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RANDOM STIRRING OF THE REAL LINE

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Random stirring of the real line R_1 is defined. This notion is derived from a generalization of the nearest-neighbor simple exclusion model on the one-dimensional lattices discussed by Spitzer and by Harris. Under the random stirring, the motion of an infinite particle system is Markovian and has a Poisson process as an invariant probability measure. An ergodic theorem is established concerning the convergence of a system to a Poisson process.

1. Introduction and summary. In his paper [12], Spitzer formulated several models concerning the motions of infinite particle systems with interactions. The configurations of the systems can be described by certain Markov processes with invariant measures which are identified as some of the classical measures in statistical mechanics. See also ([6], [8]). A special case of these is the simple exclusion model on lattices with nearest neighbor assumption. In the one-dimensional case, this can be described roughly in the following way. Consider infinitely many particles on the integers Z. They move in such a way that (a) no point of Z can be occupied by more than one particle and (b) at time t a particle can only jump to an unoccupied neighboring site with probability $dt + o(dt)$ in the time interval $(t, t + dt)$. Harris [6] considered the transition of a particle at x to a neighboring site y to be caused by switching the end points of the link $\langle xy\rangle$ joining x and y. This suggests the possibility of regarding the motion of the particles as induced by a transformation on Z. In fact, let T be the transformation on Z such that $T(0) = 1$, $T(1) = 0$, $T(u) = u$ for every $u \neq 0, 1$ and let $T_x(u) = x + T(u - x)$. Then switching the end points of the link $\langle xy\rangle$ is just the same as applying T_x or T_y to Z. If at time t each T_x has a probability $dt + o(dt)$ of being applied during the time interval $(t, t + dt)$, then the counting measure describing the occupation of points of Z has the same law as for the simple exclusion model, although the motion of individual particles is different. A first step generalization of this is to allow T to be a transformation which is a permutation on a set of integers $\{k \in Z: -m \leqq k \leqq m\}$ for some $m \geqq 1$, and leaves integers outside this set unchanged. Further generalization to consideration of particles distributed on the real line leads us to the following problem. For some of the notations see Section 2.

We will consider the effect of repeated stirring of the real line R_1 induced by a measure-preserving bijection T at random times and places, the randomness

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being controlled by the Poisson process P_0 on R_{1+}. More precisely, suppose that $\omega \in \Omega_0$ is a realization of P_0. Then at any time t if there is an atom (x, t) of ω, T_x is applied to R_1; in this case we also say that a stirring of R_1 by T occurs at x at time t. A strirring of R_1 occurs at a point in the interval $(x, x + dx)$ during the time interval $(t, t + dt)$ with probability $dxdt + o(dx\,dt)$.

We might also think of our process as representing a kind of turbulence, with little disturbances popping up at random places and times.

The main results we have obtained are the following:

In Section 3 it is shown that under the random stirring the motion of a system of particles is Markovian. In Section 4 we show that the transition function defining the motion of a finite number of particles has Lebesgue measure as an invariant measure, while the motion of an infinite particle system has a Poisson process as an invariant probability measure; this is done in each case by constructing a reverse process. In Section 5 we give an interesting fact about the distance between two particles, namely, the distance process has λ as an invariant measure on $(0, \infty)$, and under an appropriate condition on T it is a recurrent null process with a single ergodic class of states. Hence two particles will usually be far apart after a long time. Equally important to us is a similar result for the reverse process. In Section 6 it is shown that under reasonable hypotheses the distribution of an infinite particle system will approach a Poisson process; this result is related to the random ergodic theorem ([4] pages 165–166). The same kind of question for particles moving independently is solved by Stone [13]. Finally in Section 7 the corresponding lattice model is discussed

2. Notations.

Z_+ = set of all nonnegative integers. We adopt the convention that a measure described as Z_+-valued can also take infinite values.

$(R_k, \mathscr{B}_k)$ = k-dimensional real Euclidean space, $k = 1, 2, \cdots$. For $A \in \mathscr{B}_k$, $\lambda(A)$ or $|A|$ denotes the Lebesgue measure of A.

$\Xi = \{\xi : \xi$ is Z_+-valued measure on $(R_1, \mathscr{B}_1)$ such that $\xi(\{x\}) \leqq 1$ for each $x \in R_1$ and $\xi(A) < \infty$ for each bounded $A \in \mathscr{B}_1\}$. $\mathscr{S} = \sigma$-field of Ξ generated by all subsets of Ξ of the form $\{\xi : \xi(A) \leq k\}$, where $k \in Z_+$ and $A \in \mathscr{B}_1$. We can regard each ξ as the configuration of a system of indistinguishable particles, an atom of ξ (i.e., a point $x \in R$ with $\xi(\{x\}) = 1$) being identified as a particle. Thus the words "point" and "particle" will be used interchangeably.

$R_{1+} = R_1 \times (0, \infty)$. $\mathscr{B}_{1+}$ = set of all Borel sets in R_{1+}.

$\Omega = \{\omega : \omega$ is Z_+-valued measure on $\mathscr{B}_{1+}$ such that $\omega(R_1 \times \{t\}) \leqq 1$ for each $t \in (0, \infty)$ and $\omega(A) \leqq \infty$ for each bounded set $A \in \mathscr{B}_1\}$.

$\Omega_0 = \{\omega \in \Omega :$ for each $t > 0$, there are infinitely many positive and negative integers m such that $\omega([3m, 3(m + 1)) \times (0, t]) = 0$; $\omega(I \times (0, \infty)) = \infty$ for every interval I in $R_1\}$. The reason that we take Ω_0 is that we will deal with a Poisson process on R_{1+}, which is obviously concentrated on Ω_0. $\mathscr{F}_0 = \sigma$-field generated by all subsets of Ω_0 of the form $\{\omega : \omega(A) \leqq k\}$, where $k \in Z_+$ and

$A \in \mathscr{B}_{1+}$. $P_0 =$ Poisson process on R_{1+} with constant intensity 1 per unit of area. $(\Omega_0, \mathscr{F}_0, P_0)$ is our basic probability space.

The letter T denotes a bijective, bi-measurable and (Lebesgue) measure-preserving transformation from $(R_1, \mathscr{B}_1)$ onto itself such that $T(x) = x$ for each $x \notin [-1, 1]$. Also we will exclude the uninteresting case that $T(x) = x$, λ-a.e. For each $x \in R_1$, T_x is the transformation on R_1 defined by $T_x(y) = x + T(y - x)$.

