)ji - j\theta(a)i - \theta(a)i\theta(b) \\ &\quad + \theta(b)\theta(a)i - \theta(a)ij + j\theta(a)i \\ &= \theta(a)D_I(i) + \theta(a)\big(\theta(b)i - i\theta(b)\big) + \theta(a)(ji - ij) \\ &= \theta(a)D_I(i) - \theta(a)\,\mathrm{ad}_{\theta(b)}(i) - \theta(a)\,\mathrm{ad}_j(i) \\ &= \theta(a)T(i) = a\cdot T(i). \end{aligned}$$

Likewise, $T(i \cdot a) = T(i) \cdot a$ for $a \in A$ and $i \in I$. We now have

$$D(a,i) - D_T(a,i) = \big(\mathrm{ad}_b(a), \mathrm{ad}_j(a) + \mathrm{ad}_{j+\theta(b)}(i)\big) = \mathrm{ad}_{(b,j)}(a,i),$$

and so $[D] = [D_T]$. This shows that $\eta(T) = [D]$; that is, η is surjective. Now we calculate the kernel of η:

$$\begin{aligned}\ker\eta &= \big\{T \in \mathcal{Z}^1(I,I) \cap \mathrm{Hom}_A(I,I) : [D_T] = [0]\big\} \\ &= \big\{T \in \mathcal{Z}^1(I,I) \cap \mathrm{Hom}_A(I,I) : D_T \text{ is inner}\big\} \\ &= \big\{T \in \mathcal{Z}^1(I,I) \cap \mathrm{Hom}_A(I,I) : T = \mathrm{ad}_{i+\theta(a)}, a \in Z(A), i \in Z_{\theta(A)}(I)\big\} \\ &= C_A(I,I),\end{aligned}$$

where we used Proposition 3.2 in the third equality. Therefore, as vector spaces, we have

$$\mathcal{H}^1(A \bowtie^\theta I, A \bowtie^\theta I) \cong \frac{\mathcal{Z}^1(I,I) \cap \mathrm{Hom}_A(I,I)}{C_A(I,I)}. \qquad \square$$

We remark that if $\overline{I^2} = I$, especially if I is weakly amenable or if I has a one-sided approximate identity, then the only A-bimodule homomorphism $T : I \to A$ that satisfies $T|_{I^2} = 0$ is $T = 0$.

Let G be a locally compact group, let $M(G)$ be the measure algebra of G, let $L^1(G)$ be the group algebra of G which has a bounded approximate identity, let $l^1(G)$ be the space of discrete measures in $M(G)$, let $M_c(G)$ be the space of continuous measures in $M(G)$, and let $M_s(G)$ be the subspace of $M(G)$ consisting of singular measures with respect to the Haar measure on G. The measure algebra is a Banach algebra under the convolution product of measures, $L^1(G)$ and $M_c(G)$ are closed ideals of $M(G)$, $l^1(G)$ is a closed subalgebra of $M(G)$, and $M_s(G)$ is a closed G-invariant subspace of $M(G)$. Also, $M(G) = L^1(G)$ if and only if $M(G) = M_c(G)$, if and only if $M(G) = l^1(G)$, and if and only if G is discrete. Furthermore, we have the following decomposition as linear spaces:

$$M(G) = l^1(G) \oplus_1 M_c(G) = l^1(G) \oplus_1 L^1(G) \oplus_1 M_s(G)$$

(see [4, Theorem 2.3.36]). As we mentioned in Example 2.1(vii), as Banach algebras we have $M(G) = l^1(G) \ltimes M_c(G)$.

