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ON L_p-NORMS OF MULTIVARIATE DENSITY ESTIMATORS

BY LAJOS HORVÁTH

University of Utah

Central limit theorems are proven for L_p-norms ($1 \le p < \infty$) of multivariate density estimators.

1. Introduction. Let $\mathbf{X} = (X^{(1)}, X^{(2)}), \mathbf{X}_1, \mathbf{X}_2, \dots$ be independent, identically distributed bivariate random vectors with distribution function F and density function f. We consider kernel density estimators f_n, defined by

$$f_n(\mathbf{t}) = \frac{1}{nh^2(n)} \sum_{1 \le i \le n} K\left(\frac{\mathbf{t} - \mathbf{X}_i}{h(n)}\right), \tag{1.1}$$

where K is a bivariate function satisfying certain regularity conditions and $h(n)$ is a sequence of positive numbers.

Density estimators of the form (1.1) were introduced by Rosenblatt (1956) and Parzen (1962) and have been extensively studied since then. For a recent survey on density estimators, we refer to the monograph of Prakasa Rao (1983). In this article we consider the asymptotic properties of

$$I_n(p) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_n(\mathbf{t}) - f(\mathbf{t})|^p \, d\mu(\mathbf{t}),$$

where μ is a weight function. The mean integrated square error, that is, $EI_n(2)$, is a very popular measure of the distance of f_n from f. The other well-investigated case is $p = 1$. The choice of the L_1-norm is motivated by invariance under monotone transformations of coordinate axes and the fact that it is well defined for $\mu(\mathbf{t}) = t_1 t_2$ [$\mathbf{t} = (t_1, t_2)$]. Devroye and Györfi (1985) obtained necessary and sufficient conditions for $I_n(1)$ going to 0 in probability or almost surely.

Concerning limit laws for the global deviations of density function estimators, only the supremum and L_2-norms have been considered up to now [Bickel and Rosenblatt (1973) and Rosenblatt (1975, 1976)]. It has usually been assumed that the support of μ is finite or that of f is finite. The proofs of the central limit theorems for $I_n(2)$ are based on the Karhunen–Loève expansion of Gaussian processes and therefore they cannot be generalized for the general case $1 \le p < \infty$. Using martingale techniques, Hall (1984) also obtained central limit theorems for $I_n(2)$.

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Before we state our main result we list all the assumptions used throughout this article. Let $1 \le p < \infty$ and assume:

(C1) K is bounded and vanishes outside of a compact set;
(C2) $\iint_{\mathbb{R}^2} K(\mathbf{u})\,d\mathbf{u} = 1$;
(C3) $D = \iint_{\mathbb{R}^2} K^2(\mathbf{u})\,d\mathbf{u} > 0$;
(C4) $\iint_{\mathbb{R}^2} u_1 K(\mathbf{u})\,d\mathbf{u} = \iint_{\mathbb{R}^2} u_2 K(\mathbf{u})\,d\mathbf{u} = 0$ $[\mathbf{u} = (u_i, u_2)]$;
(C5) $d\mu(\mathbf{t}) = \omega(\mathbf{t})\,d\mathbf{t}$, where $\omega(\mathbf{t})$ is uniformly bounded, continuous with respect to the Lebesgue measure and $\omega(\mathbf{t}) = 0$ if $\mathbf{t} \notin A$, where A is a finite rectangle;
(C6) $0 < \inf_{\mathbf{x} \in A} f(\mathbf{x}) \le \sup_{\mathbf{x} \in A} f(\mathbf{x}) < \infty$;
(C7) $\partial f(\mathbf{x})/\partial x_i, \partial^2 f(\mathbf{x})/\partial x_i\,\partial x_j$, $1 \le i, j \le 2$, are uniformly bounded and continuous on A;
(C8) $h(n) = O(n^{-1/6})$, $n \to \infty$;
(C9) $h(n)n^{1/4} \to \infty$, $n \to \infty$.

Let

$$r(\mathbf{t}) = \frac{1}{D}\iint_{\mathbb{R}^2} K(\mathbf{x})K(\mathbf{x}+\mathbf{t})\,d\mathbf{x},$$

$$\varphi(u) = (2\pi)^{-1/2}\exp(-u^2/2),$$

$$\psi(t, s; u) = (2\pi)^{-1}(1-u^2)^{-1/2}\exp\bigl(-(t^2 - 2uts + s^2)/(2(1-u^2))\bigr),$$

$$f_{(n)}(\mathbf{x}) = Ef_n(\mathbf{x}) = \frac{1}{h^2(n)}\iint_{\mathbb{R}^2} K((\mathbf{x}-\mathbf{y})/h(n))\,f(\mathbf{y})\,d\mathbf{y}$$

and

$$m_n(\mathbf{x}) = n^{1/2}h(n)\bigl(f_{(n)}(\mathbf{x}) - f(\mathbf{x})\bigr).$$

Now we define the asymptotic expected value and variance of $(n^{1/2}h(n))^p I_n(p)$,

$$e(n) = \int_{-\infty}^{\infty}\iint_{\mathbb{R}^2}\bigl|f^{1/2}(\mathbf{x})D^{1/2}u + m_n(\mathbf{x})\bigr|^p\omega(\mathbf{x})\varphi(u)\,d\mathbf{x}\,du$$

and

$$\sigma^2(n) = \iint_{\mathbb{R}^6}\bigl|f(\mathbf{x})Duv + f^{1/2}(\mathbf{x})D^{1/2}(u+v)m_n(\mathbf{x}) + m_n^2(\mathbf{x})\bigr|^p \times\omega^2(\mathbf{x})\bigl(\psi(u, v; r(\mathbf{y})) - \varphi(u)\varphi(v)\bigr)\,du\,dv\,d\mathbf{x}\,d\mathbf{y}.$$

It is easy to check that

$$0 < \liminf_{n\to\infty}\sigma^2(n) \le \limsup_{n\to\infty}\sigma^2(n) < \infty. \tag{1.2}$$

