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STRONG LAWS FOR THE MAXIMA OF STATIONARY GAUSSIAN PROCESSES

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Let $\{X_n\}$ be a stationary Gaussian sequence with $EX_0 = 0$, $EX_0^2 = 1$ and $EX_0X_n = r(n)$. Let $c_n = (2\ln n)^{\frac{1}{2}}$ and set $M_n = \max_{0\leqq k\leqq n} X_k$. It is presently known that if $r(n)\ln n = O(1)$,

$$(1) \qquad \liminf \frac{2c_n(M_n - c_n)}{\ln\ln n} = -1 \qquad \text{and} \qquad \limsup \frac{2c_n(M_n - c_n)}{\ln\ln n} = 1$$

with probability 1. Related results are obtained here assuming $r(n) = o(1)$ and $(r(n)\ln n)^{-1}$ is monotone for large n and $o(1)$. Subject to some regularity in $r(n)$, it is shown that if $r(n)\ln n/(\ln\ln n)^2 = o(1)$, then a.s.

$$(2) \qquad \liminf \frac{2c_n(M_n - (1-r(n))^{\frac{1}{2}}c_n - Z_n)}{\ln\ln n} = -1 \quad \text{and}$$
$$\limsup \frac{2c_n(M_n - (1-r(n))^{\frac{1}{2}}c_n - Z_n)}{\ln\ln n} = 1$$

where Z_n is the minimum variance estimate of the mean based on $X_0, \cdots, X_n$. Furthermore if $(\ln\ln n)^2/r(n)\ln n = o(1)$, then a.s.

$$(3) \qquad \lim_{n\to\infty} r^{-\frac{1}{2}}(n)(M_n - (1-r(n))^{\frac{1}{2}}c_n - Z_n) = 0\,.$$

It is pointed out that (2) and (3) contain laws for M_n which more closely resemble the one given here in (1). Corresponding results for continuous parameter Gaussian processes are sketched.

1. Introduction. Let $\{X_n\}$ be a stationary Gaussian sequence and let M_n be the maximum term in $X_0, X_1, \cdots, X_n$. A large number of results are now known which relate the large sample behaviour of M_n to the asymptotic behaviour of the correlation sequence of the process. We give a short review of certain weak and strong results in this vein in order to point the direction of the present study.

Suppose $EX_0 = 0$, $EX_0^2 = 1$ and $EX_0X_n = r(n)$. Set $c_n = (2\ln n)^{\frac{1}{2}}$ and $b_n = c_n - (\ln(4\pi\ln n))/2c_n$. Consider first the convergence in distribution of M_n as $n\to\infty$. Berman has shown in [1] that the classical extreme value distribution for independent normal variables applies as well to processes with sufficient asymptotic independence. A precise statement is as follows.

Theorem 1.1. *If* $r(n)\ln n = o(1)$ *then*

$$(1.1) \qquad P[c_n(M_n - b_n) \leqq x] \to e^{-e^{-x}} \quad \textit{as} \quad n\to\infty \quad \textit{for all} \quad x\,.$$

In a previous paper [7] we have considered the convergence in distribution of

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M_n when $r(n) = o(1)$ but $r(n) \ln n \neq o(1)$. Given such a setting, a variety of limit distributions is possible. In particular we note here that Berman's result is about the best of its kind, for already if $r(n) \ln n = \gamma$ for $n \geqq N$, the limit distribution is not the extreme value distribution of (1.1). It is suggestive of what is to follow, that the normal is itself a limit distribution according to

THEOREM 1.2. *Suppose $r(n)$ is convex and $o(1)$. If $(r(n) \ln n)^{-1}$ is monotone for large n and $o(1)$, then*

$$P[r(n)^{-\frac{1}{2}}(M_n - (1 - r(n))^{\frac{1}{2}} b_n) \leqq x] \to \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du \quad \text{as} \quad n \to \infty \quad \text{for all} \quad x\,. \tag{1.2}$$

The convexity condition is stronger than is necessary to obtain such a result and (1.2) represents a quite general phenomenon. This point has been remarked on in [7] and it will surface again through weakened assumptions in Section 2. On the other hand, the normal limit does depend crucially on some type of smoothness in the decrease of $r(n)$ to zero (cf. [7]).

We turn next to a relevant strong convergence result about M_n. Pickands found in [8] what he termed the Iterated Logarithm Law for maxima. His conditions for validity were subsequently weakened by Mittal [5], so that one may now state

THEOREM 1.3. *If $r(n) \ln n = O(1)$ then with probability* 1,

$$\liminf_{n\to\infty} \frac{2c_n(M_n - c_n)}{\ln \ln n} = -1 \quad \text{and} \quad \limsup_{n\to\infty} \frac{2c_n(M_n - c_n)}{\ln \ln n} = 1\,. \tag{1.3}$$

In the present work we are going to obtain strong laws akin to (1.3) when $r(n) = o(1)$ but $r(n) \ln n \neq O(1)$.

Section 2 begins with an assumed form (2.1) for $r(n)$ and a related representation of the underlying process. This material follows some recent work of Berman [2] that has enabled us to greatly simplify our present tasks. Based on the representation of the process, we define a sequence $\{I_n\}$ of random variables which is to figure in the conclusions of the theorems of Sections 3 and 4.

Section 3 deals with "moderate" correlations—those which are suitably bounded above and below as $n \to \infty$. The main result there is that

$$\liminf \frac{2c_n(M_n - (1 - r(n))^{\frac{1}{2}} c_n - I_n)}{\ln \ln n} = -1 \quad \text{and} \tag{1.4}$$

$$\limsup \frac{2c_n(M_n - (1 - r(n))^{\frac{1}{2}} c_n - I_n)}{\ln \ln n} = 1$$

with probability 1.

In Section 4 we treat large correlations—those for which $r(n) \ln n/(\ln \ln n)^2 \neq O(1)$. The conclusion of Theorem 4.1 is that

$$\lim_{n\to\infty} r^{-\frac{1}{2}}(n)(M_n - (1 - r(n))^{\frac{1}{2}} c_n - I_n) = 0 \tag{1.5}$$

with probability 1. Now I_n is a normal variable with variance $r(n)$ so a considerably strengthened version of (1.2) is visible in (1.5).

The results indicated in (1.4) and (1.5) refer to a particular representation of the process through which the I_n's are defined. In Section 5 we point out that, with a few exceptional cases, (1.4) and (1.5) apply in general if I_n is replaced by Z_n, the Markov estimate of the mean, or $\bar{X}_n$, the least squares estimate of the mean. It is also noted there that iterated logarithm type laws for M_n follow from (1.4) and (1.5) (see (5.11) for the statement of one of these).

