The Annals of Probability
1977, Vol. 5, No. 6, 829-860

REFLECTION GROUPS, GENERALIZED SCHUR FUNCTIONS, AND THE GEOMETRY OF MAJORIZATION

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Let G be a closed subgroup of the orthogonal group $O(n)$ acting on R^n. A real-valued function f on R^n is called *G-monotone* (decreasing) if $f(y) \geqq f(x)$ whenever $y \precsim x$, i.e., whenever $y \in C(x)$, where $C(x)$ is the convex hull of the G-orbit of x. When G is the permutation group $\mathscr{P}_n$ the ordering $\precsim$ is the majorization ordering of Schur, and the $\mathscr{P}_n$-monotone functions are the Schur-concave functions. This paper contains a geometrical study of the convex polytopes $C(x)$ and the ordering $\precsim$ when G is any closed subgroup of $O(n)$ that is generated by reflections, which includes $\mathscr{P}_n$ as a special case. The classical results of Schur (1923), Ostrowski (1952), Rado (1952), and Hardy, Littlewood and Polya (1952) concerning majorization and Schur functions are generalized to reflection groups. It is shown that a smooth G-invariant function f is G-monotone iff $(r'x)(r'\nabla f(x)) \leqq 0$ for all $x \in R^n$ and all $r \in R^n$ such that the reflection across the hyperplane $\{z \mid r'z = 0\}$ is in G. Furthermore, it is shown that the convolution (relative to Lebesgue measure) of two nonnegative G-monotone functions is again G-monotone. The latter extends a theorem of Marshall and Olkin (1974) concerning $\mathscr{P}_n$, and has applications to probability inequalities arising in multivariate statistical analysis.

1. Introduction. Let $O(n)$ denote the group of $n \times n$ orthogonal matrices acting on R^n, and suppose G is a closed subgroup of $O(n)$. For $x \in R^n$ let $C(x) \equiv C_G(x)$ denote the convex hull of the G-orbit $\{gx \mid g \in G\}$ of x. The group G determines a partial ordering $\precsim$ on R^n as follows:

DEFINITION 1.1. $y \precsim x$ iff $y \in C(x)$.

Geometrically, $y \precsim x$ implies that y is in some sense closer to 0 than x (although $C(x)$ need not contain 0—see Lemma 2.1). When G is the permutation group $\mathscr{P}_n$ acting on R^n, the ordering $\precsim$ is exactly the majorization ordering of Schur (see Example 4.1; also see Rado (1952), Berge (1963), or Marshall and Olkin (1979)). When G is the group $\mathscr{B}_n$ generated by all permutations and sign changes of coordinates acting on R^n, the ordering $\precsim$ is related to the weak majorization ordering of Marshall and Olkin (1979).

Received September 8, 1976; revised May 16, 1977.

[1] Support for this research was provided in part by National Science Foundation Grant No. NSF GP-34482.

[2] Support for this research was provided in part by National Science Foundation Grant No. MCS 72-04364 A04 and by U.S. Energy Research and Development Administration Contract No. E(11-1)-2751.

AMS 1970 *subject classifications.* Primary 26A84, 26A86, 50B35; Secondary 52A25, 62H99.

Key words and phrases. Orthogonal transformations, groups generated by reflections, Coxeter groups, roots, fundamental region, G-orbit, convex hull, convex polytope, convex polyhedral cone, extreme point, extreme ray, edge, G-monotone, majorization, Schur-concave function, convolution.

DEFINITION 1.2. An extended real-valued function $f\colon R^n \to [-\infty, \infty]$ is *G-monotone decreasing*, abbreviated as *G-monotone*, if $y \precsim x$ implies that $f(y) \geqq f(x)$.

Since $x \in C(gx)$ and $gx \in C(x)$ for every $g \in G$, a G-monotone function is necessarily G-invariant. When $G = \mathscr{P}_n$, the G-monotone functions are the so-called Schur-concave functions.

DEFINITION 1.3. Let $\mathscr{F} \equiv \mathscr{F}_G$ denote the class of all G-monotone functions $f\colon R^n \to [0, \infty]$ which are integrable over R^n with respect to Lebesgue measure. Clearly, $\mathscr{F}_G$ is a convex cone of functions which is closed under minimum and maximum. A central question concerning $\mathscr{F}_G$ is the following:

QUESTION 1.1. Under what conditions on the group $G \subseteqq \mathrm{O}(n)$ is $\mathscr{F}_G$ closed under convolution (integrating with respect to Lebesgue measure on R^n)?

Our primary motivation for posing this question has been an attempt to extend the following result, essentially due to Anderson (1955) and Mudholkar (1966) (see also Sherman (1955)). Notice that no restrictions on $G \subseteqq \mathrm{O}(n)$ are imposed here:

THEOREM 1.1. *Let f_1, f_2 be nonnegative Lebesgue-integrable functions on R^n which are G-invariant. Suppose that $K_i(c) \equiv \{x \mid f_i(x) \geqq c\}$ is a convex set for each $c > 0$, $i = 1, 2$. Then the convolution $f_1 * f_2$ is in $\mathscr{F}_G$.*

An important application of Theorem 1.1 has been the derivation of probability inequalities leading to unbiasedness and monotonicity properties of the power functions of statistical tests for multivariate hypotheses (e.g., Anderson, Das Gupta, and Mudholkar (1964), Cohen and Strawderman (1971), Eaton and Perlman (1974)). In these applications one studies convolutions of the form

$$(I_A * f)(y) = \int_A f(y - x)\, dx$$

where f is a probability density on R^n and $A \subseteqq R^n$ is the acceptance region of a statistical test. To apply Theorem 1.1, A and $\{x \mid f(x) \geqq c\}$ must be convex sets. In many testing problems, however, A is not convex. For example, Matthes and Truax (1967) have shown that for testing problems in multivariate exponential families with nuisance parameters, the class of acceptance regions A having convex sections is essentially complete; however, such regions need not be convex. Nonetheless, I_A may be G-monotone. If $\mathscr{F}_G$ is closed under convolution, monotonicity results for power functions still can be obtained.

The convexity and invariance assumptions in Theorem 1.1 imply that $f_i \in \mathscr{F}_G$, $i = 1, 2$, but $f_i \in \mathscr{F}_G$ need not imply that $K_i(c)$ is convex. The convexity assumption is crucial in the proofs of Anderson and Mudholkar, which are based on the Brünn–Minkowski inequality. Without some restriction on the group G, the convexity and invariance assumptions on f_i in Theorem 1.1 cannot be weakened to the condition that $f_i \in \mathscr{F}_G$, $i = 1, 2$. For example, take $G = \{\pm I\}$, $f_1 = I_{Q \cup (-Q)}$, where $Q = \{(x_1, \cdots, x_n) : 0 \leqq x_1, \cdots, x_n \leqq 2\}$, and $f_2 = I_B$ where B is the unit ball; then $f_1, f_2 \in \mathscr{F}_G$ but $f_1 * f_2$ is not G-monotone. (Throughout this paper, I without subscripts denotes the identity transformation on R^n, while I_A denotes the indicator function of the set A.)

In a recent paper Marshall and Olkin (1974) have proved that $\mathscr{F}_G$ is closed under convolution when G is the permutation group $\mathscr{P}_n$. It is easy to show that $\mathscr{F}_G$ is closed under convolution when G is the group of all sign changes on coordinates; when $n = 1$ this has been proved by Wintner (1938), and the general case is an easy consequence. Also, if G acts transitively on $\mathscr{S}_{n-1} \equiv \{x : \|x\| = 1\}$, then $C_G(x) = \{y : \|y\| \leqq \|x\|\}$ and either Theorem 1.1 or a direct argument shows that $\mathscr{F}_G$ is closed under convolution. Thus, there is reason to believe that a non-trivial answer to Question 1.1 may be obtainable.

A second question of interest is that of characterizing the smooth G-monotone functions f in terms of the gradient vector ∇f. When $G = \mathscr{P}_n$, for example, Schur (1923) and Ostrowski (1952) have shown that a $\mathscr{P}_n$-invariant function f having a differential is $\mathscr{P}_n$-monotone (i.e., Schur-concave) iff

$$(1.1) \qquad (x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \leqq 0\,, \qquad 1 \leqq i, j \leqq n\,.$$

For a proof see Berge (1963) or Marshall and Olkin (1979).

It is important to note that when $G = \mathscr{P}_n$, both the convolution theorem of Marshall and Olkin and the differential characterization (1.1) of Schur-concave functions can be proved by applying a basic lemma of Hardy, Littlewood, and Polya (1952, page 47) concerning majorization. This lemma states that if $y \precsim x$ then y can be obtained from x by successive applications of a finite number of transformations of the form $\lambda I + (1 - \lambda)S_{ij}$, where $0 \leqq \lambda < 1$ and S_{ij} is the permutation which interchanges the ith and jth coordinates. Geometrically, this implies the existence of a polygonal path from x to y such that the endpoints of each directed line segment in the path differ in exactly two coordinates. Furthermore, if for a given segment these two coordinates are, say, the ith and jth, then the line segment is perpendicular to the hyperplane (subspace) $H_{ij} = \{z \in R^n \mid z_i = z_j\}$, and the initial point of the segment is further from H_{ij} than the terminal point. The lemma enables one to show that a function is $\mathscr{P}_n$-monotone by showing that it is monotone on these special directed line segments. For example, (see the proof of Theorem 2.1 of Marshall and Olkin (1974)), this reduces the convolution theorem for $\mathscr{F}_{\mathscr{P}_n}$ to the monotonicity of the convolution of two symmetric unimodal functions of a *single* real variable, which has been proved by Wintner (1938).

The hyperplanes H_{ij} are intimately related to the permutation group $\mathscr{P}_n$, in that the set of reflections S_{ij} across the H_{ij} generate $\mathscr{P}_n$, i.e., any permutation is the product of permutations interchanging only two coordinates. The main purpose of this paper is to extend the theory of majorization and Schur functions from $\mathscr{P}_n$ to an *arbitrary reflection group* G, i.e., a subgroup of $\mathrm{O}(n)$ generated by reflections (across $(n - 1)$-dimensional hyperplanes containing the origin). For such G we study the generalized majorization ordering $\precsim$ of Definition 1.1 and the generalized Schur (concave) functions of Definition 1.2. Furthermore, we establish the convolution theorem for $\mathscr{F}_G$ and differential characterizations of G-monotonicity extending (1.1). These results rely on Lemmas 4.2 and 4.5, which are extensions of the basic path lemma of Hardy, Littlewood, and Polya to finite reflection groups.

Section 2 contains preliminary results. There it is shown that for *any* subgroup $G \subseteqq O(n)$, the convolution theorem and a differential characterization of G-monotonicity will follow provided that G contains enough reflections and that a path lemma for G can be established (see Corollary 2.1 and Proposition 2.3). It is also shown that if the convolution theorem and differential characterization are valid for $G_1, \cdots, G_k$ acting on $R^{n_1}, \cdots, R^{n_k}$ respectively, then these results also hold for the direct product $G_1 \times \cdots \times G_k$ acting on $R^{n_1+\cdots+n_k}$, thereby providing a useful reduction of the two problems.

The structure of groups generated by reflections is reviewed in Section 3. Section 4, the core of this paper, is devoted to a detailed study of the ordering $\precsim$ and the geometric structure of the convex polytopes $C(x)$, which leads to the basic path lemmas for finite reflection groups, Lemmas 4.2 and 4.5. As interesting by-products of this study, we obtain two new results about the geometric and algebraic structure of finite reflection groups, Proposition 4.1 and Theorem 4.3. The convolution theorem for $\mathscr{F}_G$ and the differential characterizations of G-monotonicity are summarized in Section 5.

2. **Preliminary results.** Throughout this paper R^n denotes Euclidean n-space and G is a closed subgroup of the orthogonal group $O(n)$ which preserves the usual inner product $(x, y) \equiv \sum x_i y_i$ on R^n. Elements of R^n are represented by column vectors. Subsets of the unit sphere $\mathscr{S}_{n-1} \equiv \{x \in R^n : ||x|| = 1\}$ will be denoted by Δ and Π, with or without subscripts. The transpose of a vector or matrix a is denoted by a'.

DEFINITION 2.1. If $r \in \mathscr{S}_{n-1}$, the linear transformation $S_r = I - 2rr'$ is called a *reflection*.

Clearly $S_r \in O(n)$, $S_r = S_{-r}$, and $S_r = S_r' = S_r^{-1}$. Geometrically, S_r reflects points across the $(n-1)$-dimensional hyperplane (actually subspace) $H_r = \{x \in R^n \mid r'x = 0\}$.

DEFINITION 2.2. If $S_r \in G$, r is called a *root* of G. The *root system* of G is $\Delta_G = \{r \in \mathscr{S}_{n-1} \mid S_r \in G\}$.

REMARK 2.1. If $r \in \Delta_G$ then also $gr \in \Delta_G$ for each $g \in G$, since $S_{gr} = gS_r g' \in G$; $gS_r g' \equiv gS_r g^{-1}$ is a *conjugate* of S_r in G.

DEFINITION 2.3. A nonnegative function ψ on R^1 is *symmetric* and *unimodal* about $\eta_0 \in R^1$ if $\psi(\eta_0 + \eta) = \psi(\eta_0 - \eta)$ for all $\eta \in R^1$ and $\psi(\eta_0 + \eta)$ is nonincreasing for $\eta \geqq 0$.

PROPOSITION 2.1. *Suppose $f_1, f_2 \in \mathscr{F} \equiv \mathscr{F}_G$ and suppose $r \in \Delta_G$. Let*

$$h(x) = \int_{R^n} f_1(x-y) f_2(y)\, dy \equiv (f_1 * f_2)(x) .$$

For fixed $x \in R^n$ *the function* $\psi(\eta) \equiv h(x + \eta r)$, $\eta \in R^1$, *is symmetric and unimodal about* $-r'x$.

PROOF. First note that

$$x + \eta r = \tfrac{1}{2}(x + S_r x) + (\eta + r'x)r .$$

If we set $\beta = \eta + r'x$, it is sufficient to show that

$$\psi_0(\beta) \equiv h(\tfrac{1}{2}(x + S_r x) + \beta r)$$

is symmetric and unimodal about 0. Set $u = \frac{1}{2}(x + S_r x)$, so

$$\psi_0(\beta) = \int f_1(u + \beta r - y) f_2(y)\, dy .$$

Let $v_1, \cdots, v_n$ be an orthogonal basis for R^n such that $r = v_1$ and let $\alpha_i = v_i'y$, so $y = \sum_{i=1}^n \alpha_i v_i$. Then

$$\begin{aligned}\psi_0(\beta) = \int_{R^{n-1}} [\int_{R^1} f_1(u + (\beta - \alpha_1)v_1 - \textstyle\sum_2^n \alpha_i v_i) \\ \times f_2(\alpha_1 v_1 + \textstyle\sum_2^n \alpha_i v_i)\, d\alpha_1]\, d\alpha_2 \cdots d\alpha_n .\end{aligned} \tag{2.1}$$

Next, for $\gamma \in R^1$ define

$$\begin{aligned}\tilde{f}_1(\gamma) &= f_1(u + \gamma v_1 - \textstyle\sum_2^n \alpha_i v_i) , \\ \tilde{f}_2(\gamma) &= f_2(\gamma v_1 + \textstyle\sum_2^n \alpha_i v_i) .\end{aligned}$$

Note that $S_r v_1 = -v_1$, $S_r v_i = v_i$ for $2 \leqq i \leqq n$, and $S_r u = u$. Since f_i is G-invariant and $S_r \in G$, this implies that $\tilde{f}_i(\gamma) = \tilde{f}_i(-\gamma)$, $i = 1, 2$, i.e., $\tilde{f}_i$ is symmetric about 0. To show that $\tilde{f}_2$ is unimodal about 0, it must be verified that $\tilde{f}_2(\gamma_1) \geqq \tilde{f}_2(\gamma_2)$ whenever $0 \leqq \gamma_1 < \gamma_2$. Since $f_2 \in \mathscr{F}$, it suffices to show that $z_1 \in C(z_2)$, where $z_j = \gamma_j v_1 + \sum_2^n \alpha_i v_i$, $j = 1, 2$. However, $z_1 = \lambda z_2 + (1 - \lambda) S_r z_2$, where $\lambda \equiv (\gamma_1 + \gamma_2)/(2\gamma_2) \in (0, 1)$. In exactly the same way it is shown that $\tilde{f}_1$ is unimodal about 0. Thus, by Wintner's theorem,

$$\int_{R^1} \tilde{f}_1(\beta - \alpha_1) \tilde{f}_2(\alpha_1)\, d\alpha_1$$

is symmetric and unimodal about $\beta = 0$. The result now follows from (2.1).

