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MINIMUM COST TREND-FREE RUN ORDERS OF FRACTIONAL FACTORIAL DESIGNS[1]

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Run orders of fractional factorial designs which minimize a cost function based on the number of times the factors change levels during the time sequence in which the runs are performed and which simultaneously have all factor main effects components orthogonal to a polynomial time trend are found for a wide variety of factorial plans. A construction technique based on a generalized foldover scheme is presented.

1. Introduction. Suppose an experiment is to be performed according to a given fractional factorial plan. In some cases, the time order in which the runs or treatment combinations are performed need not be randomized. Instead, certain systematic run orders may be preferred. For example, if the runs are made in some time or space sequence, each observation may be affected by a trend which is a function of time or position. In the presence of a time trend, a nonrandomized run order may improve the efficiency with which factor effects are estimated. A design objective of full efficiency is attained when the factor effects are orthogonal to the time trend effects.

The cost of conducting an experiment is often of practical importance. A second design criterion is a cost function based on the number of times each factor changes levels. The practical interpretation is that it costs a certain amount to change the levels of each factor, for example, to reset a measurement instrument, change the fertilizer on a field trial, restart an industrial plant and so on. If all level changes are equally expensive, run orders that minimize the total number of factor level changes are optimal with respect to this second criterion.

Cox (1951) began the study of systematic designs, for replicated variety trials, with the single criterion of efficient estimation of treatment effects in the presence of a smooth polynomial trend. Certain 2^n factorial designs robust to both linear and quadratic trends were found by Daniel and Wilcoxon (1966). The cost criterion was introduced by Draper and Stoneman (1968) in their exhaustive searches of some eight-run factorial plans. Dickinson (1974) extended the work of Draper and Stoneman to 2^4 and 2^5 complete factorial plans with the search restricted to minimum cost run orders. Joiner and Campbell (1976) took an approach in which each factor changed levels from one run to the next with a given probability. More expensive factors were assigned smaller probabilities of changing levels. In an unpublished report, P. W. M. John extended the method

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of Daniel and Wilcoxon to certain designs for factors at two and three levels and discussed the foldover properties of such systematic run orders. Cheng (1985) gave a theoretical description of the cost structure in two-level factorial designs and provided some examples of run orders optimal with respect to both our design criteria. The theory presented in Section 4 extends Cheng's results and provides an algorithm for constructing optimal orders for many fractional factorial designs. In particular, our results may be applied to the designs listed in two National Bureau of Standards tables (1957, 1959). A majority of these designs can be optimally ordered with respect to both design criteria.

In Section 2, we briefly summarize the definition and group properties of fractional factorial designs and the notation we use to describe these designs. The design criteria are defined in Section 3 while the main results are presented in Section 4. Proofs are left until the Appendix. Section 5 contains applications of the construction results of Section 4. A summary discussion appears in Section 6.

2. Fractional factorial designs. Attention is restricted to designs in which all factors are at the same number of levels. Consider n factors, each at s levels where s is a prime power. Let the s levels of each factor be the s elements of the Galois field of order s, GF(s). We denote the s factor levels by $0, 1, \dots, s-1$, with 0 the additive identity and 1 the multiplicative identity in GF(s).

A complete factorial design in all n factors requires s^n runs. Let $G = (s_r^{n-p})$ denote a s^{-p} fraction of the complete factorial, blocked in s^r blocks each of size s^{n-p-r}. Let $N = s^{n-p}$ be the number of runs in the design G. Let $R = s^{n-p-r}$ be the size of each block.

DEFINITION 1. A design G is defined by a set of $(p+r)$ linearly independent vectors whose elements are in GF(s), say $\alpha_{ij} \in \mathrm{GF}(s)$, $i = 1, \dots, p+r$, $j = 1, \dots, n$. If $\boldsymbol{\alpha}_i = (\alpha_{i1}, \dots, \alpha_{in})^{\mathrm{T}}$, the treatment combinations in the principal block are the R solutions $\mathbf{z} = (\zeta_1, \dots, \zeta_n)^{\mathrm{T}}$, $\zeta_j \in \mathrm{GF}(s)$, $j = 1, \dots, n$, to the system of equations

$$\boldsymbol{\alpha}_i^{\mathrm{T}}\mathbf{z} = 0, \qquad i = 1, \dots, p+r. \tag{2.1}$$

The remaining $s^r - 1$ blocks, each of size R, are cosets of the principal block and correspond to solutions $\mathbf{z}$ of the first p equations only in system (2.1).

The n-tuples $\boldsymbol{\alpha}_1, \dots, \boldsymbol{\alpha}_p$ represent the p independent defining effects of the fraction while $\boldsymbol{\alpha}_{p+1}, \dots, \boldsymbol{\alpha}_{p+r}$ are the blocking effects. The group operations involved in solving system (2.1) are those of addition and multiplication in the field GF(s). To find the R solutions to system (2.1), it is sufficient to find $h = n - p - r$ independent solutions $\mathbf{z}_1, \dots, \mathbf{z}_h$ and from them form all linear combinations

$$b_1\mathbf{z}_1 + \cdots + b_h\mathbf{z}_h, \quad \text{for all } b_j \in \mathrm{GF}(s),\ j = 1, \dots, h. \tag{2.2}$$

If $\mathbf{z}_{h+1}, \dots, \mathbf{z}_{n-p}$ are r independent solutions of the first p equations of system (2.1), but not of all $(p+r)$ equations, they may be used to find the cosets of the principal block by forming the s^r runs

$$b_{h+1}\mathbf{z}_{h+1} + \cdots + b_{n-p}\mathbf{z}_{n-p}, \quad \text{for all } b_j \in \mathrm{GF}(s),\ j = h+1, \dots, n-p, \tag{2.3}$$

and adding each resulting run to all R treatment combinations in the principal block.

The notation we use to describe the treatment combinations of the design G is as follows. Let the n factors be named $a_1, \dots, a_n$. If $\mathbf{z}$ is in design G, we write run $\mathbf{z}$ equivalently as

$$\mathbf{g} = a_1^{\zeta_1} a_2^{\zeta_2} \cdots a_n^{\zeta_n}. \tag{2.4}$$

Then design G is a group $\{\mathbf{g}_1, \dots, \mathbf{g}_N\}$. Without loss of generality, G is generated by $\{\mathbf{g}_1, \dots, \mathbf{g}_{n-p}\}$, the first h of which are independent solutions to all $p + r$ equations of system (2.1) and generate the principal block. From expression (2.4), these h principal block generators are in one-to-one correspondence with the independent solutions $\mathbf{z}_1, \dots, \mathbf{z}_h$ of (2.2). We call $\{\mathbf{g}_1, \dots, \mathbf{g}_h\}$ the within-block generators. The between-block generators $\mathbf{g}_{h+1}, \dots, \mathbf{g}_{n-p}$ correspond to solutions $\mathbf{z}_{h+1}, \dots, \mathbf{z}_{n-p}$ of (2.3). Any treatment combination in G is of the form

$$\mathbf{g} = \mathbf{g}_1^{b_1} \mathbf{g}_2^{b_2} \cdots \mathbf{g}_{n-p}^{b_{n-p}}, \qquad b_j \in \mathrm{GF}(s),\ j = 1, \dots, n-p.$$

Write $\mathbf{g} = \mathbf{1}$ to denote the treatment combination corresponding to all factors at level 0. We assume that any design G is at least a main effects plan, that is, the $p + r$ n-tuples $\{\boldsymbol{\alpha}_i\}$ of Definition 1 are chosen to ensure that no main effect is aliased with another main effect or confounded with block effects.

