The Annals of Probability
1976, Vol. 4, No. 6, 1030–1032

THE IFRA CLOSURE PROBLEM[1]

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The class of IFRA distributions is closed under convolution.

1. Introduction. The class of distributions with increasing failure rate average (IFRA) has been of great importance in reliability theory. This class was introduced by Birnbaum, Esary and Marshall (1966) (and was called by these authors the class of distributions with increasing hazard rate average (IHRA)). The importance of this class, and properties thereof, are discussed in the recent text by Barlow and Proschan (1975), whose notation and terminology are followed here.

An important unresolved question concerning the IFRA class is whether or not it is closed under convolutions. Such is the case and will be proven here. Two proofs are given. Each proof requires a lemma. The result used in the first proof, Lemma 2.1, gives a new characterization of IFRA distributions and should be useful in other contexts. Lemma 3.1 which is used in the second proof gives an inequality involving distribution functions. This inequality generalizes an inequality used in the proof of the closure theorem for coherent systems (Barlow and Proschan (1975), page 85) and so relates the present theorem to that result.

(1.1) Definition. Let F be a (right-continuous) distribution function such that $F(0+) = 0$. F is said to be an *IFRA distribution* iff for all $0 \leqq \alpha \leqq 1$ and $0 \leqq x$

$$\bar{F}(\alpha x) \geqq \bar{F}^{\alpha}(x) .$$

(1.2) Theorem. *The class of* IFRA *distributions is closed under convolution.*

2. First proof.

(2.1) Lemma. *F is* IFRA iff

$$\int h(x)\, dF(x) \leqq \{\int h^{\alpha}(x/\alpha)\, dF(x)\}^{1/\alpha}$$

for all $0 < \alpha < 1$ and all nonnegative nondecreasing functions h.

Proof. Suppose that F is IFRA. It follows from the definition that for $0 < \alpha < 1$, $t \geqq 0$,

$$\int I_{(t,\infty)}(x)\, dF(x) \leqq \{\int I^{\alpha}_{(t,\infty)}(x/\alpha)\, dF(x)\}^{1/\alpha} . \tag{2.2}$$

Received November 25, 1975; revised May 14, 1976.

[1] Research has been partially supported by the U.S. Nuclear Regulatory Commission under contract AT(49-24)-0147 and ONR contract N00014-75-C-0781 with the University of California and ONR contract N00014-76-C-0839 with the University of Pittsburgh.

AMS 1970 *subject classifications.* Primary 60K10; Secondary 62N05.

Key words and phrases. IFRA distribution, convolution.

More generally, let

$$\phi(x) = \sum_{i=1}^{n} a_i I_{(t_i,\infty)}(x) \tag{2.3}$$

where $a_i \geqq 0$, $0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n < \infty$. It follows from (3.2) that

$$\int \phi(x)\, dF(x) \leqq \sum_{i=1}^{n} \{\int a_i^{\alpha} I^{\alpha}_{(t_i,\infty)}(x/\alpha)\, dF(x)\}^{1/\alpha} .$$

Using the Minkowski inequality for $0 < \alpha < 1$ (cf. Hewitt and Stromberg (1965), Theorem 13.9, page 192), we obtain

$$\begin{aligned}\int \phi(x)\, dF(x) &\leqq \sum_{i=1}^{n} ||a_i I_{(t_i,\infty)}(\cdot/\alpha)||_{\alpha} \\ &\leqq ||\sum_{i=1}^{n} a_i I_{(t_i,\infty)}(\cdot/\alpha)||_{\alpha} = \{\int \phi^{\alpha}(x/\alpha)\, dF(x)\}^{1/\alpha} .\end{aligned}$$

Consequently we have proven (2.1) for functions of the form (2.3). But it is known that any nonnegative nondecreasing function can be obtained as a non-decreasing limit of such functions. Thus the result follows by the Lebesgue monotone convergence theorem. To prove the result in the other direction just take $h(x) = I_{(t,\infty)}(x)$.

