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ON THE UNIQUE REPRESENTATION OF NON-GAUSSIAN LINEAR PROCESSES

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In this paper, we prove the uniqueness of linear i.i.d. representations of non-Gaussian linear processes on a countable abelian group under a basic invertibility condition, without requiring the existence of higher than second moments.

1. Introduction. Let x_t be a stationary linear process on a countable abelian group G:

$$x_t = w_t * u_t = \sum_{s \in G} w_s u_{t-s}, \tag{1.1}$$

where u_t is an independent and identically distributed random series with $Eu_t = 0$, $Eu_t^2 = \sigma^2 > 0$, w_t is a square-summable constant sequence. We give the condition

$$W(\gamma) \neq 0, \qquad d\gamma \text{ (a.e.)}, \tag{1.2}$$

where $W(\gamma)$ is the Fourier transform of w_t [for related symbols, see Cheng (1990)].

The uniqueness of linear representations of non-Gaussian processes plays an important role in the theory and application of time series modeling. Donoho [(1981), pages 569 and 575, $G = \mathbb{Z}$], Lii and Rosenblatt [(1982), $G = \mathbb{Z}$], Rosenblatt [(1985), pages 46 and 235, $G = \mathbb{Z}$ and $\mathbb{Z}^2$], Findley [(1986), $G = \mathbb{Z}$; (1990)] and Cheng (1990) have established uniqueness results under (1.2) in combination with other conditions. These supplementary conditions involve either the existence of a nonzero kth order cumulant ($k > 2$) of the random series u_t and/or a stronger summability condition on the constant sequence w_t.

In this paper, we prove the uniqueness theorem only under the basic condition (1.2).

2. The uniqueness theorem. Now we will prove the uniqueness theorem.

THEOREM 2.1 (The uniqueness theorem). *Let*

$$x_t = w_t * u_t = w'_t * u'_t, \qquad t \in G, \tag{2.1}$$

where u_t and u'_t are i.i.d. and w_t and w'_t are square-summable sequences

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satisfying (1.2). *If* x_t *is non-Gaussian, then*

$$u'_t = a u_{t-t_0}, \qquad w'_t = \frac{1}{a} w_{t+t_0}, \tag{2.2}$$

where a *is a nonzero constant and* t_0 *is an element of* G.

Proof. If (2.1) holds, then by formula (3.5) of Cheng (1990), we have

$$u'_t = c_t * u_t = \sum_s c_s u_{t-s}, \tag{2.3}$$

$$u_t = d_t * u'_t = \sum_s d_s u'_{t-s}. \tag{2.4}$$

According to the theory of infinite product probability spaces [see Loéve (1963)], we can construct the random variables $z(t,s)$, $t, s \in G$, on certain probability space which are i.i.d. with the same probability distribution as u'_t. Set

$$y_t = \sum_s d_s z(t,s). \tag{2.5}$$

It is evident that y_t is an i.i.d. series. Comparing (2.5) with (2.4), we see that y_t has the same probability distribution as u_t. Let us consider

$$z \triangleq \sum_s c_s y_s = \sum_s \sum_t c_s d_t z(s,t). \tag{2.6}$$

By comparison with (2.3), we know that z and u'_t have the same distribution. Since x_t is non-Gaussian, u'_t is non-Gaussian too. Hence in (2.6), $z(s,t)$ are i.i.d. and non-Gaussian with the same probability distribution as z. Thus, (2.6) implies $\sum_s \sum_t (c_s d_t)^2 = 1$. According to Theorems 5.6.1 and 3.3.1 in Kagan, Linnik and Rao (1973), it follows from (2.6) that there exists (t_0, t_1) such that $c_{t_0} d_{t_1} \neq 0$, $c_s d_t = 0$ $[(s,t) \neq (t_0, t_1)]$. This leads to $c_s = 0$ $(s \neq t_0)$. From (2.3), we have $u'_t = a u_{t-t_0}$, where $a = c_{t_0}$. From this and (2.1), we get (2.2). □

Corollary 2.1. *Let* x_t *and* u_t *be i.i.d. and* $x_t = w_t * u_t$, x_t *be non-Gaussian, then*

$$w_t = a\delta_{t-t_0},$$

where a *is a nonzero constant,* t_0 *is an element of* G, $\delta_0 = 1$ *and* $\delta_t = 0$ *for* $t \neq 0$.

The proof of the corollary is immediate.

Remark. In this paper, we do not require the existence of higher than second moments. If we keep the assumption of the existence of higher than second moments, the independence and identical distribution assumptions can be weakened, which was pointed out by Findley (1990), after carefully reading and analyzing the proofs in Findley (1986) and Cheng (1990).

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