3. Optimal design criteria.

Both the polynomial time trends and the values taken by the main effects components of the n factors in the design matrix are defined in terms of systems of orthogonal polynomials. We begin with a definition.

DEFINITION 2. The system of orthogonal polynomials on m equally spaced points $i = 0, \dots, m-1$ is the set $\{P_{km},\ k = 0, 1, 2, \dots, m-1\}$ of polynomials satisfying

$$\sum_{i=0}^{m-1} P_{km}(i) = 0, \quad \text{for all } k \geq 1, \tag{3.1}$$

$$\sum_{i=0}^{m-1} P_{km}(i) P_{k'm}(i) = 0, \quad \text{for all } k \neq k', \tag{3.2}$$

where $P_{0m}(i) = 1$ and $P_{km}(i)$ is a polynomial of degree k. We assume that each polynomial in the system is scaled so that its values are always integers.

Note that if Q_{km} is any polynomial of degree $k \leq m-1$ on m equally spaced points $i = 0, \dots, m-1$, then, for some $\{w_0, \dots, w_k\}$, Q_{km} may be expressed as

$$Q_{km}(i) = \sum_{j=0}^{k} w_j P_{jm}(i). \tag{3.3}$$

DEFINITION 3 (Factor effects). The s coefficients of the jth main effects component of each factor, $1 \le j \le s-1$, are $P_{js}(i)$, $0 \le i \le s-1$, the values of the orthogonal polynomial of degree j on s equally spaced points.

DEFINITION 4 (Trend effects). The R values of a polynomial trend of degree j, $1 \le j \le R-1$, in a block of size R are $P_{jR}(i)$, $0 \le i \le R-1$, the values of the orthogonal polynomial of degree j on R equally spaced points.

The linear model for the N observations is

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{3.4}$$

where $\boldsymbol{\varepsilon}$ is an N-vector of zero mean, uncorrelated random errors. Let $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_\tau) = (x_{ij})$, $i = 1, \dots, N$, $j = 1, \dots, \tau$. Each column $\mathbf{x}_j$ of the $N \times \tau$ design matrix $\mathbf{X}$ is either a factor, trend or block effect. The first R rows of $\mathbf{X}$ correspond to the R treatment combinations in the principal block, the next R rows to the runs in the second block and so on. There is one column in $\mathbf{X}$ for each block of G. Without loss of generality, these are the last s^r columns of $\mathbf{X}$. For any block column $\mathbf{x}_j$, $\tau - s^r + 1 \le j \le \tau$, $x_{ij} = 1$, if run i is in block $j - (\tau - s^r)$, otherwise $x_{ij} = 0$.

Let the first q_1 columns of $\mathbf{X}$ correspond to the factor effects in the model. Unless otherwise stated, we assume that interactions are negligible. Then $q_1 = n(s-1)$. By Definitions 1 and 3, if column $\mathbf{x}_j$ represents the mth main effects component of a factor a_t which is at level $u \in \mathrm{GF}(s)$ in run i of G, then

$$x_{ij} = P_{ms}(u). \tag{3.5}$$

We assume that the same polynomial time trend of degree k is present in each block. Let columns $\mathbf{x}_{q_1+1}, \dots, \mathbf{x}_{q_1+k}$ of $\mathbf{X}$ represent such a time trend, that is, the coefficients in column $q_1 + m$, $1 \le m \le k$, are given by the polynomial of degree m from the orthogonal system defined in Definition 2 for the R equally spaced positions in each block. By Definition 4, for $j = q_1 + m$, if run position $i \equiv i_0 \pmod{R}$, then $x_{ij} = P_{mR}(i_0)$.

Partition the design matrix $\mathbf{X}$ into two parts, $(\mathbf{X}_1, \mathbf{X}_2)$, where $\mathbf{X}_1$ is the $N \times q_1$ matrix of factor effects and $\mathbf{X}_2$ the $N \times q_2$, $q_2 = k + s^r$, matrix of trend and block effects. Partition the parameter vector $\boldsymbol{\beta}$ similarly into two vectors $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ of dimensions q_1 and q_2, respectively. The following facts are immediate: The q_1 columns of $\mathbf{X}_1$ are orthogonal; the q_2 columns of $\mathbf{X}_2$ are orthogonal; the q_1 columns of $\mathbf{X}_1$ are orthogonal to the s^r block effects columns of $\mathbf{X}_2$.

For any main effect column $\mathbf{x}_1$ of $\mathbf{X}_1$ and trend column $\mathbf{x}_2$ of $\mathbf{X}_2$, we define the time count between factor effect $\mathbf{x}_1$ and trend component $\mathbf{x}_2$ as $\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_2$. The design criterion based on efficient factor effect estimation in the presence of a smooth polynomial time trend may now be defined using the orthogonal polynomial structure of the linear model described previously. The objective is to eliminate the effect of the time trend by finding run orders for which all the main effects components of all n factors are orthogonal to the k trend columns of $\mathbf{X}_2$. Such run orders are said to be k-trend free. If the time counts between all factor effects and trend effects are 0, the run order is optimal with respect to our first design criterion. If this is achieved, $\mathbf{X}_1$ will be orthogonal to $\mathbf{X}_2$ and the factor effects will be estimated with full efficiency.

As stated in Section 1, our second optimality criterion is a cost function based on the number of times each factor changes levels. We assume that all factor level changes are equally expensive. Then a run order is optimal if it minimizes the total number of level changes. The compatibility of this cost function with the group structure of a fractional factorial design is used in Section 4 to produce a construction method that generates run orders optimal with respect to both design criteria. The restrictive assumption of equally expensive factor level changes is discussed in Section 6.

4. Construction of optimal run orders. We present conditions under which the main effects components of each factor become or remain orthogonal to a polynomial time trend during a stepwise construction of a run order of a design. We begin by assuming that design G is run in a single block of size N. Later in this section, we present results that allow this restriction to be dropped. In addition, the construction method is adapted to produce run orders that are optimal with respect to our second design criterion. Proofs of all the results in this section are given in the Appendix.