PROOF OF THEOREM. Let F and G be IFRA distributions and let $H = F*G$ be their convolution. Let h be any nonnegative nondecreasing function and take $0 < \alpha < 1$. Consider then

$$\int h(z)\, dH(z) = \int\int h(x+y)\, dF(x)\, dG(y) .$$

Since F is IFRA and $h(x+y)$ is, for fixed y, nonnegative and nondecreasing in x it follows from (2.1) that

$$\int h(z)\, dH(z) \leqq \int [\{\int h^{\alpha}((x/\alpha)+y)\, dF(x)\}^{1/\alpha}]\, dG(y) .$$

But the function inside the brackets [ ] is also nonnegative and nondecreasing (in y). Since G is IFRA we have by (2.1) that

$$\begin{aligned}\int h(z)\, dH(z) &\leqq \{\int\int h^{\alpha}((x+y)/\alpha)\, dF(x)\, dG(y)\}^{1/\alpha} \\ &= \{\int h^{\alpha}(z/\alpha)\, dH(z)\}^{1/\alpha} .\end{aligned}$$

This finishes the proof.

(2.4) REMARK. If F is an IFRA distribution, then it follows that $\lambda_t^{1/t}$ is nonincreasing in $t \geqq 0$, where $\lambda_t = \mu_t/\Gamma(t+1)$, $\mu_t = \int x^t\, dF(x)$ and Γ is the gamma function (Barlow and Proschan (1975), page 112). As an immediate consequence of Lemma 2.1 we can show that $(1/t)\mu_t^{1/t}$ is nonincreasing in $t \geqq 0$: just take $0 < s < t$, $h(x) = x^t$ and set $\alpha = s/t$.

3. Second proof.

(3.1) LEMMA. *For distributions F_1 and F_2 such that $F_1(0) = F_2(0) = 0$ and $0 < \alpha \leqq 1$*

$$\int_0^{\infty} \bar{F}_1^{\alpha}(x-x_2)d(1-\bar{F}_2^{\alpha}(x_2)) \geqq (\int_0^{\infty} \bar{F}_1(x-x_2)d(1-\bar{F}_2(x_2)))^{\alpha} .$$

PROOF. Lemma 2.3, page 84 of Barlow and Proschan (1975) generalizes to give that for $0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_n$, $y_i \geqq 0$ for $i = 1, 2, \cdots, n$, $\sum_{i=1}^{n} y_i > 0$, and $0 \leqq \alpha \leqq 1$

$$(\sum_{i=1}^n x_i y_i)^\alpha \leqq \sum_{i=1}^n x_i^\alpha[(\sum_{k=i}^n y_k)^\alpha - (\sum_{k=i+1}^n y_k)^\alpha] \tag{3.2}$$

where $\sum_{k=n+1}^n y_i = 0$. From this, the conclusion can be obtained by a standard limiting argument.

PROOF OF THEOREM. For $0 < \alpha \leqq 1$ and $0 < x$ using integration by parts and the definition of IFRA it can be shown that

$$P\{X_1 + X_2 > \alpha x\} \geqq \int_0^\infty \bar{F}_1^\alpha(x - x_2)d(1 - \bar{F}_2^\alpha(x)) .$$

By Lemma 2.1 the quantity on the right dominates

$$(\int_0^\infty \bar{F}_1(x - x_2)d(1 - \bar{F}_2(x_2)))^\alpha = (P\{X_1 + X_2 > x\})^\alpha .$$

The cases $\alpha = 0$ or $x = 0$ are simple and so $X_1 + X_2$ is IFRA.

Acknowledgment. The two proofs were obtained independently: the proof in Section 3 by H. Block in Berkeley (November 1975) and the proof in Section 2 by T. Savits in Pittsburgh (January 1976). The authors would like to acknowledge Frank Proschan who visited Pittsburgh during the summer of 1975 and motivated them to work on this problem.

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