Example 3.4. Let G be a locally compact group. We know that $\mathcal{H}^1(l^1(G), l^1(G)) = 0$ (see [12]). We show that $\mathcal{H}^1(l^1(G), L^1(G)) = 0$. Let $D : l^1(G) \to L^1(G)$ be a derivation. Then, as readily checked, $\tilde{D} : M(G) \to M(G)$ defined by $\tilde{D}(\mu) = D(\mu_d)$ is a derivation, where $\mu = \mu_d + \mu_a + \mu_s$, $\mu_d \in l^1(G)$, $\mu_a \in L^1(G)$, and $\mu_s \in M_s(G)$. Since $\mathcal{H}^1(M(G), M(G)) = 0$, there is $\nu \in M(G)$ such that $\tilde{D} = \mathrm{ad}_\nu$. So, for each $\mu \in M(G)$,

$$\begin{aligned}D(\mu_d) = \tilde{D}(\mu) = \tilde{D}(\mu_d) &= \mu_d * \nu - \nu * \mu_d \\ &= (\mu_d * \nu_d - \nu_d * \mu_d) + (\mu_d * \nu_a - \nu_a * \mu_d) + (\mu_d * \nu_s - \nu_s * \mu_d).\end{aligned}$$

Since $D(\mu_d) - (\mu_d * \nu_a - \nu_a * \mu_d) \in L^1(G) \cap (l^1(G) \oplus M_s(G)) = \{0\}$, we have $D = \mathrm{ad}_{\nu_a}$, showing $\mathcal{H}^1(l^1(G), L^1(G)) = 0$. Now let $D : L^1(G) \to L^1(G)$ be a derivation which is also an $l^1(G)$-module homomorphism. Then, by [12], there is $\mu \in M(G)$ such that $D = \mathrm{ad}_\mu$. If $f \in L^1(G)$ and $x \in G$, then

$$\delta_x * f * \mu - \delta_x * \mu * f = \delta_x * D(f) = D(\delta_x * f) = \delta_x * f * \mu - \mu * \delta_x * f,$$

and so $\delta_x * \mu * f = \mu * \delta_x * f$, where δ_x is the Dirac measure at x. Let (e_α) be a bounded approximate identity for $L^1(G)$. Then $\delta_x * \mu * e_\alpha = \mu * \delta_x * e_\alpha$ and since $e_\alpha \to \delta_e$ in the w^*-topology of $M(G) = \mathcal{C}_0(G)^*$, we have $\delta_x * \mu = \mu * \delta_x$ for each $x \in G$, where e is the identity of G and $\mathcal{C}_0(G)$ is the space of continuous functions on G vanishing at infinity. It follows from the strict density of $l^1(G)$ in $M(G)$ that $\mu \in Z(M(G))$. Hence $D = \mathrm{ad}_\mu = 0$. Therefore, by Theorem 3.3, we have

$$\mathcal{H}^1\big(l^1(G) \ltimes L^1(G), l^1(G) \ltimes L^1(G)\big) = \mathcal{H}^1\big(l^1(G) \bowtie^\iota L^1(G), l^1(G) \bowtie^\iota L^1(G)\big) = 0,$$

where $\iota : l^1(G) \to M(G)$ is the inclusion map. Similarly,

$$\mathcal{H}^1\big(l^1(G) \ltimes M(G), l^1(G) \ltimes M(G)\big) = \mathcal{H}^1\big(l^1(G) \bowtie^\iota M(G), l^1(G) \bowtie^\iota M(G)\big) = 0.$$

Example 3.5. Let G be a locally compact group. We claim that the only $l^1(G)$-bimodule homomorphism $T : M_c(G) \to l^1(G)$ vanishing on $M_c(G)^2$ is $T = 0$, although $\overline{M_c(G)^2} \neq M_c(G)$ in general. In fact, we show that $\mathrm{Hom}_{l^1(G)}(M_c(G), l^1(G)) = 0$. Let $T : M_c(G) \to l^1(G)$ be an $l^1(G)$-bimodule homomorphism. Define $\tilde{T} : M(G) \to M(G)$ by $\tilde{T}(\mu) = T(\mu_c)$, where $\mu = \mu_d + \mu_c$, $\mu_d \in l^1(G)$, and $\mu_c \in M_c(G)$. Since $l^1(G)$ is a subalgebra of $M(G)$ and $M_c(G)$ is an ideal in $M(G)$, $\tilde{T}$ is an $l^1(G)$-module homomorphism and $\tilde{T}|_{l^1(G)} = 0$. Because of strict density of $l^1(G)$ in $M(G)$, $\mathrm{Hom}_{l^1(G)}(M(G), M(G))$ is equal to $\mathrm{Hom}_{M(G)}(M(G), M(G))$, and the latter is isomorphic to $Z(M(G))$ since $M(G)$ is unital. Thus, there is $\nu \in Z(M(G))$ such that $\tilde{T}(\mu) = \mu * \nu$ for each $\mu \in M(G)$. Hence,