Consider a single factor, a_1 say. Let $U = (u_{\xi+1}, \dots, u_{\xi+sv})$, $u_i \in \mathrm{GF}(s)$, $i = \xi + 1, \dots, \xi + sv$, be a sequence of sv consecutive levels of a_1 in rows $\xi + 1, \dots, \xi + sv$ of design matrix $\mathbf{X}$. Usually, v is a power of s. Let $\mathbf{x}$ be the column of $\mathbf{X}_1$ representing the main effects component of degree q of a_1. By (3.5), $x_i = P_{qs}(u_i)$.

DEFINITION 5. Factor a_1 is k-trend free over U if

(a) each of the s levels of a_1 appears v times in U and
(b) all $s - 1$ main effects components of a_1 are orthogonal to trend components $P_{0N}, \dots, P_{kN}$ over the sv runs of U.

Let i_{tm}, $m = 1, \dots, v$, be the v run positions in U at which a_1 is at level t, for each $t = 0, \dots, s - 1$. Suppose a_1 is k-trend free over U, for some $k \geq 0$. For each main effects component of degree $q = 1, \dots, s - 1$ and each trend of degree $j = 0, \dots, k$, we have, by Definition 5,

$$0 = \sum_{t=0}^{s-1} \sum_{m=1}^{v} P_{qs}(t) P_{jN}(i_{tm}) = \sum_{t=0}^{s-1} P_{qs}(t) W(t; j, N), \tag{4.1}$$

where $W(t; j, N)$ is the sum of the values of the jth trend over the v runs of U in which a_1 is at level t. The term $W(t; j, N)$ is simplified by Lemma 1. Then with Definition 6, Theorem 1 is true.

LEMMA 1. *If a_1 is k-trend free over U, then $W(t; j, N) = W(j, N)$ is independent of the level t, for $j = 1, \dots, k$.*

DEFINITION 6. For sequence of levels U as before and for some $e \in \mathrm{GF}(s)$, let $U(e)$ be another sequence of sv levels of factor a_1 located at run positions

$\xi' + 1, \ldots, \xi' + sv$, where the level of factor a_1 at position $\xi' + i$ is given by $u_{\xi+i} + e$.

THEOREM 1. *Let a_1 be k-trend free over U, for some $k \geq 0$. Then a_1 is also k-trend free over $U(e)$.*

We may now define the generalized foldover of U in terms of some nonzero element $e \in \mathrm{GF}(s)$. Then Theorem 2 which follows Definition 7 provides the main method for constructing trend-free orders optimal with respect to the first design criterion.

DEFINITION 7 (Generalized foldover of U). For U as before, the generalized foldover of U is the sequence of $s^2 v$ levels of a_1 given by

$$U^*(e) = (U, U(e), U(2e), \ldots, U((s-1)e)).$$

THEOREM 2. *Suppose a_1 is k-trend free over U. Let $U^*(e)$ be the generalized foldover of U with respect to $e \neq 0 \in \mathrm{GF}(s)$. Then a_1 is $(k+1)$-trend free over U^*.*

We will give a scheme that allows k-trend-free run orders of G to be constructed. We assume that any run order of G begins with the run $\mathbf{1}$ in which all factors are at level 0. We employ the notation of expression (2.4) and write the runs of G as $\{\mathbf{g}_1, \ldots, \mathbf{g}_N\}$. Recall that by $\mathbf{g}^t$ for $t = 0, \ldots, s-1$ we mean the multiplication of each exponent of a factor name by t according to the operation of group multiplication in $\mathrm{GF}(s)$.

At the beginning of this section, we assumed that design G would be run in a single block of size s^{n-p}. We now reinstate the block structure. There are s^r blocks of size $R = s^h$, where $h = n - p - r$. Recall that by a within-block generator we mean a run $\mathbf{g}$ that is in the principal block and is used, along with $h-1$ other independent principal block runs, to generate the principal block by (2.2) whereas a between-block generator is one of the r independent runs from r distinct blocks, other than the principal block, used to generate the $s^r - 1$ cosets of the principal block by (2.3).

Let $\{\mathbf{g}_1, \ldots, \mathbf{g}_{n-p}\}$ be $n - p$ independent generators of G, the first h of which generate the principal block. Suppose G is generated as follows: set $U_0 = \mathbf{1}$. Then let

$$U_i = U_{i-1}^*(\mathbf{g}_i), \qquad i = 1, \ldots, n-p, \tag{4.2}$$

where $\mathbf{g}_i = a_1^{e_1} \cdots a_n^{e_n}$. Then factor a_j is folded over according to Definition 7 with respect to level e_j.

Theorem 3 shows how k-trend-free orders may be constructed. We precede Theorem 3 by a result that exploits the block structure of the design and the assumption that the trend components in every block of G are identical.

LEMMA 2. *Using generalized foldover scheme* (4.2), *if a factor is at a nonzero level in at least one between-block generator $\{\mathbf{g}_{h+1}, \ldots, \mathbf{g}_{n-p}\}$, that* *factor is orthogonal to all the polynomial trend components present in linear model* (3.4).

THEOREM 3. *For G generated according to system* (4.2), *G is k-trend free if each factor appears at least $(k+1)$ times at nonzero levels in the sequence of generators or, for any factor appearing fewer than $(k+1)$ times at a nonzero level, that factor is at a nonzero level in at least one between-block generator. Note that these $(k+1)$ appearances at nonzero levels do not have to be at the same level.*

EXAMPLE 1. Consider the design $G = 2_0^{3-0}$, a complete 2^3 factorial in factors a, b and c run in one block of eight runs. Then $G = \{1, a, b, c, ab, ac, bc, abc\}$ and if we choose $\mathbf{g}_1 = ab$, $\mathbf{g}_2 = abc$, $\mathbf{g}_3 = ac$ each nonzero factor level appears at least twice and the resulting run order constructed according to the scheme (4.2) is linear trend free or 1-trend free. This order is $G = \{1, ab, abc, c, ac, bc, b, a\}$ and was found by Draper and Stoneman in their exhaustive search of all 8! run orders.

Note that with the generalized foldover scheme (4.2), the last run of the first s^i runs, $i = 1, \dots, n-p$, is given by

$$\mathbf{g}_1^{s-1}\mathbf{g}_2^{s-1} \cdots \mathbf{g}_i^{s-1}. \tag{4.3}$$

We turn now to the second design criterion: a cost function given by the number of factor level changes. Recall the assumption that all factor level changes are equally expensive. Cheng (1985) gives a method for constructing minimum cost run orders of two-level fractional factorial designs. We will present a generalization of Cheng's arguments to fractional factorial designs at s levels, where s is a prime power. A method based on the generalized foldover scheme defined previously is shown to produce minimum cost run orders of designs G. Among all possible minimum cost run orders generated by the foldover method, one is sought that meets the trend elimination conditions of Theorem 3. If such an order exists, it is optimal with respect to both our design criteria.