3. Markov processes. The proofs of the results in this section are standard and will be omitted.

(3.1) *The motion of a point.* For each $x \in R_1$ and $\omega \in \Omega_0$, we construct two sequences $x_0, x_1, \cdots$, where each $x_i \in R_1$, and $0 = t_0 < t_1 < \cdots$, in the following way. Take $x_0 = x$ and $t_0 = 0$. Suppose that for some $n \geqq 0$ we have determined $x_0, \cdots, x_n$ and $t_0 < \cdots < t_n$. Choose t_{n+1} such that for some $\tilde{x}_{n+1} \in R_1$, $(\tilde{x}_{n+1}, t_{n+1})$ is the atom of ω in $[x_n - 1, x_n + 1] \times (t_n, \infty)$ having the smallest t-coordinate. Then put $x_{n+1} = T_{\tilde{x}_{n+1}}(x_n)$ and continue the construction inductively. Next for the same x and ω, define

(3.2) $$X_t(x, \omega) = x_i\,, \qquad t_i \leqq t < t_{i+1}\,, \qquad i = 0, 1, 2, \cdots .$$

PROPOSITION 3.3. *Under the Poisson process P_0 for each fixed $x \in R_1$, $X_t(x, \cdot)$ is a Markov process with initial value x, state space $(R_1, \mathscr{B}_1)$ and stationary transition function $Q_{t,1}(y, A) = P_0\{\omega : X_t(y, \omega) \in A\}$, where $t \geqq 0$, $y \in R_1$ and $A \in \mathscr{B}_1$.*

$X_t(x, \omega)$ describes the motion of a particle initially at x. It will be the basis for constructing other Markov processes that are interesting to us. Note that for any $h > 0$,

(3.4) $$Q_{t,1}(x + h, A + h) = Q_{t,1}(x, A)\,, \qquad \text{where} \quad A + h = \{a + h : a \in A\}\,.$$

(3.5) *The motion of a finite number of particles.* For $k = 1, 2, \cdots$, let $(x_1, \cdots, x_k) \in S_k = \{(y_1, \cdots, y_k) \in R_k : y_i \neq y_j \text{ for } i \neq j\}$. It is clear from Proposition 3.3 that under P_0, $(X_t(x_1, \cdot), \cdots, X_t(x_k, \cdot))$ is a Markov process with initial value $(x_1, \cdots, x_k)$, state space $(S_k, \mathscr{B}_k \cap S_k)$ and stationary transition function $Q_{t,k}(y_1, \cdots, y_k; \Gamma) = P_0\{\omega : (X_t(y_1, \omega), \cdots, X_t(y_k, \omega)) \in \Gamma\}$, where $(y_1, \cdots, y_k) \in S_k$ and $\Gamma \in \mathscr{B}_k \cap S_k$. The following easy observation will be useful later.

(3.6) (i) $Q_{t,k}(x_1, \cdots, x_k; \Gamma) = Q_{t,k}(x_{\sigma(1)}, \cdots, x_{\sigma(k)}; \Gamma_\sigma)$, where σ is a permutation on $\{1, \cdots, k\}$ and

$$\Gamma_\sigma = \{(y_{\sigma(1)}, \cdots, y'_{\sigma(k)}) : (y_1, \cdots, y_k) \in \Gamma\}\,.$$

(ii) $Q_{t,k}(x_1 + s, \cdots, x_k + s; \Gamma + s) = Q_{t,k}(x_1, \cdots, x_k; \Gamma)$ where $\Gamma + s = \{(y_1 + s, \cdots, y_k + s) : (y_1, \cdots, y_k) \in \Gamma\}$.

(3.7) *The motion of a particle system.* For each $\xi \in \Xi$, $\omega \in \Omega_0$ and $t \geqq 0$, define $\xi_t(\xi, \omega)(A) = \xi\{x \in R_1 : X_t(x, \omega) \in A\}$, for $A \in \mathscr{B}_1$. If we interpret ξ as the configuration of a system of indistinguishable particles, then $\xi_t(\xi, \omega)$ can be regarded as the configuration of the system at time t.

PROPOSITION 3.8. *For each $\xi \in \Xi$ under P_0, $\xi_t(\xi, \cdot)$ is a Markov process with initial value ξ, state space $(\Xi, \mathscr{S})$ and stationary transition function $W_t(\eta, E) = P_0\{\omega : \xi_t(\eta, \omega) \in E\}$, where $t \geqq 0$, $\eta \in \Xi$ and $E \in \mathscr{S}$.*

(3.9) *The distance process between two points.* Consider the motion of two points. Denote the set of all strictly positive real numbers by R_+ and the set of all Borel sets in R_+ by $\mathscr{B}_+$. Let $\varphi : S_2 \to R_+$ be the function $\varphi(x, y) = |x - y|$. It is clear from (3.6) that $Q_{t,2}(x, y; \varphi^{-1}(A)) = Q_{t,2}(|x - y|, 0; \varphi^{-1}(A))$, for every $A \in \mathscr{B}_+$. This justifies the following definition.

DEFINITION 3.10. The distance process $D_t(u, \omega)$, $t \geqq 0$, $u \in R_+$, $\omega \in \Omega_0$, is defined as $D_t(u, \omega) = |X_t(u, \omega) - X_t(0, \omega)|$. It represents the distance at time t of two particles initially at a distance u apart. For $u \in R_+$, let $q(u) = \lambda\{x \in R_1 : |T(u - x) - T(-x)| \neq u\}$.

PROPOSITION 3.11. *For each fixed $u \in R_+$, $D_t(u, \cdot)$ is a jump type Markov process with initial value u, state space $(R_+, \mathscr{B}_+)$, stationary transition function $\Delta_t(\alpha, A) = Q_{t,2}(\alpha, 0; \varphi^{-1}(A))$ and intensity $q(\alpha)$, where $\alpha \in R_+$ and $A \in \mathscr{B}_+$.*

4. Reverse processes and invariant measures. In the symmetric simple exclusion model discussed by Spitzer, one important feature is the symmetry of the stochastic kernel defining the motion of a finite number of particles. In our case this symmetry is not necessarily present, but as in the case of some Markov processes, we can still define a reverse process, which will help us to investigate the original one. The following definition is adopted from [10].