THEOREM. *Let* $1 \le p < \infty$. *If* (C1)–(C9) *hold, then*

$$\frac{1}{\sigma(n)h(n)}\left\{\left(n^{1/2}h(n)\right)^p I_n(p) - e(n)\right\} \to_{\mathscr{D}} N(0,1),$$

where $N(0,1)$ *is a standard normal random variable.*

We can get a simpler form of the theorem if we assume slightly stronger conditions on $h(n)$. Let

$$g(\mathbf{x}) = \frac{1}{2}\sum_{1\le i,j\le 2}\frac{\partial^2 f(\mathbf{x})}{\partial x_i\,\partial x_j}\int\!\!\int_{\mathbb{R}^2} y_i y_j K(\mathbf{y})\,d\mathbf{y},$$

$$S^2 = D^p\int\!\!\int_{\mathbb{R}^2} f^p(\mathbf{x})\omega^2(\mathbf{x})\,d\mathbf{x}$$
$$\times\int\!\!\int_{\mathbb{R}^4}|uv|^p\left(\psi(u,v;r(\mathbf{y})) - \varphi(u)\varphi(v)\right)d\mathbf{y}\,du\,dv$$

and

$$S_\lambda^2 = \int\!\!\int_{\mathbb{R}^6}\left|f(\mathbf{x})Duv + D^{1/2}f^{1/2}(\mathbf{x})\lambda^{1/2}g(\mathbf{x})(u+v) + \lambda g^2(\mathbf{x})\right|^p$$
$$\times\omega^2(\mathbf{x})\left(\psi(u,v;r(\mathbf{y})) - \varphi(u)\varphi(v)\right)du\,dv\,d\mathbf{x}\,d\mathbf{y}.$$

We note that $S^2 = S_0^2$.

COROLLARY. *Let* $1 \le p < \infty$. *We assume that* (C1)–(C9) *hold.*

(i) *If* $nh^6(n) \to 0$, *then*

$$\frac{1}{Sh(n)}\left\{\left(n^{1/2}h\right)^p I_n(p) - e(n)\right\} \to_{\mathscr{D}} N(0,1).$$

(ii) *If* $nh^6(n) \to \lambda > 0$, *then*

$$\frac{1}{S_\lambda h(n)}\left\{\left(n^{1/2}h\right)^p I_n(p) - e(n)\right\} \to_{\mathscr{D}} N(0,1).$$

[$N(0,1)$ *stands for a standard normal random variable.*]

The method of the proof is different from that used in Csörgő and Horváth (1988) and involve a Poissonization of the sample size. Csörgő and Horváth (1988) applied approximations of the uniform empirical process with suitably constructed Gaussian processes. However, the rate of approximation is rather poor in the multidimensional case [Borisov (1982)] and, therefore, the Gaussian approximation cannot cover some important cases. We use Poisson approximation of the bivariate empirical process and prove central limits theorem for L_p-norms of kernel-transformed bivariate Poisson processes. Using this technique, we can obtain central limit theorems for L_p-norms, when

$\lim_{n\to\infty} nh^6(n) = \lambda > 0$. This choice of $h(n)$ is called optimal, because it minimizes the mean square error [cf. Prakasa Rao (1983), page 182].

We considered only the estimation of bivariate densities. However, the results and the methods can be generalized immediately for the general multivariate case.

2. Proofs. The proofs of our result will be based on a series of lemmas. We assume without loss of generality that all random variables and processes are defined on the same probability space [de Acosta (1982)]. Throughout the proofs of the lemmas, C stands for a generic constant whose value may differ from line to line. The support of K is denoted by J. Also, we assume $1 \le p < \infty$.

Before we start with the lemmas, we give a brief outline of the proofs. We consider

$$\frac{1}{nh^2(n)} \sum_{i=1}^{\eta} K\left(\frac{\mathbf{t} - \mathbf{X}_i}{h(n)}\right),$$

the Poissonized version of the density estimator. The first and second moments of the L_p-distance between the Poissonized estimate and the true density are calculated. This is done in Lemma 4 using normal approximations. The second step is to show that

$$\frac{1}{nh^2(n)} \sum_{i=\eta+1}^{N} K\left(\frac{\mathbf{t} - \mathbf{X}_i}{h(n)}\right)$$

can be ignored. Lemma 5 handles this problem, by comparison of first and second moments with those found in Lemma 4. Finally, the central limit theorem is obtained for the Poissonized L_p-norm. This is accomplished by breaking the region of integration up into separated blocks. Liapounov's central limit theorem is applied to a sum of weakly dependent random variables, where each random variable represents an integral over one block.

Let $(Y, Z), (Y_1, Z_1), \dots, (Y_K, Z_K)$ be independent, identically distributed random vectors and define $\alpha = EY$, $\beta = EZ$, $\gamma^2 = \operatorname{Var} Y$, $\delta^2 = \operatorname{Var} Z$ and $\rho = \operatorname{Cov}(Y, Z)/\gamma\delta$.

LEMMA 1.

(i) *If $E(Y)^{p+2} < \infty$, then there is a constant $C_1 = C_1(p)$ such that*

$$\left| E\left|\sum_{i=1}^{K} (Y_i - \alpha) + K^{1/2}\gamma\mu\right|^p - K^{p/2}\gamma^p E|N + \mu|^p \right| \le C_1\left(1 + |\mu|^{p-1}\right)\left\{K^{(p-1)/2}\gamma^{p-3}E|Y - \alpha|^3 + \gamma^{-2}E|Y - \alpha|^{p+2}\right\}, \tag{2.1}$$

where N is a standard normal random variable.