For convenience, we employ the same notation as Cheng. The reader is referred to Cheng (1985) for details. Begin by defining a cost or distance function between any two subsets A and B of G by

$$d(A, B) = \min_{\omega \in A,\, \nu \in B} d(\omega, \nu),$$

where $d(\omega, \nu)$ is the number of factor level changes between runs ω and ν. In the notation of (2.4), if $\omega = a_1^{\omega_1} \cdots a_n^{\omega_n}$ and $\nu = a_1^{\nu_1} \cdots a_n^{\nu_n}$, $\omega_i, \nu_i \in \mathrm{GF}(s)$, then $d(\omega, \nu) = \sum I(\omega_i \neq \nu_i)$, where $I(\omega_i \neq \nu_i)$ equals 1 if $\omega_i \neq \nu_i$, and is 0 otherwise. In particular, $d(\mathbf{1}, \omega)$ is the number of factors at a nonzero level in run ω. In what follows, assume that the first block of G is the principal block, denoted by B_1, a subgroup of G. Blocks $B_2, \dots, B_{s^r}$ are cosets of B_1 in G.

LEMMA 3. *Let $\{\mathbf{g}_1, \dots, \mathbf{g}_{n-p}\}$ generate G by the generalized foldover scheme of Theorem 3. Let*

$$d_i = d\left(\mathbf{g}_i, \prod_{j=0}^{i-1} \mathbf{g}_j^{s-1}\right), \qquad i = 1, \dots, n-p. \tag{4.4}$$

Then the cost of the run order so generated is

$$C = \sum_{i=1}^{n-p} (s-1) s^{n-p-i} d_i. \tag{4.5}$$

Consider the following group structured decomposition of the principal block, B_1. Beginning with $H_1^{(0)} = \{\mathbf{1}\}$, we iteratively define a sequence of quotient groups $G_i = B_1 / H_1^{(i)}$ and subgroups $H_1^{(i+1)}$ of B_1 along with a set of minimum within-block costs $\{c_{i+1}\}$, $i = 0, 1, \dots, t-1$, by

$$c_{i+1} = \min_{H, K \in G_i,\, H \neq K} d(H, K) \quad \text{and} \quad H_1^{(i+1)} = \bigcup_{H \in S_1^{(i)}} H,$$

where $S_1^{(i)}$ is the subgroup of G_i generated by $\{H\colon\ d(H_1^{(i)}, H) = c_{i+1}\}$. Let $m_i = |S_1^{(i-1)}| = s^{r_i}$, $N_i = N_{i-1}/m_i$ and $N_0 = s^{n-p}$. Note that $G_0 = B_1$.

Each N_i equals s^r multiplied by the number of cosets of $H_1^{(i)}$ in B_1, where for convenience we count $H_1^{(i)}$ as a coset of itself, each coset being of size $m_1 m_2 \cdots m_i$, whereas r_{i+1} is the number of independent generators of $S_1^{(i)}$, the subgroup of the quotient group G_i generated by those elements of G_i distance c_{i+1} from the current subgroup $H_1^{(i)}$ of B_1. The elements of $S_1^{(i)}$ are cosets of $H_1^{(i)}$. The $H_1^{(i)}$'s form a nested sequence of subgroups, of strictly increasing size, of B_1. The sequence of costs $\{c_i,\ i = 1, \dots, t\}$ is strictly increasing. The iterations terminate when $N_t = s^r$ for some t at which time $H_1^{(t)} = B_1$. Note that $r_1 + \cdots + r_t = n - p - r$. At each stage $i = 0, \dots, t-1$, there are arrangements of the $s^{r_{i+1}}$ elements of $S_1^{(i)}$ that have cost c_{i+1} between any two adjacent elements in the arrangement. This produces a minimum cost ordering of the elements of $S_1^{(i)}$. Theorem 4 shows how the generalized foldover scheme may be used to find such arrangements. When the principal block has been minimally ordered, we repeat the previous induction, starting with $H_1^{(t)} = B_1$ and G replacing B_1, until some $N_{t+t'} = 1$ and $H_1^{(t+t')} = G$. The between-block minimum costs $\{c_{t+1}, \dots, c_{t+t'}\}$ found from this second iterative procedure, although strictly increasing, may be less than the within-block costs found when ordering B_1.

This cost structured decomposition of G may be combined with the generalized foldover scheme to produce minimum cost run orders as follows. At each stage $i = 1, \dots, t + t'$, suppose $S_1^{(i-1)}$ is generated by $\{K_{i1}, \dots, K_{ir_i}\} \in G_{i-1}$. By definition of $S_1^{(i-1)}$, there must exist independent runs $\mathbf{z}_{ij} \in K_{ij}$, $j = 1, \dots, r_i$, each distance c_i from run $\mathbf{1}$. Thus, at each stage, $\mathbf{z}_{ij}$ has the minimum possible number of factors at a nonzero level. Setting $r_0 = 1$ and $\mathbf{z}_{01} = \mathbf{1}$, define a set of

$n - p$ independent generators of G by

$$\mathbf{g}_{ij} = \left(\prod_{q=1}^{i-1} \prod_{j_1=1}^{r_q} \mathbf{g}_{qj_1}^{s-1} \right) \left(\prod_{j_1=1}^{j-1} \mathbf{g}_{ij_1}^{s-1} \right) \mathbf{z}_{ij}, \qquad j = 1, \dots, r_i,\ i = 1, \dots, t + t'. \tag{4.6}$$

Note that $\mathbf{g}_{ij}$ is $\mathbf{z}_{ij}$ multiplied by the product of all previous generators raised to the power $(s - 1)$. Since the $\mathbf{z}_{ij}$ are independent in $H_1^{(i)}$, the collection

$$\{\mathbf{g}_{ij},\ j = 1, \dots, r_i,\ i = 1, \dots, t + t'\} \tag{4.7}$$

are $n - p$ independent generators of G. With the help of Lemma 3, the following theorem is true.