$$T(\mu_c) = \tilde{T}(\mu) = \tilde{T}(\mu_c) = \mu_c * \nu \in l^1(G) \cap M_c(G) = \{0\},$$

and so $T = 0$. Now we claim that $\mathcal{H}^1(l^1(G), M_c(G)) = 0$. Let $D : l^1(G) \to M_c(G)$ be a derivation. Then $\tilde{D} : M(G) \to M(G)$ defined by $\tilde{D}(\mu) = D(\mu_d)$ is a derivation, too. Therefore, there is $\nu \in M(G)$ such that $\tilde{D} = \mathrm{ad}_\nu$. So, for each $\mu \in M(G)$,

$$D(\mu_d) = \tilde{D}(\mu_d) = \mu_d * \nu - \nu * \mu_d = (\mu_d * \nu_d - \nu_d * \mu_d) + (\mu_d * \nu_c - \nu_c * \mu_d).$$

Since $D(\mu_d) - (\mu_d * \nu_c - \nu_c * \mu_d) \in M_c(G) \cap l^1(G) = \{0\}$, we have $D = \mathrm{ad}_{\nu_c}$ and hence $\mathcal{H}^1(l^1(G), M_c(G)) = 0$.

It can be easily verified that $C_{l^1(G)}(M_c(G), M_c(G)) = 0$. Therefore, by Theorem 3.3, we have

$$\begin{aligned}&\mathcal{H}^1\big(l^1(G) \ltimes M_c(G), l^1(G) \ltimes M_c(G)\big)\\ &\quad = \mathcal{Z}^1\big(M_c(G), M_c(G)\big) \cap \mathrm{Hom}_{l^1(G)}\big(M_c(G), M_c(G)\big).\end{aligned}$$

Similar to Example 3.4, we observe that

$$\mathcal{Z}^1\big(M_c(G), M_c(G)\big) \cap \mathrm{Hom}_{l^1(G)}\big(M_c(G), M_c(G)\big) = \{0\},$$

and thus

$$\mathcal{H}^1\big(l^1(G) \ltimes M_c(G), l^1(G) \ltimes M_c(G)\big) = 0.$$

In particular, by noting that $M(G) = l^1(G) \ltimes M_c(G)$, we get $\mathcal{H}^1(M(G), M(G)) = 0$, which is unfortunately used in the calculation of $\mathcal{H}^1(l^1(G), M_c(G))$ above. So, if one could show that $\mathcal{H}^1(l^1(G), M_c(G)) = 0$ by another technique that does not rely on the innerness of derivations on $M(G)$, then this would solve the derivation problem of Johnson (see [4, Question 5.6.B, p. 746]), which is solved by V. Losert in [12] (a short proof is given in [2]).

Example 3.6. Let G be a locally compact group such that $\mathcal{H}^1(L^1(G), L^1(G)) = 0$ (e.g., G can be a discrete group or an abelian group). Let $i : L^1(G) \to M(G)$ be the usual inclusion map. Then a similar calculation as in Examples 3.4 and 3.5 shows that

$$\mathcal{H}^1\big(L^1(G) \bowtie^i M(G), L^1(G) \bowtie^i M(G)\big) = 0.$$

In [12], V. Losert proved that derivations from $L^1(G)$ to $M(G)$ are inner. We show that derivations from $M(G)$ to $L^1(G)$ are also inner, which is needed in next example.

Lemma 3.7. *Let G be a locally compact group, and let*

$$M_0(G) = \big\{\nu \in M(G) : \mu * \nu - \nu * \mu \in L^1(G) \text{ for all } \mu \in M(G)\big\}.$$

Then $M_0(G) = Z(M(G)) + L^1(G)$.