The results mentioned above at (1.1), (1.2) and (1.3) have analogues for continuous parameter Gaussian processes (see [10], [7] and [6]). In such extensions, one requires a local condition on the process as well as some form of asymptotic independence. The local condition is generally reflected in the conclusions obtained and ensures that the extensions are nontrivial. We will not carry through such a full-fledged program here although it is possible to do so. We do however handle the continuous parameter problems of Section 2 with no extra effort. In Section 3 we simply state a continuous version of Theorem 3.1 while in Section 4 we sketch a proof of a version of Theorem 4.1.

2. Preliminaries. The object under study is a stationary Gaussian process with mean zero and correlation function r. Through the present section the indexing parameter t may be real-valued or merely integer-valued. We first state and discuss our assumptions on the function r. A particular representation of the process is then brought forward and some simple consequences of it are noted for future use.

Let f be a density function on the real line and set

$$A_t = \{(x, y) \mid -\infty < x < \infty, 0 \leqq y < \infty \text{ and } f(x+t) > y\}.$$

The function r has the special form

$$(2.1) \qquad r(t) = \int_{-\infty}^{\infty} f(x) \wedge f(x+t)\,dx = \int_{-\infty}^{\infty} \int_0^{\infty} 1_{A_t}(x, y) 1_{A_0}(x, y)\,dy\,dx = |A_t \cap A_0| .$$

In (2.1) it will be further assumed that

$$(2.2) \qquad \text{For large } t, \quad A_t \cap A_0 \subseteqq A_s \cap A_0 \quad \text{for} \quad 0 \leqq s \leqq t .$$

Lastly, a growth condition is imposed on r through

$$(2.3) \qquad \text{For large } t, \quad (r(t) \ln t)^{-1} \text{ is monotone in } t \text{ and } o(1) .$$

In connection with (2.1), Berman has shown in [2] that r is a characteristic function and that this might be viewed as a generalization of Pólya's result that suitable convex (on $[0, \infty)$) functions are positive definite. In particular, convex functions are realized in (2.1) by choices of decreasing densities supported on $[0, \infty)$. In such cases (2.2) is automatically satisfied. As a simple indication of the greater freedom available under (2.2), take f to be supported on $[0, \infty)$ and suppose there is a Q so that $f(x) \geqq f(Q)$ on $[0, Q]$ with f decreasing on $[Q, \infty)$.

(2.2) now applies as soon as $t \geqq Q$. The corresponding r is convex on $[Q, \infty)$ but need not be so on $[0, \infty)$.

With regard to (2.3), observe that the $o(1)$ character of the function $(r(t) \ln t)^{-1}$ is natural in light of (1.3). That this function be monotone for large t proves useful through the inequalities below. For large s and $t \geqq s$,

$$\text{(2.4)} \qquad \begin{aligned} &\text{(i)} \quad r(s) - r(t) \leqq r(t)\left(\frac{\ln t}{\ln s} - 1\right), \\ &\text{(ii)} \quad r(s) - r(t) \leqq r(s) \ln \frac{\ln t}{\ln s}. \end{aligned}$$

The first of these inequalities is immediate from (2.3) while the second follows as in the derivation of (2.15) of [7].

We shall make use of a representation given in [2] of a Gaussian process with correlation function (2.1). Let Z be a white noise process on the plane, set

$$\text{(2.5)} \qquad X_t = \textstyle\int_{A_t}\int Z(dx \times dy)$$

and observe that this process has correlation function r. If (2.2) is now invoked for large T, we may define random variables $Y_t{}^T$, $0 \leqq t \leqq T$, and I_T through

$$\text{(2.6)} \qquad \begin{aligned} X_t &= \textstyle\int\int_{A_t \cap (A_0 \cap A_T)^c} Z + \int\int_{A_0 \cap A_T} Z \\ &= (1 - r(T))^{\frac{1}{2}} Y_t{}^T + I_T . \end{aligned}$$

The factorization (2.6) of X_t, $0 \leqq t \leqq T$, into the independent factors $Y_t{}^T$ and I_T is to be used crucially through Sections 3 and 4. The covariance properties of these new processes are as follows: for large T

$$\text{(2.7)} \qquad \begin{aligned} &\text{(i)} \quad EY_s{}^T Y_t{}^T = \frac{r(s-t) - r(T)}{1 - r(T)}, \qquad 0 \leqq s, t \leqq T, \\ &\text{(ii)} \quad EX_t I_T = r(T) \quad \text{for} \quad t \leqq T \quad \text{and if} \quad I_t \quad \text{is defined,} \\ &\qquad EI_t X_T = EI_t I_T = r(T) . \end{aligned}$$

In connection with (2.7 ii) note that if $\{B_t\}$ is a standard Brownian motion, one can represent I_T as $B_{r(T)}$.

3. Moderate correlations. Let $\{X_n\}$ be the discrete parameter process given by (2.6), set $M_n = \max_{0 \leq k \leq n} X_k$ and $c_n = (2 \ln n)^{\frac{1}{2}}$. The main purpose of the section is to prove

THEOREM 3.1. *Assume* (2.1), (2.2) *and* (2.3) *and suppose that* $r(n) \ln n/(\ln \ln n)^2 = O(1)$. *Then with probability* 1,

$$\text{(3.1)} \qquad \begin{aligned} &\liminf \frac{2c_n(M_n - (1 - r(n))^{\frac{1}{2}} c_n - I_n)}{\ln \ln n} = -1 \qquad \text{and} \\ &\limsup \frac{2c_n(M_n - (1 - r(n))^{\frac{1}{2}} c_n - I_n)}{\ln \ln n} = 1 . \end{aligned}$$

For notational purposes we will sometimes write $M_n - I_n = (1 - r(n))^{\frac{1}{2}} M_n{}'$

where M_n' is the maximum of the variables Y_k^n, $0 \leqq k \leqq n$, defined at (2.6). In these terms we must conclude that with probability 1,

$$\text{(3.2)} \qquad \liminf \frac{2c_n(M_n' - c_n)}{\ln \ln n} = -1 \qquad \text{and} \qquad \limsup \frac{2c_n(M_n' - c_n)}{\ln \ln n} = 1 .$$

The proof of Theorem 3.1 is rather long and is broken up into segments as follows. In Lemma 3.1 it will be shown that the lim inf in (3.2) is at least -1 while the lim sup is at most 1. The most arduous segment is completed in Lemma 3.2 where we find that the lim sup of (3.2) is at least 1. Before beginning these lemmas note that M_n' is the maximum of $n + 1$ standard normal variables with nonnegative correlations between variables (2.7 i). By Slepian's lemma

$$\text{(3.3)} \qquad P\left[M_n' \leqq c_n - \frac{(1-\varepsilon)\ln \ln n}{2c_n}\right] \geqq P\left[M_n^* \leqq c_n - \frac{(1-\varepsilon)\ln \ln n}{2c_n}\right]$$

where M_n^* is the maximum of $n + 1$ independent standard normal variables. But the right side of (3.3) $\to 1$ as $n \to \infty$ by (1.1). Hence $M_n' \leqq c_n - (1 - \varepsilon) \ln \ln n/(2c_n)$ infinitely often and the lim inf in (3.2) is no larger than -1.