DEFINITION 4.1. Let $\{P_t\}_{t \geqq 0}$ and $\{P_t^*\}_{t \geqq 0}$ be two transition functions defined on a measurable space $(Y, \mathscr{C})$. They are called *reverses* of each other with respect to a measure m on $(Y, \mathscr{C})$ if $\int_{C_2} P_t(y, C_1)m(dy) = \int_{C_1} P_t^*(y, C_2)m(dy)$, for every $C_1, C_2 \in \mathscr{C}$ and $t \geqq 0$.

REMARK 4.2. If $\{P_t\}_{t \geqq 0}$ has a reverse with respect to some measure m, then m is an invariant measure for $\{P_t\}_{t \geqq 0}$.

DEFINITION 4.3. If we use $T^* = T^{-1}$ instead of T, we will obtain a different random stirring of the real line. This is called the *reverse* of the original one. The reason for this will be clear later. All the quantities defined for T also make sense for T^* and will be denoted by adding an asterisk.

For each $t > 0$, let $H_t : R_{1+} \to R_{1+}$ be the map

$$\begin{aligned} H_t(x, r) &= (x, t - r) \qquad && 0 < r < t,\ x \in R_1 \\ &= (x, r) && r \geqq t,\ x \in R_1 . \end{aligned}$$

H_t is a measurable bijection and defines a map $H_t^* : \Omega_0 \to \Omega_0$ by $H_t^*(\omega) = \omega H_t^{-1}$ for $\omega \in \Omega_0$. H_t^* in turn defines a measurable transformation on point processes on Ω_0 by mapping P into PH_t^{*-1} for each point process P on Ω_0. An easy observation is that $P_0 = P_0 H_t^{*-1}$. In the following we will drop the dependence on t and simply write ωH_t^{*-1} and PH_t^{*-1} as ω^* and P^* respectively.

Finally, let μ be a probability measure on $(\Xi, \mathscr{S})$. For $r = 1, 2, \cdots$, we define $p_\mu{}^r(A_1, \cdots, A_r; k_1, \cdots, k_r) = \mu\{\xi \in \Xi : \xi(A_i) = k_i, 1 \leqq i \leqq r\}$, where $k_i \in Z_+$ and A_i are bounded sets in $\mathscr{B}_1$, $1 \leqq i \leqq r$. It is known that μ is uniquely determined by the functions $p_\mu{}^r$.

THEOREM 4.4. *For $k = 1, 2, \cdots$,*

(i) *$\{Q_{t,k}\}_{t\geqq 0}$ and $\{Q^*_{t,k}\}_{t\geqq 0}$ are reverses of each other with respect to λ;*
(ii) *λ is invariant for $\{Q_{t,k}\}_{t\geqq 0}$.*

PROOF. (i) The case $k = 2$ is sufficiently representative. In this case it is enough to prove that

$$\int\int_{\Gamma_1} Q_{t,2}(x, y; \Gamma_2)\, dx\, dy = \int\int_{\Gamma_2} Q^*_{t,2}(x, y; \Gamma_1)\, dx\, dy \tag{4.5}$$

for $\Gamma_1 = A_1 \times A_2$ and $\Gamma_2 = B_1 \times B_2$, where A_1, A_2, B_1, and B_2 are bounded sets in $\mathscr{B}_1$ such that $A_1 \cap A_2 = \phi$ and $B_1 \cap B_2 = \phi$. For such sets (let us drop the index $k = 2$), the left-hand side of (4.5) is

$$\int_{\Omega_0} P_0(d\omega) \int\int_{A_1 \times A_2} I_{B_1 \times B_2}(X_t(x, \omega), X_t(y, \omega))\, dx\, dy \tag{4.6}$$

by definition of Q_t. Let ω and t be fixed. Since A_1, A_2, B_1 and B_2 are bounded, we can find a finite sequence $x_1, \cdots, x_n$ in R_1 such that $X_t(z, \omega) = T_{x_n} \circ \cdots \circ T_{x_1}(z)$ for $z \in A_1 \cup A_2$, and $X_t{}^*(z, \omega^*) = T^*_{x_1} \circ \cdots \circ T^*_{x_n}(z)$ for $z \in B_1 \cup B_2$. Noting that for each x_i, T_{x_i} and $T^*_{x_i}$ preserve λ, we have

$$\int\int_{A_1 \times A_2} I_{B_1 \times B_2}(X_t(x, \omega), X_t(y, \omega))\, dx\, dy$$
$$= \int\int_{B_1 \times B_2} I_{A_1 \times A_2}(X_t{}^*(x, \omega^*), X_t(y, \omega^*))\, dx\, dy\,.$$

Accordingly, since $P^* = P$, (4.7) is equal to

$$\int_{\Omega_0} P_0(d\omega) \int\int_{B_1 \times B_2} I_{A_1 \times A_2}(X_t{}^*(x, \omega), X_t{}^*(y, \omega))\, dx\, dy$$

and (4.5) is proved. (ii) follows from (i) by Remark 4.2.

Using similar argument, we can prove

THEOREM 4.7. *Suppose that μ is a probability measure on $(\Xi, \mathscr{S})$ such that for each $r = 1, 2, \cdots$, and each sequence $A_1, \cdots, A_r$ of bounded disjoint sets in $\mathscr{B}_1$, $p_\mu{}^r$ depends on $A_1, \cdots, A_r$ only through their Lebesgue measures. Then*

(i) *$\{W_t\}_{t\geqq 0}$ and $\{W_t{}^*\}_{t\geqq 0}$ are reverses of each other with respect to μ;*
(ii) *μ is invariant for $\{W_t\}_{t\geqq 0}$. In particular, the results are true if μ is a Poisson process.*

THEOREM 4.8. (i) *$\{\Delta_t\}_{t\geqq 0}$ and $\{\Delta_t{}^*\}_{t\geqq 0}$ are reverses of each other with respect to λ on R_+;*
(ii) *λ on R_+ is invariant for $\{\Delta_t\}_{t\geqq 0}$.*