Proof. Without loss of generality we may assume that G is nondiscrete. Obviously, $L^1(G) + Z(M(G)) \subseteq M_0(G)$. Let $\nu \in M_0(G)$. Then $\nu = \nu_d + \nu_a + \nu_s$ for some $\nu_d \in l^1(G)$, $\nu_a \in L^1(G)$, and $\nu_s \in M_s(G)$. Since $M_s(G)$ is invariant under translations by elements of G, for each $x \in G$ we have $\delta_x * \nu_s, \nu_s * \delta_x \in M_s(G)$. Thus

$$\begin{aligned}(\delta_x * \nu - \nu * \delta_x) - (\delta_x * \nu_a - \nu_a * \delta_x) &= \delta_x * (\nu_d + \nu_s) - (\nu_d + \nu_s) * \delta_x \\ &= (\delta_x * \nu_d - \nu_d * \delta_x) + (\delta_x * \nu_s - \nu_s * \delta_x) \\ &\in L^1(G) \cap \big(l^1(G) \oplus M_s(G)\big) = \{0\},\end{aligned}$$

and so $\delta_x * (\nu_d + \nu_s) = (\nu_d + \nu_s) * \delta_x$ for all $x \in G$. Since $l^1(G)$ is dense in $M(G)$ with respect to the strict topology, it follows that $\nu_d + \nu_s \in Z(M(G))$. Therefore, $\nu \in L^1(G) + Z(M(G))$. □

Theorem 3.8. *Let G be a locally compact group. Then $\mathcal{H}^1(M(G), L^1(G)) = 0$.*

Proof. Let $\Phi : \mathcal{Z}^1(M(G), L^1(G)) \to \frac{M(G)}{Z(M(G))+L^1(G)}$ be defined by $\Phi(D) = [\mu]$, where $D = \mathrm{ad}_\mu$ for some $\mu \in M(G)$, and $[\mu]$ denotes the equivalence class of $\mu \in M(G)$ in the quotient space $\frac{M(G)}{Z(M(G))+L^1(G)}$. It is easy to see that $\ker \Phi = \mathcal{B}^1(M(G), L^1(G))$. So

$$\mathcal{H}^1\big(M(G), L^1(G)\big) \cong \frac{M_0(G)}{Z(M(G)) + L^1(G)},$$

and hence by Lemma 3.7, $\mathcal{H}^1(M(G), L^1(G)) = 0$. □

Example 3.9. Let G be a locally compact group. By Theorem 3.8, all of the conditions of Theorem 3.3 hold. By Wendel's theorem [4, Theorem 3.3.40], $\mathrm{Hom}_{M(G)}(L^1(G), L^1(G)) = Z(M(G))$. Thus, the only derivation $D : L^1(G) \to L^1(G)$ which is an $M(G)$-bimodule homomorphism is $D = 0$. Therefore, by Theorem 3.3, $\mathcal{H}^1(M(G) \ltimes L^1(G), M(G) \ltimes L^1(G)) = 0$.

Corollary 3.10. *Let A be a dual Banach algebra with predual A_*. As we explained in Example 2.1(vi), $A^{**} = A \ltimes A_*^\perp = A \bowtie^t A_*^\perp$. So, if $\mathcal{H}^1(A, A) = \mathcal{H}^1(A, A_*^\perp) = 0$ and the only A-bimodule homomorphism $T : A_*^\perp \to A$ vanishing on $(A_*^\perp)^2$ is $T = 0$, then*

$$\mathcal{H}^1(A^{**}, A^{**}) \cong \frac{\mathcal{Z}^1(A_*^\perp, A_*^\perp) \cap \mathrm{Hom}_A(A_*^\perp, A_*^\perp)}{C_A(A_*^\perp, A_*^\perp)}.$$