The following corollary is a main tool for proving the convolution theorem for finite reflection groups. Notice that the hypotheses of this corollary imply that $y \in C_G(x)$.

COROLLARY 2.1. *Consider* $x, y \in R^n$. *Assume there exists a sequence of points* $z_0, z_1, \cdots, z_m$ *such that* $z_0 = y$, $z_m = x$, *and*

$$z_{j-1} = [\lambda_j I + (1 - \lambda_j) S_{r_j}] z_j , \qquad 1 \leqq j \leqq m ,$$

where $0 \leqq \lambda_j < 1$ *and* $r_j \in \Delta_G$. *If* $f_1, f_2 \in \mathscr{F}_G$ *and* $h = f_1 * f_2$, *then* $h(y) \geqq h(x)$.

PROOF. By Proposition 2.1 the function $\psi(\eta) = h(z_j + \eta r_j)$ is symmetric and unimodal about $\eta_0 = -r_j'z_j$. Hence, $\psi(\eta) \geqq \psi(0)$ for any point η in the interval J with endpoints 0 and $-2r_j'z_j$. However, $z_{j-1} = z_j + \eta^* r_j$, where $\eta^* = -2(1 - \lambda_j) r_j' z_j \in J$, so $h(z_{j-1}) = \psi(\eta^*) \geqq \psi(0) = h(z_j)$. Therefore $h(y) \geqq h(x)$, as claimed.

We turn now to the characterization of smooth G-monotone functions f via conditions on the gradient vector ∇f. The following is a necessary condition for G-monotonicity.

PROPOSITION 2.2 (Eaton (1975)). *If f is G-monotone on R^n and if f has a differential at $x \in R^n$, then*

$$(gx - x)'\nabla f(x) \geqq 0 \qquad \text{for all} \quad g \in G . \tag{2.2}$$

PROOF. Since f is G-monotone,

$$\phi(\alpha) \equiv f((1-\alpha)x + \alpha gx) \geqq f(x)$$

for all $\alpha \in [0, 1]$. Expand ϕ in a Taylor series about $\alpha = 0$, so

$$\phi(\alpha) = \phi(0) + \phi'(0)\alpha + o(\alpha) .$$

Since $\phi(\alpha) \geqq \phi(0)$ and $\phi'(0) = (gx - x)'\nabla f(x)$, we have

$$\alpha(gx - x)'\nabla f(x) + o(\alpha) \geqq 0 .$$

Dividing by α and letting $\alpha \to 0$ yields (2.2).

When $G = \mathscr{P}_n$, (2.2) implies (1.1) (take g to be the permutation interchanging x_i and x_j), so (2.2) is both necessary and sufficient for $\mathscr{P}_n$-monotonicity (=Schur concavity) of a smooth $\mathscr{P}_n$-invariant function on R^n. The sufficiency of (2.2) for G-monotonicity can be verified for a variety of particular groups, but the sufficiency in general is an open question. The following proposition will be applied in Section 5 to show that if G is a reflection group, the validity of (2.2) for all $x \in R^n$ is a necessary and sufficient condition for the G-monotonicity of a smooth G-invariant function f. We shall use the identity

$$(S_r z - z)'\nabla f(z) = -2(r'z)(r'\nabla f(z)) . \tag{2.3}$$

PROPOSITION 2.3. *Let x, y satisfy the hypotheses of Corollary 2.1. Suppose f is a G-invariant function possessing a differential on R^n. If*

$$(r_j'z)(r_j'\nabla f(z)) \leqq 0 \tag{2.4}$$

for all $1 \leqq j \leqq m$ and all z in the polygonal path $y \equiv z_0 \to z_1 \to \cdots \to z_m \equiv x$, then $f(y) \geqq f(x)$.

PROOF. Fix j, $1 \leqq j \leqq m$, and for $-\frac{1}{2} \leqq \delta \leqq \frac{1}{2}$ define

$$\begin{aligned} z(\delta) &= \tfrac{1}{2}(z_j + S_{r_j} z_j) + \delta(S_{r_j} z_j - z_j) , \\ \gamma(\delta) &= f(z(\delta)) . \end{aligned}$$

Note that $z(\frac{1}{2}) = z_j$, $z(\frac{1}{2} - \lambda_j) = z_{j-1}$, and $-\frac{1}{2} \leqq \frac{1}{2} - \lambda_j \leqq \frac{1}{2}$. Since

$$\gamma(-\delta) = f(S_{r_j} z(\delta)) = f(z(\delta)) = \gamma(\delta) ,$$

γ is symmetric about 0. Also, for $|\frac{1}{2} - \lambda_j| \leqq \delta \leqq \frac{1}{2}$,

$$\begin{aligned}\gamma'(\delta) &= (S_{r_j} z_j - z_j)' \nabla f(z(\delta)) \\ &= -\frac{1}{2\delta} [S_{r_j} z(\delta) - z(\delta)]' \nabla f(z(\delta)) \\ &= \frac{1}{\delta} [r_j' z(\delta)]' [r_j' \nabla f(z(\delta))] \\ &\leqq 0 .\end{aligned}$$

We conclude that

$$f(z_{j-1}) \equiv \gamma(\tfrac{1}{2} - \lambda_j) = \gamma(|\tfrac{1}{2} - \lambda_j|) \geqq \gamma(\tfrac{1}{2}) \equiv f(z_j) ,$$

and hence that $f(y) \geqq f(x)$.

REMARK 2.2. If f is G-invariant and smooth, then

$$\nabla f(gz) = g \nabla f(z) \tag{2.5}$$

for all $g \in G$. Therefore,

$$[(gr)'(gz)][(gr)' \nabla f(gz)] = (r'z)(r' \nabla f(z)) \tag{2.6}$$

for all $g \in G$, which often simplifies the verification of (2.4) (see Corollary 5.3 and (5.4)). We also remark that in Proposition 2.3 the assumption that f possesses a differential *everywhere* on R^n obviously can be weakened. For example, the differential need only exist in a neighborhood of the polygonal path from y to x.

In the next two propositions it is shown that when G is a direct product $G_1 \times \cdots \times G_k$ acting on $R^{n_1} \times \cdots \times R^{n_k}$ coordinate-wise, G-monotonicity and the convolution theorem for $\mathscr{F}_G$ are consequences of the corresponding properties for each G_i. It suffices to consider $k = 2$. Let $G_i \subseteqq O(n_i)$ act on R^{n_i}, $i = 1, 2$, so that $G_1 \times G_2$ acts on $R^{n_1} \times R^{n_2}$ via $(g_1, g_2)(x_1, x_2) = (g_1 x_1, g_2 x_2)$. The easy proof of the next proposition is omitted.

PROPOSITION 2.4. *Consider* $f\colon R^{n_1} \times R^{n_2} \to [-\infty, \infty]$. *The following are equivalent*:

(i) f *is* $G_1 \times G_2$-*monotone*;
(ii) (a) $f(\cdot, x_2)$ *is* G_1-*monotone, for each* $x_2 \in R^{n_2}$;
  (b) $f(x_1, \cdot)$ *is* G_2-*monotone, for each* $x_1 \in R^{n_1}$.

PROPOSITION 2.5. *Suppose that* $\mathscr{F}_{G_1}$ *and* $\mathscr{F}_{G_2}$ *are closed under convolution. Then* $\mathscr{F}_{G_1 \times G_2}$ *is closed under convolution.*

PROOF. Suppose $\phi, \psi\colon R^{n_1} \times R^{n_2} \to [0, \infty]$ are both in $\mathscr{F}_{G_1 \times G_2}$ and consider $h = \phi * \psi$, i.e.,

$$h(x_1, x_2) = \int_{R^{n_2}} \int_{R^{n_1}} \phi(y_1, y_2) \psi(x_1 - y_1, x_2 - y_2)\, dy_1\, dy_2 .$$

Fix x_2 and write

$$h(\cdot, x_2) = \int_{R^{n_2}} k(\cdot, y_2)\, dy_2 ,$$

where

$$k(x_1, y_2) = \int_{R^{n_1}} \phi(y_1, y_2)\psi(x_1 - y_1, x_2 - y_2)\, dy_1 .$$

Since $\phi(\cdot, y_2) \in \mathscr{F}_{G_1}$ and $\psi(\cdot, x_2 - y_2) \in \mathscr{F}_{G_1}$, also $k(\cdot, y_2) \in \mathscr{F}_{G_1}$. Since $\mathscr{F}_{G_1}$ is a convex cone, $h(\cdot, x_2) \in \mathscr{F}_{G_1}$. Similarly, $h(x_1, \cdot) \in \mathscr{F}_{G_2}$. By Proposition 2.4 we conclude that $h \in \mathscr{F}_{G_1 \times G_2}$.

This section concludes with some preliminary results about the structure of the convex sets $C(x) \equiv C_G(x)$, with implications concerning the boundedness and continuity of functions in $\mathscr{F}_G$ and their convolutions. As before, G denotes a closed subgroup of $O(n)$ acting on R^n. If V is a subspace of R^n, V is called *G-invariant* if $gV = V$ for every $g \in G$. Let $V(x) \equiv V_G(x)$ denote the linear subspace spanned by the G-orbit of x (equivalently, by $C(x)$). Then $V(x)$ is the smallest G-invariant subspace containing x.

DEFINITION 2.4. Suppose V is G-invariant. We say G acts *effectively* on V if $M_G(V) = \{0\}$, where $M_G(V)$ is the subspace of V defined as

$$M_G(V) = \{x \in V \mid gx = x \text{ for every } g \in G\} . \tag{2.7}$$

We abbreviate $M_G(R^n)$ as M_G, and say G is *effective* if G acts *effectively* on R^n, i.e., if $M_G = \{0\}$.

REMARK 2.3. M_G and $M_G{}^\perp$ are G-invariant subspaces. Obviously G acts effectively on $M_G{}^\perp$, and G does not act effectively on any subspace which properly contains $M_G{}^\perp$. The elements of M_G are minimal elements under the ordering $\precsim$ determined by G, since $C(x) = \{x\}$ for $x \in M_G$.

DEFINITION 2.5. Suppose V is G-invariant. We say G acts *irreducibly* on V if V contains no proper G-invariant subspace. If G acts irreducibly on R^n we say G is *irreducible*.

REMARK 2.4. If G acts irreducibly on V then $M_G(V) = \{0\}$ or V, so G acts effectively on V except in the trivial case where dimension $(V) = 1$ and $G = \{I\}$.

LEMMA 2.1.

(i) *If $0 \in C(x)$ then $0 \in$ relative interior of $C(x)$.*

(ii) *Let V be a G-invariant subspace of R^n. Then G acts effectively on V iff $0 \in C(x)$ for each $x \in V$.*

(iii) *Suppose G acts effectively on V, and set $d =$ dimension (V). Then G acts irreducibly on V iff $C^0(x) \neq \varnothing$ for all $0 \neq x \in V$, where $C^0(x)$ denotes the (d-dimensional) interior of $C(x)$ in V. In this case, $0 \in C^0(x)$.*

PROOF. (i) If $x = 0$ the result is trivial. If $x \neq 0$ then $0 \in C(x)$ implies that $0 = \sum_{i=1}^k \alpha_i g_i x$ for some integer $k \geqq 2$, where $g_i \in G$, $\alpha_i > 0$, $\sum \alpha_i = 1$. If $0 \notin$ relative interior of $C(x)$, there must exist a nonzero vector $a \in V(x)$ such that the $(d-1)$-dimensional hyperplane $H_a \equiv \{y \in V(x) \mid y'a = 0\} \subset V(x)$ supports $C(x)$ at 0, i.e., $C(x) \subseteqq \{y \in V(x) \mid y'a \geqq 0\}$. In particular, $(gx)'a \geqq 0$ for every $g \in G$. Since

$$0 = (gg_1^{-1}0)'a = \alpha_1(gx)'a + \sum_{i=2}^k \alpha_i(gg_1^{-1}g_i x)'a$$

it follows that $(gx)'a = 0$ for every $g \in G$, so the G-orbit of x, and hence $C(x)$, is contained in H_a, a proper subspace of $V(x)$. This is impossible, however, since $V(x) = \text{span}\,(C(x))$.

(ii) If G is not effective, choose $x \in M_G$, $x \neq 0$. Then $C(x) = \{x\}$ so $0 \notin C(x)$. Conversely, if $0 \notin C(x)$ for some $x \in R^n$, let $c_0 \neq 0$ be the unique point in $C(x)$ closest to 0. Since $gc_0 \in C(x)$ and $\|gc_0\| = \|c_0\|$ for each $g \in G$, the uniqueness of c_0 implies that $gc_0 = c_0$. Thus $c_0 \in M_G$, so G is not effective.

(iii) Suppose W is a proper G-invariant subspace of V. If $0 \neq x \in W$ then $C(x) \subset W$, so $C^0(x) = \emptyset$. Conversely, suppose G acts irreducibly on V and fix $0 \neq x \in V$. That $C^0(x) \neq \emptyset$ follows from the fact that $V(x)$ is a G-invariant subspace of V and is spanned by $C(x)$. Finally, part (i) implies that $0 \in C^0(x)$.

REMARK 2.5. Each $x \in R^n$ may be represented uniquely as $x = x^* + x^{**}$, where $x^* \in M_G^\perp$ and $x^{**} \in M_G$. For each $g \in G$ one has $gx = gx^* + x^{**}$, so $C(x) = C(x^*) + x^{**} \subseteqq M_G^\perp + x^{**} = M_G^\perp + x$, and dimension $(C(x)) \leqq$ dimension $(M_G^\perp) \equiv n^* \leqq n$. Since G acts effectively on $M_G^\perp$, Lemma 2.1 implies that $0 \in$ relative interior of $C(x^*)$, hence $x^{**} \in$ relative interior of $C(x)$. Conversely,

$$C(x) \cap M_G = (C(x^*) + x^{**}) \cap M_G \subseteqq (M_G^\perp + x^{**}) \cap M_G = \{x^{**}\} .$$

Therefore, $C(x) \cap M_G = \{x^{**}\}$, so x^{**} is the unique minimal element in $C(x)$ under the ordering $\precsim$. Finally, Lemma 2.1 (iii) implies that if G acts irreducibly on $M_G^\perp$ then dimension $(C(x))$ = dimension $(C(x^*)) = n^*$ for every $x \notin M_G$.

REMARK 2.6. Lemma 2.1 (ii) implies that if G acts effectively on R^n, a G-monotone function decreases along every ray emanating from 0.

If f_1, f_2 are nonnegative, Lebesgue-integrable functions on R^n, their convolution $f_1 * f_2$ is also integrable, but need not be continuous. By means of Lemma 2.1, additional boundedness and continuity properties for f_1, f_2 and $f_1 * f_2$ can be deduced when $f_1, f_2 \in \mathscr{F}_G$. We consider only the case where G acts effectively and irreducibly on R^n, but similar arguments apply in other cases.