LEMMA 3.1. *Under the assumptions of Theorem* 3.1,

$$\text{(3.4)} \qquad \liminf \frac{2c_n(M_n' - c_n)}{\ln \ln n} \geqq -1 \qquad \textit{and} \qquad \limsup \frac{2c_n(M_n' - c_n)}{\ln \ln n} \leqq 1 \quad \text{a.s.}$$

PROOF. We use the method of Pickands [10] to argue that it is sufficient to establish (3.4) along suitable subsequences. Assume $r(n) \ln n/((\ln \ln n)^2) \leqq K^2$ for some $K > 2$. Fix $\varepsilon > 0$ and take $n_m = [e^{\varepsilon m}]$. Now let

$$D_m = \left\{ M_{n_m} - I_{n_m} < (1 - r(n_m))^{\frac12} c_{n_m} + \frac{(1+\varepsilon)\ln \ln n_m}{2c_{n_m}} \right\},$$

$$E_m = \left\{ M_N - I_N > (1 - r(N))^{\frac12} c_N + \frac{(1 + 8K\varepsilon)\ln \ln N}{2c_N} \right.$$
$$\left. \text{for some} \quad n_m \leqq N < n_{m+1} \right\},$$

$$F_m = D_m \cap D_{m+1} \cap E_m .$$

We will conclude that the lim sup in (3.4) is $\leqq 1$ by showing $P(E_m \text{ i.o.}) = 0$. This, in turn, is done by showing

$$\text{(3.5)} \qquad \text{(i)} \quad P(F_m \ \text{ i.o.}) = 0 ,$$
$$\text{(ii)} \quad P(D_m^c \ \text{ i.o.}) = 0 .$$

We address (3.5 i) first. Note that if F_m occurs,

$$(1 - r(n_{m+1}))^{\frac12} c_{n_{m+1}} + \frac{(1+\varepsilon)\ln \ln n_{m+1}}{2c_{n_{m+1}}}$$
$$> M_{n_{m+1}} - I_{n_{m+1}} > M_N - I_N + I_N - I_{n_{m+1}}$$
$$> (1 - r(N))^{\frac12} c_N + \frac{(1 + 8K\varepsilon)\ln \ln N}{2c_N} + I_N - I_{n_{m+1}}$$

for some $n_m \leqq N < n_{m+1}$. For that N,

$$I_N - I_{n_{m+1}} < [(1 - r(n_{m+1}))^{\frac{1}{2}} c_{n_{m+1}} - (1 - r(N))^{\frac{1}{2}} c_N] + \frac{1+\varepsilon}{2}\left(\frac{\ln\ln n_{m+1}}{c_{n_{m+1}}} - \frac{\ln\ln N}{c_N}\right) - \frac{4K\varepsilon \ln\ln N}{c_N}. \tag{3.6}$$

The second term on the right side of (3.6) is negative and

$$\frac{(1 - r(n_{m+1}))c^2_{n_{m+1}} - (1 - r(N))c_N^2}{(1 - r(n_{m+1}))^{\frac{1}{2}} c_{n_{m+1}} + (1 - r(N))^{\frac{1}{2}} c_N} < \frac{2 \ln n_{m+1} - 2 \ln n_m}{c_N} < \frac{4\varepsilon}{c_N}$$

if $r(n)c_n^2$ is nondecreasing for $n \geqq n_m$. Thus the occurrence of F_m for large enough m means there is an N, $n_m \leqq N < n_{m+1}$, so that

$$I_N - I_{n_{m+1}} < -\frac{2K\varepsilon \ln\ln N}{c_N} < -\frac{2K\varepsilon \ln\ln n_{m+1}}{c_{n_{m+1}}}.$$

Thus F_m implies

$$\max_{n_m \leq N < n_{m+1}} (I_{n_{m+1}} - I_N) > \frac{2K\varepsilon \ln\ln n_{m+1}}{c_{n_{m+1}}}.$$

Now $I_{n_{m+1}} - I_N$ has normal independent increments so by III, Theorem 2.2 of [4], the probability of this latter event is no more than

$$\begin{aligned} 2P&\left[(I_{n_{m+1}} - I_{n_m}) > \frac{2K\varepsilon \ln\ln n_{m+1}}{c_{n_{m+1}}}\right] \\ &\leqq 2P\left[\frac{I_{n_{m+1}} - I_{n_m}}{(r(n_m) - r(n_{m+1}))^{\frac{1}{2}}} > \frac{2K\varepsilon \ln\ln n_{m+1}}{c_{n_{m+1}} r(n_{m+1})^{\frac{1}{2}}((\ln n_{m+1}/\ln n_m) - 1)^{\frac{1}{2}}}\right] \\ &\leqq 2\Phi(-\varepsilon m^{\frac{1}{2}}) \end{aligned}$$

and the last bound is summable in m. Thus (3.5i) is established. For (3.5ii),

$$\begin{aligned} P[D_m{}^c] &\leqq P\left[M'_{n_m} \geqq c_{n_m} + \frac{(1+\varepsilon)\ln\ln n_m}{2c_{n_m}}\right] \\ &\leqq n_m\left(1 - \Phi\left(c_{n_m} + \frac{(1+\varepsilon)\ln\ln n_m}{2c_{n_m}}\right)\right) \leqq e^{-(1+\varepsilon)\ln\ln n_m} \end{aligned}$$

and this too is summable. The latter portion of (3.4) has been verified.

To establish the first half of (3.4), we proceed through the subsequence $\{n_m\}$ as above. That this is sufficient (corresponding to (3.5i)) is shown by a parallel argument to the one just given. These details will therefore be omitted. We complete matters by indicating an analogue to (3.5ii). Namely, if

$$G_m = \left\{M'_{n_m} > c_{n_m} - \frac{(1+\varepsilon)\ln\ln n_m}{2c_{n_m}}\right\},$$

then $P(G_m{}^c \text{ i.o.}) = 0$. But since $r(n) \ln n/((\ln\ln n)^2)$ is bounded,

$$P(G_m{}^c) < P[M'_{n_m} \leqq b_{n_m} - \varepsilon' r(n_m)^{\frac{1}{2}}] \tag{3.7}$$

where it is to be recalled that $b_n = c_n - \ln(4\pi \ln n)/(2c_n)$. Now the right side of (3.7) appears as (2.13) in [7] and is shown there to be $O(e^{-(\ln n_m)^{\frac{1}{2}}})$ under the assumption that r_n is convex. This can be extended to apply when r_n is positive, and monotone for large n, as is noted in the remarks at the end of Theorem 2 of [7]. Thus we claim the right side of (3.7) is summable in m, which is all we need in order to finish the proof.