THEOREM 4. *If a run order of G is constructed by the generalized foldover scheme* (4.2) *applied to the sequence of generators* (4.7), *the resulting run order has minimum cost given by*

$$C_{\min} = \sum_{i=1}^{t+t'} (N_{i-1} - N_i) c_i. \tag{4.8}$$

EXAMPLE 2. Consider the design $G = 2_1^{8-4}$, a design for eight factors in two blocks of size 8, defined by $I = ABEGH = ACFG = ABCD = ABEF$ with blocking effect ACE. (Note that this design is too small to be of much practical use and serves only as an example here.) The principal block contains three runs, **abcd, acfg** and **bdfg**, each with four factors at a nonzero level. Any two of these three runs are independent. Thus $c_1 = 4$, $r_1 = 2$, $m_1 = 4$ and $N_1 = 4$. Choosing $\mathbf{z}_{11} = \mathbf{abcd}$ and $\mathbf{z}_{12} = \mathbf{bdfg}$, by (4.6) $\mathbf{g}_{11} = \mathbf{abcd}$ and $\mathbf{g}_{12} = \mathbf{acfg}$. With these generators, the subgroup $H_1^{(1)}$ and its coset $H_2^{(1)}$ are

$$H_1^{(1)} = \{\mathbf{1}, \mathbf{abcd}, \mathbf{acfg}, \mathbf{bdfg}\},$$
$$H_2^{(1)} = \{\mathbf{cdefh}, \mathbf{abefh}, \mathbf{adegh}, \mathbf{bcegh}\}.$$

Now $G_1 = B_1/H_1^{(1)}$ consists of $H_1^{(1)}$ and its coset $H_2^{(1)}$. Also, $S_1^{(1)} = G_1$ in this example. Since each run in $H_2^{(1)}$ has five factors at a nonzero level, $c_2 = 5$, $r_2 = 1$ and $N_2 = 2 = s^r$. If we choose $\mathbf{g}_{21} = \mathbf{cdefh}$, then the final minimum cost ordering of B_1 by the foldover method is $H_1^{(1)}$ followed by $H_2^{(1)}$ with the runs in the order shown. The second block of the design has three runs with three factors at a nonzero level. Any one of these may be used as the required between-block minimum cost run. Thus $c_3 = 3$. If we set $\mathbf{z}_{31} = \mathbf{bde}$, then, by (4.6), $\mathbf{g}_{31} = \mathbf{cdgh}$ and the resulting minimum cost ordering of B_2 is

$$B_2 = \{\mathbf{cdgh}, \mathbf{abgh}, \mathbf{adfh}, \mathbf{bcfh}, \mathbf{efg}, \mathbf{abcdefg}, \mathbf{ace}, \mathbf{bde}\}.$$

By (4.8), the overall minimum cost is 61 level changes.

Including the between-block costs $\{c_{t+1}, \dots, c_{t+t'}\}$ in the cost decomposition described previously implies that the observations for treatment combinations in each block are made before the next block's observations are begun. In reality, observations for runs in each block may be made concurrently and there will be no between-block costs. If this is the case, a run order will have minimum cost of level changes if each block is minimally ordered according to the within-block costs found previously and *any r* independent between-block generators may be used in the generalized foldover scheme (4.2). With this added freedom, minimum cost run orders that satisfy the preceding orthogonality design criterion are more readily found. An example is provided in Section 5. Note that expression (4.8) for the minimum cost for design G becomes

$$C_{\min} = \sum_{i=1}^{t} (N_{i-1} - N_i)c_i. \tag{4.9}$$

The previous results provide a sufficient condition under which a run order of G is optimal with respect to both design criteria: trend elimination and minimum cost of level changes. Assume that the trend is of degree k and is represented by k columns in design matrix $\mathbf{X}$ as described in Section 3. Usually, k will be small: Values of 1 and 2 are most common. Let the cost structure of G be given by

$$\{(c_1, r_1), (c_2, r_2), \dots, (c_{t+t'}, r_{t+t'})\}, \quad \text{where } \sum_{j=1}^{t+t'} r_j = n - p.$$

Let $\{\mathbf{z}_{ij},\ j = 1, \dots, r_i,\ i = 1, \dots, t + t'\}$ be some choice of $n - p$ independent minimum distance runs with respect to this cost structure. Let $\{\mathbf{g}_{ij}\}$ be formed from these as in expression (4.6). All preceding results may be combined to give the following theorem.

THEOREM 5. *If each factor appears at some nonzero level at least* $(k + 1)$ *times in the sequence of runs* $\{\mathbf{g}_{ij}\}$ *which generate G by the generalized foldover scheme* (4.2), *or at least once in a between-block generator, the resulting run order, having minimum cost* (4.8), *or* (4.9) *if the between-block costs are* 0, *and being k-trend free by Theorem* 3, *is optimal with respect to both design criteria.*

5. Examples of optimal run orders. In this section, we present some examples of series of fractional factorial designs with factors at two or three levels for which optimal orders may be found by the construction techniques of Section 4. Throughout this section, unless otherwise stated, a run order is optimal if it is *linear trend free*, that is 1-trend free, and has minimum cost of level changes. We add one further result which leads to linear trend-free two-factor interactions for designs with factors at two levels but requires more than the minimum number of factor level changes.

Before presenting specific examples, we make the following observation: When s is a prime number, group operations in GF(s) are addition and multiplication modulo s. Thus, if $\{\omega_1, \dots, \omega_{n-p}\}$ is the ordered series of minimum cost runs, in one-to-one correspondence with the runs $\{\mathbf{z}_{ij},\ j = 1, \dots, r_i,\ i = 1, \dots, t + t'\}$ used in (4.6) to find the set of generators that construct an optimal run order of

G by the generalized foldover scheme, then

$$\mathbf{g}_i = \omega_{i-1}^{(s-1)}\omega_i, \tag{5.1}$$

since $(s-1)^2 + (s-1) \equiv 0 \pmod{s}$. Only the current and previous members of $\{\omega_i\}$ are needed to find the next generator in (4.6). With this simplification of (4.6), whenever a sequence of generators for a particular design is presented in the following discussion, only the set of minimum cost runs $\{\omega_i,\ i = 1, \dots, n-p\}$ is shown.

By Theorem 3, for a two-level factor to be linear trend free it must be at its high level, level 1, in at least two of the generators in sequence (4.7). An equivalent form of this requirement is: Following the first appearance of the factor at its high level in, say, run ω_i it must be at its low level, 0, in some subsequent minimum cost run ω_j, $j > i$. It is this condition that is most easily checked for some choice of $\{\omega_i\}$.

Cheng gives an example of a series of fractional factorial designs that have an optimal order. The series he proves can be optimally ordered is $\{G = 2_0^{n-1}, n \geq 5\}$, with defining relation $I = A_1 \cdots A_n$. This is the series of 1/2 replicates of a complete 2^n in one block of size 2^{n-1} with the highest-order interaction confounded. All $(n-1)$ independent minimum cost runs have cost 2 so a minimum cost run order requires $2(2^{n-1}-1)$ level changes, by (4.8). We may reproduce Cheng's result by using the sequence of minimum cost runs

$$a_1a_2,\ a_3a_4, \dots,\ a_{n-3}a_{n-2},\ a_{n-1}a_n,\ a_1a_3\ \cdots,$$

if n is even, where the remaining $n/2 - 1$ minimum cost runs may be any other independent cost 2 runs, and if n is odd the slight modification

$$a_1a_2,\ a_3a_4, \dots,\ a_{n-4}a_{n-3},\ a_{n-2}a_{n-1},\ a_na_1,\ a_2a_3\ \cdots,$$

where any $(n-1)/2 - 3$ other independent cost 2 runs may be used after a_2a_3.

We give two other examples of series of designs, each member of which may be optimally ordered, to illustrate how readily Theorem 5 may be used.