Example 3.11. Let $A = l^1$ be the usual Banach sequence algebra which is a commutative dual Banach algebra under pointwise multiplication with predual $A_* = c_0$. Then $(l^1)^{**} = l^1 \ltimes c_0^\perp$. We show that l^1 satisfies conditions of Corollary 3.10 and then calculate $\mathcal{H}^1((l^1)^{**}, (l^1)^{**})$. Since l^1 is a commutative weakly amenable Banach algebra and l^1 and $c_0^\perp$ are symmetric l^1-bimodules, we have $\mathcal{H}^1(l^1, l^1) = \mathcal{H}^1(l^1, c_0^\perp) = 0$. It can be easily shown that $F \,\square\, G = G \,\square\, F = 0$ for all $F \in c_0^\perp$ and $G \in (l^1)^{**}$, where $\square$ denotes the first Arens product. Thus, if $T : c_0^\perp \to l^1$ is an l^1-bimodule homomorphism, for all $a \in l^1$ and $F \in c_0^\perp$ we have $0 = T(a \,\square\, F) = a \,\square\, T(F) = a \cdot T(F)$, which implies $T = 0$. Therefore, by Corollary 3.10, we have

$$\mathcal{H}^1\big((l^1)^{**}, (l^1)^{**}\big) = \frac{\mathcal{Z}^1(c_0^\perp, c_0^\perp) \cap \mathrm{Hom}_{l^1}(c_0^\perp, c_0^\perp)}{C_{l^1}(c_0^\perp, c_0^\perp)}.$$

Since the action of $(l^1)^{**}$ on $c_0^\perp$ is trivial, $C_{l^1}(c_0^\perp, c_0^\perp) = 0$ and every bounded linear map on $c_0^\perp$ is a derivation and an l^1-module homomorphism; that is, $\mathcal{Z}^1(c_0^\perp, c_0^\perp) = \mathrm{Hom}_{l^1}(c_0^\perp, c_0^\perp) = \mathcal{B}(c_0^\perp)$, where $\mathcal{B}(c_0^\perp)$ is the space of all bounded operators on $c_0^\perp$. Hence

$$\mathcal{H}^1\big((l^1)^{**}, (l^1)^{**}\big) = \mathcal{B}(c_0^\perp).$$

Example 3.12. Let A be a dual Banach algebra with predual A_* such that $\mathcal{H}^1(A, A) = \mathcal{H}^1(A, A^{**}) = 0$ (e.g., A can be a von Neumann algebra or the group algebra of a discrete group), and let the only A-bimodule homomorphism $T : A_*^\perp \to A$ vanishing on $(A_*^\perp)^2$ be $T = 0$. We show that A satisfies the conditions of Corollary 3.10. So we have to show that $\mathcal{H}^1(A, A_*^\perp) = 0$. Let $D : A \to A_*^\perp$ be a derivation. Then $\iota \circ D : A \to A^{**}$ is also a derivation, where $\iota : A_*^\perp \to A^{**}$ is the inclusion map. Since $\mathcal{H}^1(A, A^{**}) = 0$, there is $F \in A^{**}$ such that $\iota \circ D = \mathrm{ad}_F$. Let $F = F_1 + F_2$, where $F_1 \in A$ and $F_2 \in A_*^\perp$. Thus, for each $a \in A$, $D(a) = (a \cdot F_1 - F_1 \cdot a) + (a \cdot F_2 - F_2 \cdot a)$. Since $A_*^\perp$ is an ideal in A^{**}, $a \cdot F_1 - F_1 \cdot a = D(a) - (a \cdot F_2 - F_2 \cdot a) \in A_*^\perp$ and so $a \cdot F_1 - F_1 \cdot a = 0$ for each $a \in A$. Hence $D = \mathrm{ad}_{F_2}$, which shows that $\mathcal{H}^1(A, A_*^\perp) = 0$.