(ii) *If* $E|Y|^{2p+2} < \infty$ *and* $E|Z|^{2p+2} < \infty$, *then there is a constant* $C_2 = C_2(p)$ *such that if*

$$E(|Y - \alpha|/\gamma)^{2p+2} \le K^p \tag{2.2}$$

and

$$E(|Z - \beta|/\delta)^{2p+2} \le K^p, \tag{2.3}$$

then we have

$$\begin{aligned}\Bigg|E\Bigg|\Bigg\{\sum_{i=1}^{K}(Y_i - \alpha) + K^{1/2}\gamma\mu\Bigg\}\Bigg\{\sum_{i=1}^{K}(Z_i - \beta) + K^{1/2}\delta\nu\Bigg\}\Bigg|^p \\ -K^p\gamma^p\delta^p E|(N_1 + \mu)(N_2 + \nu)|^p\Bigg| \\ \le C_2(1 + |\mu|^{p-1})(1 + |\nu|^{p-1}) \\ \times\Big\{K^{-1/2}\big(E(|Y - \alpha|/\gamma)^3 + E(|Z - \beta|/\delta)^3\big) \\ +K^{-p}\big(E(|Y - \alpha|/\gamma)^p + E(|Z - \beta|/\delta)^p\big)\Big\}K^p\gamma^p\delta^p,\end{aligned} \tag{2.4}$$

where (N_1, N_2) *is a bivariable normal random vector with* $EN_1 = EN_2 = 0$, $EN_1^2 = EN_2^2 = 1$ *and* $EN_1N_2 = \rho$.

PROOF. The first part follows from Theorem 13 in Petrov (1975), page 125, and Theorem 17.6 in Bhattacharya and Ranga Rao (1976), page 171, implies (2.4). □

We use Lemma 1 with

$$Y = K((\mathbf{X} - \mathbf{y})/h(n)), \qquad Z = K((\mathbf{X} - \mathbf{z})/h(n)) \tag{2.5}$$

and

$$Y_i = K((\mathbf{X}_i - \mathbf{y})/h(n)), \qquad Z_i = K((\mathbf{X}_i - z)/h(n)). \tag{2.6}$$

LEMMA 2. *We assume that* $(Y, Z), (Y_i, Z_i)$, $i \ge 1$, *are defined by* (2.5) *and* (2.6). *If* (C1)–(C8) *hold*, *then we have*

$$\alpha(\mathbf{y}) = h^2(n)f(\mathbf{y}) + o(h^2(n)), \tag{2.7}$$

$$\gamma^2(\mathbf{y}) = h^2(n)Df(\mathbf{y}) + o(h^2(n)) \tag{2.8}$$

uniformly on A, *and*

$$\rho = \begin{cases} 0, & \textit{if } |\mathbf{y} - \mathbf{z}| \ge Ch(n), \\ r((\mathbf{y} - \mathbf{z})/h(n)) + o(1), & \textit{if } |\mathbf{y} - \mathbf{z}| < Ch(n), \end{cases} \tag{2.9}$$

uniformly on $A \times A$.

PROOF. It is easy to see that

$$\alpha(\mathbf{y}) = h^2(n)\iint_{\mathbb{R}^2} K(\mathbf{u})\, f(\mathbf{y} + \mathbf{u}h(n))\, d\mathbf{u}, \tag{2.10}$$

$$EY^2(\mathbf{y}) = h^2(n)\iint_{\mathbb{R}^2} K^2(\mathbf{u})\, f(\mathbf{y} + \mathbf{u}h(n))\, d\mathbf{u} \tag{2.11}$$

and

$$EYZ = \iint_{\mathbb{R}^2} K((\mathbf{x} - \mathbf{y})/h(n))K((\mathbf{x} - \mathbf{z})/h(n))\, f(\mathbf{x})\, d\mathbf{x}. \tag{2.12}$$

Using one- and two-term Taylor expansion, we get (2.7)–(2.9) from (2.10)–(2.12).

Let $\eta(n)$ be a Poisson random variable with $E\eta(n) = n$, independent from $\mathbf{X}$, $\{\mathbf{X}_i,\ i \geq 1\}$. Thus one can write

$$n^{1/2}h(n)(f_n(\mathbf{y}) - f(\mathbf{y})) = \Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + \Gamma_n^{(3)}(\mathbf{y}) + m_n(\mathbf{y}), \tag{2.13}$$

where

$$\Gamma_n^{(1)}(\mathbf{y}) = \frac{1}{h(n)n^{1/2}}\left\{\sum_{i=1}^{\eta} Y_i(\mathbf{y}) - n\alpha(\mathbf{y})\right\}, \tag{2.14}$$

$$\Gamma_n^{(2)}(\mathbf{y}) = \frac{n-\eta}{h(n)n^{1/2}}\alpha(\mathbf{y}) \tag{2.15}$$

and

$$\Gamma_n^{(3)}(\mathbf{y}) = \frac{1}{h(n)n^{1/2}}\sum_{i=\eta+1}^{n} (Y_i(\mathbf{y}) - \alpha(\mathbf{y})). \tag{2.16}$$

(If $\eta = n$, then $\Sigma_{\eta+1}^{n} = 0$, and if $\eta > n$, then $\Sigma_{i=\eta+1}^{n} = -\Sigma_{i=n+1}^{\eta}$.) □

LEMMA 3. *If* (C1)–(C9) *hold, then we have*

$$E\iint_{\mathbb{R}^2} |\Gamma_n^{(3)}(\mathbf{y})|^p \omega(\mathbf{y})\, d\mathbf{y} = o(h(n)).$$