The first example is the series of 1/4 replicates of a complete 2^n for $n \geq 7$ defined by $I = A_1A_2S = A_3A_4S$ where the common stem $S = A_5 \cdots A_n$. A minimum cost run sequence that produces optimal run orders is:

$$a_1a_2,\ a_3a_4,\ a_5a_6,\ a_6a_7,\ a_7a_8, \dots,\ a_{n-1}a_n,\ a_1a_3a_5.$$

The cost structure is $\{(2, n-3), (3, 1)\}$, for the (c_i, r_i), with minimum cost $2^{n-1} - 1$ by (4.8).

The next example in this section is the series of 1/8 replicates of a complete 2^n factorial in one block of size 2^{n-3} defined by $I = A_1A_4A_5S = A_2A_4A_6S = A_3A_5A_6S$, where the common stem $S = A_7 \cdots A_n$. The cost structure is $\{(2, n-7), (3, 4)\}$ so the minimum number of level changes becomes $(2(2^{n-3} - 2^4) + 3(2^4 - 1) = 2^{n-2} + 13$. For $n \geq 8$, an optimal set of minimum cost runs is

$$a_{n-1}a_n,\ a_{n-2}a_n, \dots,\ a_7a_n,\ a_1a_2a_4,\ a_1a_3a_5,\ a_4a_5a_6,\ a_1a_2a_7.$$

As stated in the Introduction, the construction techniques of Section 4 were applied to the designs tabled in two National Bureau of Standards publications.

Of the 125 plans for factors at two levels in Applied Mathematics Series 48 (1957), 96 may be optimally ordered by the generalized foldover scheme. Furthermore, for 63 of these 96 plans, not only a linear but also a quadratic trend-free run order with minimum cost is obtainable. Similarly, all 41 plans for factors at three levels in Applied Mathematics Series 54 (1959) may be optimally ordered. Tables of minimum cost linear trend-free run sequences for all the designs with optimal orders may be obtained from the authors.

Expression (4.9) gives the minimum cost for a run order under the often realistic assumption that between-block costs are 0. To illustrate how this modification may be beneficial, consider the design $G = 2_2^{8-3}$ defined by $I = ABEGH = ACFG = ABCD$ with blocking effects $ABEF$ and ACE. This is plan 8.8.8 in Applied Mathematics Series 48 (1957) and is a modified version of Example 2 in Section 4. Note also that in Example 2, factors **b** and **e** are not linear trend free. For this design, the generalized foldover scheme does not find a minimum cost run sequence that has the main effects of all eight factors linear trend free. However, if between-block costs are 0, the minimum cost runs $\{\mathbf{z}_{ij}\}$ given by

bdfg acfg adegh bdh abcdefg

produce a minimum cost order by (4.9) and all eight factors are both linear and quadratic trend free. Of the 29 plans in AMS 48 that cannot be optimally ordered when costs are given by (4.8), there are 12 with an optimal order using (4.9).

The generalized foldover scheme may be used to find run orders of two-level fractional factorial designs for which all main effects and two factor interactions are linear trend free, although the run order is unlikely to have minimum cost. Without loss of generality, let $G = 2_0^{n-p}$ be run in a single block. We have the following construction theorem.

THEOREM 6. *Suppose a run order of G, constructed by the generalized foldover scheme* (4.2) *with generator sequence $\{\mathbf{g}_1, \dots, \mathbf{g}_{n-p}\}$, is* 1-*trend free. For each pair of factors a_1 and a_2, suppose that there exist generators $\mathbf{g}_i, \mathbf{g}_j$, $i \neq j$, $i, j \in \{1, \dots, n-p\}$, in which a_1 and a_2 are at different levels* (*that is, one is high and the other low*). *Then all $n(n-1)/2$ two-factor interactions are linear trend free.*

Applying this theorem to complete 2^n factorials gives the following corollary.

COROLLARY 1. *For $n \geq 4$, the generalized foldover scheme* (4.2) *applied to sequence $\{\omega_i, i = 1, \dots, n\}$*:

$$a_n, a_{n-1}, \dots, a_5, a_1a_2, a_1a_3, a_4, a_2, \tag{5.2}$$

from which generators $\{\mathbf{g}_i, i = 1, \dots, n\}$ may be found by (5.1), *produces a run order that has all main effects and two-factor interactions linear trend free and requires $2^n + 11$ level changes,* 12 *more than the minimum.*

EXAMPLE 3. Consider the case $n = 4$. The run order

1 ab bc ac acd bcd abd d bd ad cd abcd abc c a b

is generated by **ab**, **bc**, **acd**, **bd**, has all four main effects and six two-factor interactions linear trend free and 27 level changes.

Daniel and Wilcoxon found a run order of a complete 2^4 with all main effects 2-trend free. Their run order may be found by folding over with the generator sequence $\{abd, acd, bcd, abcd\}$. Each factor name appears at least three times so by Theorem 3 each factor is quadratic trend free. The cost of 37 level changes is well above the minimum of 15.

6. Discussion. Linear trend-free minimum cost run orders have been found for a wide variety of two- and three-level fractional factorial designs. The examples of Section 5 illustrate the construction techniques detailed in Section 4. It is important to note that as the number of factor levels and/or the number of blocks increase, by Lemma 2, it becomes easier to find run orders that are k-trend free for $k > 1$. The assumption of zero between-block costs also aids in this search.

If the two-factor interactions are not negligible, the double optimization problem becomes difficult or impossible in small designs as the requirements of Theorem 6 become harder to satisfy. When faced with this difficulty, the experimenter must decide how to compromise between the competing criteria of efficiency and cost. Additionally, if factor level changes for a subset of the factors are expensive, for example, closing down and cleaning a chemical plant between runs at different levels, while the remaining factors are essentially free, throwing a switch to change the operating temperature say, then run orders for which the first set of factors changes levels least often may be sought by finding generator sequences in which the expensive factors appear at nonzero levels in the latter generators only. The cost optimization must be attempted whenever the experimenter has a design problem with cost constraints of the type developed here. If in reality no cost minimization is required, trend elimination is even easier to achieve as a scalar optimization problem only. Similarly, if trend elimination is not required but cost minimization is necessary, the generalized foldover method guarantees an optimal run order when combined with the cost structured group decomposition described in Section 4.

The general problem of cost minimization for arbitrary, unequal costs for each factor's level changes is much more difficult. Indeed, such a minimization problem is of the travelling-salesman type for which no finite time polynomial algorithm is known. Only for certain special cost structures, such as that mentioned previously where some factors have zero cost while all others remain equally expensive, is an exact solution readily available with the foldover method. If the allowable cost of the experiment is some amount $C > C_{\min}$, then trend-free run orders may be sought using the foldover method with generator sequences that produce any total cost less than C. This added freedom in the choice of generator sequences will improve the experimenter's chances of finding k-trend-free run orders.