The establishment of (3.1) or (3.2) will be complete once we demonstrate:

LEMMA 3.2. *Under the assumptions of Theorem* 3.1,

$$\limsup \frac{2c_n(M_n' - c_n)}{\ln \ln n} \geqq 1 \quad \text{a.s.} \tag{3.8}$$

PROOF. Recall that $M_n' = \max_{0 \leqq i \leqq n} Y_i^n = \max_{0 \leqq i \leqq n} (1 - r(n))^{-\frac{1}{2}}(X_i - I_n)$ and let $\theta_n(\varepsilon) = \theta_n = c_n + (1 - \varepsilon) \ln \ln n/2c_n$. We show that

$$P[Y_i^j \leqq \theta_j,\ i = 0, 1, \cdots, j, j = n, \cdots, N] \tag{3.9}$$

$\to 0$ as $N \to \infty$ for any fixed (suitably large) n and $\varepsilon > 0$. This proceeds through a series of reductions at (3.10), (3.11) and (3.13). Let $L = L(N) = [e^{\ln N(1-1/(\ln \ln N)^2)}]$ and observe that (3.9) is bounded above by

$$\begin{aligned}
&P[X_i - I_{(2q+1)L} \leqq (1 - r((2q+1)L))^{\frac{1}{2}}\theta_{(2q+1)L},\\
&\qquad i = 2qL, \cdots, (2q+1)L, q = 0, 1, \cdots, [\tfrac{1}{2}[N/L] - \tfrac{1}{2}]]\\
&\qquad\quad = P[(X_i - I_N) \leqq (1 - r((2q+1)L))^{\frac{1}{2}}\theta_{(2q+1)L}\\
&\qquad\qquad + (I_{(2q+1)L} - I_N), i = 2qL, \cdots, (2q+1)L, q \leqq [\tfrac{1}{2}[N/L] - \tfrac{1}{2}]]\\
&\qquad\quad \leqq P[(X_i - I_N) \leqq (1 - r((2q+1)L))^{\frac{1}{2}}\theta_{(2q+1)L}\\
&\qquad\qquad + \lambda(r(L) - r(N))^{\frac{1}{2}}, i = 2qL, \cdots, (2q+1)L, q \leqq [\tfrac{1}{2}[N/L] - \tfrac{1}{2}]]\\
&\qquad\qquad + P[\max_q (I_{(2q+1)L} - I_N) > \lambda(r(L) - r(N))^{\frac{1}{2}}]\,.
\end{aligned}$$

Now $I_n - I_m$ has normal independent increments so

$$\begin{aligned}
&P[\max_q (I_{(2q+1)L} - I_N) > \lambda(r(L) - r(N))^{\frac{1}{2}}]\\
&\qquad\qquad \leqq 2P[(I_L - I_N) > \lambda(r(L) - r(N))^{\frac{1}{2}}] = 2\Phi(-\lambda)
\end{aligned}$$

and this is small by choice of λ. It therefore suffices to show that

$$\begin{aligned}
P\Big[\, &(1 - r(N))^{-\frac{1}{2}}(X_i - I_N) \leqq \left(\frac{1 - r((2q+1)L)}{1 - r(N)}\right)^{\frac{1}{2}} \theta_{(2q+1)L}\\
&+ \lambda \left(\frac{r(L) - r(N)}{1 - r(N)}\right)^{\frac{1}{2}}, i = 2qL, \cdots, 2(q+1)L, q \leqq \left[\frac{1}{2}\left[\frac{N}{L}\right] - \tfrac{1}{2}\right]\Big]
\end{aligned} \tag{3.10}$$

$\to 0$ as $N \to \infty$ for any fixed ε and λ. Moreover the bounds in (3.10) may be simplified by noting that

$$\begin{aligned}
r(L) - r(N) &= r(L) - r((2q+1)L) + r((2q+1)L) - r(N)\\
&\leqq r((2q+1)L)\left(\frac{\ln (2q+1)L}{\ln L} - 1\right) + r((2q+1)L) \ln \frac{\ln N}{\ln (2q+1)L}\\
&\leqq r((2q+1)L)\left(\frac{\ln N}{\ln L} - 1 + \ln \frac{\ln N}{\ln L}\right) = r((2q+1)L)o(1)\\
&\leqq \frac{(\ln \ln (2q+1)L)^2}{\ln (2q+1)L} \cdot o(1)
\end{aligned}$$

in which we have used (2.4) and the assumed bound on $r(n)$. The two terms in each bound in (3.10) may now be coalesced into one if ε is suitably modified in $\theta_{(2q+1)L} = \theta_{(2q+1)L}(\varepsilon)$. Thus it is sufficient to show that

$$(3.11) \qquad P\left[Y_i^N \leqq \left(\frac{1 - r((2q+1)L)}{1 - r(N)}\right)^{\frac{1}{2}} \theta_{(2q+1)L}, \quad i = 2qL, \cdots, (2q+1)L,\ q \leqq \left[\frac{1}{2}\left[\frac{N}{L}\right] - \frac{1}{2}\right]\right]$$

$\to 0$ as $N \to \infty$ for any $\varepsilon > 0$.

The variables in (3.11) consist of blocks of length $L + 1$ drawn from a stationary sequence with correlation sequence $(r(k) - r(N))/(1 - r(N))$. These blocks are separated by a distance of L so that correlations between blocks are never more than $(r(L) - r(N))/(1 - r(N))$. Compare this family of random variables with the following family. Let

$$Z_{iq} = \left(\frac{1 - r(L)}{1 - r(N)}\right)^{\frac{1}{2}} \xi_{iq} + \left(\frac{r(L) - r(N)}{1 - r(N)}\right)^{\frac{1}{2}} U, \quad i = 2qL, \cdots, (2q+1)L, \quad q \leqq \left[\frac{1}{2}\left[\frac{N}{L}\right] - \frac{1}{2}\right]$$

where the variables ξ_{iq} and U have a zero mean multivariate normal distribution for which

$$E\xi_{iq}\xi_{jq} = \frac{r(j-i) - r(L)}{1 - r(L)}, \qquad E\xi_{ip}\xi_{jq} = 0 \quad \text{for} \quad p \neq q,$$
$$E\xi_{iq}U \equiv 0, \qquad EU^2 = 1.$$

(This construction depends only on the fact that $(r(k) - r(L))/(1 - r(L))$ provides a valid correlation sequence and it is already that of Y_i^L by (2.7i).)