PROOF. Similarly to Lemma 2 one can easily establish that

$$E|Y - \alpha|^3 = h^2(n)\, f(\mathbf{y})\iint_{\mathbb{R}^2} K^3(\mathbf{u})\, d\mathbf{u} + o(h^2(n)) \tag{2.17}$$

and

$$E|Y - \alpha|^{p+2} = h^2(n)\, f(\mathbf{y})\iint_{\mathbb{R}^2} K^{p+2}(\mathbf{u})\, d\mathbf{u} + o(h^2(n)) \tag{2.18}$$

uniformly on A. Using (2.1), we get that for each $K \geq 0$,

$$\left| E\left| \sum_{i=n+1}^{K} (Y_i - \alpha) \right|^p - |K - n|^{p/2}\gamma^p E|N|^p \right| \tag{2.19}$$
$$\leq C_1\{|K - n|^{(p-1)/2}\gamma^{p-3}E|Y - \alpha|^3 + \gamma^{-2}E|Y - \alpha|^{p+2}\},$$

where N stands for a standard normal r.v. Now we apply (2.8), (2.17) and (2.18) and we obtain

$$\left| E\left| \sum_{i=n+1}^{\eta} (Y_i - \alpha) \right|^p - E|\eta(n) - n|^{p/2}\gamma^p E|N|^p \right| \tag{2.20}$$
$$\leq C\big(E|\eta - n|^{(p-1)/2}h^{p-1}(n) + O(1)\big)$$

uniformly on A. It is easy to see that

$$E|\eta - n|^{p/2} = O(n^{p/4}) \tag{2.21}$$

and

$$E|\eta - n|^{(p-1)/2} = O(n^{(p-1)/4}), \tag{2.22}$$

and therefore we get from (2.20) and (C9)

$$E\left| \sum_{i=n+1}^{\eta} (Y_i - \alpha) \right|^p = O(n^{p/4}h^p(n)),$$

uniformly on A. Hence

$$E\int\int_{\mathbb{R}^2} |\Gamma_n^{(3)}(\mathbf{y})|^p \omega(\mathbf{y})\, d\mathbf{y} = O(n^{-1/4}),$$

and Markov's inequality with (C9) implies Lemma 3. □

Lemma 4. *If* (C1)–(C9) *hold, then we have*

$$E\int\int_{\mathbb{R}^2} |\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})|^p \omega(\mathbf{y})\, d\mathbf{y} = e(n) + o(h(n)) \tag{2.23}$$

and

$$\operatorname{Var}\int\int_{\mathbb{R}^2} |\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})|^p \omega(\mathbf{y})\, d\mathbf{y} = h^2(n)\sigma^2(n) + o(h^2(n)). \tag{2.24}$$

Proof. First we note that

$$\sup_{\mathbf{y}\in A} |m_n(\mathbf{y})| \leq C. \tag{2.25}$$

Let

$$B_n^{(1)}(\mathbf{y}) = E\left|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \tag{2.26}$$

and define

$$\begin{aligned}\psi_1(K) &= E\left|\left(\frac{K}{n}\right)^{1/2}\frac{\gamma}{h(n)}N + \frac{K-n}{n^{1/2}}\frac{\alpha}{h(n)} + m_n\right|^p\\ &= \int_{-\infty}^{\infty}\left|\left(\frac{K}{n}\right)^{1/2}\frac{\gamma}{h(n)}x + \frac{K-n}{n^{1/2}}\frac{\alpha}{h(n)} + m_n\right|^p \varphi(x)\,dx.\end{aligned}$$

Using Lemma 1, we get

$$\sup_{\mathbf{y}\in A}\left|B_n^{(1)}(\mathbf{y}) - E\psi_1(\eta(n))\right| = o(h(n)). \tag{2.27}$$

By the Kómlos–Major–Tusnády approximation [cf. Theorem 2.6.2 in Csörgő and Révész (1981)], we can define a Wiener process $\{W(t),\ 0 \le t < \infty\}$ such that

$$P\{|\eta(n) - n - W(n)| > x\} \le C_3\exp(-C_4x). \tag{2.28}$$

Also,

$$n^{-1/2}W(n) =_{\mathscr{D}} N,$$

where N is a standard normal r.v. and therefore (2.28) implies

$$\begin{aligned}\sup_{y\in A}\Bigg|E&\left|\left(\frac{\eta}{n}\right)^{1/2}\frac{\gamma}{h(n)}x + \frac{\eta(n)-n}{n^{1/2}}\frac{\alpha}{h(n)} + m_n\right|^p\\ &- E\left|\frac{\gamma}{h(n)}x + N\frac{\alpha}{h(n)} + m_n\right|^p\Bigg|\\ &\le C(1+|x|)^{2p}n^{-1/2}.\end{aligned} \tag{2.29}$$

Let $c(n) = -(m_nh(n) + \gamma x)/\alpha$. Now we can write

$$\begin{aligned}E&\left|\frac{\gamma}{h(n)}x + N\frac{\alpha}{h(n)} + m_n\right|^p\\ &= \int_{-\infty}^{c(n)}\left(-\left(\frac{\gamma}{h(n)}x + m_n + \frac{\alpha}{h(n)}t\right)\right)^p\varphi(t)\,dt\\ &\quad + \int_{c(n)}^{\infty}\left(\frac{\gamma}{h(n)}x + m_n + \frac{\alpha}{h(n)}t\right)^p\varphi(t)\,dt.\end{aligned} \tag{2.30}$$

Let $c(n) > 0$. [A similar argument works, if $c(n) < 0$.] Using a two-term

Taylor expansion, we get

$$\left|\int_{-\infty}^{c(n)}\left(-\left(\frac{\gamma}{h(n)}x + m_n + t\frac{\alpha}{h(n)}\right)\right)^p \varphi(t)\,dt - \Phi(c(n))\left(-\left(\frac{\gamma}{h(n)}x + m_n\right)\right)^p\right| \le C(1+|x|^{p-1})h(n)\left|\int_{-\infty}^{c(n)} t\varphi(t)\,dt\right| + O(h^2(n)) \tag{2.31}$$

and

$$\left|\int_{c(n)}^{\infty}\left(\frac{\gamma}{h(n)}x + m_n + \frac{\alpha}{h(n)}t\right)^p \varphi(t)\,dt\right| \le C\exp(-c^2(n)/3). \tag{2.32}$$

It is easy to see that $\lim_{n\to\infty} c(n)h(n) = a_1(\mathbf{y})(x + a_2(\mathbf{y}))$ uniformly on A, where a_1, a_2 are continuous functions and $\inf_{\mathbf{y}\in A}|a_1(\mathbf{y})| > 0$. If $|x + a_2(\mathbf{y})| > (\log 1/h(n))^{-1}$, then $|c(n)| \to \infty$. Thus, if x satisfies $|x + a_2(\mathbf{y})| > (\log 1/h(n))^{-1}$ and $|x| \le h^{-1/2}(n)$, then by (2.30), (3.31) and (2.32) we have

$$\left|E\left|\frac{\gamma}{h(n)}x + m_n + N\frac{\alpha}{h(n)}\right|^p - \left|\frac{\gamma}{h(n)}x + m_n\right|^p\right| \le Ch^2(n). \tag{2.33}$$