PROOF. It is enough to show that

$$\int_A \Delta_t(u, B)\, du = \int_B \Delta_t{}^*(u, A)\, du \qquad \text{for open intervals } A, B \in R_+\,.$$

For convenience we transform the coordinates of R_2 by $\psi(x, y) = (u, v)$, where $u = (x - y)2^{-\frac{1}{2}}$ and $v = (x + y)2^{-\frac{1}{2}}$. $Q_{t,2}$ is then transformed into a transition function $Q'_{t,2}(u, v; \Gamma) = Q_{t,2}(\phi^{-1}(u, v); \phi^{-1}(\Gamma))$ satisfying

$$(4.9) \qquad Q_t'(u, v + h; \Gamma + (0, h)) = Q_t'(u, v; \Gamma) \qquad \text{for} \quad h \in R_1$$

and $Q_t'(-u, v; \bar{\Gamma}) = Q_t'(u, v; \Gamma)$, where $\bar{\Gamma} = \{(u, v) : (-u, v) \in \Gamma\}$, (we have dropped the index 2 again). Notice also that λ is again invariant for Q_t' and that $\Delta_t(2^{\frac{1}{2}}u, 2^{\frac{1}{2}}A) = Q_t'(u, 0; \tilde{A})$, where $u \in R_+$, $A \in \mathscr{B}_+$ and $\tilde{A} = (A \cup -A) \times R_1$. Similar statements hold for Q_t^*. Furthermore, the relation of two processes being reverses of each other is invariant under a coordinate transformation. Thus Q_t' has a reverse denoted by $Q_t'^*$ which is just Q_t^* transformed by the above rotation, i.e., $Q_t'^* = Q_t^{*\prime}$. Letting $I = (0, 1)$ and using (4.9), we have

$$(4.10) \qquad \begin{aligned} \int_A \Delta_t(2^{\frac{1}{2}}u, 2^{\frac{1}{2}}B)\, du &= \int_I \int_A Q_t'(u, v; \tilde{B})\, du\, dv \\ &= \iint_{\tilde{B}} Q_t^{*\prime}(u, v; A \times I)\, du\, dv \\ &= \int_B \{\int_{R_1} Q_t^{*\prime}(u, 0; (A \cup -A) \times (I - v))\, dv\}\, du\,. \end{aligned}$$

We claim that for fixed $A \in \mathscr{B}_+$,

$$\int_{R_1} Q_t^{*\prime}(u, 0; (A \cup -A) \times (I - v))\, dv \leqq Q_t^{*\prime}(u, 0; \tilde{A}) \qquad \text{for} \quad \lambda\text{-a.e. } u\,.$$

To see this,

(i) First suppose that Q is any transition function on $(R_2, \mathscr{B}_2)$ with a density $q(u, v; x, y)$ satisfying $q(u, v; x, y) = q(u, v + h; x, y + h)$ for $u, v, x, y, h \in R_1$. It is easily verified that the equality in our claim holds in this case.

(ii) Next we define for $n = 1, 2, \cdots$,

$${}_nq_t(u, v; x, y) = \iint_{R_2} Q_t^{*\prime}(u, v; dz, dw) r_n(z, x) r_n(w, y)\,,$$

where $r_n(r, s) = (n/2\pi)^{\frac{1}{2}} \exp[-(n/2)(r - s)^2]$. Using the properties of $Q_t^{*\prime}$ it is not hard to check that ${}_nq_t$ satisfies the requirement in (i). Let ${}_nQ_t$ be the transition function defined by ${}_nq_t$. Then

$$(4.11) \qquad \int_{R_1} {}_nQ_t(u, 0; (A \cup -A) \times (I - v))\, dv \leqq {}_nQ_t(u, 0; \tilde{A})\,.$$

From definition,

$${}_nQ_t(u, 0; \tilde{A}) = \iint_{R_2} Q_t^{*\prime}(u, 0; dx, dy) q_n(x; A \cup -A)\,,$$

where $q_n(x, A \cup -A) = \int_{A \cup -A} r_n(z, x)\, dz$. Thus,

$$(4.12) \qquad \begin{aligned} \lim_{n\to\infty} {}_nQ_t(u, 0; \tilde{A}) &= Q_t^{*\prime}(u, 0; \tilde{A}) \\ &\quad + 2^{-1}Q_t^{*\prime}(u, 0; \{a, -a, b, -b\} \times R_1)\,, \end{aligned}$$

where a and b are the end points of A.

For any $x_0 \in R_1$, using the invariance of λ for $Q_t^{*\prime}$ and (4.9), we obtain $\iint_{R_2} Q_t^{*\prime}(u, 0; \{x_0\} \times R_1)\, du\, dv = 0$. This implies that $Q_t^{*\prime}(u, 0; \{x_0\} \times R_1) = 0$, λ-a.e. u. Accordingly, the limit in (4.12) $= Q_t^{*\prime}(u, 0; \tilde{A})$ for λ-a.e. u. Similarly we prove that

$$\liminf_{n\to\infty} {}_nQ_t(u, 0; A \times (I - v)) \geqq Q_t^{*\prime}(u, 0; A \times (I - v))$$

and

$$\liminf_{n\to\infty} {}_nQ_t(u, 0; -A \times (I - v)) \geqq Q_t^{*\prime}(u, 0; -A \times (I - v))$$

for *every* u and v. The claim then follows from (4.11) and Fatou's lemma. Now returning to (4.10), we have

$$\int_A \Delta_t(2^{\frac{1}{2}}u, 2^{\frac{1}{2}}B)\, du \leqq \int_B \Delta_t^*(2^{\frac{1}{2}}u; 2^{\frac{1}{2}}A)\, du\,. \tag{4.13}$$

However $Q_t^{**} = Q_t$. Thus, equality actually holds in (4.13) and (i) is proved. (ii) again follows from (i).

Theorem 4.8 is quite an interesting fact for D_t and will be an important tool in the next section.

5. The null recurrence of the distance process. In simple exclusion model on lattices with nearest neighbor assumption, it can be shown that any two particles will "usually" be far apart from each other. We wish to establish the same phenomenon in our model.

Let $\hat{\Delta}$ be the transition function of the imbedded pure jump Markov process $\hat{D}_n$ of D_t, and $q(u)$ be as defined in (3.10).