By construction, the covariance matrix of Z_{iq} is at least as large as that of the Y_i^N in (3.11). Hence Slepian's lemma applies and (3.11) is no larger than

$$P\left[Z_{iq} \leqq \theta_{(2q+1)L}\left(\frac{1 - r(2q+1)L}{1 - r(N)}\right)^{\frac{1}{2}}, \quad i = 2qL, \cdots, (2q+1)L,\ q \leqq \left[\frac{1}{2}\left[\frac{N}{L}\right] - \frac{1}{2}\right]\right]$$

$$(21.3) \qquad = \int P\left[\xi_{iq} \leqq \theta_{(2q+1)L}\left(\frac{1 - r(2q+1)L}{1 - r(L)}\right)^{\frac{1}{2}} - \left(\frac{r(L) - r(N)}{1 - r(L)}\right)^{\frac{1}{2}} u, \quad i = 2qL, \cdots, (2q+1)L,\ q \leqq \left[\frac{1}{2}\left[\frac{N}{L}\right] - \frac{1}{2}\right]\right] \varphi(u)\, du$$

$$\leqq \Phi\left(-\frac{\varepsilon}{2}\frac{(\ln\ln N)^{\frac{1}{2}}}{c_N}\left(\frac{1 - r(L)}{r(L) - r(N)}\right)^{\frac{1}{2}}\right) + P\left[\xi_{iq} \leqq \theta_{(2q+1)L}\left(\frac{1 - r((2q+1)L)}{1 - r(L)}\right)^{\frac{1}{2}} + \frac{\varepsilon}{2}\frac{(\ln\ln N)^{\frac{1}{2}}}{c_N}, \quad i = 2qL, \cdots, (2q+1)L,\ q \leqq \left[\frac{1}{2}\left[\frac{N}{L}\right] - \frac{1}{2}\right]\right].$$

For the first term on the right-hand side of (3.12) note that

$$\begin{aligned}\frac{\varepsilon}{2}\frac{(\ln\ln N)^{\frac{1}{2}}}{c_N}&\left(\frac{1-r(L)}{r(L)-r(N)}\right)^{\frac{1}{2}}\\ &\geqq \frac{\varepsilon}{2}\frac{(\ln\ln N)^{\frac{1}{2}}}{c_N}\frac{(1-r(L))^{\frac{1}{2}}}{r(N)^{\frac{1}{2}}(\ln N/(\ln L)-1)^{\frac{1}{2}}}\\ &\geqq \frac{\varepsilon}{2}\frac{(\ln\ln N)^{\frac{1}{2}}}{c_N}\frac{(1-r(L))^{\frac{1}{2}}}{K\ln\ln N/(\ln N)^{\frac{1}{2}}(\ln N/(\ln L)-1)^{\frac{1}{2}}}\\ &\geqq \frac{\varepsilon}{4K}(\ln\ln N)^{\frac{1}{2}}\to\infty .\end{aligned}$$

In the second term on the right side of (3.12), the bound on ξ_{iq} may be collected into one term if ε is modified in $\theta_{(2q+1)L} \doteq \theta_{(2q+1)L}(\varepsilon)$. Thus it is sufficient for the lemma to conclude that

$$\prod_{q=0}^{[\frac{1}{2}[N/L]-\frac{1}{2}]} P\left[\xi_{iq} \leqq \theta_{(2q+1)L}\left(\frac{1-r((2q+1)L)}{1-r(L)}\right)^{\frac{1}{2}},\; i=2qL,\cdots,(2q+1)L\right] \tag{3.13}$$

$\to 0$ as $N\to\infty$ for any $\varepsilon>0$.

Consider (3.13). If the variables ξ_{iq} were totally independent, this probability would be

$$\prod_{q=0}^{[\frac{1}{2}[N/L]-\frac{1}{2}]} \Phi^{L+1}\left(\theta_{(2q+1)L}\left(\frac{1-r((2q+1)L)}{1-r(L)}\right)^{\frac{1}{2}}\right), \tag{3.14}$$

Now (3.14) does go to zero as $N\to\infty$ for any $\varepsilon>0$ as may be easily demonstrated by taking logarithms and invoking standard estimates of normal tail probabilities. The proof of the lemma is finished by showing, via a form of Berman's lemma [1], that the difference between (3.13) and (3.14) also goes to zero as $N\to\infty$. According to Lemma 1.5 of [11] and writing $\bar r(k)=(r(k)-r(L))/(1-r(L))$, the difference between (3.13) and (3.14) is no more than

$$\sum_{q=0}^{[\frac{1}{2}[N/L]-\frac{1}{2}]}\sum_{k=1}^{L}\bar r(k)L\exp[-(1+\bar r(k))^{-1}\theta^2_{(2q+1)L}] . \tag{3.15}$$

We first bound the inner summation in (3.15). Let $L_1=L_1(N)=[N^\gamma]$ for some $0<\gamma<1$ to be determined and let $L_2 \doteq L_2(N)=e^{\ln N(1-\delta/(\ln\ln N))}$ for $\delta>0$ to be determined. We have

$$\begin{aligned}\sum_{k=1}^{L}\bar r(k)&\exp[-(1+\bar r(k))^{-1}\theta^2_{(2q+1)L}]\\ &\leqq L_1L\bar r\exp[-(1+\bar r)^{-1}\theta^2_{(2q+1)L}]\\ &\quad+L_2L\bar r(L_1)\exp[-(1+\bar r(L_1))^{-1}\theta^2_{(2q+1)L}]\\ &\quad+L^2\bar r(L_2)\exp[-(1+\bar r(L_2))^{-1}\theta^2_{(2q+1)L}]\end{aligned} \tag{3.16}$$

where $\bar r=\sup_{k\geqq 1}(r(k)-r(L))/1-r(L)=(\rho-r(L))/1-r(L)<1$. We argue successively that the terms on the right in (3.16), when summed on q, are each

$o(1)$ as $N \to \infty$. First

$$L_1 L\bar{r} \exp[-(1+\bar{r})^{-1}\theta^2_{(2q+1)L}]$$
$$\leqq \exp\Big[\gamma \ln N + \ln N - \frac{\ln N}{(\ln\ln N)^2} - \frac{2}{1+\bar{r}} \ln(2q+1)$$
$$- \frac{2}{1+\bar{r}} \ln N + \frac{2}{1+\bar{r}} \frac{\ln N}{(\ln\ln N)^2}\Big]$$
$$= \exp\Big[\Big(\gamma + 1 - \frac{2}{1+\bar{r}}\Big)\ln N - \frac{2}{1+\bar{r}} \ln(2q+1) + O\Big(\frac{\ln N}{(\ln\ln N)^2}\Big)\Big].$$