Also,

$$\int_{a_2(\mathbf{y})-(\log 1/h(n))^{-1}}^{a_2(\mathbf{y})+(\log 1/h(n))^{-1}}\left|\left|\frac{\gamma}{h(n)}x + m_n\right|^p - E\left|\frac{\gamma}{h(n)}x + m_n + N\frac{\alpha}{h(n)}\right|^p\right|\varphi(x)\,dx \le \frac{Ch(n)}{\log(1/h(n))}. \tag{2.34}$$

It is easy to see that

$$\int_{-\infty}^{-h^{-1/2}(n)}\left(\left|\frac{\gamma}{h(n)}x + m_n\right|^p + E\left|\frac{\gamma}{h(n)}x + m_n + N\frac{\alpha}{h(n)}\right|^p\right)\varphi(x)\,dx \le Ch^2(n) \tag{2.35}$$

and

$$\int_{h^{-1/2}(n)}^{\infty}\left(\left|\frac{\gamma}{h(n)}x + m_n\right|^p + E\left|\frac{\gamma}{h(n)}x + m_n + N\frac{\alpha}{h(n)}\right|^p\right)\varphi(x)\,dx \le Ch^2(n). \tag{2.36}$$

Thus we proved that

$$\int\int_{\mathbb{R}^2} E\left|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \omega(\mathbf{y})\,d\mathbf{y}$$
$$= \int\int_{\mathbb{R}^2}\int_{-\infty}^{\infty}\left|\frac{\gamma(\mathbf{y})}{h(n)}x + m_n(\mathbf{y})\right|^p \varphi(x)\omega(\mathbf{y})\,dx\,d\mathbf{y} + o(h(n)), \tag{2.37}$$

and now (2.23) follows from (2.8).

Let N_n be a Poisson process on the plane with $EN_n(\mathbf{x}) = nF(\mathbf{x})$ and define $G_n(\mathbf{x}) = N_n(\mathbf{x}) - nF(\mathbf{x})$. Then

$$\left\{\sum_{i=1}^{\eta(n)} I\{X_i^{(1)} \le x_1,\, X_i^{(2)} \le x_2\},\, \mathbf{x} \in \mathbb{R}^2\right\} =_{\mathscr{D}} \{N_n(\mathbf{x}),\, \mathbf{x} \in \mathbb{R}^2\}. \tag{2.38}$$

The maximum norm in $\mathbb{R}^2$ is denoted by $|\mathbf{x}| = \max(|x_1|, |x_2|)$. The Poisson process $N_n(\mathbf{x})$ has independent increments and therefore (C1) implies that $\Gamma_n^{(1)}(\mathbf{x})$ and $\Gamma_n^{(1)}(\mathbf{y})$ are independent, if $|\mathbf{x} - \mathbf{y}| > Ch(n)$. Thus we get

$$E\left|\Gamma_n^{(1)}(\mathbf{x}) + m_n(\mathbf{x})\right|^p\left|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p$$
$$= E\left|\Gamma_n^{(1)}(\mathbf{x}) + m_n(\mathbf{x})\right|^p E\left|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \tag{2.39}$$

if $|\mathbf{x} - \mathbf{y}| > Ch(n)$. Let $A_n = \{(\mathbf{x},\mathbf{y})\colon \mathbf{x} \in A,\, \mathbf{y} \in A \text{ and } |\mathbf{x} - \mathbf{y}| \le Ch(n)\}$. Using (2.39), we obtain

$$\operatorname{Var}\int\int_{\mathbb{R}^2}\left|\Gamma_n^{(1)}(\mathbf{x}) + m_n(\mathbf{x})\right|^p \omega(\mathbf{x})\,d\mathbf{x}$$
$$= \int\int_{A_n} E\left|\Gamma_n^{(1)}(\mathbf{x}) + m_n(\mathbf{x})\right|^p\left|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \omega(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y}$$
$$-\int\int_{A_n} E\left|\Gamma_n^{(1)}(\mathbf{x}) + m_n(\mathbf{x})\right|^p E\left|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \omega(\mathbf{x})\omega(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} \tag{2.40}$$
$$= B_n^{(2)} - B_n^{(3)}.$$

Proving (2.37), we showed that

$$E\left|\Gamma_n^{(1)}(\mathbf{x}) + m_n(\mathbf{x})\right|^p = E\left|Nf^{1/2}(\mathbf{x})D^{1/2} + m_n(\mathbf{x})\right|^p + o(h(n)), \tag{2.41}$$

where N is a standard normal r.v. Hence we have

$$B_n^{(3)} = \int\int_{A_n} E\left|Nf^{1/2}(\mathbf{x})D^{1/2} + m_n(\mathbf{x})\right|^p$$
$$\times E\left|Nf^{1/2}(\mathbf{y})D^{1/2} + m_n(\mathbf{y})\right|^p \omega(\mathbf{x})\omega(\mathbf{y})\,d\mathbf{x}\,d\mathbf{y} + o(h^2(n)). \tag{2.42}$$