PROPOSITION 2.6. *Assume that G acts effectively and irreducibly on R^n. If $f \in \mathscr{F}_G$ then f is bounded outside every neighborhood of 0. If $f_1, f_2 \in \mathscr{F}_G$ then $f_1 * f_2$ is continuous on $R^n - \{0\}$.*

PROOF. For each $x \neq 0$, Lemma 2.1 (iii) implies that $0 \in C^0(x)$, where $C^0(x)$ is the (n-dimensional) interior of $C(x)$, so

$$\delta(x) \equiv \inf \{\|z\| : z \in \partial C(x)\} > 0 . \tag{2.8}$$

It can be shown that δ is a continuous function on R^n, so

$$\gamma(\varepsilon) \equiv \inf \{\delta(x) : \|x\| = \varepsilon\} > 0 \tag{2.9}$$

for every $\varepsilon > 0$. However, $y \precsim x$ whenever $\|x\| = \varepsilon$ and $\|y\| \leqq \gamma(\varepsilon)$, so by Remark 2.6,

$$\sup \{f(x) : \|x\| \geqq \varepsilon\} \leqq \inf \{f(y) : \|y\| \leqq \gamma(\varepsilon)\} . \tag{2.10}$$

Thus if $f \in \mathscr{F}_G$, (2.9) and (2.10) imply that

$$\sup\{f(x) : \|x\| \geqq \varepsilon\} < \infty \tag{2.11}$$

for every $\varepsilon > 0$, as claimed.

Next, suppose $f_1, f_2 \in \mathscr{F}_G$. Fix $0 \neq x \in R^n$ and let $B = \{z \in R^n : \|z\| \leqq \frac{1}{4}\|x\|\}$. Then for all $y \in R^n - \{0\}$ such that $\|y - x\| \leqq \frac{1}{4}\|x\|$ we have

$$\begin{aligned}(f_1 * f_2)(y) &= [f_1 * (f_2 I_B)](y) + [f_1 * (f_2 I_{B^c})](y) \\ &= [(f_1 I_{B^c}) * f_2 I_B](y) + [f_1 * (f_2 I_{B^c})](y) .\end{aligned} \tag{2.12}$$

By (2.11), however, $f_1 I_{B^c}$ and $f_2 I_{B^c}$ are bounded on R^n, hence the two convolutions on the right of (2.12) are continuous (apply Theorem 4.3c of Williamson (1962)). Thus $f_1 * f_2$ is continuous in a neighborhood of x.

REMARK 2.7. If G is reducible then for $f_1, f_2 \in \mathscr{F}_G$, $f_1 * f_2$ may be $+\infty$ on all or part of a G-invariant subspace. For example, take G to be the 4-element group generated by sign changes of coordinates on R^2, and let $f_1(x_1, x_2) = f_2(x_1, x_2) = |x_1 x_2|^{-\frac{1}{2}} \exp\{-x_1^2 - x_2^2\}$. Then $(f_1 * f_2)(x_1, x_2) = +\infty$ whenever $x_1 = 0$ or $x_2 = 0$.

3. The structure of reflection groups.

DEFINITION 3.1. A closed group $G \subseteqq O(n)$ acting on R^n is called a *reflection group* if there exists $\Delta^* \subseteqq \mathscr{S}_{n-1}$ such that G is the smallest closed subgroup of $O(n)$ containing the set of reflections $\{S_r \mid r \in \Delta^*\}$.

REMARK 3.1. Clearly, G is the closure in $O(n)$ of the group generated algebraically by $\{S_r \mid r \in \Delta^*\}$. Also, any reflection group G obviously is generated by $\{S_r \mid r \in \Delta_G\}$.

A complete enumeration of the finite irreducible reflection groups can be found in Theorem 5.3.1 of Benson and Grove (1971) (hereinafter abbreviated as B–G); see also Coxeter (1963) and Coxeter and Moser (1972). Examples of reflection groups include $O(n)$ itself, the permutation group $\mathscr{P}_n$ (cf. B–G, pages 65–66), the group of all sign changes of coordinates in R^n, the group $\mathscr{B}_n$ generated by all permutations and sign changes of coordinates in R^n (cf. B–G, pages 66–68), and the symmetry groups of regular polyhedra.

PROPOSITION 3.1. *Suppose G is a reflection group acting on R^n and suppose M is a proper G-invariant subspace. Let $\Delta_1 = \Delta_G \cap M$ and $\Delta_2 = \Delta_G \cap M^\perp$. For $r \in \Delta_1$ let $S_r^{(1)}$ denote the restriction of S_r to M, and for $r \in \Delta_2$ let $S_r^{(2)}$ denote the restriction of S_r to $M^\perp$. Let G_i be the reflection group generated by $\{S_r^{(i)} \mid r \in \Delta_i\}$, $i = 1, 2$, so that G_1 acts on M and G_2 acts on $M^\perp$. Then there is an isomorphism $G \leftrightarrow G_1 \times G_2$ such that if $x = x_1 + x_2$ with $x_1 \in M$ and $x_2 \in M^\perp$, and if $g \leftrightarrow (g_1, g_2)$, then $g(x) = g_1 x_1 + g_2 x_2$.*

PROOF. The proposition follows from the elementary fact that M is invariant under a reflection S_r if and only if either $r \in M$ or $r \in M^\perp$. Thus, $\Delta_G = \Delta_1 \cup \Delta_2$. The remainder of the argument is similar to that on page 56 of B–G (with their Π_i replaced by our Δ_i, $i = 1, 2$).

PROPOSITION 3.2. *Suppose $G \subseteqq O(n)$ is a reflection group acting on R^n. Then G is isomorphic to $G_1 \times G_2 \times \cdots \times G_k$ acting on $M_1 \oplus M_2 \oplus \cdots \oplus M_k$ ($1 \leqq k \leqq n$), where $M_1, \cdots, M_k$ are mutually orthogonal subspaces of R^n with $\sum$ dimension $(M_i) = n$, and G_i is a reflection group acting irreducibly on M_i.*

PROOF. Apply Proposition 3.1 until the component groups have no invariant subspaces.

REMARK 3.2. G is effective iff $G_i \neq \{I\}$ for each $i = 1, \cdots, k$. The permutation group $\mathscr{P}_n$ does not act effectively on R^n, for $M_{\mathscr{P}_n} = \{x \in R^n \mid x_1 = x_2 = \cdots = x_n\}$ is of dimension 1, but $\mathscr{P}_n$ acts effectively on $M_{\mathscr{P}_n}^{\perp} = \{x \in R^n \mid \sum x_i = 0\}$, an $(n-1)$-dimensional subspace of R^n (see Definition 2.4).

By Propositions 2.4, 2.5 and 3.2, in order to establish the convolution theorem and the differential characterization of G-monotonicity for reflection groups, it suffices to establish these results for irreducible reflection groups. In view of the next theorem and Remark 3.3, these results are easily established for *infinite* irreducible reflection groups.

THEOREM 3.1. *If $G \subseteqq O(n)$ is an infinite irreducible reflection group then $G = O(n)$.*

PROOF. See Eaton and Perlman (1978).

REMARK 3.3. Theorem 3.1 shows that when G is an infinite irreducible reflection group, $C(x) = \mathscr{S}_{n-1}$ for each $x \in \mathscr{S}_{n-1}$. In particular, the only G-monotone functions are the decreasing radial functions.

In the remainder of this section we briefly review those geometrical properties of *finite* reflection groups G acting on R^n which will be applied in Section 4 to study the structure of $C(x)$ and obtain the basic path lemmas. The geometry of finite *effective* reflection groups acting on R^n, called *Coxeter groups*, is discussed in Chapter 4 of B–G, and that discussion carries over to non-effective finite reflection groups G with only trivial changes. Indeed, if one defines $n^* =$ dimension $(M_G{}^{\perp}) \leqq n$ and identifies $M_G{}^{\perp}$ with R^{n^*}, then G acting on R^{n^*} is a Coxeter group, and G acts trivially on M_G. We have stated our results for general (not necessarily effective) finite reflection groups in order that these results be directly applicable to the permutation group $\mathscr{P}_n$, which does not act effectively on R^n, and hence that these results be direct generalizations of the classical results concerning majorization.

A unit vector $r \in \mathscr{S}_{n-1}$ is called a *root* of G if the reflection S_r is in G; the root system $\Delta \equiv \Delta_G$ of G is the (finite) set of all roots of G. Note that $r \in \Delta$ iff $-r \in \Delta$, and $r \in \Delta$ implies that $gr \in \Delta$ for every $g \in G$ (see Remark 2.1). Clearly $\Delta \subset M_G{}^{\perp}$; in fact span$(\Delta) = M_G{}^{\perp}$. Define the open set $T \equiv T_G \subset R^n$ by

$$T \equiv T_G \equiv \{t \in R^n \mid r't \neq 0 \text{ for each } r \in \Delta\} = \bigcap \{H_r{}^c \mid r \in \Delta\},$$

so

$$T^c = \bigcup \{H_r \mid r \in \Delta\}.$$

Note that $T + M_G = T$ and $T^c + M_G = T^c$, i.e., T and T^c are cylinder sets parallel to M_G, with bases in $M_G^\perp$. For $t \in T$ let $\Delta_t^+ \subset \Delta$ denote the set of all *t-positive* roots, i.e.,

$$\Delta_t^+ = \{r \in \Delta \mid r't > 0\},$$

and let $\Delta_t^- = -\Delta_t^+$. Clearly $\Delta = \Delta_t^+ \cup \Delta_t^-$ and $|\Delta_t^+| = \frac{1}{2}|\Delta|$, where $|A|$ denotes the cardinality of a finite set A. Let $K_t \subset M_G^\perp$ denote the closed convex cone generated by Δ_t^+, so that K_t is a pointed polyhedral cone, and let $\Pi_t\,(\subseteqq \Delta_t^+)$ denote the set of t-positive roots which determine the extreme rays ($\equiv$ frame vectors) of K_t; thus, K_t is also generated by Π_t. By Theorem 4.1.7 of B–G, Π_t contains exactly n^* ($\equiv$ dimension $(M_G^\perp) \leqq n$) roots, say $\Pi_t = \{r_1, \cdots, r_{n^*}\}$, and these form a basis for $M_G^\perp \equiv R^{n^*}$; Π_t is called the *t-base* for Δ, By Proposition 4.1.5 of B–G, $r_i'r_j \leqq 0$ if $i \neq j$.

A main result in the theory of finite reflection groups is that $\{S_{r_i} \mid 1 \leqq i \leqq n^*\}$ comprises a set of *fundamental reflections* for G, i.e., a minimal set of reflections generating G (B–G, Theorem 4.1.12). Furthermore, every reflection in G is conjugate to some S_{r_i}, i.e., every root $r \in \Delta$ is of the form $r = gr_i$ for at least one $g \in G$ (B–G, Theorem 4.2.5).

Let $\Pi_t^* \equiv \{s_1, \cdots, s_{n^*}\} \subset M_G^\perp$ be the dual basis to Π_t in $M_G^\perp$, i.e., $r_i's_j = \delta_{ij}$ for $1 \leqq i, j \leqq n^*$. Let $F_t^* \subseteqq M_G^\perp$ be the relatively open convex cone generated by Π_t^*, i.e.,

$$F_t^* = \{\textstyle\sum_{i=1}^{n^*} \lambda_i s_i \mid \lambda_i > 0,\ 1 \leqq i \leqq n^*\},$$

and let $\bar{F}_t^* \subseteqq M_G^\perp$ be the closure of F_t^*, so

$$\bar{F}_t^* = \{\textstyle\sum_{i=1}^{n^*} \lambda_i s_i \mid \lambda_i \geqq 0,\ 1 \leqq i \leqq n^*\}.$$

Equivalently,

$$F_t^* = \{x \in M_G^\perp \mid r_i'x > 0,\ 1 \leqq i \leqq n^*\}$$

and

$$\begin{aligned}\bar{F}_t^* &= \{x \in M_G^\perp \mid r_i'x \geqq 0,\ 1 \leqq i \leqq n^*\}\\ &= \{x \in M_G^\perp \mid z'x \geqq 0 \ \text{ for all } \ z \in K_t\}\\ &\equiv \text{dual}^*\,(K_t),\end{aligned}$$

where dual* (K_t) denotes the dual cone of K_t in $M_G^\perp$. By Theorem 4.2.6 of B–G, $s_i's_j \geqq 0$ for $1 \leqq i, j \leqq n^*$, so

$$\bar{F}_t^* \subseteqq \text{dual}^*\,(F_t^*) = \text{dual}^*\,(\text{dual}^*\,(K_t)) = K_t$$

(see Rockafellar (1970), Theorem 14.1); hence $\bar{F}_t^*$ is also a pointed polyhedral cone.

Define the open convex cone $F_t \subseteqq R^n$ by

$$F_t = F_t^* \oplus M_G = \{\textstyle\sum_{i=1}^{n^*} \lambda_i s_i + u \mid u \in M_G,\ \lambda_i > 0,\ 1 \leqq i \leqq n^*\}$$

and let $\bar{F}_t \subseteqq R^n$ be the closure of F_t, so

$$\bar{F}_t = \bar{F}_t^* \oplus M_G = \{\textstyle\sum_{i=1}^{n^*} \lambda_i s_i + u \mid u \in M_G,\ \lambda_i \geqq 0,\ 1 \leqq i \leqq n^*\}.$$

Equivalently,

$$F_t = \{x \in R^n \mid r_i'x > 0,\ 1 \leqq i \leqq n^*\} \tag{3.1}$$

and

$$\begin{aligned} \bar{F}_t &= \{x \in R^n \mid r_i'x \geqq 0,\ 1 \leqq i \leqq n^*\} \\ &= \{x \in R^n \mid z'x \geqq 0 \text{ for all } z \in K_t\} \\ &\equiv \operatorname{dual}(K_t)\,, \end{aligned} \tag{3.2}$$

where $\operatorname{dual}(K_t)$ denotes the dual cone of K_t in R^n. The cone $\bar{F}_t$ is pointed iff $M_G = \{0\}$, i.e., iff G acts effectively on R^n.

The convex polyhedral cone F_t has the following properties (B–G, Theorem 4.2.4):

$$F_t \text{ is open in } R^n\,; \tag{3.3}$$

$$F_t \cap gF_t = \varnothing \quad \text{if} \quad I \neq g \in G\,; \tag{3.4}$$

$$R^n = \bigcup \{g\bar{F}_t \mid g \in G\}\,. \tag{3.5}$$

A set F satisfying (3.3), (3.4) and (3.5) is called a *fundamental region* for G in R^n. A set F is a fundamental region for G in R^n iff $F^* \equiv F \cap M_G{}^\perp$ is a fundamental region for G in $M_G{}^\perp$, so if $t \in T$ then $F_t{}^*$ is a fundamental region for G in $M_G{}^\perp$. If F is a fundamental region for G then so is gF for each $g \in G$. The fundamental reflections S_{r_i}, $1 \leqq i \leqq n^*$, are the reflections through the bounding hyperplanes H_{r_i} of F_t. Thus (B–G, page 46) every finite reflection group G acting on R^n is generated by the reflections through the n^* ($\leqq n$) walls of a convex polyhedral cone $\bar{F}$; G acts effectively on R^n iff $\bar{F}$ has n walls. Figure 4.3 of Coxeter and Moser (1972) will help the reader visualize these geometric properties of G.

We pause to illustrate these concepts with the permutation group $\mathscr{P}_n$.