PROPOSITION 5.1. (i) $0 < q(u) \leqq 4$, *for every* $u \in R_+$; $q(u)$ *and* $\hat{\Delta}(u, A)$ *are independent of* u *for* $u > 2$.

(ii) *Let* $\sigma(A) = \int_A q(u)\, du$ *for* $A \in \mathscr{B}_+$. *Then* σ *is an invariant measure for* $\hat{\Delta}$.

PROOF. In (i) the only nonobvious statement is $q(u) > 0$. This can be shown as follows: Let

$$x_0 = \sup\{x : x \in R_1 \text{ and } T(y) \leqq y \text{ for } \lambda\text{-a.e. } y \leqq x\}\,. \tag{5.2}$$

Since T is measure-preserving from $[-1, 1]$ onto $[-1, 1]$, $\int_{[-1,1]}[T(x)-x]\,dx = 0$. This implies that x_0 is in $[-1, 1)$ because the case $T(x) = x$ λ-a.e. is excluded and $T(x) = x$ for $x \notin [-1, 1]$. Also it is obvious that $T(y) \leqq y$ for λ-a.e. $y \leqq x_0$ and that

(5.3) for every $u > 0$, there exist $1 \geqq K_u > 0$ and $L_u > 0$ such that $|B_u| \geqq L_u$, where $B_u = \{x \in (x_0, x_0 + u) : T(x) \geqq x + K_u\}$.

Next note that $T(-x) \leqq -x$ and $T(u - x) \geqq u - x + K_u$ for λ-a.e. $x \in B_u + u$. Hence, $q(u) \geqq |-B_u + u| = |B_u| \geqq L_u > 0$.

(ii) follows directly from the Kolmogorov forward equation for Δ_t.

In what follows, the terminology can be found in ([11], pages 4 and 12). For convenience, we state the following fact ([3] and [11]).

(5.4) Let $\{Z_n\}$ be a Markov process with state space $(Y, \mathscr{C})$ and stationary transition function $P(y, C)$ and let m be a nontrivial σ-finite measure on $(Y, \mathscr{C})$. Suppose that

(i) $\mathscr{C}$ is separable,

(ii) for each m-null set C, $P(y, C) = 0$ m-a.e. and

(iii) $\{Z_n\}$ is m-recurrent on a stochastically closed set A such that $m(A^c) = 0$.

Then the following hold.

(i) $\{Z_n\}$ has a unique (up to a multiplicative constant) nontrivial σ-finite invariant measure π stronger than m.

(ii) If $\pi(Y) = \infty$, then there exists an m-null set B such that $\lim_{n\to\infty} P^{(n)}(y, C) = \lim_{n\to\infty} P^{*(n)}(y, C) = 0$ for every C with $\pi(C) < \infty$ and every $y \notin B$. (P^* denotes a reverse of P with respect to π.)

Next we will impose the following condition on T.

$[D]$. There exists $0 < \delta < 1$ such that

$[D]_1$. For each Borel set A in $(-\delta, 0)$, $|(T - \theta)^{-1}(A)| \geqq \beta|A|$ for some $\beta > 0$ independent of A, where θ denotes the identity function on R_1.

$[D]_2$. For each Borel set A in $(-2, -2 + \delta)$ with $|A| > 0$, $|(T - \theta)^{-1}(A)| > 0$.

See (5.10) below for examples of T satisfying these conditions.

REMARK 5.5. $\hat{\Delta}$ can be expressed in terms of T by $\hat{\Delta}(u, A) = |F_u^{-1}(A)|q(u)^{-1}$, where $u \in R_+$, $A \in \mathscr{B}_+$ with $u \notin A$ and $F_u(x) = |T(u - x) - T(-x)|$ for $x \in R_1$. Therefore, $[D]$ implies the following conditions on $\hat{\Delta}$(δ and β are as in $[D]$).

$[\hat{D}]_1$. For $u > 2$, $\hat{\Delta}(u, A) \geqq \beta|A|4^{-1}$ for every Borel set A in $(u - \delta, u)$.

$[\hat{D}]_2$. For $u > 2$, $\hat{\Delta}(u, A) > 0$ for every Borel set A in $(u - 2, u - 2 + \delta)$ with $|A| > 0$.

LEMMA 5.6. (i) *Under Condition* $[D]_1$, R_+ *is the union of two disjoint sets* R_0 *and* I_0 *such that* R_0 *is stochastically closed*, $\{\hat{D}_n\}$ *is* λ*-recurrent on* R_0 *and* $|I_0 \cap (2, \infty)| = 0$.

(ii) *If furthermore* $[D]_2$ *holds, then* $|I_0 \cap (0, 2)| = 0$.

PROOF. We will divide the proof into several stages. First observe that the size of a jump of $\hat{D}_n$ is at most equal to 2.

(i) Any interval $I = (a, b)$ in $[2, \infty)$ with $|I| > 2$ is essential. For if such an interval I is inessential, then

$$\Pr\{\hat{D}_n \in I \text{ infinitely often} \mid \hat{D}_0 = \alpha\} = 0 \qquad \text{for every} \quad \alpha > 0\,. \tag{5.7}$$

On R_1 define a random walk $V_n = Y_1 + \cdots + Y_n$ where Y_i's are independent random variables with a common distribution given by the displacement random variable Z of $\hat{D}_n$ when $u > 2$, more precisely, $Z = T_x(u) - T_x(0) - u$ with $u > 2$ and x uniform on $\{x \in R_1 \colon T_x(u) - T_x(0) \neq u\}$. Then $E(Z) = 0$. This together with $[\hat{D}]_1$ shows that V_n is λ-recurrent ([2] Chapter 8 and [7]). Now for every $\alpha \in R_+$ because of (5.7) $\hat{D}_n$ is either in $(0, a]$ or $[b, \infty)$ after a finite number of steps. In the latter case, $\hat{D}_n$ would follow the same law as V_n on $[b, \infty)$ and have to visit (a, b) infinitely often almost surely, which is a contradiction. Accordingly $\lim_{n\to\infty} \hat{\Delta}^{(n)}(\alpha, (0, a]) = 1$. However since σ is invariant for $\hat{\Delta}$, $\sigma(0, a] = \int_{R_+} \hat{\Delta}^{(n)}(\alpha, (0, a])\sigma(d\alpha)$, $n \in Z_+$, and so Fatou's lemma gives $\sigma(0, a] \geqq \sigma(R_+)$. This is a contradiction [(5.1)].