The computation of $B_n^{(2)}$ requires more effort. Two-term Taylor expansion shows that

$$E(|Y-\alpha|/\gamma)^{2p+2} \le Ch^{-2p}(n) \tag{2.43}$$

and

$$E(|Z-\beta|/\delta)^{2p+2} \le Ch^{-2p}(n). \tag{2.44}$$

Thus

$$\eta(n) \ge C^{1/p}/h^2(n) \tag{2.45}$$

implies that

$$E(|Y-\alpha|/\gamma)^{2p+2} \le \eta^p(n) \tag{2.46}$$

and

$$E(|Z-\beta|/\delta)^{2p+2} \le \eta^p(n). \tag{2.47}$$

Let I_n be the indicator of the event $\eta(n) \ge C^{1/p}/h^2(n)$. Let

$$\begin{aligned}\psi_2(K) = E\Big|&\big(N_1 + (K/n)^{-1/2}(h(n)/\gamma)m_n(\mathbf{y})\big)\\ &\times\big(N_2 + (K/n)^{-1/2}(h(n)/\delta)m_n(\mathbf{z})\big)\Big|^p,\end{aligned}$$

where (N_1, N_2) is a bivariate normal random vector with $EN_1 = EN_2 = 0$, $EN_1^2 = EN_2^2 = 1$ and $EN_1N_2 = \rho$. Using Lemma 1, we get

$$\begin{aligned}&\left|E\left|\left(\sum_{i=1}^{\eta(n)}(Y_i-\alpha) + \eta^{1/2}h(n)m_n(\mathbf{y})\right)^p\right.\right.\\ &\qquad\left.\left.\times\left(\sum_{i=1}^{\eta(n)}(Z_i-\beta) + \eta^{1/2}h(n)m_n(\mathbf{z})\right)^p\right|I_n - E\eta^p(n)\gamma^p\delta^pI_n\psi_2(\eta(n))\right|\\ &\quad= o(h(n)n^p\gamma^p\delta^p).\end{aligned} \tag{2.48}$$

It is easy to see that

$$E(1-I_n) = O(n^{-\kappa}) \tag{2.49}$$

for every $\kappa > 0$, and therefore

$$\begin{aligned}&|E\eta^p(n)\gamma^p\delta^pI_n\psi_2(\eta(n)) - E\eta^p(n)\gamma^p\delta^p\psi_2(\eta(n))|\\ &\quad= o(h(n)n^p\gamma^p\delta^p).\end{aligned} \tag{2.50}$$

Applying (2.28), we obtain

$$|E\eta^p(n)\gamma^p\delta^p\psi_2(\eta(n)) - n^p\gamma^p\delta^p\psi_2(n)| = O(h(n)n^p\gamma^p\delta^pn^{-1/2}). \tag{2.51}$$

The Cauchy–Schwarz inequality and (2.49) give

$$\begin{aligned}E&\left|\left(\sum_{i=1}^{\eta(n)}(Y_i-\alpha) + n^{1/2}h(n)m_n(\mathbf{y})\right)^p\right.\\ &\left.\times\left(\sum_{i=1}^{\eta(n)}(Z_i-\beta) + n^{1/2}h(n)m_n(\mathbf{z})\right)^p\right|(1-I_n) = O(n^{-\kappa})\end{aligned} \tag{2.52}$$

for every $\kappa > 0$. By (2.48), (2.50)–(2.52) and Lemma 2, we have

$$B_n^{(2)} = \int\int_{A_n} E\left|\left(N_1 f^{1/2}(\mathbf{x}) D^{1/2} + m_n(\mathbf{x})\right)\left(N_2 f^{1/2}(\mathbf{y}) D^{1/2} + m_n(\mathbf{y})\right)\right|^p \tag{2.53}$$
$$\times \omega(\mathbf{x})\omega(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y} + o(h^2(n)).$$

Change of variables and (2.9) implies

$$\begin{aligned} B_n^{(2)} - B_n^{(3)} = \int\int_{\mathbb{R}^2}\int\int_{A_n} &\left|D f^{1/2}(\mathbf{x}) f^{1/2}(\mathbf{y}) ts + f^{1/2}(\mathbf{x}) D^{1/2} m_n(\mathbf{y}) t \right. \\ &\left. + f^{1/2}(\mathbf{y}) D^{1/2} m_n(\mathbf{x}) s + m_n(\mathbf{x}) m_n(\mathbf{y})\right|^p \\ &\times \omega(\mathbf{x})\omega(\mathbf{y})\left(\psi(t, s; \rho(\mathbf{x}, \mathbf{y})) - \varphi(t)\varphi(s)\right) d\mathbf{x}\, d\mathbf{y}\, dt\, ds \\ &+ o(h^2(n)) \\ = {} & h^2(n)\sigma^2(n) + o(h^2(n)). \end{aligned}$$

This also completes the proof of Lemma 4. □

LEMMA 5. *If* (C1)–(C9) *hold, then we have*

$$\left|\int\int_{\mathbb{R}^2}\left|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \omega(\mathbf{y})\, d\mathbf{y} - \int\int_{\mathbb{R}^2}\left|\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \omega(\mathbf{y})\, d\mathbf{y}\right| = o_P(h(n)).$$

PROOF. First we note that

$$\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) = \frac{1}{n^{1/2} h(n)} \sum_{i=1}^{\eta(n)} (Y_i - \alpha),$$

and therefore similarly to (2.23) Lemma 1 implies

$$E\int\int_{\mathbb{R}^2}\left|\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \omega(\mathbf{y})\, d\mathbf{y} = e(n) + o(h(n)). \tag{2.54}$$

Next we show that

$$\operatorname{Var}\left(\int\int_{\mathbb{R}^2}\left|\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + m_n(\mathbf{y})\right|^p \omega(\mathbf{y})\, d\mathbf{y}\right) = h^2(n)\sigma^2(n) + o(h^2(n)) \tag{2.55}$$

and

$$\begin{aligned} (2.56)\qquad &\operatorname{Cov}\biggl(\int\int_{\mathbb{R}^2} \bigl|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\bigr|^p \omega(\mathbf{y})\, d\mathbf{y}, \\ &\qquad \int\int_{\mathbb{R}^2} \bigl|\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + m_n(\mathbf{y})\bigr|^p \omega(\mathbf{y})\, d\mathbf{y}\biggr) \\ &\qquad = h^2(n)\sigma^2(n) + o(h^2(n)). \end{aligned}$$