EXAMPLE 3.1. Let $G = \mathscr{P}_n$ acting on R^n. The subspaces $M_{\mathscr{P}_n}$ and $M_{\mathscr{P}_n}^\perp$ have been described in Remark 3.2; note that $n^* = n - 1$. The root system of $\mathscr{P}_n$ (cf. B–G, page 66) is

$$\Delta_{\mathscr{P}_n} = \{e_i - e_j \mid 1 \leqq i \neq j \leqq n\} \subset M_{\mathscr{P}_n}^\perp\,, \tag{3.6}$$

where e_i is the ith coordinate vector (we temporarily drop the convention that a root have length 1), so

$$T \equiv T_{\mathscr{P}_n} = \{t = (t_1, \cdots, t_n) \mid t_i \neq t_j,\ 1 \leqq i \neq j \leqq n\}\,. \tag{3.7}$$

If we select $t \in T$ such that $t_1 > t_2 > \cdots > t_n$, then

$$\begin{aligned} \Delta_t^+ &= \{e_i - e_j \mid 1 \leqq i < j \leqq n\}\,, \\ \Pi_t &= \{e_i - e_{i+1} \mid 1 \leqq i \leqq n - 1 \equiv n^*\}\,; \end{aligned} \tag{3.8}$$

note that Π_t is a basis for $M_{\mathscr{P}_n}^\perp$. Set $r_i = e_i - e_{i+1}$, so $\Pi_t = \{r_1, \cdots, r_{n-1}\}$. The closed convex cone $K_t \subset M_{\mathscr{P}_n}^\perp$ generated by Π_t is given by

$$\begin{aligned} K_t &= \{\textstyle\sum_{i=1}^{n-1} c_i r_i \mid c_i \geqq 0\} \\ &= \{x = (x_1, \cdots, x_n) \mid \textstyle\sum_{i=1}^{k} x_i \geqq 0,\ 1 \leqq k \leqq n - 1,\ \sum_{i=1}^{n} x_i = 0\}\,. \end{aligned} \tag{3.9}$$

(This last representation of K_t is closely related to the classical definition of majorization—see Example 4.1.) The reflection S_{r_i} is the permutation which interchanges the coordinates x_i and x_{i+1}. It is a well-known fact that $S_{r_1}, \cdots, S_{r_{n-1}}$ generate $\mathscr{P}_n$, and hence constitute a set of fundamental reflections for $\mathscr{P}_n$. Next,

$$F_t = \{x = (x_1, \cdots, x_n) \mid x_1 > x_2 > \cdots > x_n\} \tag{3.10}$$

is a fundamental region for $\mathscr{P}_n$. The convex polyhedral cone F_t is open and is not pointed, for $\bar{F}_t$ contains the 1-dimensional subspace $M_{\mathscr{P}_n}$. Note that T is the union of the $n!$ images of F_t under $\mathscr{P}_n$. (This discussion of $\mathscr{P}_n$ is continued in Examples 4.1 and 4.4.)

The following facts about fundamental regions for a Coxeter group G will be used frequently. Let $t, \tau \in T$ and $g \in G$, and let $\Pi_t = \{r_1, \cdots, r_{n^*}\}$.

FACT 3.1. *$t \in F_t \subset T$.*

PROOF. Since $\Pi_t \subseteqq \Delta_t^+$, $r_i't > 0$ for $1 \leqq i \leqq n^*$, so $t \in F_t$. Next, since $\Delta = \Delta_t^+ \cup \Delta_t^-$, every root r in Δ is a nonzero linear combination of $r_1, \cdots, r_{n^*}$ with coefficients either all nonnegative or all nonpositive. Thus for $x \in F$, (3.1) implies that $r'x \neq 0$, so $x \in T$.

FACT 3.2. *If $\tau \in F_t$ then $F_\tau = F_t$.*

PROOF. Since $\tau \in F_t$, (3.1) implies that $\Delta_t^+ \subseteqq \Delta_\tau^+$. Hence $\Delta_t^+ = \Delta_\tau^+$ because $|\Delta_t^+| = \frac{1}{2}|\Delta| = |\Delta_\tau^+|$. Thus $\Pi_t = \Pi_\tau$, so $F_t = F_\tau$.

FACT 3.3 (B–G, Proposition 4.2.2). $\Delta_{gt}^+ = g\Delta_t^+$; $\Pi_{gt} = g\Pi_t$; $K_{gt} = gK_t$; $F_{gt} = gF_t$; $\bar{F}_{gt} = g\bar{F}_t$.

FACT 3.4. *$F_\tau = gF_t$ for some $g \in G$. The collection $\mathscr{A} \equiv \mathscr{A}_G \equiv \{gF_t \mid g \in G\}$ consists of distinct fundamental regions and does not depend on $t \in T$. Also, $T = \bigcup \{F \mid F \in \mathscr{A}\}$, and $R^n = \bigcup \{\bar{F} \mid F \in \mathscr{A}\}$. Each point in the G-orbit of t lies in exactly one member of $\mathscr{A}$.*

PROOF. Fact 3.3 and (3.5) imply that $\tau \in \bar{F}_{gt}$ for some $g \in G$. Furthermore, $\tau \in T$ implies that $\tau \notin \partial F_{gt}$, so $\tau \in F_{gt}$. By Fact 3.2, $F_\tau = F_{gt} = gF_t$. The second statement in Fact 3.4 follows from the first and (3.4). The third statement follows from Fact 3.1 and (3.5), while the last statement is obvious.

FACT 3.5. *Let $x, y \in R^n$. Then $x, y \in F$ for some $F \in \mathscr{A}$* iff *$(r'x)(r'y) > 0$ for each $r \in \Delta$.*

PROOF. Clearly, $[(r'x)(r'y) > 0$ for each $r \in \Delta]$ iff $[x, y \in T$ and $\Delta_x^+ = \Delta_y^+]$ iff $[x, y \in T$ and $F_x = F_y]$ iff $[x, y \in F$ for some $F \in \mathscr{A}]$.

LEMMA 3.1. *Let $x, y \in R^n$ be distinct. The following are equivalent:*

(i) *$x, y \in \bar{F}$ for at least one $F \in \mathscr{A}$;*

(ii) *for every $r \in \Delta$, x and y lie on the same side of H_r,* i.e., $(r'x)(r'y) \geqq 0$;

(iii) *for every $r \in \Delta$, either $H_r \cap [x, y]^0 = \emptyset$ or $[x, y] \subseteqq H_r$, where $[x, y]([x, y]^0)$ is the closed (open) line segment connecting x and y;*

(iv) *for every $r \in \Delta$, $H_r \cap [x, y]^0$ is not a single point.*

PROOF. The equivalence of (ii), (iii), and (iv) is elementary. The implication (i) $\Rightarrow$ (ii) follows from (3.2) and the fact that every $r \in \Delta$ is a nonzero linear combination of $r_1, \cdots, r_{n^*}$ with coefficients either all nonnegative or all nonpositive, where $\{r_1, \cdots, r_{n^*}\} = \Pi_t$ with $t \in T$ selected so that $F_t = F$. Conversely, assume that (ii) holds and choose $t \in T$ such that $z \equiv \frac{1}{2}(x+y) \in \bar{F}_t$. Since $r_i'z \geqq 0$ for each $r_i \in \Pi_t$ and $r_i'z = \frac{1}{2}(r_i'x + r_i'y)$, (ii) implies that $r_i'x \geqq 0$, so $x, y \in \bar{F}_t$.

The following technical lemma will be applied repeatedly in the proof of Lemma 4.1.

LEMMA 3.2. *Suppose that* $\rho \in \Delta$, $x_0 \in T$, $x_1 = S_\rho x_0$, *and* $v_1 \in H_\rho$. *Furthermore, assume that*

(i) $v_0 \in \bar{F}_{x_0}$;
(ii) $v_1 \in \bar{F}_{x_1} (= S_\rho \bar{F}_{x_0})$;
(iii) $(\rho' v_0)(\rho' v_2) < 0$;
(iv) $v_1 \notin H_r$ *for each* $r \in \Delta$ *such that* $H_r \neq H_\rho$;
(v) $(r'v_1)(r'v_2) \geqq 0$ *for every* $r \in \Delta$.

Then $v_2 \in \bar{F}_{x_1}$.

PROOF. It must be shown that $r_i'v_2 \geqq 0$ for each $r_i \in \Pi_{x_1}$. Assumption (ii) implies $r_i'v_1 \geqq 0$. If $r_i'v_1 > 0$ then $r_i'v_2 \geqq 0$ by (v), as required. If $r_i'v_1 = 0$ then $v_1 \in H_{r_i}$. Therefore $H_{r_i} = H_\rho$ by (iv), so $r_i = \pm\rho$ and $S_{r_i} = S_\rho$. Hence by (iii),

$$(r_i'v_0)(r_i'v_2) < 0 . \tag{3.11}$$

Also, since $x_0 \notin H_{r_i}$ and $x_1 = S_{r_i} x_0$,

$$(r_i'x_0)(r_i'x_1) < 0 . \tag{3.12}$$

Finally, (i) and Lemma 3.1 imply $(r_i'x_0)(r_i'v_0) \geqq 0$; however, by (3.11) and the assumption $x_0 \in T$, this implies that

$$(r_i'x_0)(r_i'v_0) > 0 . \tag{3.13}$$

Taken together, (3.11), (3.12) and (3.13) show that $(r_i'x_1)(r_i'v_2) > 0$. However, $r_i \in \Pi_{x_1}$ implies $r_i'x_1 > 0$, so $r_i'v_2 > 0$, which completes the proof.

This section concludes with several comments about the dimension of the convex sets $C(x) \equiv C_G(x)$ when G is a finite reflection group acting on R^n. Let $n^* =$ dimension $(M_G{}^\perp) \leqq n$. If $x \in T \equiv T_G$, then $C(x)$ contains the line segments $[x, S_{r_i}x]$, $1 \leqq i \leqq n^*$, where $\{r_1, \cdots, r_{n^*}\} = \Pi_x$. Since $x \in T$, the vectors $x - S_{r_i}x = 2(r_i'x)r_i$, $1 \leqq i \leqq n^*$, are nonzero and linearly independent. Thus, dimension $(C(x)) = n^*$ when $x \in T$.

If $x \notin T$ little can be said at this point about dimension $(C(x))$ (see Remark 4.6). From Remark 2.5, however, if G acts irreducibly on $M_G{}^\perp$ then dimension $(C(x)) = n^*$ for all $x \notin M_G$. For example, consider $G = \mathscr{P}_n$. Since there does not exist a proper subspace M of $M_{\mathscr{P}_n}^\perp$ such that $\Delta_{\mathscr{P}_n} = (\Delta_{\mathscr{P}_n} \cap M) \cup (\Delta_{\mathscr{P}_n} \cap M^\perp)$ (see Remark 3.2 and (3.6)), it follows that $\mathscr{P}_n$ acts irreducibly on $M_{\mathscr{P}_n}^\perp$ (see the proof of Proposition 3.1). Therefore, dimension $(C(x)) = n^* \equiv n - 1$ for each

$x \notin M_{\mathscr{P}_n}$. The representation $x = x^* + x^{**}$ of Remark 2.5 here takes the form $x = (x_1 - \bar{x}, \cdots, x_n - \bar{x}) + (\bar{x}, \cdots, \bar{x})$, where $x = (x_1, \cdots, x_n)$ and $\bar{x} = n^{-1} \sum x_i$. From Remark 2.5, $x^{**} \equiv (\bar{x}, \cdots, \bar{x}) \precsim x$ for every $x \in R^n$, and x^{**} lies in the ($(n-1)$-dimensional) relative interior of $C(x)$ whenever $x^* \neq 0$.

4. The structure of $C(x)$ and the basic path lemmas. Throughout this section G denotes a finite reflection group acting on R^n. The notation of Section 3 is continued here: for $t \in T$ let $\Pi_t = \{r_1, \cdots, r_{n^*}\}$, where $n^* =$ dimension $(M_G{}^\perp)$, and let $K_t \subseteqq M_G{}^\perp$ denote the convex cone generated by Π_t; $\bar{F}_t$ is the dual cone of K_t in R^n, and the fundamental region F_t is the interior of $\bar{F}_t$. The results in this section are based on the following fundamental geometric property of reflection groups.

LEMMA 4.1. *If $u, v \in \bar{F} \equiv \bar{F}_t$ and $g \in G$, then*

$$(gu)'v \leqq u'v \;. \tag{4.1}$$

REMARK 4.1. When G is the permutation group $\mathscr{P}_n$ and F_t is given by (3.10), (4.1) reduces to a well-known *rearrangement* inequality: if $u_1 \geqq u_2 \geqq \cdots \geqq u_n$ and $v_1 \geqq v_2 \geqq \cdots \geqq v_n$ and if $(\gamma(1), \cdots, \gamma(n))$ is a permutation of $(1, \cdots, n)$, then $\sum u_{\gamma(i)} v_i \leqq \sum u_i v_i$.

PROOF OF LEMMA 4.1. By continuity, it is sufficient to show that (4.1) holds whenever $u \in F \equiv F_t (= F_u)$, $v \in \bar{F}$, and $g \in G$. By means of Lemma 3.1 it is easy to show that (4.1) holds for $g = S_r$, $r \in \Delta$. To demonstrate (4.1) for arbitrary $g \in G$, we will find a finite sequence of reffections $\{S_{\rho_j} \mid 1 \leqq j \leqq k\}$ with $\rho_j \in \Delta$, such that $g = S_{\rho_k} S_{\rho_{k-1}} \cdots S_{\rho_1}$ and such that the sequence of points $\{x_j \mid 0 \leqq j \leqq k\}$ defined by $x_0 = u$ and $x_{j+1} = S_{\rho_{j+1}} x_j$ satisfies

$$x'_{j+1} v \leqq x_j{}' v \,, \qquad 0 \leqq j \leqq k-1 \,. \tag{4.2}$$

Since $x_k = gu$, this will imply (4.1).

CLAIM 1. There exists $z \in gF (= gF_u = F_{gu})$ such that the line segment $L \equiv [u, z]$ satisfies $L \cap H_r \cap H_{\tilde{r}} = \varnothing$ for every pair of distinct hyperplanes H_r, $H_{\tilde{r}}$ such that $r, \tilde{r} \in \Delta$.

PROOF. Let $\{P_1, P_2, \cdots, P_M\}$ be the set of all $(n-2)$-dimensional subspaces $H_r \cap H_{\tilde{r}}$ such that $r, \tilde{r} \in \Delta$, $H_r \neq H_{\tilde{r}}$. Let $L^* = [u, gu]$ and let Q be the $(n-1)$-dimensional hyperplane perpendicular to L^* and containing gu. Denote the projection operator onto Q along L^* by π. Since each πP_i is of dimension at most $n-2$, there exists $z \in Q \cap (gF)$ such that $z \neq gu$ and $[gu, z] \cap \pi P_i$ is either empty or the single point $\{gu\}$ for $1 \leqq i \leqq M$. Let $L = [u, z]$, so $\pi L = [gu, z]$. If there existed some point $w \in L \cap P_i$, then $\pi w \in \pi L \cap \pi P_i = [gu, z] \cap \pi P_i = \{gu\}$. This would imply $w \in L^* \cap L \cap P_i$ so $w = u$, but this is impossible since $u \in F \subset T$ implies $u \notin P_i$. Thus $L \cap P_i = \varnothing$ for $1 \leqq i \leqq M$ as claimed.

Because u and z are in different fundamental regions, L must intersect at least one H_r, $r \in \Delta$, by Lemma 3.1. Let $H_{\tilde{\rho}_1}, \cdots, H_{\tilde{\rho}_k}$, $\tilde{\rho}_i \in \Delta$, be the hyperplanes intersected by L as one moves from u to z, listed in order. By Claim 1 the $H_{\tilde{\rho}_j}$

and their order of appearance are uniquely determined. The intersection $L \cap H_{\tilde\rho_j}$ consists of a single point, denoted by v_j. Set $x_0 = u$ and $x_{j+1} = S_{\tilde\rho_{j+1}} x_j$ for $0 \leqq j \leqq k-1$. Notice that each $x_j \in F_{x_j} \subset T$. To complete the proof of Lemma 4.1 it remains to prove that $g = S_{\tilde\rho_k} S_{\tilde\rho_{k-1}} \cdots S_{\tilde\rho_1}$ and that (4.2) is true.

