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AN UNEXPECTED EXPECTATION

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It is shown that, while the value of the expectation $E(X+Y)$ always depends on the random variables X and Y only through their marginal distributions, the same kind of statement cannot be made for $E(X+Y+Z)$.

Because of the linearity property for expectations, it is generally accepted that the values of expectations such as $E(X+Y)$ and $E(X+Y+Z)$ depend on the random variables within their parentheses only through their marginal distributions. Below, it is shown that this conclusion is valid for $E(X+Y)$ but is *not* valid for $E(X+Y+Z)$. Specifically, we shall illustrate three random variables X, Y and Z, which have fixed marginal distributions, with the property that $X+Y+Z=0$ almost surely under one possible joint distribution and $X+Y+Z>0$ almost surely under another possible joint distribution. We shall begin by showing that such paradoxical behavior is impossible in the case of two random variables.

THEOREM. *If $E(X+Y)$ is defined (in the sense that $E(X+Y)^+$ and/or $E(X+Y)^-$ is finite), then its value depends on X and Y only through their marginal distributions.*

REMARKS. Of course the theorem's conclusion is obvious when the formula

$$E(X+Y)=EX+EY \tag{1}$$

is applicable. The strength of the theorem is that the conclusion holds even when EX and EY are not defined, or when one of them equals $+\infty$ and the other equals $-\infty$. The theorem partially generalizes a result previously reported by the author (1976) which states that $E(X-Y)=0$ when the expectation is defined and one of the following two conditions holds:

(a) X and Y have the same distribution.
(b) X and Y are symmetric random variables.

(Neither (a) nor (b) implies $X-Y$ is a symmetric random variable.) It should not be inferred from the theorem that the existence of $E(X+Y)$ depends only on the marginal distributions. This clearly is not the case.

PROOF. For any random variable W, let W^c denote the truncated random variable which equals $-c$, W or c as $W<-c$, $-c\leqq W\leqq c$ or $W>c$, respectively. Observe that $(X^c+Y^c)^+\uparrow(X+Y)^+$ and $(X^c+Y^c)^-\uparrow(X+Y)^-$ as $c\to\infty$.

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Thus, by the monotone convergence theorem,

$$E(X^c + Y^c) \to E(X + Y) \qquad \text{as} \quad c \to \infty$$

when the latter is defined. The conclusion then follows from (1), which holds for the truncated random variables X^c and Y^c. □

To see that the theorem does not generalize to three random variables, consider the following two ways of defining X, Y and Z. Both ways result in the same set of marginal distributions. Let U denote a uniform variable on [0, 1] and observe that $1 - U$ and $V = |2U - 1|$ are uniform variables on [0, 1] as well.

First definition. $X = Y = \tan(\pi/2)U$ and $Z = -2X$.

Clearly $X + Y + Z \equiv 0$ and $E(X + Y + Z) = 0$.

Second definition. $X = \tan(\pi/2)U$, $Y = \tan(\pi/2)(1-U)$ and $Z = -2\tan(\pi/2)V$.

By standard trigonometric identities, it follows that $Y = \cot(\pi/2)U$ and that $Z = \tan(\pi/2)U - \cot(\pi/2)U$ when $0 < U < \frac{1}{2}$ and equals $\cot(\pi/2)U - \tan(\pi/2)U$ when $\frac{1}{2} < U < 1$. Thus $X + Y + Z = 2\tan(\pi/2)U = 2X$ when $0 < U < \frac{1}{2}$ and equals $2\cot(\pi/2)U = 2Y$ when $\frac{1}{2} < U < 1$. It follows that $X + Y + Z > 0$ almost surely. It is an easy matter to calculate $E(X + Y + Z)$ which equals $(4/\pi)\log 2$.

We have no explanation why the theorem given above should hold for two random variables but fail for three. In spite of this, the theorem does say something about the three random variable case. Let $X' = X + Y$ and $Y' = Z$. It follows from the theorem that the value of $E(X + Y + Z) = E(X' + Y')$, when defined, depends on X' and Y' only through their marginal distributions. This means that the value of $E(X + Y + Z)$ depends on X, Y and Z only through the bivariate distribution of X and Y and the marginal distribution of Z. A similar observation can be made for $E(W + X + Y + Z)$. We do not know whether more than knowledge of the bivariate distributions is needed when the number of random variables is raised to five.

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