Let

$$B_n^{(3)} = \bigl(\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})\bigr)\bigl(\Gamma_n^{(1)}(\mathbf{z}) + m_n(\mathbf{z})\bigr)$$

and

$$B_n^{(4)} = \bigl(\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y})\bigr)\bigl(\Gamma_n^{(1)}(\mathbf{z}) + m_n(\mathbf{z}) + \Gamma_n^{(2)}(\mathbf{z})\bigr).$$

A two-term Taylor expansion gives

$$\begin{aligned} (2.57)\qquad &\Bigl|E\bigl|B_n^{(3)}\bigr|^p - E\bigl|B_n^{(4)}\bigr|^p \\ &\quad + pE\bigl|B_n^{(4)}\bigr|^{p-1} \operatorname{sign}\bigl(B_n^{(4)}\bigr)\bigl\{-2\Gamma_n^{(2)}(\mathbf{y})\Gamma_n^{(2)}(\mathbf{z}) \\ &\qquad + \bigl(\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + m_n(\mathbf{y})\bigr)\Gamma_n^{(2)}(\mathbf{z}) \\ &\qquad + \bigl(\Gamma_n^{(1)}(\mathbf{z}) + m_n(\mathbf{z}) + \Gamma_n^{(2)}(\mathbf{z})\bigr)\Gamma_n^{(2)}(\mathbf{y})\bigr\}\Bigr| \\ &= o(h^2(n)) \end{aligned}$$

uniformly on $A \times A$. The Cauchy–Schwarz inequality implies

$$(2.58)\qquad E\bigl|B_n^{(4)}\bigr|^{p-1}\bigl|\Gamma_n^{(2)}(\mathbf{y})\Gamma_n^{(2)}(\mathbf{z})\bigr| = o(h^2(n)).$$

By definition

$$\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) = \frac{1}{n^{1/2}h(n)} \sum_{i=1}^{\eta} (Y_i - \alpha).$$

Also conditionally on $\eta(n)$, $\{\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}), \mathbf{y} \in \mathbb{R}^2\}$ and $\{\Gamma_n^{(2)}(\mathbf{y}), \mathbf{y} \in \mathbb{R}^2\}$ are independent.

Let

$$\begin{aligned} \psi_3(K) = E\Bigl|&\bigl(N_1 + (K/n)^{1/2}(h(n)/\gamma)m_n(\mathbf{y})\bigr) \\ &\times\bigl(N_2 + (K/n)^{1/2}(h(n)/\delta)m_n(\mathbf{z})\bigr)\Bigr|^{p-1} \\ &\times\Bigl\{\operatorname{sign}\bigl(N_1 + (K/n)^{1/2}(h(n)/\gamma)m_n(\mathbf{y})\bigr) \\ &\qquad\times\bigl(N_2 + (K/n)^{1/2}(h(n)/\delta)m_n(\mathbf{z})\bigr)\Bigr\} \\ &\times\bigl(N_1 + (K/n)^{1/2}(h(n)/\gamma)m_n(\mathbf{y})\bigr), \end{aligned}$$

where (N_1, N_2) is a bivariate normal random vector with $EN_1 = EN_2 = 0$, $EN_1^2 = EN_2^2 = 1$ and $EN_1N_2 = \rho$. As in the proof of Lemma 3, I_n is the indicator of the event $\eta(n) \geq C^{1/p}/h^2(n)$. Using Theorem 17.6 in Bhattacharya and Ranga Rao (1976), page 171 [cf. (2.4)], we get

$$\begin{aligned}\Big| E|B_n^{(4)}|^{p-1}\,\mathrm{sign}(B_n^{(4)})(\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + m_n(\mathbf{y}))I_n\Gamma_n^{(2)}(\mathbf{z})& \\ -E\psi_3(\eta(n))I_n\Gamma_n^{(2)}(\mathbf{z})\Big|& \\ \leq CE\frac{1}{\eta^{1/2}h}|\Gamma_n^{(2)}(\mathbf{z})| = o(1/n^{1/2}) = O(1/h^2(n)).&\end{aligned}\tag{2.59}$$

We apply (2.28) and get

$$E|\psi_3(\eta(n)) - \psi_3(n)| = O(1/n^{1/2}),\tag{2.60}$$

and therefore by (2.49) we have

$$E|\psi_3(\eta(n))I_n\Gamma_n^{(2)}(\mathbf{z}) - \psi_3(n)\Gamma_n^{(2)}(\mathbf{z})| = O(1/n^{1/2}).\tag{2.61}$$

Also, $E\Gamma_n^{(2)}(\mathbf{z}) = 0$, and thus we obtain from (2.59)–(2.62) that

$$\begin{aligned}&E|B_n^{(4)}|^{p-1}\,\mathrm{sign}(B_n^{(4)})(\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + m_n(\mathbf{y}))I_n\Gamma_n^{(2)}(\mathbf{z}) \\ &\quad = o(h^2(n)).\end{aligned}\tag{2.62}$$

Using (2.49), we get

$$\begin{aligned}&E|B_n^{(4)}|^{p-1}\,\mathrm{sign}(B_n^{(4)})(\Gamma_n^{(1)}(\mathbf{y}) + \Gamma_n^{(2)}(\mathbf{y}) + m_n(\mathbf{y}))(1 - I_n)\Gamma_n^{(2)}(\mathbf{z}) \\ &\quad = o(h^2(n)).\end{aligned}\tag{2.63}$$

Now (2.57), (2.62), (2.63) and (2.23) imply (2.55).

The proof of (2.56) is very similar to that of (2.55) and therefore it is omitted. Lemma 5 follows immediately from Chebyshev's inequality and (2.54)–(2.56). □

In the following lemma we prove a central limit theorem for the L_p-norm of kernel-transformed Poisson processes. We follow the proof of a similar result in Rosenblatt (1975), pages 7–9.