First, note that by construction $[x_0, v_1]^0 \cap H_r = \varnothing$ for all $r \in \Delta$. Hence by Lemma 3.1, x_0 and v_1 are in $\bar F_{x_0} (= \bar F)$. Thus $x_1 = S_{\tilde\rho_1} x_0$ and $v_1 = S_{\tilde\rho_1} v_1$ are in $\bar F_{x_1} = S_{\tilde\rho_1} \bar F_{x_0}$. Apply Lemma 3.2 (with $(\rho, x_0, x_1, v_0, v_1, v_2)$ of that lemma replaced by $(\tilde\rho_1, x_0, x_1, x_0, v_1, v_2)$ here) to see that $v_2 \in \bar F_{x_1}$. Since x_1 and $v_2 \in \bar F_{x_1}$ and $v_2 = S_{\tilde\rho_2} v_2$, x_2 and v_2 lie in $\bar F_{x_2}$. Apply Lemma 3.2 (with $(\rho, x_0, x_1, v_0, v_1, v_2)$ replaced by $(\tilde\rho_2, x_1, x_2, v_1, v_2, v_3)$) to see that $v_3 \in \bar F_{x_2}$. By induction we find that

$$\begin{aligned} x_{j-1} \quad &\text{and} \quad v_j \quad \text{lie in} \quad \bar F_{x_{j-1}}, \\ x_j \quad &\text{and} \quad v_j \quad \text{lie in} \quad \bar F_{x_j} \end{aligned}$$

for $1 \leqq j \leqq k$. Thus x_k and $v_k \in \bar F_{x_k}$. By definition of v_k, $[v_k, z]^0 \cap H_r = \varnothing$ for all $r \in \Delta$, so $(r'v_k)(r'z) \geqq 0$. Thus a final application of Lemma 3.2 (with $(\rho, x_0, x_1, v_0, v_1, v_2)$ replaced by $(\tilde\rho_k, x_{k-1}, x_k, v_{k-1}, v_k, z)$) shows that $z \in \bar F_{x_k}$. However, since $z \in F_{gu}$, it must be that $F_{x_k} = F_{gu}$. Since $x_k = S_{\tilde\rho_k} \cdots S_{\tilde\rho_1} u$, we conclude that $g = S_{\tilde\rho_k} \cdots S_{\tilde\rho_1}$. (This argument presumes that $k \geqq 2$; if $k = 1$ only the final application of Lemma 3.2 is required, with v_0 taken to be x_0.)

Lastly, we shall establish (4.2). Since

$$x_{j+1} = S_{\tilde\rho_{j+1}} x_j = x_j - 2(\tilde\rho'_{j+1} x_j) \tilde\rho_{j+1},$$

it must be shown that

$$(\tilde\rho'_{j+1} x_j)(\tilde\rho'_{j+1} v) \geqq 0, \quad 0 \leqq j \leqq k-1. \tag{4.3}$$

For $1 \leqq i \leqq k$ consider the triangle with vertices x_{i-1}, v_i, and x_i. If $l \neq i$, the hyperplane $H_{\tilde\rho_l}$ does not contain x_{i-1}, v_i, or x_i, and $H_{\tilde\rho_l}$ cannot intersect $[x_{i-1}, v_i]^0$ (since $v_i \in \bar F_{x_{i-1}}$) or $[v_i, x_i]$ (since $v_i \in \bar F_{x_i}$). Hence

$$H_{\tilde\rho_l} \cap [x_{i-1}, x_i] = \varnothing, \quad 1 \leqq i \neq l \leqq k. \tag{4.4}$$

For $0 \leqq j \leqq k-1$ consider the polygonal path

$$\Lambda_j : v \to x_0 \to x_1 \to \cdots \to x_j$$

and define $\psi(w) = \tilde\rho'_{j+1} w$ for $w \in \Lambda_j$. Since $x_0 \equiv u$ and v lie in $\bar F (= \bar F_u)$, Lemma 3.1 implies that ψ has no zero on $[v, x_0]^0$, and (4.4) implies that ψ has no zero on the rest of Λ_j. Hence ψ does not change sign on Λ_j, so $\psi(v)\psi(x_j) \geqq 0$. Thus (4.3) holds and (4.2) is established, so the proof of Lemma 4.1 is complete.

LEMMA 4.2 (First Path Lemma). *Suppose $x, y \in F \equiv F_t$. The following are equivalent*:

(i) $y \in C(x)$;

(ii) $x - y \in K_t$, i.e., $x - y = \sum_{i=1}^{n^*} c_i r_i$ *where* $\Pi_t = \{r_1, \cdots, r_{n^*}\}$ *and each* $c_i \geqq 0$.

(iii) $x - y = \sum_{i=1}^{k} \eta_i r_{(i)}$ *for some integer k, where each $\eta_i > 0$, each $r_{(i)} \in \Pi_t$, and* $z_j \equiv y + \sum_{i=1}^{j} \eta_i r_{(i)} \in F_t$ *for* $1 \leqq j \leqq k$.

The implication (ii) $\Rightarrow$ (i) *remains valid if the assumption that* $x, y \in F_t$ *is weakened to* $x, y \in \bar{F}_t$. *The implication* (i) $\Rightarrow$ (ii) *remains valid if* $x, y \in F_t$ *is weakened to* $x \in \bar{F}_t$.

PROOF. Clearly (iii) $\Rightarrow$ (ii). That (ii) $\Rightarrow$ (iii) when $x, y \in F_t$ follows by dividing the line segment $[x, y] \subset F_t$ into sufficiently small subsegments and arguing as in the proof of Theorem 2 of Marshall, Walkup, and Wets (1967).

To show that (iii) $\Rightarrow$ (i), set

$$\delta = \eta_{j+1}/2(\eta_{j+1} + r'_{(j)} z_j)\,,$$

so that

$$z_j = (1-\delta)(z_j + \eta_{j+1} r_{(j+1)}) + \delta S_{r_{(j+1)}}(z_j + \eta_{j+1} r_{(j+1)})\,.$$

Since $0 < \delta < \frac{1}{2}$ this implies that $z_j \in C(z_{j+1})$, $1 \leqq j \leqq k-1$. Similarly $y \in C(z_1)$, so that $y \in C(z_k) \equiv C(x)$.

Next we show that (i) $\Rightarrow$ (ii), assuming only that $x \in \bar{F}_t$. Since $y \in C(x)$ we have $y = \sum_{\alpha=1}^{|G|} \lambda_\alpha g_\alpha x$, where $G = \{g_\alpha \mid 1 \leqq \alpha \leqq |G|\}$, $\lambda_\alpha \geqq 0$, $\sum \lambda_\alpha = 1$. Apply Lemma 4.1 with $u \equiv x \in \bar{F}_t$ and $v \in \bar{F}_t$ to deduce that

$$(\sum \lambda_\alpha g_\alpha x)'v \leqq x'v\,.$$

Thus $(x-y)'v \geqq 0$ for each $v \in \bar{F}_t$, so $x - y$ is in the dual cone of $\bar{F}_t$, namely K_t.

It remains to show that (ii) $\Rightarrow$ (i) if $x, y \in \bar{F}_t$. (This implication is already established for the case $x, y \in F_t$.) Since t is an arbitrary point of F_t, without loss of generality we can assume that x, y and t are distinct points and consider the triangle which they determine. Choose $x_m \in [x, t]^0 \subset F_t$ and $y_m \in [y, t]^0 \subset F_t$ so that $x_m \to x$, $y_m \to y$, and $x_m - y_m$ is parallel to $x - y$. Thus $x_m - y_m \in K_t$ and $x_m, y_m \in F_t$ so $y_m \in C(x_m)$, i.e.,

$$y_m = \sum_{\alpha=1}^{|G|} \lambda_\alpha^{(m)} g_\alpha x_n\,, \tag{4.5}$$

where $(\lambda_1^{(m)}, \cdots, \lambda_{|G|}^{(m)})$ lies in the probability simplex $\Lambda \subset R^{|G|}$. Since Λ is compact there is a subsequence $\{m'\} \subset \{m\}$ and a point $(\lambda_1, \cdots, \lambda_{|G|}) \in \Lambda$ such that $\lambda_\alpha^{(m')} \to \lambda_\alpha$ as $m' \to \infty$, $1 \leqq \alpha \leqq |G|$. Replace m by m' in (4.5) and let $m' \to \infty$ to obtain that $y \in C(x)$.

REMARK 4.2. When $x, y \in F_t$ and $y \in C(x)$, the sequence $\{z_j\}$ constructed in (iii) satisfies the hypotheses of Corollary 2.1. However, the implication (i), (ii) $\Rightarrow$ (iii) in Lemma 4.2 is not valid if it is only assumed that $x, y \in \bar{F}_t$, even if the requirement that $z_j \in F_t$ is weakened to $z_j \in \bar{F}_t$ in (iii). For example, take $n = 2$ and consider the group $\mathscr{B}_2$ generated by permutation and sign changes of coordinates in R^2, i.e., the group generated by the reflections $S_{e_1-e_2}$ and S_{e_1}, where $e_1 = (1, 0)$, $e_2 = (0, 1)$. The group $\mathscr{B}_2$ acts effectively on R^2, so $n^* = 2$. This group has eight roots: $\pm e_1$, $\pm e_2$, $\pm e_1 \pm e_2$, and $|G| = 8$. For $t = (2, 1)$, we find that $\Pi_t = \{e_2, e_1 - e_2\}$, $\Delta_t^+ = \{e_1, e_2, e_1 \pm e_2\}$, and $\bar{F}_t = \{(x_1, x_2) \mid x_1 \geqq x_2 \geqq 0\}$. If $0 \neq x \in \bar{F}_t$ and $y = 0\ (\in \partial F_t)$, then (i) and (ii) hold but (iii) fails, for $z_1 \equiv \eta_1 r_{(1)}$ cannot lie in F_t if $\eta_1 > 0$ and $r_{(1)} \in \Pi_t$. (Although (iii) fails, there is a polygonal path between x and y satisfying the hypotheses of Corollary 2.1—see Lemma 4.5.)

REMARK 4.3. Lemma 4.2 implies that for $x, y \in \bar{F}_t$, $y \precsim x$ iff $x - y \in K_t$, so the ordering $\precsim$ induced on $\bar{F}_t$ by G is the cone ordering determined by K_t (see Marshall, Walkup, and Wets (1967) for special cases of this remark).

EXAMPLE 4.1. Return to Example 3.1 where $G = \mathscr{P}_n$ and let K_t and F_t be as in (3.9) and (3.10). Assume that $x \equiv (x_1, \cdots, x_n)$ and $y \equiv (y_1, \cdots, y_n)$ are in $\bar{F}_t$, i.e., $x_1 \geqq \cdots \geqq x_n$, $y_1 \geqq \cdots \geqq y_n$. By Remark 4.3 and (3.9), $y \precsim x$ iff $\sum_{i=1}^k x_i \geqq \sum_{i=1}^k y_i$ for $1 \leqq k \leqq n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Next, for arbitrary $z \equiv (z_1, \cdots, z_n)$ in R^n define $\tilde{z} = (z_{(1)}, \cdots, z_{(n)})$ where $z_{(1)} \geqq \cdots \geqq z_{(n)}$ are the ordered components of z. Since $\tilde{z} = gz$ for some $g \in \mathscr{P}_n$ it follows that $C(z) = C(\tilde{z})$. Therefore for arbitrary $x, y \in R^n$, $y \in C(x)$ iff $\tilde{y} \in C(\tilde{x})$, so $y \precsim x$ iff $\sum_{i=1}^k x_{(i)} \geqq \sum_{i=1}^k y_{(i)}$ for $1 \leqq k \leqq n-1$ and $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$. Thus when $G = \mathscr{P}_n$, the ordering $\precsim$ is exactly the classical majorization ordering.

Lemma 4.2 shows that the structure of $C(x)$ is related to that of the convex polyhedral cones K_t. This relation is explicitly stated in the following corollary.

COROLLARY 4.1. *For $x \in R^n$, choose $\tau \equiv \tau(x) \in T$ such that $x \in \bar{F}_\tau$ (F_τ is not unique unless $x \in T$, in which case $F_\tau = F_x$). Then*

$$C(x) = \bigcap \{g(x - K_\tau) \mid g \in G\} \equiv \bigcap \{(gx - K_{g\tau}) \mid g \in G\}, \tag{4.6}$$

where $x - K_\tau$ is the convex cone with vertex x defined by

$$x - K_\tau = \{z \mid z = x - u, u \in K_\tau\}.$$

PROOF. By Lemma 4.2 ((i) $\Rightarrow$ (ii)), $C(x) \subseteqq x - K_\tau$. Since $C(x)$ is G-invariant, this implies that $C(x) \subseteqq \bigcap \{g(x - K_\tau) \mid g \in G\}$. Conversely, suppose that $y \in \bigcap \{g(x - K_\tau) \mid g \in G\}$ and choose $g \in G$ such that $y \in g\bar{F}_\tau$ (by (3.5)). Since $x, g^{-1}y \in \bar{F}_\tau$ and $x - g^{-1}y \in K_\tau$, Lemma 4.2 ((ii) $\Rightarrow$ (i)) implies that $y \in C(x)$, as required.

Corollary 4.1 states that $C(x)$ is an intersection of the congruent convex polyhedral cones $g(x - K_\tau)$, $g \in G$. The vertices of these cones are the points gx in the G-orbit of x, and are not necessarily distinct unless $x \in T$. The second path lemma for reflection groups, Lemma 4.5, requires the determination of the *edges* of the convex polytope $C(x)$, namely, that each edge is parallel to some root $r \in \Delta$, the root system of G. The case $x \in T$ will be considered first (Theorem 4.1). Here we may take $\tau(x) = x$ and rewrite (4.6) as

$$C(x) = \bigcap \{g(x - K_x) \mid g \in G\} = \bigcap \{gx - K_{gx} \mid g \in G\}, \tag{4.7}$$

where now the points $\{gx \mid g \in G\}$ are distinct. It will be shown that the edges of $C(x)$ emanating from gx lie along the extreme rays of $g(x - K_x)$, so $C(x)$ and $g(x - K_x)$ coincide in a neighborhood of gx.

Recall that a subset A of a convex polytope $C \subseteqq R^n$ is called a *face* of C if there exists an $(n-1)$-dimensional supporting hyperplane Q for C such that $A = Q \cap C$. A 1-dimensional face of C is called an *edge*, and a 0-dimensional face is called an *extreme point*, or *vertex*. (The reader is referred to Grünbaum

(1967) or Rockafellar (1970) for basic results concerning convex sets and polytopes.)

THEOREM 4.1 (Structure of $C(x)$ when $x \in T$). *The convex polytope $C(x)$ has exactly $|G|$ extreme points (vertices), the members of the G-orbit of x. Exactly n^* edges emanate from the vertex x, namely, the line segments $[x, S_{r_i}x]$ where $\{r_1, \cdots, r_{n^*}\} = \Pi_x$. Similarly, the n^* edges emanating from the extreme point gx are exactly the g-images $[gx, gS_{r_i}x] = [gx, S_{gr_i}gx]$ of these segments. The edge $[gx, gS_{r_i}x]$ is parallel to $g(x - S_{r_i}x) = 2(r_i'x)gr_i$, a nonzero vector in the direction of the root $gr_i \in \Delta$. The polytope $C(x)$ has exactly $\frac{1}{2}|G|n^*$ edges.*

EXAMPLE 4.2. Take $n = 2$ and consider the group $\mathscr{B}_2$ of Remark 4.2. For $x = (2, 1) \in T$, $C(x)$ is an octagon whose edges are line segments with slopes either 0, ± 1, or ∞ (see Figure 1 of Eaton and Perlman (1974)). Hence, each edge of $C(x)$ is parallel to one of the roots of G.