(ii) R_+ is properly essential. We assume the contrary. Then R_+ is a countable union of inessential sets. Take the intervals $I_1 = (5, 8)$ and $I_2 = (2, 11)$, and then pick an inessential set A such that $|A^c \cap I_2| < \delta 2^{-1}$, where δ is the number in condition $[D]$. From (1), I_1 is essential. Hence there exists $\alpha_0 \in R_+$ such that

$\Pr\{\hat{D}_n \in I_1 \text{ infinitely often} \mid \hat{D}_0 = \alpha_0\} > 0$. However for each $\alpha \in I_1$, $\hat{\Delta}(\alpha, A) \geqq \hat{\Delta}(\alpha, [\alpha - \delta, \alpha] \cap A) \geqq \beta 4^{-1}|[\alpha - \delta, \alpha] \cap A| \geqq \beta\delta 8^{-1} > 0$. Hence if I_1 is visited infinitely often with nonzero probability, so is A. This is impossible since A is inessential.

(iii) For every α and b in R_+, there exist positive integer n and $r > b$ such that $\hat{\Delta}^{(n)}(\alpha, [b, r]) > 0$. This is clear for $\alpha \geqq b$. Hence we may assume that $\alpha < b$. For $u \geqq \alpha$, let $C_{u,\alpha} = \{x \in (x_0, x_0 + u) : T(x) \geqq x + K_\alpha\}$, where x_0 and K_α are defined in (5.2) and (5.3). Then for λ-a.e. $x \in -C_{u,\alpha} + u$, $T(u - x) - T(-x) \geqq u + K_\alpha$. Thus,

$$\lambda\{x : T(u - x) - T(-x) \geqq u + K_\alpha\} \geqq |C_{u,\alpha}| \geqq |B_\alpha| \geqq L_\alpha ,$$

where B_α and L_α are defined in (5.3). This implies that for $u \geqq \alpha$, $\hat{\Delta}(u, [u + K_\alpha, \infty)) \geqq L_\alpha 4^{-1}$. Next let n_0 be the smallest positive integer such that $\alpha + n_0 K_\alpha > b$. During the time interval $[0, n_0]$, from α $\hat{D}_n$ can move at most to $\alpha + 2n_0$. Then $\hat{\Delta}^{(n_0)}(\alpha, [b, \alpha + 2n_0]) \geqq (L_\alpha/4)^{n_0}$, and this proves (iii).

(iv) R_+ is indecomposable. Let A be a stochastically closed set in R_+. From (iii) it follows that A contains an arbitrarily large positive number. Pick $a \in A$ with $a > 2$. Partition $[2, a]$ into finitely many subintervals J_i each of which has length $< \delta 2^{-1}$. Then from $[\hat{D}]_1$ it is easily seen that $|J_i \cap A^c| = 0$ for each J_i and so $|[2, a] \cap A^c| = 0$. Accordingly,

(5.8) $\quad |A^c \cap [2, \infty)| = 0$ for each stochastically closed set A, and (iv) follows.

(v) Since $\mathscr{B}_+$ is separable, a theorem in ([11] page 39) says that R_+ has the decomposition stated in Lemma 5.6 (i). That $|I_0 \cap [2, \infty)| = 0$ follows from (5.8).

(vi) Finally to prove (ii) subdivide $(0, 2)$ into finitely many intervals J_j each of length $< \delta 2^{-1}$. Because of $[\hat{D}]_2$ for each J_j we can find an interval L_j in $[2, \infty)$ such that $\hat{\Delta}(u, A) > 0$ for every $u \in L_j$ and A in J_j with $|A| > 0$. From (v) we know that $L_j \cap R_0 \neq \phi$. Thus $|J_j \cap R_0{}^c|$ must be zero for each J_j, whence $|I_0 \cap (0, 2)| = 0$.

COMMENTS 5.9. Condition $[D]$ can be modified so that Lemma 5.6 still holds. For example, 1° replace $(T - \theta)$ in $[D]_1$ by $\theta - T$ or $(-\delta, 0)$ by $(0, \delta)$; 2° $[D]_2$ also can be replaced by

$[D]_2'$ For every Borel set A in $(-1, -1 + \delta)$ with $|A| > 0$, $|(T - \theta)^{-1}(A) \cap (-1, 0)| > 0$.

EXAMPLES 5.10. Quite a number of transformations T can be constructed to satisfy $[D]$. In fact, $[D]_1$ is satisfied by any T such that for some $a \in (-1, 1]$ and $\varepsilon > 0$, $T(a) = a$ and T is continuous and linear with slope—1 (abbreviated as CLS-1) on $[a - \varepsilon, a]$. In the following examples if the values of T on some part of $[-1, 1]$ are not specified, it is understood that they can be arbitrarily defined as long as T is as in Section 2.

(i) $T(x) = -x, x \in [-1, 1]$. This is a continuous analogue of the nearest neighbor simple exclusion model on lattices.

(ii) $T(x) = -x$, for $x \in [-1, -1 + \varepsilon)$, $T(-1 + \varepsilon) = -1 + \varepsilon$, and T is CLS-1 on $[-1 + \varepsilon, -1 + 2\varepsilon]$, where $0 < \varepsilon < 2^{-1}$.

(iii) $T(0) = -1$, $T(2^{-1}) = 2^{-1}$, and T is CLS-1 on $[-\varepsilon, 0]$, and $[2^{-1} - \varepsilon, 2^{-1})$, where $0 < \varepsilon < 4^{-1}$.

(i) and (ii) satisfy $[D]_1$ and $[D]_2$, while (iii) satisfies $[D]_1$ and $[D]_2'$.

Condition $[D]$ is assumed from now on.

COROLLARY 5.11. *For each $h > 0$, the set R_0 is stochastically closed with respect to the discrete-time transition function Δ_h. Δ_h is λ-recurrent on R_0.*

THEOREM 5.12. *For each bounded set $A \in R_+$, $\lim_{t\to\infty} \Delta_t(\alpha, A) = \lim_{t\to\infty} \Delta_t^*(\alpha, A) = 0$ for λ-a.e. $\alpha \in R_+$.*

PROOF. It follows from (5.4) and (5.11) that for each $h > 0$ there is a λ-null set B (depending on Δ_h) such that

(5.13) $\quad \lim_{n\to\infty} \Delta_{nh}(\alpha, A) = 0$ for every bounded set $A \in \mathscr{B}_+$ and every $\alpha \notin B$.