LEMMA 6. *If* (C1)–(C9) *hold, then we have*

$$\frac{1}{\sigma(n)h(n)}\left\{\iint_{\mathbb{R}^2}|\Gamma_n^{(1)}(\mathbf{y}) + m_n(\mathbf{y})|^p\omega(\mathbf{y})\,dy - e(n)\right\} \to_{\mathscr{D}} N(0,1),$$

where $N(0,1)$ *is a standard normal random variable.*

PROOF. Let

$$\Gamma_n^{(4)}(\mathbf{y}) = \frac{1}{n^{1/2}h(n)}\iint_{\mathbb{R}^2}K((\mathbf{x} - \mathbf{y})/h(n))\,dG_n(\mathbf{y}),$$

where $G_n(\mathbf{y})$ is a centralized Poisson process on the plane with $EG_n(\mathbf{y}) = 0$ and

$EG_n^2(\mathbf{y}) = nF(\mathbf{y})$. By (2.38) and (2.23) it is enough to show that

$$(2.64)\quad \frac{1}{\sigma(n)h(n)}\int\int_{\mathbb{R}^2}\Big(\big|\Gamma_n^{(4)}(\mathbf{y}) + m_n(\mathbf{y})\big|^p - E\big|\Gamma_n^{(4)}(\mathbf{y}) + m_n(\mathbf{y})\big|^p\Big)\omega(\mathbf{y})\,d\mathbf{y} \to_{\mathscr{D}} N(0,1).$$

We can assume without loss of generality that $A \subseteq [-a,a]\times[-a,a]$ and $J \subseteq [-1,1]\times[-1,1]$. We define

$$\begin{aligned}
T_{i,j} &= [-a + ih(n), -a + (i+1)h(n)]\\
&\qquad \times[-a + jh(n), -a + (j+1)h(n)],\\
T_{k_0,j} &= [-a + k_0h(n), a] \times [-a + jh(n), -a + (j+1)h(n)],\\
T_{i,k_0} &= [-a + ih(n), -a + (i+1)h(n)] \times [-a + k_0h(n), a],\\
&\qquad\qquad 1 \le i, j \le k_0 - 1,\ k_0 = [2a/h(n)]
\end{aligned}$$

and

$$T_{k_0,k_0} = [-a + k_0h(n), a] \times [-a + k_0h(n), a].$$

Now we consider

$$\begin{aligned}
&\int\int_{\mathbb{R}^2}\Big(\big|\Gamma_n^{(4)}(\mathbf{y}) + m_n(\mathbf{y})\big|^p - E\big|\Gamma_n^{(4)}(\mathbf{y}) + m_n(\mathbf{y})\big|^p\Big)\omega(\mathbf{y})\,d\mathbf{y}\\
(2.65)\quad &= \sum_{1\le i,j\le k_0}\int\int_{T_{i,j}}\Big(\big|\Gamma_n^{(4)}(\mathbf{y}) + m_n(\mathbf{y})\big|^p - E\big|\Gamma_n^{(4)}(\mathbf{y}) + m_n(\mathbf{y})\big|^p\Big)\omega(\mathbf{y})\,d\mathbf{y}\\
&= \sum_{1\le i,j\le k_0}\xi_{i,j}.
\end{aligned}$$

Let $M_1 = h^{-\nu}(n)$, $0 < \nu < 1$, and define

$$\mu_{i,j} = \sum_{(l,k)\in c_{i,j}}\xi_{l,k},$$

where

$$\begin{aligned}
c_{i,j} = \{(l,k)\colon &(i-1)(M_1+4) + 1 \le l \le (i-1)(M_1+4) + 1 + M_1,\\
&(j-1)(M_1+4) + 1 \le k \le (j-1)(M_1+4) + 1 + M_1\},\\
&1 \le i,j \le M_2 = [(k_0+3)/(M_1+4)] \le Ch^{\nu-1}(n).
\end{aligned}$$

Also, we introduce

$$(2.66)\quad \mu = \sum_{1\le i,j\le k_0}\xi_{i,j} - \sum_{1\le i,j\le M_2}\mu_{i,j}.$$

First, we note that similarly to (2.24) we get from Lemma 1,

$$(2.67)\quad E\xi_{i,j}^2 \le Ch^4(n).$$

If $|i-k| > 4$ or $|j-l| > 4$, then $\xi_{i,j}$ and $\xi_{k,l}$ are independent random variables, and therefore (2.67) implies

$$E\mu^2 \le C(M_2 k_0 + M_1 k_0) h^4(n) = o(h^2(n)). \tag{2.68}$$

Similarly,

$$E\left|\mu \sum_{1\le i,j\le M_2} \mu_{i,j}\right| = o(h^2(n)). \tag{2.69}$$

It follows from (2.68), (2.69) and Lemma 2 that

$$\sum_{1\le i,j\le M_2} E\mu_{i,j}^2 = h^2(n)\sigma^2(n) + o(h^2(n)). \tag{2.70}$$

Now we must estimate $E|\xi_{i,j}|^4$. Using again Theorem 17.6 in Bhattacharya and Ranga Rao (1976), page 171 [or a calculation similar to (39) in Rosenblatt (1976)], we get

$$E|\xi_{i,j}|^4 \le Ch^8(n). \tag{2.71}$$

Now (2.71) yields

$$\left(\sum_{1\le i,j\le M_2} E|\mu_{i,j}|^4\right)^{1/4} \le C(M_1^4 M_2^2)^{1/4} h^2(n) = o(h(n)). \tag{2.72}$$

By (2.70) and (2.71) we can apply the Liapounov central limit theorem, and hence we obtain (2.64). □

PROOF OF THEOREM. It follows immediately from (2.13) and Lemmas 3, 5 and 6. □

PROOF OF COROLLARY. Using a two-term Taylor expansion, we get

$$Ef_n(\mathbf{y}) - f(\mathbf{y}) = h^2(n) g(\mathbf{x}) + o(h^2(n))$$

uniformly on A. Hence

$$\lim_{n\to\infty} \sigma^2(n) = \begin{cases} S^2, & \text{if } nh^6(n) \to 0,\\ S_\lambda^2, & \text{if } nh^6(n) \to \lambda, \end{cases}$$

and the corollary follows from the theorem. □

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
SALT LAKE CITY, UTAH 84112