PROOF OF THEOREM 4.1. Since $||gx|| = ||x||$ for each $g \in G$, each point gx must be an extreme point of $C(x)$ and these points gx, $g \in G$, are distinct. As the endpoints of each edge of $C(x)$ are extreme points of $C(x)$ (Grünbaum (1967), Theorem 5, page 33), each edge emanating from x must be of the form $[x, gx]$ for some g. Since $gx \in C(x) \subseteqq x - K_x$, it must be that

$$x - gx = \textstyle\sum_{i=1}^{n^*} c_i r_i \equiv \sum_{i=1}^{n^*} c_i^*(x - S_{r_i}x)\,, \tag{4.8}$$

where each $c_i \geqq 0$ and $c_i^* \equiv c_i/2(r_i'x) \geqq 0$. Since $x \neq gx$ at least one c_i must be positive, say $c_1 > 0$. By the definition of an edge there exists a nonzero vector $a \in R^n$ such that

$$\begin{aligned} z \in [x, gx] &\Rightarrow a'z = 1\,, \\ z \in C(x) \cap [x, gx]^0 &\Rightarrow a'z < 1\,. \end{aligned} \tag{4.9}$$

From (4.8) and (4.9) we have that

$$0 = \textstyle\sum_{i=1}^{n^*} c_i^*(1 - a'S_{r_i}x)\,.$$

Since $c_1^* > 0$, it follows that $a'S_{r_1}(x) = 1$, i.e., $S_{r_1}x \in [x, gx]$. Clearly, $S_{r_1}x \neq x$ since $r_1'x > 0$, while $S_{r_1}x \notin [x, gx]^0$ since $S_{r_1}x$ is an extreme point of $C(x)$. Hence $S_{r_1}x = gx$. Thus each edge of $C(x)$ emanating from x is of the form $[x, S_{r_i}x]$ for some $r_i \in \Pi_x$.

Conversely, it remains to show that each segment $[x, S_{r_i}x]$ is an edge of $C(x)$, $1 \leqq i \leqq n^*$. This follows from the facts that

$$2(r_i'x)r_i = x - S_{r_i}x \in C(x) \subseteqq K_x$$

and that r_i determines an extreme ray of the cone K_x. The rest of Theorem 4.1 follows readily.

When $x \notin T$, $x = S_r x$ for those roots $r \in \Delta$ such that $x \in H_r$, so the structure of $C(x)$ is different than in the case $x \in T$. Now $C(x)$ will have fewer than $|G|$ vertices, since the G-orbit of x contains fewer than $|G|$ distinct points; also, the number of edges of $C(x)$ emanating from each vertex may be greater or smaller than n^*. It will still be true that each edge of $C(x)$ is parallel to some root of G (Theorem 4.2), although not every root need be parallel to some edge.

Before proceeding we present two preliminary lemmas.

LEMMA 4.3. *Let $x \in R^n$, $g \in G$, $F \in \mathscr{A}$. If x, $gx \in \bar{F}$ then $x = gx$.*

PROOF. By Lemma 4.1 and the Cauchy–Schwartz inequality,

$$||x||^2 = ||gx||^2 = (gx)'gx \leqq x'gx \leqq ||x||^2 .$$

This implies that $x = gx$, as claimed.

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$$G_x = \{g \mid g \in G, gx = x\} ,$$

a subgroup of G, and define

$$\mathscr{A}_x = \{F \mid F \in \mathscr{A}, x \in \bar{F}\} ,$$

a subcollection of the set of fundamental regions for G. By (3.5), $\mathscr{A}_x$ is non-empty; in fact, when $x \notin T$, $\mathscr{A}_x$ has at least two members. (When $x \in T$, $\mathscr{A}_x = \{F_x\}$ and $G_x = \{I\}$.) The following lemma is an easy consequence of Lemma 4.3, (3.4), and Fact 3.4.

LEMMA 4.4. *There is a 1—1 correspondence between G_x and $\mathscr{A}_x$. Specifically, let F_τ be a fixed but arbitrary member of $\mathscr{A}_x$ as in Corollary 4.1, so that $x \in \bar{F}_\tau$. Then*

$$\mathscr{A}_x = \{gF_\tau \mid g \in G_x\} ,$$
$$G_x = \{g \mid gF_\tau \in \mathscr{A}_x\} \equiv \{g \mid x \in g\bar{F}_\tau\} .$$

REMARK 4.4. Lemma 4.4 implies that $G_x = \{g \mid g\mathscr{A}_x = \mathscr{A}_x\}$. In Theorem 4.3 it will be shown that G_x is itself a reflection group, and that its system of fundamental regions is essentially $\mathscr{A}_x$.

In order to study the structure of $C(x)$ when $x \notin T$, we must first extend the definitions of K_x, $\Delta_x{}^+$, $\bar{F}_x$, and Π_x to this case. Let $\tau \equiv \tau(x)$ and F_τ be as in Corollary 4.1 and Lemma 4.4, and define

$$\begin{aligned} K_x &= \bigcap \{K_t \mid F_t \in \mathscr{A}_x\} \equiv \bigcap \{K_{g\tau} \mid g \in G_x\} \subset M_G{}^\perp \\ \Delta_x{}^+ &= \bigcap \{\Delta_t{}^+ \mid F_t \in \mathscr{A}_x\} \equiv \bigcap \{\Delta_{g\tau}^+ \mid g \in G_x\} \subset M_G{}^\perp \\ \bar{F}_x &= \bigcup \{\bar{F}_t \mid F_t \in \mathscr{A}_x\} \equiv \bigcup \{\bar{F}_{g\tau} \mid g \in G_x\} \subseteqq R^n . \end{aligned} \tag{4.10}$$

Next, write Π_τ as

$$\Pi_\tau = \{\rho_1, \cdots, \rho_q, \rho_{q+1}, \cdots, \rho_{n^*}\} ,$$

where $\rho_i'x = 0$, $1 \leqq i \leqq q$, and $\rho_i'x > 0$, $q + 1 \leqq i \leqq n^*$. Since $x \notin T$, x lies in at least one wall of $\bar{F}_\tau$, so $q \geqq 1$; $q = n^*$ iff $x \in M_G$. Any other $F_t \in \mathscr{A}_x$ is of the form $F_t = gF_\tau$ for some $g \in G_x$, so

$$\Pi_t = g\Pi_\tau = \{g\rho_1, \cdots, g\rho_q, g\rho_{q+1}, \cdots, g\rho_{n^*}\} , \quad g \in G_x . \tag{4.11}$$

Since $gx = x$ and $g' = g^{-1}$, it follows that $(g\rho_i)'x = 0$ for $1 \leqq i \leqq q$ and $(g\rho_i)'x > 0$

for $q+1 \leqq i \leqq n^*$. The roots $g\rho_i$, $1 \leqq i \leqq q$, are called the *x-internal roots* in $g\Pi_\tau$ $(g \in G_x)$; the corresponding walls

$$H_{g\rho_i} \cap g\bar{F}_\tau \equiv g(H_{\rho_i} \cap \bar{F}_\tau)\,, \qquad 1 \leqq i \leqq q\,,$$

are called the *x-internal walls* of $g\bar{F}_\tau$ and contain x. The roots $g\rho_i$, $q+1 \leqq i \leqq n^*$, are called the *x-external roots* in $g\Pi_\tau$ $(g \in G_x)$ and the corresponding walls of $g\bar{F}_\tau$ are the *x-external walls* of $g\bar{F}_\tau$; these do not contain x. Under the action of any $g \in G_x$, x-internal (external) roots and walls are sent to x-internal (external) roots and walls, respectively. (It can happen that $g\rho_i = \rho_i$ for an x-external root ρ_i and $g \in G_x$.) The set of all x-external roots is denoted by Π_x, i.e.,

$$\Pi_x = \{g\rho_i \mid q+1 \leqq i \leqq n^*,\, g \in G_x\} \subset M_G^\perp\,. \tag{4.12}$$

The definitions of K_x, Δ_x^+, $\bar{F}_x$, and Π_x in (4.10) and (4.12) do not depend on the choice of $\tau \equiv \tau(x)$, and reduce to the original definitions when $x \in T$, since in that case $G_x = \{I\}$ and we may take $\tau(x) = x$. We shall show below (see Proposition 4.1) that, somewhat surprisingly, all inter-relationships among K_x, Δ_x^+, $\bar{F}_x$, Π_x carry over without change from the case $x \in T$ to the case $x \notin T$. In particular, it will be shown that $\bar{F}_x$ is a convex cone containing x as an interior point, K_x is the dual cone of $\bar{F}_x$, and the extreme rays of K_x are determined by the roots in Π_x. As in the proof of Theorem 4.1 (see (4.8)), this last fact will enable us to show in Theorem 4.2 that each edge of $C(x)$ is parallel to some root of G.

EXAMPLE 4.3. Return to the Coxeter group $\mathscr{B}_2$ acting on R^2 considered in Remark 4.2 and Example 4.2. If $x = (1, 1) \notin T_{\mathscr{B}_2}$, we find that $G_x = \{I, \tilde{g}\}$ where $\tilde{g}$ is the permutation $S_{e_1-e_2}$. We may take $\tau(x) = (2, 1)$, so that $F_\tau = \{(x_1, x_2) \mid x_1 > x_2 > 0\}$ and $\mathscr{A}_x = \{F_\tau, \tilde{g}F_\tau\}$. Furthermore, $K_x = \{(x_1, x_2) \mid x_1 \geqq 0, x_2 \geqq 0\} = \bar{F}_x$, $\Delta_x^+ = \{e_1, e_2, e_1 + e_2\}$, and $\Pi_x = \{e_1, e_2\}$. Finally, $C(x)$ is the square with vertices $(\pm 1, \pm 1)$ (not an octagon, as in Example 4.2 where $x \in T_{\mathscr{B}_2}$ was chosen), so that each edge of $C(x)$ is again parallel to some root of G, in fact, to an x-external root ($\pm e_1$ or $\pm e_2$).

Because examples in R^2 such as the above do not adequately illustrate the general case, the reader is urged to consider the Coxeter group G acting on R^3, whose fundamental regions are represented in Figure 4.3 on page 38 of Coxeter and Moser (1972). Take x to be a boundary point of one of the spherical triangles in that figure (the vertices are particularly interesting) and consider the quantities G_x, $\mathscr{A}_x$, $\bar{F}_x$, K_x, Δ_x^+, and Π_x, the last of which requires consideration of the x-external roots. It will be seen that $\bar{F}_x$ is a convex cone such that $\bar{F}_x \subseteqq K_x$, and that $\bar{F}_x$ may have more than n $(= 3)$ walls, each of which is perpendicular to some x-external root. It is not as easy to envision the convex cone K_x (dual to $\bar{F}_x$), to see that the extreme rays of K_x are determined by the x-external roots, nor to see that each edge of $C(x)$ is parallel to some x-external root. (Also see Example 4.4.)

Return now to the general case and consider the definitions (4.10) and (4.12).

The set K_x is a closed, pointed, convex polyhedral cone in R^n, while $\bar{F}_x$ is a closed, positively homogeneous set. It is not immediately apparent that Δ_x^+ is nonempty. By Corollary 4.1

$$C(x) \subseteqq \bigcap \{x - K_{g\tau} \mid g \in G_x\} \equiv x - K_x \,, \tag{4.13}$$

which suggests a relationship between the edges of $C(x)$ emanating from x and the extreme rays of K_x. Recall that for $t \in T$, $K_t = \mathrm{co}\,(\Delta_t^+)$, where $\mathrm{co}\,(A)$ denotes the closed convex cone generated by a set $A \subset R^n$. Therefore

$$K_x = \bigcap \{\mathrm{co}\,(\Delta_{g\tau}^+) \mid g \in G_x\} \supseteqq \mathrm{co}\,(\bigcap \{\Delta_{g\tau}^+ \mid g \in G_x\}) \equiv \mathrm{co}\,(\Delta_x^+) \,. \tag{4.14}$$

Next,

$$\begin{aligned} \mathrm{dual}\,(K_x) &\equiv \mathrm{dual}\,(\bigcap \{K_{g\tau} \mid g \in G_x\}) \supseteqq \bigcup \{\mathrm{dual}\,(K_{g\tau}) \mid g \in G_x\} \\ &\equiv \bigcup \{\bar{F}_{g\tau} \mid g \in G_x\} \equiv \bar{F}_x \,, \end{aligned} \tag{4.15}$$

where $\mathrm{dual}\,(K)$ denotes the dual cone $\{z \mid z'y \geqq 0 \text{ for all } y \in K\}$ of K in R^n. Since $\mathrm{dual}\,(\mathrm{dual}\,(K)) = K$ (Rockafellar (1970), Theorem 14.1), (4.14) and (4.15) yield

$$\mathrm{co}\,(\Delta_x^+) \subseteqq K_x \subseteqq \mathrm{dual}\,(\bar{F}_x) \,. \tag{4.16}$$

The next result shows that all inclusions in (4.14)—(4.16) are in fact equalities.

PROPOSITION 4.1. *Let* $\{\rho_i \mid q + 1 \leqq i \leqq n^*\}$ *be the x-external roots in* Π_τ $(\tau = \tau(x))$.

(i) $\bar{F}_x = \mathrm{dual}\,(\mathrm{co}\,(\Pi_x)) \equiv \{z \in R^n \mid z'(g\rho_i) \geqq 0 \textit{ for all } q + 1 \leqq i \leqq n^*, g \in G_x\}$. *Hence*, $\mathrm{dual}\,(\bar{F}_x) = \mathrm{co}\,(\Pi_x)$.

(ii) $\Pi_x \subseteqq \Delta_x^+$.

(iii) $\mathrm{co}\,(\Pi_x) = \mathrm{co}\,(\Delta_x^+) = K_x = \mathrm{dual}\,(\bar{F}_x)$.

(iv) *The extreme rays of* K_x *are exactly determined by the members of* Π_x, i.e., *the x-external roots.*

PROOF.

(i) If $z \in \bar{F}_x$ then $z = g_1 u$ for some $g_1 \in G_x$ and some $u \in \bar{F}_\tau$. Fix ρ_i $(q + 1 \leqq i \leqq n^*)$ and $g \in G_x$, and let $g_2 = g'g_1 \in G_x$. Since $g_2 u$ and $g_2 x \equiv x$ both lie in $g_2\bar{F}_\tau = \bar{F}_{g_2\tau}$, and since $\rho_i'x > 0$, Lemma 3.1 implies that $0 \leqq \rho_i'(g_2 u) = (g\rho_i)'z$. Thus $\bar{F}_x \subseteqq \mathrm{dual}\,(\mathrm{co}\,(\Pi_x))$.

Conversely, suppose $z \notin \bar{F}_x$. By Lemma 3.1 there exists at least one point w in the open line segment $[x, z]^0$ such that $\{w\} = [x, z]^0 \cap H_r$ for some (not necessarily unique) $r \in \Delta$. Since Δ is finite there are only finitely many such points w in $[x, z]^0$; let w_0 denote the point closest to x. Therefore $[x, z]^0 \cap H_{r_0} = \{w_0\}$ for at least one root $r_0 \in \Delta$, which implies that $r_0'x \neq 0$, while $r_0'w_0 = 0$. Furthermore, for every $r \in \Delta$, $[x, w_0]^0 \cap H_r$ does not consist of a single point. Thus by Lemma 3.1, there exists some fundamental region F_t such that $x, w_0 \in \bar{F}_t$. This implies that $F_t \in \mathscr{A}_x$, so $F_t = F_{g\tau}$ for some $g \in G_x$. Without loss of generality assume that $r_0 \in \Delta_t^+ \equiv g\Delta_\tau^+$ (otherwise replace r_0 by $-r_0$) so there exist real numbers $c_i \geqq 0$ such that

$$r_0 = \textstyle\sum_{i=1}^{q} c_i g\rho_i + \sum_{i=q+1}^{n^*} c_i g\rho_i$$

(see (4.11)). Since $(g\rho_i)'x = 0$ for $1 \leqq i \leqq q$ and $(g\rho_i)'x > 0$ for $q + 1 \leqq i \leqq n^*$, while $r_0'x \neq 0$, it follows that $c_j > 0$ for at least one $j \geqq q + 1$. Also, $w_0 \in \bar{F}_t \equiv g\bar{F}_\tau$ implies that $(g\rho_i)'w_0 \geqq 0$ for $1 \leqq i \leqq n^*$. Since $r_0'w_0 = 0$, it follows that $(g\rho_j)'w_0 = 0$. Finally, since $(g\rho_j)'x > 0$ and $w_0 \in [x, z]^0$, we conclude that $(g\rho_j)'z < 0$, so $z \notin \operatorname{dual}(\operatorname{co}(\Pi_x))$. This proves (i).