As an application, let A be a von Neumann algebra such that the only A-bimodule homomorphism $T : A_*^\perp \to A$ vanishing on $(A_*^\perp)^2$ is $T = 0$. Since A^{**} is also a von Neumann algebra, we have $\mathcal{H}^1(A^{**}, A^{**}) = 0$. Therefore, $\mathcal{Z}^1(A_*^\perp, A_*^\perp) \cap \mathrm{Hom}_A(A_*^\perp, A_*^\perp) = C_A(A_*^\perp, A_*^\perp)$; that is, every derivation $D : A_*^\perp \to A_*^\perp$ which is an A-bimodule homomorphism is of the form ad_{a+F} where $a \in Z(A)$ and $F \in Z_A(A_*^\perp)$.

3.1. **Derivations on $A \oplus_\phi B$.** In the following theorem, we reformulate Theorem 3.3 for $A \oplus_\phi B$. We write B_ϕ for the Banach algebra B as a symmetric Banach A-bimodule with the action $a \cdot b = b \cdot a = \phi(a)b$ for $a \in A$ and $b \in B$. Also, $\sigma(A)$ denotes the space of nonzero characters of A.

Theorem 3.13. *Let $\mathcal{H}^1(A, A) = \mathcal{H}^1(A, B_\phi) = 0$, and let the only A-bimodule homomorphism $T : B_\phi \to A$ vanishing on B^2 be $T = 0$. Then, as vector spaces,*

$$\mathcal{H}^1(A \oplus_\phi B, A \oplus_\phi B) \cong \mathcal{H}^1(B, B).$$

Proof. It is enough to note that $\mathrm{Hom}_A(B_\phi, B_\phi) = \mathcal{B}(B_\phi)$ and $C_A(B_\phi, B_\phi) = \mathcal{B}^1(B_\phi, B_\phi)$. □

Corollary 3.14. *Let A be a weakly amenable commutative Banach algebra, $\phi \in \sigma(A)$, and let B be a Banach algebra such that the only A-bimodule homomorphism $T : B_\phi \to A$ vanishing on B^2 is $T = 0$. Then $\mathcal{H}^1(A \oplus_\phi B, A \oplus_\phi B) = \mathcal{H}^1(B, B)$.*

Without any assumption on A and B, we have the following theorem.

Theorem 3.15. *$\mathcal{H}^1(B, B)$ embeds in $\mathcal{H}^1(A \oplus_\phi B, A \oplus_\phi B)$, and so $\mathcal{H}^1(A \oplus_\phi B, A \oplus_\phi B) \neq 0$ if $\mathcal{H}^1(B, B) \neq 0$.*

Proof. For $D \in \mathcal{Z}^1(B, B)$, define $\tilde{D} : A \oplus_\phi B \to A \oplus_\phi B$ by $\tilde{D}(a, b) = (0, D(b))$. Then $\tilde{D}$ is a derivation, and it is inner if and only if D is inner. Now the mapping $D \mapsto \tilde{D}$ from $\mathcal{Z}^1(B, B)$ into $\mathcal{Z}^1(A \oplus_\phi B, A \oplus_\phi B)$ gives the desired embedding. □

Corollary 3.16. *Let A and B be commutative Banach algebras. If $\mathcal{H}^1(B, B) \neq 0$, then $A \oplus_\phi B$ is not n-weakly amenable for each $n \in \mathbb{N}$.*

Proof. This follows from Theorem 3.15 and [5, p. 23]. □

Example 3.17. Let $A = l^1$, let $B = c_0$ or l^∞, and let $\phi \in \sigma(l^1)$. Then $\mathcal{H}^1(l^1, l^1) = \mathcal{H}^1(l^1, c_0) = \mathcal{H}^1(c_0, c_0) = \mathcal{H}^1(l^1, l^\infty) = \mathcal{H}^1(l^\infty, l^\infty) = 0$, and so by Theorem 3.13,

$$\mathcal{H}^1(l^1 \oplus_\phi c_0, l^1 \oplus_\phi c_0) = \mathcal{H}^1(l^1 \oplus_\phi l^\infty, l^1 \oplus_\phi l^\infty) = 0.$$