In certain problems it may be necessary to *maximize* the number of factor level changes to meet some other optimality condition. For example, for n factors each at two levels, if there is a correlated error structure represented by a first-order autoregression with positive correlation, a run order that maximizes the number of factor level changes may lead to a D-optimal design. Such run orders may be constructed by applying the same generalized foldover scheme (4.2) to sequences of maximum length generators.

APPENDIX

Proofs of the results presented in Sections 4 and 5 follow.

PROOF OF LEMMA 1. Fix j and N and suppress them in the expressions that follow. A polynomial of degree at most $(s-1)$ may be fitted exactly through the s points $\{(t, W(t)),\ t = 0,\dots,s-1\}$. By the remark following Definition 2 and expression (3.3), we may express this polynomial as a weighted sum of the orthogonal polynomials $P_{j_1 s}$, $j_1 = 0,\dots,s-1$, defined by (3.1) and (3.2), that is,

$$W(t) = \sum_{j_1=0}^{s-1} w_{j_1} P_{j_1 s}(t).$$

For each component $q = 1,\dots,s-1$, expression (4.1) becomes

$$\begin{aligned} 0 &= \sum_{t=0}^{s-1} \left\{ P_{qs}(t) \sum_{j_1=0}^{s-1} w_{j_1} P_{j_1 s}(t) \right\} \\ &= \sum_{j_1=0}^{s-1} \left\{ w_{j_1} \sum_{t=0}^{s-1} P_{qs}(t) P_{j_1 s}(t) \right\} \\ &= w_q \sum_{t=0}^{s-1} \left(P_{qs}(t) \right)^2, \quad \text{by Definition 3.} \end{aligned}$$

Hence $w_q = 0$ for $q = 1,\dots,s-1$ and $W(t) = w_0 P_{0s}(t) =$ constant by Definition 2. □

PROOF OF THEOREM 1. By the group properties of addition in GF(s), each level of factor a_1 appears equally often in $U(e)$. In particular, level t appears in run positions $i_{tm} - \xi + \xi' = i_{tm} + \xi_1$ say. The time count over $U(e)$ of the qth main effects component of a_1 against the jth trend is

$$\sum_{t=0}^{s-1} \sum_{m=1}^{v} P_{qs}(t+e) P_{jN}(i_{tm} + \xi_1). \tag{A.1}$$

Now $P_{jN}(i_{tm} + \xi_1)$ is a polynomial of degree j in i_{tm}. By (3.3), this polynomial may be written as

$$P_{jN}(i_{tm} + \xi_1) = \sum_{j_1=0}^{j} w_{j_1} P_{j_1 N}(i_{tm}),$$

where the coefficients w_{j_1} depend on the constant ξ_1 only. Substituting this expression into (A.1) gives a time count of

$$\text{(A.2)} \qquad \begin{aligned} &\sum_{t=0}^{s-1} \sum_{m=1}^{v} \left\{ P_{qs}(t+e) \sum_{j_1=0}^{j} w_{j_1} P_{j_1 N}(i_{tm}) \right\} \\ &\quad = \sum_{j_1=0}^{j} \left\{ w_{j_1} \sum_{t=0}^{s-1} P_{qs}(t+e) W(t;\, j_1, N) \right\}. \end{aligned}$$

By the assumptions of the theorem and Lemma 1, $W(t;\, j_1, N) = W(j_1, N)$, so (A.2) becomes

$$\sum_{j_1=0}^{j} \left\{ w_{j_1} W(j_1, N) \sum_{t=0}^{s-1} P_{qs}(t+e) \right\} = 0.$$

The preceding is true for each $j = 0, \dots, k$ and hence a_1 is k-trend free over $U(e)$. □

PROOF OF THEOREM 2. Without loss of generality, U is in run positions $1, \dots, sv$ of G. By Theorem 1, a_1 is k-trend free over U^*. As t ranges over $\mathrm{GF}(s) - \{0\}$, so too does te, $t, e \neq 0$. As before, assume that a_1 is at level t in positions i_{tm}, $m = 1, \dots, v$, of U. Then a_1 is at level $(t + qe)$ in these same run positions of $U(qe)$, $q = 1, \dots, s-1$. Each level of a_1 is represented by some $(t + qe)$ as q ranges from 0 to $s - 1$ for fixed t and similarly as t ranges from 0 to $s - 1$ for fixed q.

Let the level of factor a_1 be fixed at t. The contribution to the time count of the lth main effects component of a_1 against a trend of degree $(k + 1)$ over the run positions $\{i_{tm}\}$ in each $U(qe)$ of U^* is

$$\text{(A.3)} \qquad \sum_{q=0}^{s-1} \sum_{m=1}^{v} P_{ls}(t + qe) P_{k+1, N}(qsv + i_{tm}).$$

Now

$$P_{k+1, N}(qsv + i_{tm}) = w_{k+1} P_{k+1, N}(i_{tm}) + \sum_{j_1=0}^{k} w_{j_1}(q) P_{j_1 N}(i_{tm}),$$

where w_{k+1} is a constant not depending on q and $w_{j_1}(q)$ is a polynomial in q of degree at most $(k + 1)$. Then (A.3) becomes

$$\sum_{q=0}^{s-1} \left\{ P_{ls}(t + qe) \sum_{m=1}^{v} \left\{ w_{k+1} P_{k+1, N}(i_{tm}) + \sum_{j_1=0}^{k} w_{j_1}(q) P_{j_1, N}(i_{tm}) \right\} \right\}.$$

In the preceding expression, $\Sigma w_{k+1} P_{k+1,N}(i_{tm})$ depends on t but not q and summing $P_{ls}(t+qe)$ over q ranging from 0 to $s-1$ yields 0 by the previous discussion so the first inner term vanishes and (A.3) simplifies to

$$\sum_{j_1=0}^{k} \sum_{q=0}^{s-1} \left\{ w_{j_1}(q) P_{ls}(t+qe) \sum_{m=1}^{v} P_{j_1,N}(i_{tm}) \right\}$$
$$= \sum_{j_1=0}^{k} \sum_{q=0}^{s-1} w_{j_1}(q) P_{ls}(t+qe) W(j_1, N).$$

In this last expression, the terms $P_{ls}(t+qe)$ sum to 0 over $q = 0, \dots, s-1$, for each fixed t and j_1. So the total time count over U^* of the lth component of a_1 against $P_{k+1,N}$ is 0 for each $l = 1, \dots, s-1$ and a_1 is $(k+1)$-trend free over U^*. □