Next fix $F_t \in \mathscr{A}_x$, so $\bar{F}_t \subseteqq \bar{F}_x$. By (i) and (4.16)

$$\Pi_x \subset \operatorname{co}(\Pi_x) = \operatorname{dual}(\bar{F}_x) \subseteqq \operatorname{dual}(\bar{F}_t) = K_t = \operatorname{co}(\Delta_t^+) .$$

Hence each root in Π_x must be t-positive, so $\Pi_x \subseteqq \Delta_t^+$. Thus $\Pi_x \subseteqq \bigcap \{\Delta_t^+ \mid F_t \in \mathscr{A}_x\} \equiv \Delta_x^+$, proving (ii). By (i) and (ii), $\operatorname{dual}(\bar{F}_x) = \operatorname{co}(\Pi_x) \subseteqq \operatorname{co}(\Delta_x^+)$. Combining this with (4.16) yields (iii). Next, since $K_x = \operatorname{co}(\Pi_x)$, each extreme ray of K_x must lie along some root in Π_x. Conversely, to show that each x-external root $g\rho_i$ ($g \in G_x$, $q + 1 \leqq i \leqq n^*$) determines an extreme ray of K_x, note that $g\rho_i \in \Pi_x \subseteqq K_x \subseteqq K_{g\tau}$. Since $g\rho_i$ determines an extreme ray of $K_{g\tau}$, it must also determine an extreme ray of K_x. Thus (iv) is established.

REMARK 4.5. If $x \notin M_G$ then there exists at least one x-external root, so Π_x and Δ_x^+ are nonempty, $K_x \neq \{0\}$, and $\bar{F}_x \neq R^n$.

Proposition 4.1 (i) implies that $\bar{F}_x$, defined as the union of convex polyhedral cones $\bar{F}_{g\tau}$, $g \in G_x$, is itself a closed convex polyhedral cone whose interior contains x and whose walls are formed from the collection of all x-external walls. If we define $F_x = \operatorname{interior}(\bar{F}_x)$, it is easy to see that for any $g \in G$, $F_{gx} = gF_x$ (indeed, Fact 3.3 remains true when $t\ (\in T)$ is replaced by $x\ (\notin T)$); in particular, $gF_x = F_x$ when $g \in G_x$. Let $\mathscr{L}$ be the collection of all left cosets of G_x in G, so that $\mathscr{L} = \{g_\alpha G_x \mid 1 \leqq \alpha \leqq d\}$ for some (nonunique) $g_1, \cdots, g_d \in G$, where $d = |G|/|G_x|$. The G-orbit of x is $\{g_\alpha x \mid 1 \leqq \alpha \leqq d\}$. Using Lemmas 4.3 and 4.4 it can be shown that the system $\mathscr{A}^x \equiv \{gF_x \mid g \in G\}$ has exactly d members, i.e., $\mathscr{A}^x = \{F_{g_\alpha x} \mid 1 \leqq \alpha \leqq d\}$, and that this system has properties analogous to (3.3)—(3.5) (with t replaced by x) concerning the system $\mathscr{A}$ (in (3.4), $g \neq I$ must be replaced by $g \notin G_x$). However, $\mathscr{A}^x$ is not necessarily a system of fundamental regions for some finite reflection group acting on R^n, since F_x may have more than n walls. (By contrast, it will be shown in Theorem 4.3 that G_x is a reflection group, and $\mathscr{A}_x$ is closely related to its system of fundamental regions.)

We may now apply Corollary 4.1 and the notation we have established (see (4.13) and the preceding paragraph) to extend (4.7) to the case $x \notin T$:

$$(4.17) \qquad C(x) = \bigcap \{g_\alpha(x - K_x) \mid 1 \leqq \alpha \leqq d\} = \bigcap \{g_\alpha x - K_{g_\alpha x} \mid 1 \leqq \alpha \leqq d\} .$$

The points $\{g_\alpha x \mid 1 \leqq \alpha \leqq d\}$ are distinct, and the edges of $C(x)$ emanating from $g_\alpha x$ lie along the extreme rays of $g_\alpha(x - K_x)$ (see Theorem 4.2), so $C(x)$ and $g_\alpha(x - K_x)$ coincide in a neighborhood of $g_\alpha x$. Since the extreme rays of $g_\alpha(x - K_x)$ are determined by the roots $g_\alpha \Pi_x$, this yields the desired characterization of the edges of $C(x)$ in the case $x \notin T$.

Let $\nu = |\Pi_x|$, the number of x-external roots, so $\nu \leqq (n^* - q)|G_x|$. The proof of the following theorem is identical to that of Theorem 4.1 (simply replace g by g_α and n^* by ν in that proof), hence is omitted.

THEOREM 4.2 (Structure of $C(x)$ when $x \notin T$). *The convex polytope $C(x)$ has exactly $d \equiv |G|/|G_x|$ extreme points (vertices), the points $\{g_\alpha x \mid 1 \leqq \alpha \leqq d\}$ of the G-orbit of x. Exactly $\nu \equiv |\Pi_x|$ edges emanate from the vertex x, namely, the line segments $[x, S_{r_i}x]$ where $\{r_i \mid 1 \leqq i \leqq \nu\} = \Pi_x (\equiv \{g\rho_i \mid q+1 \leqq i \leqq n^*, g \in G_x\})$ are the x-external roots. Similarly, the ν edges emanating from the vertex $g_\alpha x$ are exactly the g_α-images $[g_\alpha x, g_\alpha S_{r_i} x] \equiv [g_\alpha x, S_{g_\alpha r_i} g_\alpha x]$ of these segments. The edge $[g_\alpha x, g_\alpha S_{r_i} x]$ is parallel to $g_\alpha(x - S_{r_i}x) = 2(r_i'x)g_\alpha r_i$, a nonzero vector in the direction of the root $gr_i \in \Delta$. The polytope $C(x)$ has exactly $\frac{1}{2}d\nu$ $(\leqq \frac{1}{2}|G|(n^* - q))$ edges.*

REMARK 4.6. A partial answer to the question raised at the end of Section 3, concerning dimension $(C(x))$ when $x \notin T$, can be obtained from (4.13), Proposition 4.1 (iv), and Theorem 4.2: dimension $(C(x))$ is the dimension of the subspace spanned by the collection Π_x of all x-external roots.

Theorems 4.1 and 4.2 lead immediately to our generalization of the basic path lemma of Hardy, Littlewood, and Polya (1952, page 47) from the permutation group to a general finite reflection group G.

LEMMA 4.5 (Second Path Lemma). *Suppose $y \in C(x) \equiv C_G(x)$, $y \neq x$. There exists a sequence (not necessarily unique) of points $z_0, z_1, \cdots, z_m$ such that $z_0 = y$, $z_m = x$, and*

$$z_{j-1} = [\lambda_j I + (1-\lambda_j)S_{\bar r_j}]z_j\,, \qquad 1 \leqq j \leqq m\,,$$

where $\bar r_j \in \Delta$ and $0 \leqq \lambda_j < 1$.

PROOF. We appeal to several basic results concerning convex polytopes (cf. Theorem 18.2 of Rockafellar (1970) and Theorem 5, page 33, of Grünbaum (1967)). The closed convex polytope $C(x)$ is the union of its faces. Each face C of $C(x)$ is itself a closed convex polytope, and each face of C is a face of $C(x)$. There exists a unique face C_0 of $C(x)$ such that $z_0 \equiv y$ lies in the relative interior of C_0. Let $d_0 =$ dimension (C_0), so $0 \leqq d_0 \leqq n^*$.

If $d_0 = 0$, i.e., if z_0 is an extreme point of $C(x)$, proceed to the next paragraph. If $d_0 \geqq 1$, select any edge E_0 of C_0. Since E_0 is also an edge of $C(x)$, Theorems 4.1 and 4.2 imply that E_0 is parallel to some root $\bar r_1 \in \Delta$. Assume that $\bar r_1' z_0 \geqq 0$ (otherwise, replace $\bar r_1$ by $-\bar r_1 \in \Delta$). Define $\delta_1 = \sup\{\delta \mid z_0 + \delta \bar r_1 \in C_0\} \geqq 0$ and $z_1 = z_0 + \delta_1 \bar r_1$. Then $z_0 = [\lambda_1 I + (1-\lambda_1)S_{\bar r_1}]z_1$, where

$$\lambda_1 \equiv 1 - \frac{\delta_1}{2(\bar r_1' z_0 + \delta_1)}$$

satisfies $\frac{1}{2} \leqq \lambda_1 \leqq 1$. Since z_0 is in the relative interior of C_0 and since the line $\{z_0 + \delta \bar r_1 \mid -\infty < \delta < \infty\}$ is contained in the affine hull of C_0 (i.e., the d_0-dimensional flat containing C_0), it follows that $\delta_1 > 0$, so in fact $\frac{1}{2} \leqq \lambda_1 < 1$ and $z_1 \neq z_0$. Since z_1 must lie in the relative boundary of C_0, there exists a unique face of C_0, say C_1, such that z_1 is in the relative interior of C_1. Let $d_1 =$ dimension (C_1), so $0 \leqq d_1 < d_0 \leqq n^*$. If $d_1 = 0$, proceed to the next paragraph. If $d_1 \geqq 1$, since C_1 is itself a face of $C(x)$, the preceding argument may be repeated to show the existence of a root $\tilde{r}_2 \in \Delta$ and a point z_2 in the relative boundary of C_1 such that $z_1 = [\lambda_2 I + (1 - \lambda_2)S_{\tilde{r}_2}]z_2$ for some $0 \leqq \lambda_2 < 1$.

Proceeding by induction one obtains a finite nested sequence $C_0 \supset C_1 \supset \cdots \supset C_\mu$ of distinct faces of $C(x)$ and a sequence of distinct points $y \equiv z_0, z_1, \cdots, z_\mu$, such that (i) dimension $(C_\mu) = 0$; (ii) $0 \leqq \mu \leqq n^*$; (iii) for $0 \leqq j \leqq \mu$, z_j is a relative interior point of C_j; (iv) for $1 \leqq j \leqq \mu$, z_j is a relative boundary point of C_{j-1}; (v) for $1 \leqq j \leqq \mu$, $z_{j-1} = [\lambda_j I + (1 - \lambda_j)S_{\tilde{r}_j}]z_j$ for some $\tilde{r}_j \in \Delta$ and $\frac{1}{2} \leqq \lambda_1 < 1$. Therefore z_μ is an extreme point of $C(x)$, so z_μ must lie in the G-orbit of x, i.e., $z_\mu = gx$ for some $g \in G$. However, since G is a reflection group, there exists a finite sequence $\tilde{r}_{\mu+1}, \cdots, \tilde{r}_m$ of roots in Δ such that $g = S_{\tilde{r}_{\mu+1}} \cdots S_{\tilde{r}_m}$. Define $z_{\mu+1}, \cdots, z_m$ by $z_j = S_{\tilde{r}_j} z_{j-1}$, $\mu + 1 \leqq j \leqq m$, so that $z_{j-1} = S_{\tilde{r}_j} z_j$, and take $\lambda_{\mu+1} = \cdots = \lambda_m = 0$. Then $z_m = x$, so the proof is complete.

The geometric construction used in the proof of Lemma 4.1 to represent an arbitrary $g \in G$ explicitly as a product of reflections in G appears to be a powerful tool for the study of reflection groups. As an example, this construction yields an easy proof of the fact that, for each $x \in R^n$, G_x is itself a finite reflection group (see Theorem 4.3). This result extends a theorem of Witt (see Theorem 5.4.1 of B–G).

Referring to the notation and terminology introduced in the paragraph containing (4.10) and (4.11), define

$$\hat{\Pi}_\tau = \{\rho_1, \cdots, \rho_q\},$$

the set of all x-internal roots in Π_τ, and define

$$\hat{\Delta}_x = \{g\rho_i \mid 1 \leqq i \leqq q, g \in G_x\},$$

the set of all x-internal roots; $\hat{\Delta}_x$ does not depend on the choice of $\tau \equiv \tau(x)$. The root system of G_x is given by

$$\Delta_{G_x} \equiv \{r \in \mathscr{S}_{n-1} \mid S_r \in G_x\} = \{r \in \Delta_G \mid r'x = 0\} = \Delta_G \cap G_x .$$

Define

$$\hat{\Delta}_\tau^+ = \{r \in \Delta_{G_x} \mid r'\tau > 0\} = \{\text{all } \tau\text{-positive roots in } \Delta_{G_x}\},$$
$$\hat{K}_\tau = \text{co}(\hat{\Delta}_\tau^+),$$
$$\hat{F}_\tau = \{z \in R^n \mid \rho_i' z > 0, 1 \leqq i \leqq q\},$$
$$\hat{\mathscr{A}}_x = \{g\hat{F}_\tau \mid g \in G_x\}.$$

Clearly $\hat{\Delta}_x \subseteqq \Delta_{G_x}$, $\hat{\Pi}_\tau = \hat{\Delta}_\tau^+ \cap \Pi_\tau$, $\hat{K}_\tau \subseteqq K_\tau$, and $\hat{F}_\tau \supseteqq F_\tau$. The walls of the convex polyhedral cone $\hat{F}_\tau$ are determined by the x-internal walls of F_τ. In Theorem 4.3 it is shown that $\hat{\Delta}_x = \Delta_{G_x}$, $\hat{\mathscr{A}}_x = \mathscr{A}_{G_x}$, and that the quantities $\hat{\Pi}_\tau$, $\hat{\Delta}_\tau^+$, $\hat{K}_\tau$, $\hat{F}_\tau$, and q bear the same relation to G_x as Π_τ, Δ_τ^+, K_τ, F_τ, and n^* bear to G.

THEOREM 4.3. (i) *G_x (acting on R^n) is the finite reflection group generated by* $\{S_r \mid r \in \hat{\Delta}_x\}$.

(ii) $\hat{K}_\tau = \text{co}(\hat{\Pi}_\tau)$ *and* $\hat{\Pi}_\tau$ *exactly determine the extreme rays of* $\hat{K}_\tau$, *so* $\hat{\Pi}_\tau$ *is the* τ*-base for the root system* Δ_{G_x}. *Hence* $S_{\rho_1}, \cdots, S_{\rho_q}$ *is a set of fundamental reflections for* G_x, *and dimension* $(M_{G_x}^\perp) = q$. *Furthermore,* $\hat{\Delta}_x = \Delta_{G_x}$. *The open convex polyhedral cone* $\hat{F}_\tau$ *is a fundamental region for* G_x *in* R^n, *its closure is the dual cone of* $\hat{K}_\tau$ *in* R^n, *and* $\hat{\mathscr{A}}_x$ *is the system of fundamental regions of* G_x; $\hat{\mathscr{A}}_x$ *is in* 1—1 *correspondence with* $\mathscr{A}_x$.

PROOF. (i) Let $\hat{G}$ denote the reflection group generated by $\{S_r \mid r \in \hat{\Delta}_x\}$. Since $\hat{\Delta}_x \subseteqq \Delta_{G_x}$, $\hat{G} \subseteqq G_x$. Conversely, suppose $g \in G_x$. Let $u = \tau$ and $F = F_\tau$, and let $L \equiv [u, z]$ be the line segment constructed in Claim 1 of the proof of Lemma 4.1. Since $u \in F$ and $z \in gF$, both u and z lie in the interior of $\bar{F}_x$ (see (4.10)). Since $\bar{F}_x$ is a convex cone (Proposition 4.1 (i)), L must lie entirely in the interior of $\bar{F}_x$, and hence cannot intersect any of the x-external walls. Therefore the roots $\tilde{\rho}_1, \cdots, \tilde{\rho}_k$ in the paragraph following the proof of Claim 1 must be x-internal roots. Since $g = S_{\tilde{\rho}_k} S_{\tilde{\rho}_{k-1}} \cdots S_{\tilde{\rho}_1} \in \hat{G}$, we conclude that $\hat{G} = G_x$.