Also the Kolmogorov backward equation for Δ_t implies that $\Delta_t(\alpha, A)$ is uniformly continuous in t. From this and (5.13) the result for Δ_t follows. The proof for Δ_t^* is similar.

6. A limit theorem. In this section μ with or without subscript denotes a probability measure on $(\Xi, \mathscr{S})$.

Consider the Markov process $\xi_t(\xi_0, \cdot)$ defined in (3.8). Suppose that μ_0 is the initial distribution of ξ_0. Then the distribution of ξ_t at time t is given by

$$\mu_t(E) = \textstyle\int_\Xi \mu_0(d\xi) W_t(\xi, E) \qquad \text{for every} \quad E \in \mathscr{S}, \tag{6.1}$$

where W_t is defined in Proposition 3.8.

(It is assumed that the initial value ξ_0 is picked according to μ_0 independent of $\omega \in \Omega_0$.) From Theorem 4.7 we know that if μ_0 is a Poisson process on R_1, then μ_0 is an equilibrium state i.e., $\mu_t = \mu_0$ for every $t \geqq 0$. Our problem in this section is to find a reasonable condition on μ_0 such that μ_t will converge to equilibrium.

DEFINITION 6.2. Let $k = 1, 2, \cdots$. A kth order *product density* of μ is a finite nonnegative Borel measurable function f_k on $(S_k, \mathscr{B}_k \cap S_k)$ such that for every nonnegative Borel measurable function g on $(S_k, \mathscr{B}_k \cap S_k)$,

$$E(\mu, g, k) = \int \cdots \int_{S_k} g(x_1, \cdots, x_k) f_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k ,$$

where $E(\mu, g, k)$ denotes the expectation of $\int \cdots \int_{S_k} g(x_1, \cdots, x_k)\xi(dx_1) \cdots \xi(dx_k)$ with respect to μ. Obviously if f_k exists, it is unique (up to a λ-null set).

The following lemma will be useful. Its proof follows readily from Fubini's theorem.

LEMMA 6.3. *Let $\{f_k\}_{k=1,2,\cdots}$ be product densities of μ_0, and g a nonnegative measurable function on $(S_k, \mathscr{B}_k \cap S_k)$. Then*

$$E(\mu_t, g, k) = \int \cdots \int_{S_k} G(x_1, \cdots, x_k) f_k(x_1, \cdots, x_k)\, dx_1 \cdots dx_k\,,$$

where μ_t is defined in (6.1) *and*

$$G(x_1, \cdots, x_k) = \int \cdots \int_{S_k} g(y_1, \cdots, y_k) Q_{t,k}(x_1, \cdots, x_k; dy_1, \cdots, dy_k)\,.$$

DEFINITION 6.4. For $k = 1, 2, \cdots,$ and $M = 1, 2, \cdots,$ let V_{kM} be the set $\{(x_1, \cdots, x_k) \in S_k : |x_i - x_j| \leqq M$ for some $i, j \in \{1, \cdots, k\}\}$. Then a k-tuple $(x_1, \cdots, x_k) \notin V_{kM}$ represents k particles with distances greater than M from each other.

THEOREM 6.5. *Let the transformation T satisfy the condition $[D]$ in Section 5 (see also* (5.9)*). Suppose that the initial distribution μ_0 of the Markov process $\xi_t(\xi, \omega)$ satisfies the following conditions.*

(i) *For some $\rho > 0$, $E_{\mu_0}(\xi(A)) = \rho|A|$ for every $A \in \mathscr{B}_1$.*

(ii) *μ_0 has product densities $\{f_k\}_{k=1,2,\cdots}$ such that*

(a) *for each $k = 1, 2, \cdots, f_k \leqq a_k$ for some $a_k > 0$, and*

(b) *there exists a double sequence $\{\delta_{kM}\}_{k=1,2,\cdots,M=1,2,\cdots}$ of positive numbers such that for each k, $\delta_{kM} \to 0$ as $M \to \infty$, and $\rho^k - \delta_{kM} \leqq f_k(x_1, \cdots, x_k) \leqq \rho^k + \delta_{kM}$ for every $(x_1, \cdots, x_k) \notin V_{kM}$. Then as $t \to \infty$ μ_t converges weakly to ν, the Poisson process on R_1 with intensity ρ.*

REMARK. Notice that (ii b) is a kind of mixing property, insuring almost independence at great distances.

PROOF. First let us show that for each bounded Borel measurable function g on R_k with compact support,

$$\lim_{t\to\infty} E(\mu_t, g, k) = \rho^k \int \cdots \int_{S_k} g(x_1, \cdots, x_k)\, dx_1 \cdots dx_k\,. \tag{6.6}$$

1°. For $k = 1$, (6.6) follows easily from Lemma 6.3 and (6.5 i)

2°. $k \geqq 2$. It is enough to consider non-negative g only. Let A be a compact support of g. Using Lemma 6.3, (6.5 ii), and the fact that λ is invariant for $Q_{t,k}$, it is not hard to show that

$$E(\mu_t, g, k) \leqq (\rho^k + \delta_{kM}) \int \cdots \int_{S_k} g(x_1, \cdots, x_k)\, dx_1 \cdots dx_k + G_1(t)\,, \tag{6.7}$$

where $G_1(t)$ is an integral whose absolute value is bounded by a constant multiple of

$$\sum_{1\leqq i,j\leqq k} \int \cdots \int_{B_{ij}} Q_{t,k}(x_1, \cdots, x_k; A)\, dx_1 \cdots dx_k\,,$$

where $B_{ij} = \{(x_1, \cdots, x_k) \in S_k : |x_i - x_j| \leqq M\}$. Let us consider a typical term in the sum when $i = 1$ and $j = 2$. Call this $G_{12}(t)$. We may assume that $A = A_1 \times \cdots \times A_k$, where the A_j's are one-dimensional bounded intervals. Then denoting $\{(x_3, \cdots, x_k) : x_3, \cdots, x_k$ all distinct$\}$ by S'_{k-2}, we have

$$\begin{aligned} G_{12}(t) &= \int \cdots \int_{A \cap S_k} Q^*_{t,k}(x_1, \cdots, x_k; V_{2M} \times S'_{k-2})\, dx_1 \cdots dx_k \\ &\leqq \prod_{i=3}^k |A_i| \iint_{(A_1 \times A_2) \cap S_2} Q^*_{t,2}(x_1, x_2; V_{2M})\, dx_1\, dx_2\,. \end{aligned}$$