Example 3.18. Let X be a locally compact Hausdorff space, and let $\mathcal{C}_0(X)$ be the space of continuous functions on X vanishing at ∞. Then $\mathcal{H}^1(\mathcal{C}_0(X), \mathcal{C}_0(X)) = 0$ because $\mathcal{C}_0(X)$ is a commutative semisimple C^*-algebra. Let B be an arbitrary square dense Banach algebra, and let $\phi \in \sigma(\mathcal{C}_0(X)) = X$. Then, by noting that $\mathcal{C}_0(X)$ is a commutative weakly amenable Banach algebra and that B_ϕ is a symmetric Banach $\mathcal{C}_0(X)$-bimodule, we have $\mathcal{H}^1(\mathcal{C}_0(X), B_\phi) = 0$ and thus

$$\mathcal{H}^1\big(\mathcal{C}_0(X) \oplus_\phi B, \mathcal{C}_0(X) \oplus_\phi B\big) \cong \mathcal{H}^1(B, B).$$

Moreover, if B is commutative and $\mathcal{H}^1(B, B) \neq 0$, then $\mathcal{C}_0(X) \oplus_\phi B$ cannot be n-weakly amenable for any $n \in \mathbb{N}$.

Example 3.19. Let G be a locally compact abelian group, and let $A = L^1(G)$ and $B = M(G)$. Then, for any $\phi \in \sigma(L^1(G))$,

$$\mathcal{H}^1\big(L^1(G) \oplus_\phi M(G), L^1(G) \oplus_\phi M(G)\big) = 0.$$

Example 3.20. Let G be a locally compact group, let $A = \mathrm{A}(G)$ be the Fourier algebra of G, and let $B = \mathrm{VN}(G)$ be the von Neumann algebra of G. Let also $\phi \in \sigma(A(G)) = \{\epsilon_x : x \in G\}$, where $\epsilon_x : \mathrm{A}(G) \to \mathbb{C}$ is the evaluational function at x. Since $\mathrm{A}(G)$ is a commutative semisimple Banach algebra, by [15], $\mathcal{H}^1(\mathrm{A}(G), \mathrm{A}(G)) = 0$. Now, if the connected component of the identity in G is abelian, then, by [9], $\mathrm{A}(G)$ is weakly amenable and so $\mathcal{H}^1(\mathrm{A}(G), \mathrm{VN}(G)_\phi) = 0$. Therefore,

$$\mathcal{H}^1\big(\mathrm{A}(G) \oplus_\phi \mathrm{VN}(G), \mathrm{A}(G) \oplus_\phi \mathrm{VN}(G)\big) \cong \mathcal{H}^1\big(\mathrm{VN}(G), \mathrm{VN}(G)\big) = 0.$$

4. Derivations on $A \bowtie^{\mathrm{id}} A$

In this section, we characterize explicitly the first cohomology group of $A \bowtie^{\mathrm{id}} A$ in terms of that of A.

Proposition 4.1. *Let $D : A \to A$ be a derivation. Then $\hat{D} : A \bowtie^{\mathrm{id}} A \to A \bowtie^{\mathrm{id}} A$ defined by $\hat{D}(a, b) = (D(a), D(b))$ is a derivation. Moreover, $\hat{D}$ is inner if and only if D is inner. In this case, $\hat{D} = \mathrm{ad}_{(a,b)}$ if and only if $D = \mathrm{ad}_a$ and $b \in Z(A)$.*

Proof. It can be routinely checked that $\hat{D}$ is a derivation. Now, if $\hat{D} = \mathrm{ad}_{(a,b)}$, then, for each $c, d \in A$,

$$\begin{aligned}\big(D(c), D(d)\big) &= (ca - ac, cb + da + db - ad - bc - bd)\\ &= \big(\mathrm{ad}_a(c), \mathrm{ad}_b(c) + \mathrm{ad}_a(d) + \mathrm{ad}_b(d)\big).\end{aligned}$$

So $D = \mathrm{ad}_a$, and by putting $d = 0$ we get $\mathrm{ad}_b(c) = 0$ for all $c \in A$; that is, $b \in Z(A)$. Now the fact that b belongs to $Z(A)$ implies $\mathrm{ad}_b = 0$ and the proof is complete. □