Now if γ is chosen smaller than $(1-\rho)/(1+\rho) < (1-\rho)/(1-\rho-2r(L)) = (1-\bar{r})/(1+\bar{r})$, this term is $(2q+1)^{-2/(1+\bar{r})} \cdot o(1)$. Summing over q leaves a term which is $o(1)$. Secondly,

$$L_2 L\bar{r}(L_1) \exp[-(1+\bar{r}(L_1))^{-1}\theta^2_{(2q+1)L}]$$
$$\leqq \exp\Big[\ln N - \frac{\delta \ln N}{\ln\ln N} + \ln N - \frac{\ln N}{(\ln\ln N)^2} - \frac{2}{1+\bar{r}(L_1)} \ln(2q+1)$$
$$- \frac{2}{1+\bar{r}(L_1)} \ln N + \frac{2}{1+\bar{r}(L_1)} \frac{\ln N}{(\ln\ln N)^2}\Big]$$
$$= \exp\Big[\ln N\Big(2 - \frac{2}{1+\bar{r}(L_1)}\Big) - \frac{\delta\ln N}{\ln\ln N}$$
$$+ o\Big(\frac{\ln N}{\ln\ln N}\Big) - \frac{2}{1+\bar{r}(L_1)} \ln(2q+1)\Big].$$

But this can again be handled appropriately since

$$\ln N\Big(2 - \frac{2}{1+\bar{r}(L_1)}\Big) = 2\ln N \frac{r(L_1) - r(L)}{1 + r(L_1) - 2r(L)}$$
$$\leqq \frac{2}{1-r(L)} \ln N r(L)\Big(\frac{\ln L}{\ln L_1} - 1\Big)$$
$$\leqq \frac{2}{1-r(L)} \ln N K^2 \frac{(\ln\ln L)^2}{\ln L} \cdot \frac{1}{\gamma}$$
$$= O((\ln\ln N)^2).$$

Finally in the third term of (3.16) use

$$\bar{r}(L_2) \leqq 2(r(L_2) - r(L)) \leqq 2r(L_2)\Big(\ln \frac{\ln L}{\ln L_2}\Big)$$
$$\leqq 2K^2 \frac{(\ln\ln L_2)^2}{\ln L_2}\left(\ln \frac{\Big(1 - \frac{1}{(\ln\ln N)^2}\Big)}{\Big(1 - \frac{\delta}{\ln\ln N}\Big)}\right)$$
$$\leqq 4K^2 \frac{(\ln\ln L_2)^2}{\ln L_2} \frac{\delta}{\ln\ln N} \leqq 8K^2\delta \frac{\ln\ln N}{\ln N}$$

to obtain

$$
\begin{aligned}
&L^2\bar{r}(L_2)\exp[-(1+\bar{r}(L_2))^{-1}\theta^2_{(2q+1)L}] \\
&\quad \leqq 8K^2\delta\exp\Big[2\ln N - 2\frac{\ln N}{(\ln\ln N)^2} + \ln\ln\ln N - \ln\ln N - \frac{2}{1+\bar{r}(L_2)}\ln N \\
&\qquad + \frac{2}{1+\bar{r}(L_2)}\frac{\ln N}{(\ln\ln N)^2} - \frac{2}{1+\bar{r}(L_2)}\ln(2q+1)\Big] \\
&\quad \leqq 8K^2\delta\exp\Big[2\ln N\Big(\frac{\bar{r}(L_2)}{1+\bar{r}(L_2)}\Big) - 2\frac{\ln N}{(\ln\ln N)^2}\Big(\frac{\bar{r}(L_2)}{1+\bar{r}(L_2)}\Big) \\
&\qquad - \ln\ln N + \ln\ln\ln N - \frac{2}{1+\bar{r}(L_2)}\ln(2q+1)\Big] \\
&\quad \leqq 8K^2\delta\exp\Big[16K^2\delta\ln\ln N - \ln\ln N + o(\ln\ln N) - \frac{2}{1+\bar{r}(L_2)}\ln(2q+1)\Big].
\end{aligned}
$$

This last expression is summable on q to give a term which is $o(1)$ provided δ is chosen sufficiently small. Thus (3.15) is $o(1)$ as $N \to \infty$ for any $\varepsilon > 0$ and, in conjunction with (3.13) and (3.14), this completes the proof of the lemma.

Next we state a continuous parameter result which corresponds to Theorem 3.1. $\{X_t\}$ is given by (2.6) and it is assumed that for some positive C, $r(t) \sim 1 - C|t|^\alpha$ for t close to zero. Under this local condition we may take X_t to be continuous and so set $M_T = \max_{0 \leqq t \leqq T} X_t$.

THEOREM 3.2. *Assume* (2.1), (2.2) *and* (2.3) *and suppose* $r(t)\ln t/(\ln\ln t)^2$ *is nonincreasing for large* t. *Then with probability* 1,

$$
\liminf \frac{c_t(M_t - (1-r(t))^{\frac12}c_t - I_t)}{\ln\ln t} = \frac{1}{\alpha} - \tfrac12 \quad \textit{and} \tag{3.17}
$$

$$
\limsup \frac{c_t(M_t - (1-r(t))^{\frac12}c_t - I_t)}{\ln\ln t} = \frac{1}{\alpha} + \tfrac12 .
$$

The details of proof will be omitted.

4. Large correlations. In the present section the correlations $r(n)$ are no longer bounded functions of $(\ln\ln n)^2/\ln n$. The conclusions drawn refer to a.s. limits so full proofs are short by Section 3 standards. Since they are also repetitive of those in Section 3, only sketches of proofs are given.

We begin with the discrete parameter result of

THEOREM 4.1. *Let* $\{X_n\}$ *be given by* (2.6) *and assume* (2.1), (2.2) *and* (2.3). *If* $(\ln\ln n)^2/r(n)\ln n = o(1)$ *then with probability* 1,

$$
\lim_{n\to\infty} r^{-\frac12}(n)(M_n - (1-r(n))^{\frac12} - I_n) = 0 . \tag{4.1}
$$

PROOF. Set $n_m = [e^{\varepsilon m}]$ for some fixed ε and consider the events

$$
\begin{aligned}
D_m &= \{M_{n_m} - I_{n_m} < (1-r(n_m))^{\frac12}c_{n_m} + \varepsilon r^{\frac12}(n_m)\} , \\
E_m &= \{M_N - I_N > (1-r(N))^{\frac12}c_N + 4\varepsilon r^{\frac12}(N) \ \text{ for some } \ n_m \leqq N < n_{m+1}\} , \\
F_m &= D_m \cap D_{m+1} \cap E_m .
\end{aligned}
$$