(ii) Since $\hat{\Pi}_\tau \subseteqq \hat{\Delta}_\tau^+$, $\text{co}(\hat{\Pi}_\tau) \subseteqq \hat{K}_\tau$. Conversely, if $r \in \hat{\Delta}_\tau^+ \subseteqq \Delta_\tau^+ \subset K_\tau \equiv \text{co}(\Pi_\tau)$, r must be of the form

$$r = \textstyle\sum_{i=1}^{q} c_i \rho_i + \sum_{i=q+1}^{n^*} c_i \rho_i$$

where each $c_i \geqq 0$. However, $r'x = 0$, so $c_{q+1} = \cdots = c_{n^*} = 0$, and $r \in \text{co}(\hat{\Pi}_\tau)$. Hence $\text{co}(\hat{\Pi}_\tau) = \text{co}(\hat{\Delta}_\tau^+) \equiv \hat{K}_\tau$, and each extreme ray of $\hat{K}_\tau$ must be determined by some $\rho_i \in \hat{\Pi}_\tau$. Since $\hat{\Pi}_\tau \subseteqq \Pi_\tau$, and Π_t exactly determines the extreme rays of K_τ, each $\rho_i \in \hat{\Pi}_\tau$ must determine an extreme ray of $\hat{K}_\tau$. Thus $\hat{\Pi}_\tau$ is a τ-base for Δ_{G_x}, and $S_{\rho_1}, \cdots, S_{\rho_q}$ is a set of fundamental reflections for G_x. That $\hat{\Delta}_x = \Delta_{G_x}$ follows from this fact and the fact that every reflection in a finite reflection group (G_x) is conjugate to one of the fundamental reflections (B–G, Theorem 4.2.5). The rest of (ii) follows from the definitions.

EXAMPLE 4.4. In order to illustrate the concepts and results introduced for the case $x \notin T$ in the second half of this section (i.e., after Theorem 4.1), again consider $G = \mathscr{P}_n$. The notation of Examples 3.1 and 4.1 is continued here. Suppose $x \equiv (x_1, \cdots, x_n) \notin T_{\mathscr{P}_n}$. Without essential loss of generality we assume $x \in \bar{F}_t$ where F_t is given by (3.10), so $x_1 \geqq \cdots \geqq x_n$ with at least one equality. Suppose, for the sake of concreteness, that

$$x_1 > \cdots > x_\alpha = \cdots = x_\beta > \cdots > x_\gamma = \cdots = x_\delta > \cdots > x_n ,$$

where $\alpha, \beta, \gamma, \delta$ are integers such that $1 < \alpha < \beta < \gamma < \delta < n$; the cases where x has only one, or more than two, "runs" of equal components are treated similarly. It is clear that G_x consists of those permutations in $\mathscr{P}_n$ which permute the αth, $\cdots$, βth components amongst themselves and the γth, $\cdots$, δth components amongst themselves, leaving the rest fixed, so

$$G_x \cong \mathscr{P}_{\beta-\alpha+1} \times \mathscr{P}_{\delta-\gamma+1}$$

and

$$|G_x| = (\beta - \alpha + 1)!\,(\delta - \gamma + 1)!\,.$$

We may choose $\tau \equiv \tau(x) = (\tau_1, \cdots, \tau_n)$ such that $\tau_1 > \cdots > \tau_n$ so Π_τ is given by (3.8). Then $q = (\beta - \alpha) + (\delta - \gamma)$ and

$$\hat{\Pi}_\tau = \{e_i - e_{i+1} \mid \alpha \leqq i \leqq \beta - 1 \text{ or } \gamma \leqq i \leqq \delta - 1\}$$
$$\hat{\Delta}_x = \Delta_{G_x} = \{e_i - e_j \mid \alpha \leqq i \neq j \leqq \beta \text{ or } \gamma \leqq i \neq j \leqq \delta\};$$

the latter is the set of all x-internal roots. Clearly G_x is the reflection group generated by $\{S_r \mid r \in \hat{\Delta}_x\}$ and also by $\{S_r \mid r \in \hat{\Pi}_\tau\}$, as claimed in Theorem 4.3. The set of x-external roots in Π_τ is

$$\{e_i - e_{i+1} \mid 1 \leqq i \leqq \alpha - 1 \text{ or } \beta \leqq i \leqq \gamma - 1 \text{ or } \delta \leqq i \leqq n - 1\}$$

and the set Π_x of all x-external roots, defined as the set of all G_x-transforms of the x-external roots in Π_τ, is given by

$$\begin{aligned}\Pi_x = \{&e_i - e_{i+1} \mid 1 \leqq i \leqq \alpha - 1 \text{ or } \beta \leqq i \leqq \gamma - 1 \text{ or } \delta \leqq i \leqq n - 1\}\\ &\cup \{e_{\alpha-1} - e_j \mid \alpha + 1 \leqq j \leqq \beta\}\\ &\cup \{e_i - e_{\beta+1} \mid \alpha \leqq i \leqq \beta - 1\}\\ &\cup \{e_{\gamma-1} - e_j \mid \gamma + 1 \leqq j \leqq \delta\}\\ &\cup \{e_i - e_{\delta+1} \mid \gamma \leqq i \leqq \delta - 1\}.\end{aligned}$$

Therefore

$$\begin{aligned}\nu \equiv |\Pi_x| &= (\alpha - 1) + (\gamma - \beta) + (n - \delta) + 2(\beta - \alpha) + 2(\delta - \gamma)\\ &= (\beta - \alpha) + (\delta - \gamma) + (n - 1).\end{aligned}$$

Since Π_x spans $M_{\mathscr{P}_n}^\perp$, Remark 4.6 implies that dimension $(C(x)) = n^* \equiv n - 1$ (also see the final paragraph in Section 3). The number of distinct points in the $\mathscr{P}_n$-orbit of x is

$$d \equiv \frac{|G|}{|G_x|} = \frac{n!}{(\beta - \alpha + 1)!\,(\delta - \gamma + 1)!},$$

which is also the number of vertices of $C(x)$, while the number of distinct edges of $C(x)$ is $\frac{1}{2}d\nu$. The convex polyhedral cones K_x and $\bar{F}_x$ are readily obtained from Π_x via the relations $K_x = \operatorname{co}(\Pi_x)$ and $\bar{F}_x = \operatorname{dual}(K_x)$.

5. The convolution theorem and differential characterizations of G-monotonicity. Throughout this section $G \subseteqq O(n)$ is a reflection group acting on R^n.

THEOREM 5.1. *$\mathscr{F}_G$ is closed under convolution.*

PROOF. By Propositions 2.5 and 3.2, it suffices to establish this theorem for irreducible reflection groups G_i. If G_i is infinite and irreducible then, by Theorem 3.1 and Remark 3.3, $\mathscr{F}_{G_i}$ contains only decreasing radial functions and hence is closed under convolution. If G_i is finite then Corollary 2.1 and Lemma 4.5 imply that $\mathscr{F}_{G_i}$ is closed under convolution. The proof is complete.

REMARK 5.1. By means of a continuity argument, the closure of $\mathscr{F}_{G_i}$ under convolution when G_i is a finite reflection group can be obtained directly from

Lemma 4.2, rather than Lemma 4.5. Let $h = f_1 * f_2$, where $f_1, f_2 \in \mathscr{F}_{G_i}$, and first assume that f_1 is bounded. By Theorem 4.3c of Williamson (1962), h is continuous on R^n. If $x, y \in T \equiv T_{G_i}$ are such that $y \in C_{G_i}(x)$, select $g \in G_i$ such that $\tilde{y} \equiv gy \in F_x$. Since $\tilde{y} \in C_{G_i}(x)$, Lemma 4.2 ((i) $\Rightarrow$ (iii)) and Corollary 2.1 together imply that $h(y) \equiv h(\tilde{y}) \geqq h(x)$. The continuity of h now implies that $h(y) \geqq h(x)$ whenever $y \in C_{G_i}(x)$, even if $x, y \notin T$. Next, if f_1 is unbounded, for $M > 0$ let $f_M = \min\{f_1, M\}$. Since $M \geqq f_M \in \mathscr{F}_{G_i}$, the preceding argument implies that $f_M * f_2$ is G_i-monotone. Now let $M \to \infty$ and apply the monotone convergence theorem to conclude that $f_1 * f_2$ is G_i-monotone, hence is in $\mathscr{F}_{G_i}$.

COROLLARY 5.1. *If $f_1 \geqq 0$ and $f_2 \geqq 0$ are G-monotone, then $h \equiv f_1 * f_2$ is G-monotone.*

PROOF. For $M > 0$ let $B_M = \{x \in R^n : ||x|| \leqq M\}$. Since

$$f_i^{(M)} \equiv \min\{f_i, M\} \cdot I_{B_M} \tag{5.1}$$

is in $\mathscr{F}_G$, $i = 1, 2$, Theorem 5.1 implies that $h_M \equiv f_1^{(M)} * f_2^{(M)} \in \mathscr{F}_G$. Now let $M \to \infty$ and apply the monotone convergence theorem.

COROLLARY 5.2. *Suppose that $f_1 \geqq 0$ and f_2 are G-monotone functions on R^n such that their convolution $h(x) \equiv (f_1 * f_2)(x)$ exists (possibly $\pm\infty$, but well-defined) for each $x \in R^n$. Then h is G-monotone.*

PROOF. First write $f_2 = f_2^+ + f_2^-$, where $f_2^+ = \max\{f_2, 0\}$ and $f_2^- = \min\{f_2, 0\}$, so f_2^+ and f_2^- are G-monotone and

$$h = (f_1 * f_2^+) + (f_1 * f_2^-) \equiv h^+ + h^- .$$

By Corollary 5.1, h^+ is G-monotone. By the monotone convergence theorem,

$$h^- \equiv f_1 * f_2^- = \lim_{M\to\infty} (f_1^{(M)} * f_{2,M}^-) ,$$

where $f_1^{(M)}$ is defined in (5.1) and $f_{2,M}^- = \max\{f_2^-, -M\}$. Since $f_1^{(M)}$ and $f_{2,M}^- + M$ are nonnegative and G-monotone, Corollary 5.1 implies that

$$f_1^{(M)} * f_{2,M}^- = f_1^{(M)} * (f_{2,M}^- + M) + (\text{constant})$$

is G-monotone. Hence h^- is G-monotone, and the proof is complete.

Next we present several differential characterizations of G-monotonicity for reflection groups, first considering the finite case.

THEOREM 5.2. *Suppose G is a finite reflection group, and let f be a G-invariant function possessing a differential on R^n. Then a necessary and sufficient condition that f be G-monotone is that*

$$(r'z)(r'\nabla f(z)) \leqq 0 \qquad \textit{for all} \quad r \in \Delta_G , \quad z \in R^n . \tag{5.2}$$

PROOF. Necessity follows from Proposition 2.2, while sufficiency follows from Proposition 2.3 and Lemma 4.5.

When $G = \mathscr{P}_n$, (5.2) is exactly the Schur–Ostrowski condition (1.1) (see (3.6) of Example 3.1). By applying (2.6), a condition easier to verify than (5.2) can be obtained.

COROLLARY 5.3. *Let G, f be as in Theorem 5.2, and let $\Delta_0 \subseteqq \Delta_G$ be such that $G\Delta_0 = \Delta_G$. A necessary and sufficient condition that f be G-monotone is that*

$$(5.3) \qquad (r'z)(r'\nabla f(z)) \leqq 0 \qquad \textit{for all} \quad r \in \Delta_0\,, \quad z \in R^n\,.$$

By Theorem 4.2.5 of B–G, Δ_0 satisfies $G\Delta_0 = \Delta_G$ iff the set of reflections $\{S_r \mid r \in G\Delta_0\}$ generates G. This holds if Δ_0 itself determines a set of generating reflections for G (e.g., take $\Delta_0 = \Pi_t$ for some $t \in T$), but it also may hold for other, perhaps smaller, sets Δ_0. For example, when $G = \mathscr{P}_n$ we may take $\Delta_0 = \{e_1 - e_2\}$ (see Example 3.1) and obtain the following necessary and sufficient condition for $\mathscr{P}_n$-monotonicity for a smooth $\mathscr{P}_n$-invariant function f, which simplifies the condition (1.1):

$$(5.4) \qquad (z_1 - z_2)\left(\frac{\partial f}{\partial z_1} - \frac{\partial f}{\partial z_2}\right) \leqq 0 \qquad \text{for all} \quad z \in R^n\,.$$

By means of Lemma 4.2, a characterization of G-monotonicity in terms of ∇f can be obtained when it is not assumed that ∇f exists everywhere.

THEOREM 5.3. *Let G be a finite reflection group and f a G-invariant function on R^n. Suppose that f is continuous on R^n and possesses a differential on T_G. Let F_t be a fundamental region for G and let $\Pi_t = \{r_1, \cdots, r_{n^*}\}$. A necessary and sufficient condition that f be G-monotone is that*

$$(5.5) \qquad (r_i'z)(r_i'\nabla f(z)) \leqq 0 \qquad \textit{for all} \quad 1 \leqq i \leqq n^*\,, \quad z \in F_t\,.$$

PROOF. Necessity follows from Proposition 2.2. Proposition 2.3 and Lemma 4.2(iii) imply that $f(y) \geqq f(x)$ for all $x, y \in F_t$ such that $y \in C(x)$, and a continuity argument extends this to $x, y \in \bar{F}_t$. The G-invariance of f now implies that $f(y) \geqq f(x)$ whenever $y \in C(x)$.

When $G = \mathscr{P}_n$ and F_t is given by (3.10), condition (5.5) takes the form

$$(5.6) \qquad \frac{\partial f}{\partial z_1} \leqq \frac{\partial f}{\partial z_2} \leqq \cdots \leqq \frac{\partial f}{\partial z_n} \qquad \text{whenever} \quad z_1 > z_2 > \cdots > z_n\,,$$

another well-known differential characterization of $\mathscr{P}_n$-monotonicity. For example, the function

$$f(z_1, \cdots, z_n) = -\textstyle\sum_{1 \leq i < j < n} |z_i - z_j|^\alpha\,, \quad \alpha \geqq 1\,,$$

is $\mathscr{P}_n$-invariant, continuous on R^n, possesses a differential on $T_{\mathscr{P}_n}$, and satisfies (5.6).

REMARK 5.2. Theorem 5.2 and Corollary 5.3 are also valid if it is only assumed that f is G-invariant, continuous on R^n, and possesses a differential on T_G, and if (5.2) and (5.3) are only assumed to hold for all $z \in T_G$; this is a consequence of Theorem 5.3. Also, Theorems 5.2 and 5.3 and Corollary 5.3 remain true, with only minor modifications, for functions f defined not on all of R^n but only on a convex G-invariant subset of R^n having nonempty interior (e.g., a ball centered at 0).

We conclude with a differential characterization of G-monotonicity for an arbitrary (not necessarily finite) reflection group G.

THEOREM 5.4. *Let f be a G-invariant function possessing a differential on R^n. Then* (5.2) *and* (5.3) *are necessary and sufficient conditions that f be G-monotone.*

PROOF. To show sufficiency, by Propositions 2.4 and 3.2 it suffices to consider irreducible reflection groups. If G is infinite and irreducible, sufficiency follows from Theorem 3.1, Remark 3.3, and the fact that for any decreasing radial function f, $\nabla f(z)$ is proportional to $-z$.

Acknowledgment. The authors wish to thank Paul Sally for helpful discussions concerning the orthogonal group and its subgroups.

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