The inequality holds because $Q^*_{t,k}(x_1, \cdots, x_k; V_{2M} \times S'_{k-2}) \leqq Q^*_{t,2}(x_1, x_2; V_{2M})$. From Theorem 5.12, $\lim_{t\to\infty} Q^*_{t,2}(x_1, x_2; V_{2M}) = \lim_{t\to\infty} \Delta_t{}^*(|x_1 - x_2|, (0, M]) = 0$ for λ-a.e. $(x_1, x_2) \in S_2$, and so $\lim_{t\to\infty} G_{12}(t) = 0$. This together with (6.7) shows that

(6.8) $\qquad \limsup_{t\to\infty} E(\mu_t, g, k) \leqq \rho^k \int \cdots \int_{S_k} g(x_1, \cdots, x_k)\, dx_1 \cdots dx_k$

by letting t and M tend to infinity respectively in (6.7). Similarly, we can prove that $\liminf_{t\to\infty} E(\mu_t, g, k) \geqq$ right-hand side of (6.8). Hence (6.6) holds.

Now to finish the proof of the theorem, let f be a continuous function with a compact support B on R_1, $\xi(f) = \int_{R_1} f(x)\xi(dx)$ and $\mu^{(k)} = E_\mu(\xi(f)^k)$. Under the Poisson process ν, $\xi(B)$ is a Poisson variable. Hence

$$\limsup_{k\to\infty} |\nu^{(k)}|^{1/k} k^{-1} \leqq C \limsup_{k\to\infty} [E_\nu(\xi(B)^k)]^{1/k} k^{-1} < \infty , \qquad \text{where } |f| \leqq C .$$

Also it is clear that $\mu^{(k)}$ is a finite sum of terms of the form $E(\mu, g, k)$. Accordingly from (6.6) $\lim_{t\to\infty} \mu_t{}^{(k)} = \nu^{(k)}$, and so the distribution of $\xi(f)$ under μ_t converges to that under ν([1] pages 181–182). The theorem then follows from ([6], Theorem 2.1).

7. The lattice model. In this section we will discuss briefly a similar model in Z_1, the set of all integers. All of the results can be carried over without trouble. We will state two main results without proofs.

(7.1) *Notations.* Let $\bar{Z}$ stand for $Z_1 \times R_+$ with the product topology and measure $\bar{\lambda}$ = product of the counting measure on Z_1 and λ. Ξ, $\mathscr{S}$, Ω_0 and $\mathscr{F}_0$ are defined as in Section 2 with R_1 replaced by Z_1 and R_{1+} by $\bar{Z}$. T denotes a bijective transformation on Z_1 such that $T(i) = i$ for every i outside the set $\{k \in Z_1 : -m \leqq k \leqq m\}$ for some $m \geqq 1$. Z_n represents the set of all n-dimensional lattice points.

Then as in Section 3 we can define Markov processes $\{X_t(x, \omega)\}$ and $\{\xi_t(\xi, \omega)\}$ under the Poisson process $\bar{P}_0$ and $\bar{Z}$ with respect to the measure $\bar{\lambda}$. Let $S_k = \{(r_1, \cdots, r_k) \in Z_k : r_i \neq r_j \text{ for } i \neq j\}$, $k = 1, 2, \cdots$. For $(x_1, \cdots, x_k)$ and $(y_1, \cdots, y_k)$ in S_k,

$$Q_{t,k}(x_1, \cdots, x_k; y_1, \cdots, y_k) = \bar{P}_0\{\omega : X_t(x_1, \omega) = y_1, \cdots, X_t(x_k, \omega) = y_k\}$$

describes the motion of k particles initially at $(x_1, \cdots, x_k)$. Denote by Δ_t the transition function of the distance process defined by $Q_{t,2}$ as in the continuous case.

THEOREM 7.2. *Let N be the set of all positive integers. For every finite subset A of N and every $u \in N$,* $\lim_{t\to\infty} \Delta_t(u, A) = 0$.

REMARK. Unlike the continuous case, this is true for any bijection T mentioned in (7.1).

Next we state the corresponding limit theorem in the lattice model. For $n = 1, 2, \cdots$, and $M = 1, 2, \cdots$, let

$$V_{nM} = \{(r_1, \cdots, r_n) \in S_n : |r_i - r_j| \leqq M \text{ for some } i, j, 1 \leqq i, j \leqq n\} .$$

THEOREM 7.3. *Let T be a transformation defined in* (7.1), μ_0 *the initial distribution of the Markov process* $\{\xi_t(\xi, \omega)\}$ *and* μ_t *the distribution of* ξ_t *at time* t. *Assume that* μ_0 *satisfies the following conditions.*

(i) *There exists* $0 < \rho < 1$ *such that* $E_{\mu_0}(\xi(x)) = \rho$ *for every* $x \in Z_1$.

(ii) *There exists a double sequence* $\{\delta_{nM}\}_{n=1,2,\cdots,M=1,2,\cdots}$ *of positive numbers such that for fixed* n, $\delta_{nM} \to 0$ *as* $M \to \infty$, *and*

$$\rho^n - \delta_{nM} \leqq \mu_0(\{\xi : \xi(r_1) = 1, \cdots, \xi(r_n) = 1\}) \leqq \rho^n + \delta_{nM} ,$$

for $(r_1, \cdots, r_n) \notin V_{nM}$.

Then as $t \to \infty$, μ_t *converges weakly to a probability measure* μ *on* $(\Xi, \mathscr{S})$ *defined by* $\mu(\{\xi : \xi(r_1) = 1, \cdots, \xi(r_n) = 1\}) = \rho^n$, *for every* $(r_1, \cdots, r_n) \in S_n$ *and every integer* $n \geqq 1$.

REMARK. A result related to Theorem 7.3 is obtained recently by Liggett [9].

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