As at (3.5), $P(E_m \text{ i.o.}) = 0$ follows from $P(F_m \text{ i.o.}) = 0$ and $P(D_m{}^c \text{ i.o.}) = 0$. But exactly as below (3.5), F_m implies

$$\max_{n_m \leqq N \leqq n_{m+1}} (I_{n_{m+1}} - I_N) > \varepsilon r^{\frac{1}{2}}(n_{m+1}) .$$

The probability of this event is summable in m as below (3.6). Further,

$$\begin{aligned} P(D_m{}^c) &\leqq P[M'_{n_m} \geqq c_{n_m} + \varepsilon r^{\frac{1}{2}}(n_m)] \\ &\leqq n_m(1 - \Phi(c_{n_m} + \varepsilon r^{\frac{1}{2}}(n_m))) \leqq e^{-2\ln\ln n_m} \end{aligned} \tag{4.2}$$

for m suitably large, and this will be summable in m. With minor modifications this same argument reduces the problem of showing

$$M_N - I_N < (1 - r(N))^{\frac{1}{2}} c_N - 4\varepsilon r^{\frac{1}{2}}(N) \quad \text{i.o.}$$

with probability 0 to showing instead that

$$M_{n_m} - I_{n_m} < (1 - r(n_m))^{\frac{1}{2}} c_{n_m} - \varepsilon r^{\frac{1}{2}}(n_m) \quad \text{i.o.} \tag{4.3}$$

with probability 0. This latter fact is covered at (3.7) and below.

We next state a continuous parameter version of Theorem 4.1 and give a few remarks about its proof.

THEOREM 4.2. *Let* $\{X_t\}$ *be given by* (2.6) *and assume* (2.1), (2.2) *and* (2.3). *Suppose* $r(t) \sim 1 - C|t|^\alpha$ *for* t *close to zero,* $C > 0$, *and* $(\ln\ln t)^2/r(t)\ln t = o(1)$ *as* $t \to \infty$. *Then with probability* 1,

$$\lim_{t\to\infty} r^{-\frac{1}{2}}(t)(M_t - (1 - r(t))^{\frac{1}{2}} c_t - I_t) = 0 . \tag{4.4}$$

The proof of this result is reduced to a consideration of the behaviour of $M_t - I_t$ along a subsequence $\{t_m\} = \{e^{\varepsilon m}\}$ as was the case in Theorem 4.1. Arguments sufficient to establish analogues to (4.2) and (4.3) may be found in Section 3 of [7].

5. Variants. Here we point out how Theorems 3.1 and 4.1 apply to a general Gaussian process with correlation function r. Since I_N is not determined by the process $\{X_n\}$ of (2.6), we identify variables that are so determined and which can take the role of I_N in (3.1) and (4.1). Subsequently, more recognizable iterated logarithm type laws are derived from (3.1) and (4.1). Our discussion is limited to the discrete case but can be carried over to the continuous case as well.

Suppose $\{X_n\}$ is given by (2.6). Let $\mathbf{X}_n = (X_0, \cdots, X_n)'$ and let $\mathbf{1}_n$ be the $(n+1)$-vector with all entries equal to 1. $\bar{X}_n$ will denote the least squares estimate of the mean, $\bar{X}_n = \mathbf{1}_n'\mathbf{X}_n/(n+1)$, and Z_n is to be the Markov estimate of the mean, i.e., Z_n is the linear combination of minimum variance amongst all $\mathbf{c}_n'\mathbf{X}_n$ satisfying $\mathbf{c}_n'\mathbf{1}_n = 1$. If we set $\lambda(n) = EZ_n{}^2$, the minimum variance property of Z_n implies

$$\lambda(n) = EZ_n{}^2 = EZ_iZ_n = EX_iZ_n , \qquad i \leqq n . \tag{5.1}$$

A second feature of $\lambda(n)$ is that it is the largest λ so that $EX_nX_n' - \lambda\mathbf{1}_n\mathbf{1}_n'$ is nonnegative definite. The two properties of $\lambda(n)$, in conjunction with (2.7i), insure that for all large n

$$\operatorname{Var}(\bar{X}_n) \geqq \lambda(n) \geqq r(n)\,. \tag{5.2}$$

The Markov estimate Z_n will take over the role of I_n in Theorems 3.1 and 4.1. A difficulty here is that $\lambda(n) = EZ_n^2$ is not readily computable, but it will suffice for our purposes to estimate the difference between the left and right-hand sides of (5.2).

LEMMA 5.1. *If* (2.1), (2.2) *and* (2.3) *apply*, $\operatorname{Var}\bar{X}_n - r(n) = O(r(n)/\ln n)$.

PROOF. Take $\xi(n) = [(n/2)e^{-(\ln n)^{\frac{1}{2}}}]$ to be suitably large. Now

$$\begin{aligned}
&\operatorname{Var}(\bar{X}_n) - r(n)\\
&\quad = \left(\frac{1}{n+1}\right)^2 \textstyle\sum_{i=0}^n \sum_{j=0}^n r(j-i) - r(n)\\
&\quad = \left(\frac{1}{n+1}\right)^2 \textstyle\sum_{i=0}^n \sum_{j=0}^n \left[r(j-i) - r\left(\left[\frac{n}{2}\right]\right)\right] + \left[r\left(\left[\frac{n}{2}\right]\right) - r(n)\right]\\
&\quad \leqq \frac{2}{n+1} \textstyle\sum_{i=0}^{[n/2]} \left[r(i) - r\left(\left[\frac{n}{2}\right]\right)\right] + \left[r\left(\left[\frac{n}{2}\right]\right) - r(n)\right]\\
&\quad \leqq \frac{2}{n+1}\xi(n) + \frac{2}{n+1} \textstyle\sum_{i=\xi(n)}^{[n/2]} \left[r(i) - r\left(\left[\frac{n}{2}\right]\right)\right]\\
&\qquad + \left[r\left(\left[\frac{n}{2}\right]\right) - r(n)\right].
\end{aligned} \tag{5.3}$$

The first term on the right side of (5.3) is $O(r(n)/\ln n)$ and the last term is $\leqq r(n)(\ln n/\ln [n/2] - 1) = O(r(n)/\ln n)$ by (2.4i). It remains to be shown that the same order applies to the second term. Use (2.4 ii) and sum by parts to obtain

$$\begin{aligned}
\frac{2}{n+1} \textstyle\sum_{i=\xi(n)}^{[n/2]} \left[r(i) - r\left(\left[\frac{n}{2}\right]\right)\right] &\leqq \frac{2}{n+1} r(\xi(n)) \textstyle\sum_{i=\xi(n)}^{[n/2]} \ln \frac{\ln [n/2]}{\ln i}\\
&\leqq C\frac{r(\xi(n))}{\ln \xi(n)} \leqq C\frac{r(n)\ln n}{(\ln \xi(n))^2} = O\left(\frac{r(n)}{\ln n}\right).
\end{aligned}$$

The proof is complete.