Theorem 4.2. *$\mathcal{H}^1(A, A)$ embeds in $\mathcal{H}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A)$, and so $\mathcal{H}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A) \neq 0$ if $\mathcal{H}^1(A, A) \neq 0$.*

Proof. The mapping $D \mapsto \hat{D}$ from $\mathcal{Z}^1(A, A)$ into $\mathcal{Z}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A)$ gives the desired embedding by Proposition 4.1. □

Corollary 4.3. *Let A be a commutative Banach algebra such that $\mathcal{H}^1(A, A) \neq 0$. Then the Banach algebra $A \bowtie^{\mathrm{id}} A$ cannot be n-weakly amenable for each $n \in \mathbb{N}$.*

Now we characterize $\mathcal{H}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A)$ in a more general case. A Banach algebra A is termed *left (right) faithful* if $aA = 0$ ($Aa = 0$) for some $a \in A$ implies $a = 0$.

Proposition 4.4. *Let A be a left (or right) faithful Banach algebra, and let the only A-bimodule homomorphism $T : A \to A$ vanishing on A^2 be $T = 0$. Then $D \in \mathcal{Z}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A)$ if and only if*

$$D(a, b) = \big(D_1(a), D_2(a) + D_1(b) + D_2(b)\big) \quad (a, b \in A),$$

for some $D_1, D_2 \in \mathcal{Z}^1(A,A)$. *Moreover,* $D = \mathrm{ad}_{(a,b)}$ *if and only if* $D_1 = \mathrm{ad}_a$ *and* $D_2 = \mathrm{ad}_b$.

Proof. This proposition follows from the faithfulness of A and condition (iv) (or (v)) of Proposition 3.1. □

Theorem 4.5. *If A is left (or right) faithful and A^2 is dense in A, then, as vector spaces, we have*

$$\mathcal{H}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A) \cong \mathcal{H}^1(A,A) \oplus \mathcal{H}^1(A,A).$$

Proof. Define $\eta : \mathcal{Z}^1(A,A) \oplus \mathcal{Z}^1(A,A) \to \mathcal{H}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A)$ by $\eta(D_1, D_2) = [D]$, where $D \in \mathcal{Z}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A)$ is defined by

$$D(a,b) = \big(D_1(a), D_2(a) + D_1(b) + D_2(b)\big),$$

and $[D]$ denotes the equivalence class of D in $\mathcal{H}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A)$. By Proposition 4.4, η is a well-defined surjective linear map and

$$\begin{aligned}\ker \eta &= \big\{(D_1, D_2) \in \mathcal{Z}^1(A,A) \oplus \mathcal{Z}^1(A,A) : [D] = [0]\big\}\\ &= \big\{(D_1, D_2) \in \mathcal{Z}^1(A,A) \oplus \mathcal{Z}^1(A,A) : D \text{ is inner}\big\}\\ &= \big\{(D_1, D_2) \in \mathcal{Z}^1(A,A) \oplus \mathcal{Z}^1(A,A) : D_1 \text{ and } D_2 \text{ are inner}\big\}\\ &= \mathcal{B}^1(A,A) \oplus \mathcal{B}^1(A,A),\end{aligned}$$

which implies

$$\mathcal{H}^1(A \bowtie^{\mathrm{id}} A, A \bowtie^{\mathrm{id}} A) \cong \mathcal{H}^1(A,A) \oplus \mathcal{H}^1(A,A)$$

as vector spaces. □

Corollary 4.6. *If A has a left (or right) approximate identity, then the Banach algebra $A \bowtie^{\mathrm{id}} A$ has automatically continuous derivations if and only if A has automatically continuous derivations.*

Example 4.7. Let G be an infinite locally compact group. Then

$$\mathcal{H}^1\big(M(G) \bowtie^{\mathrm{id}} M(G), M(G) \bowtie^{\mathrm{id}} M(G)\big) = 0$$

and

$$\mathcal{H}^1\big(A(G) \bowtie^{\mathrm{id}} A(G), A(G) \bowtie^{\mathrm{id}} A(G)\big) = 0.$$

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