Proof of Lemma 2. Suppose that factor a_1 is at nonzero level e in between-block generator $\mathbf{g}_{h+m}$, so s^{m-1} blocks have been generated so far, $m \in \{1, \dots, r\}$. Recall that we assume that each level of a factor appears equally often in every block. Suppose that a_1 is at level t in run position i_0 of an already existing block, B_j, for some $j = 1, \dots, s^{m-1}$. When generator $\mathbf{g}_{h+m}$ is used with the generalized foldover scheme (4.2), factor a_1 will be at level $t + qe$, $q = 1, \dots, s-1$, in position i_0 in some $s-1$ new blocks $B_{j_1}, \dots, B_{j_{s-1}}$. Again, as q ranges over the set $0, \dots, s-1$, so too does $t + qe$. Hence, by Definitions 2 and 3, the time count with respect to the trend component of degree l contributed by this starting run position i_0 in block B_j for the ith main effects component is

$$\sum_{q=0}^{s-1} P_{lR}(i_0) P_{is}(t+qe) = P_{lR}(i_0) \sum_{q=0}^{s-1} P_{is}(t+qe) = 0.$$

So each starting level of factor a_1 in any starting position in an already existing block contributes zero to the time count with respect to any trend component in the model. So factor a_1 is orthogonal to any trend component in the model. □

Proof of Theorem 3. By Theorem 2, if the run order is constructed by applying the generalized foldover scheme (4.2) to the sequence of generators $\{\mathbf{g}_1, \dots, \mathbf{g}_{n-p}\}$, a factor a_i is k-trend free over G if a nonzero level of this factor appears in at least $(k+1)$ of the generators. From this and Lemma 2, Theorem 3 follows. □

Proof of Lemma 3. By (4.3), d_i is the number of factor level changes between the ith generator $\mathbf{g}_i$ and the last run of the first s^{i-1} runs. Hence, by the definition of the generalized foldover scheme (4.2), d_i is the number of factor level changes between each adjacent pair of groups of s^{i-1} runs within each group of s^i runs. There are s^{n-p-i} groups of s^i runs, and s groups of s^{i-1} runs within each such group of s^i runs. Thus, the number of factor level changes between groups of size s^{i-1} within groups of size s^i is $s^{n-p-i}(s-1)d_i$. Summing over $i = 1, \dots, n-p$ gives the result (4.5). □

PROOF OF THEOREM 4. Set $R_0 = 0$, $R_i = r_1 + \cdots + r_i$, $i = 1, \dots, n-p$. By (4.3), generator $\mathbf{g}_{ij}$ of (4.6) is $\mathbf{z}_{ij}$ multiplied by the last run in the first $s^{R_{i-1}+j-1}$ runs. So the number of level changes between these two runs is $d(\mathbf{1}, \mathbf{z}_{ij}) = c_i$ by the definition of the runs $\{\mathbf{z}_{ij}\}$. Hence, at each stage $i = 1, \dots, n-p$, the number of level changes between each group of s^{i-1} runs within each group of s^i runs is the minimum possible. Therefore, the resulting run order has minimum cost. The $\{d_{i_1}, i_1 = 1, \dots, n-p\}$ of (4.4) are given by

$$(\text{A.4}) \qquad d_{i_1} = c_i, \qquad i_1 = \sum_{j=1}^{i-1} r_j + 1, \dots, \sum_{j=1}^{i} r_j, \qquad i = 1, \dots, t+t'.$$

Note that $s^{R_i} = N_0/N_i$. From (4.5) and (A.4), the minimum cost of this run order is

$$\begin{aligned}\sum_{i=1}^{n-p} (s-1)s^{n-p-i} d_i &= \sum_{i=1}^{t+t'} \sum_{i_1 = R_{i-1}+1}^{R_i} c_i (s-1) s^{n-p-i_1} \\ &= \sum_{i=1}^{t+t'} (s-1) c_i \{ s^{n-p-R_i}(1 - s^{r_i})/(1-s) \} \\ &= \sum_{i=1}^{t+t'} c_i N_0 N_i / N_0 (s^{r_i} - 1) \\ &= \sum_{i=1}^{t+t'} c_i (N_{i-1} - N_i),\end{aligned}$$

which gives (4.8). □

PROOF OF THEOREM 5. Follows directly from Theorems 3 and 4 for the stated choice of minimum cost generator sequence. □

PROOF OF THEOREM 6. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the columns of the design matrix corresponding to the main effect of any two factors a_1 and a_2. All entries in $\mathbf{x}_1$ and $\mathbf{x}_2$ are either $+1$ or -1. Then the two-factor interaction column $\mathbf{x}$ has ith entry $x_{i1} \times x_{i2}$. We assume that the interaction is estimable when no time trend is present.

Without loss of generality, both factors are at the same level in $\mathbf{g}_1, \dots, \mathbf{g}_{k-1}$; a_1 is high, a_2 low in $\mathbf{g}_k$; both are at the same level in $\mathbf{g}_{k+1}, \dots, \mathbf{g}_{m-1}$ and a_1 is high, a_2 low in $\mathbf{g}_m$, $m > k$. Then the interaction column $\mathbf{x}$ contains $+1$ in the first 2^{k-1} rows and -1 in the next 2^{k-1} rows giving a time count of $2^{2(k-1)}$ over the first 2^k runs. Note that since the trend is linear, we have shifted the values of the trend polynomial to $1, \dots, R$ rather than $P_{1R}(i)$, $i = 0, \dots, R-1$. This results in a linear rescaling of the time count but does not affect the result stated here. This same time count is contributed by each of the first 2^{m-k-1} groups of 2^k runs. So the time count after 2^{m-1} runs is 2^{m+k-3}. When $\mathbf{g}_m$ is used, the entries in the interaction column are all multiplied by -1 and the second group of 2^{m-1} runs contributes a time count of exactly -2^{m+k-3} and hence the time count for the interaction effect becomes 0 after 2^m runs. This time count remains 0 in all future foldovers by Theorem 2. So interaction column $\mathbf{x}$ is orthogonal to a linear trend. By the assumptions of the theorem, this is true for all two-factor interactions. □

PROOF OF COROLLARY 1. The n runs in (5.2) are independent and so generate the complete factorial design. Referring to (4.5), the runs (5.2) have cost $d_i = 1$ if $i \neq n-3, n-2$, and $d_{n-3} = d_{n-2} = 2$. By (4.5), the cost of the run order is $\Sigma_{i=1}^{n}(1 \times 2^{n-i}) + 2^3 + 2^2$ which gives $2^n + 11$ as required.

The generator sequence $\{\mathbf{g}_1, \dots, \mathbf{g}_n\}$ found from runs (5.2) by (4.6) is

$$a_n,\ a_n a_{n-1}, \dots,\ a_6 a_5,\ a_1 a_2 a_5,\ a_2 a_3,\ a_1 a_3 a_4,\ a_2 a_4,$$

namely, the ith generator is the product of runs $\mathbf{z}_i$ and $\mathbf{z}_{i-1}$, as stated in Section 5. Inspecting this sequence shows that, for any two factors, two generators in which only one factor name appears may be found. The conditions of Theorem 6 are met so all two-factor interactions are linear trend free. □

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