Now consider the covariance structure of I_n as given by (2.7 ii) and that of Z_n as noted in (5.1). One sees thereby that $(r(n)/\lambda(n))Z_n$ is the projection of I_n on the linear manifold generated by $\mathbf{X}_n$. According to (5.2) and Lemma 5.1,

$$\begin{aligned}
E\left(\frac{r(n)}{\lambda(n)} Z_n - I_n\right)^2 &= r(n)\left(1 - \frac{r(n)}{\lambda(n)}\right) \leqq \lambda(n) - r(n)\\
&\leqq \operatorname{Var}\bar{X}_n - r(n) = O\left(\frac{r(n)}{\ln n}\right).
\end{aligned} \tag{5.4}$$

Lemma 5.1 also implies

$$E\left(\frac{r(n)}{\lambda(n)} Z_n - Z_n\right)^2 = \left(\frac{r(n)}{\lambda(n)} - 1\right)^2 \lambda(n) \leqq \frac{(\lambda(n) - r(n))^2}{r(n)} = O\left(\frac{r(n)}{(\ln n)^2}\right). \tag{5.5}$$

With these details disposed of, we proceed to a translation of (3.1) and (4.1).

For (3.1), use (5.4) and (5.5) to conclude that

$$(5.6)\qquad E\left(\frac{c_n}{\ln\ln n}(Z_n - I_n)\right)^2 \leqq C\frac{r(n)\ln n}{(\ln\ln n)^2}\frac{1}{\ln n}.$$

As the $Z_n - I_n$ are normal variables, $(c_n/\ln\ln n)(Z_n - I_n) \to_{\text{a.s.}} 0$ if the right side of (5.6) is $o(1/\ln n)$. Thus (2.1), (2.2), (2.3) and $r(n)\ln n/(\ln\ln n)^2 = o(1)$ imply

$$(5.7)\qquad \liminf \frac{2c_n(M_n - (1-r(n))^{\frac{1}{2}}c_n - Z_n)}{\ln\ln n} = -1 \quad \text{and}$$

$$\limsup \frac{2c_n(M_n - (1-r(n))^{\frac{1}{2}}c_n - Z_n)}{\ln\ln n} = 1$$

with probability 1.

Now the conclusion to Theorem 4.1 is that

$$(5.8)\qquad r^{-\frac{1}{2}}(n)\left(M_n - (1-r(n))^{\frac{1}{2}}c_n - \frac{r(n)}{\lambda(n)}Z_n\right) + r^{-\frac{1}{2}}(n)\left(\frac{r(n)}{\lambda(n)}Z_n - I_n\right) \to_{\text{a.s.}} 0.$$

Since the two terms in (5.8) are independent, each $\to 0$ with probability 1. Furthermore, $E(r^{-\frac{1}{2}}(n)(r(n)/\lambda(n) - 1)Z_n)^2 = O((1/\ln n)^2)$ by (5.5). Thus under the assumptions of Theorem 4.1,

$$(5.9)\qquad r^{-\frac{1}{2}}(n)(M_n - (1-r(n))^{\frac{1}{2}}c_n - Z_n) \to_{\text{a.s.}} 0.$$

The modified results (5.7) and (5.9) apply to processes with correlation r which are not given specifically by (2.6). Because of (5.1) the sequence $\{Z_n\}$ may be thought of as $\{B_{\lambda(n)}\}$ so it is probabilistically as pleasant as $\{I_n\}$. Again, however, $\lambda(n)$ is not easily computed. Before proceeding, we note that Z_n may be replaced in (5.7) by $\bar{X}_n$ through the same reasoning that led us to (5.7). In Theorem 4.1 we can show that $\bar{X}_n$ is a suitable replacement for I_n when it is further assumed that $r(n)$ is convex for large n. The proof of this is long and will not be given.

Finally it is to be noted that (3.1) and (4.1) conceal iterated logarithm type laws for M_n of the more usual variety. First consider (3.1) and the "excess term" $2c_nI_n/\ln\ln n$ there. Recall that I_n may be represented as $B_{r(n)}$. The law of the iterated logarithm for Brownian motion tells us that

$$\left|\frac{2c_nI_n}{\ln\ln n}\right| < (1+\varepsilon)(2\ln\ln r^{-1}(n))^{\frac{1}{2}}\frac{2c_nr^{\frac{1}{2}}(n)}{\ln\ln n}$$

for some N on, with probability 1. Consequently if

$$r(n)\ln\ln r^{-1}(n) = o\left(\frac{(\ln\ln n)^2}{\ln n}\right),$$

(3.1) obtains without the I_n term. This is the case if, for example, $r(n) \sim \ln\ln n/\ln n$ for large n.

Now consider (4.1) and write

$$(5.10)\qquad \frac{r^{-\frac{1}{2}}(n)(M_n - (1-r(n))^{\frac{1}{2}}c_n)}{(2\ln\ln r^{-1}(n))^{\frac{1}{2}}} = \frac{r^{-\frac{1}{2}}(n)I_n}{(2\ln\ln r^{-1}(n))^{\frac{1}{2}}} + o((2\ln\ln r^{-1}(n))^{-\frac{1}{2}})$$

with probability 1. The lim sup of the right-hand side of (5.10) is a.s. $\leqq 1$ by the iterated logarithm law. To see that is $\geqq 1$ we follow the proof in [3]. There it is shown that $B_{t_n} > (1-\varepsilon)(2t_n \ln\ln t_n^{-1})^{\frac{1}{2}}$ i.o. for $t_n = q^n$ and q sufficiently close to zero. For such a q, let $t_n' = r(m_n)$ where $r(m_n) \geqq q^n > r(m_n+1)$. Then

$$q^n \leqq t_n' = r(m_n+1) + r(m_n) - r(m_n+1) \leqq q^n \frac{\ln(m_n+1)}{\ln m_n} = q^n(1+o(1))$$

by (2.4i). Minor modifications in the proof given in [3] ensure that $B_{t_n'} > (1-\varepsilon)(2t_n' \ln\ln t_n'^{-1})^{\frac{1}{2}}$ i.o. Thus under the assumptions of Theorem 4.1,

$$\liminf \frac{r^{-\frac{1}{2}}(n)(M_n - (1-r(n))^{\frac{1}{2}}c_n)}{(2\ln\ln r^{-1}(n))^{\frac{1}{2}}} = -1 \quad \text{and} \tag{5.11}$$

$$\limsup \frac{r^{-\frac{1}{2}}(n)(M_n - (1-r(n))^{\frac{1}{2}}c_n)}{(2\ln\ln r^{-1}(n))^{\frac{1}{2}}} = 1$$

with probability 1.

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