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LIMIT THEOREMS WITHOUT MOMENT HYPOTHESES FOR SUMS OF INDEPENDENT RANDOM VARIABLES[1]

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Let $\{S_n\}$ be the partial sums of a sequence of independent random variables and let $\{a_n\}$ be a nondecreasing, divergent real sequence. Necessary and sufficient conditions for $\limsup_{n\to\infty} S_n/a_n < \infty$ a.s. are given under mild conditions on $\{S_n\}$; these conditions do not involve the existence of any moments. These results are employed to widen the scope of the law of the iterated logarithm.

1. Introduction. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X_1, X_2, \cdots$ denote a sequence of independent random variables (rv). Assume that $\{a_n\}$ is a real sequence satisfying

$$(1) \qquad 0 < a_1 \leqslant a_2 \leqslant \cdots \uparrow \infty.$$

For each $n \geqslant 1$, let $S_n = \sum_{j=1}^n X_j$. This article will investigate circumstances under which

$$(2) \qquad \limsup_{n\to\infty} S_n/a_n < \infty \text{ almost surely (a.s.).}$$

Conditions will also be given under which the (a.s. constant) value of the lim sup in (2) can be ascertained. Necessary and sufficient conditions for (2) will be developed in Section 3 under rather modest hypotheses which do not include any assumptions about the existence of moments of any order.

This problem has been tackled by a number of authors. The case in which the X_n's have a common distribution has been considered from various perspectives by Chow and Robbins (1961), Feller (1968), Heyde (1969), Baum, Katz and Stratton (1971), Kesten (1972), Klass (1976, 1977), and Klass and Teicher (1977). Some results of Petrov (1972, 1974), Kruglov (1974), Teicher (1975), Martikainen and Petrov (1977), Volodin and Nagaev (1977) and Volodin (1977) deal with random variables which need not be identically distributed.

The importance of the main results of this paper, especially Theorem 3, is three-fold: (a) conditions tantamount to (2) are established with no assumptions concerning either existence of moments or properties of $|S_n|$ or $|S_n - S_m|$; (b) the results apply to all sequences obeying (1), no matter how rapid or pathological the rate of divergence; and (c) information about the precise value in (2) is obtained.

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It will prove convenient to define $S_0 = a_0 = n_0 = 0$. $\to_P$ will denote convergence in probability, and "i.o." will abbreviate "infinitely often". The infimum of the empty set is ∞.

2. Admissible sequences and some useful lemmas. Let $\{a_n\}$ satisfy (1). An increasing sequence $\{n_k\}$ of positive integers will be called *admissible* if and only if (iff)

$$(3) \qquad a_{n_{k+1}} \geqslant c a_{n_k} \quad \text{for some} \quad c > 1 \quad \text{and all} \quad k \geqslant 1.$$

Clearly, admissible sequences exist for every $c > 1$, and every subsequence of an admissible sequence is itself admissible.

Let $\{a_n\}$ and $\{S_n\}$ be as given in Section 1. The first lemma is based on a technique used by Hartman (1941).

LEMMA 1. *Suppose* $\{n_k\}$ *satisfies* (3) *and that the series*

$$(4) \qquad \Sigma_{k=1}^{\infty} P[S_{n_k} > r a_{n_k}]$$

diverges for some r. *Let* $M \geqslant 1$ *be any number. Then one may assume that* (3) *holds with* $c > M$ *and also that* (4) *diverges.*

PROOF. For each $k \geqslant 1$, define $j_{1k} = n_{2k-1}$ and $j_{2k} = n_{2k}$. From the hypotheses, $\Sigma_{k=1}^{\infty} P[S_{j_{ik}} > r a_{j_{ik}}] = \infty$ for either $i = 1$ or $i = 2$, or both. But (3) implies that $a_{j_{i,k+1}} \geqslant c^2 a_{j_{ik}}$ in either case. Hence, if p is an integer satisfying $c^{2^p} > M$, then p repetitions of this procedure will yield an admissible sequence satisfying (3) with c^{2^p} in lieu of c. $\square$

The next lemma is a reformulation of one given by Petrov (1972).

LEMMA 2. *If, for some* $b \geqslant 0$ *and* $\lambda > 1$, *and for all* $m \leqslant j \leqslant n$, $P[S_n - S_j > -b] \geqslant \lambda^{-1}$, *then* $P[\max_{m \leqslant j \leqslant n} S_j > x] \leqslant \lambda P[S_n > x - b]$ *for all real* x.

3. The main results. Throughout this section, $X_1, X_2, \cdots$ is a sequence of independent rv, $S_n = \Sigma_{j=1}^{n} X_j$ and $\{a_n\}$ satisfies (1). Define the constants

$$\alpha \equiv \inf\{r | \Sigma_{k=1}^{\infty} P[S_{n_k} > r a_{n_k}] < \infty \text{ for all admissible } \{n_k\}\};$$

$$\beta \equiv \inf\{r | \Sigma_{k=1}^{\infty} P[S_{n_k} - S_{n_{k-1}} > r a_{n_k}] < \infty \text{ for all admissible } \{n_k\}\};$$

and

$$\delta \equiv \limsup_{n\to\infty} S_n / a_n.$$

Note that any of the above could be $\pm\infty$.

THEOREM 1. (i) *If* $\beta < \infty$ *and if, for some* $\varepsilon > 0$,

$$(5) \qquad \Sigma_{k=1}^{\infty} P[S_{n_{k-1}} > \varepsilon a_{n_k}] < \infty \text{ for all admissible } \{n_k\},$$

then $\alpha \leqslant \beta + \varepsilon$.

(ii) *If* $\alpha < \infty$ *then* $\alpha \leqslant \beta$.

(iii) *If* $\alpha < \infty$ *and* $\liminf_{n\to\infty} S_n/a_n > -\infty$ a.s. *then* $\limsup_{n\to\infty} S_n/a_n \geqslant \alpha$ a.s.

PROOF. Note, first, that

$$P[S_{n_k} > ra_{n_k}] \leqslant P[S_{n_k} - S_{n_{k-1}} > (r - \varepsilon)a_{n_k}] + P[S_{n_{k-1}} > \varepsilon a_{n_k}]. \tag{6}$$

If $\beta < \infty$ and (5) holds, then choose r so large that $r - \varepsilon > \beta$. It is clear from (6) that $\alpha \leqslant \beta + \varepsilon$.

If $\alpha = -\infty$ then both (ii) and (iii) are obvious, so assume $|\alpha| < \infty$. Let $r < \alpha$, $\varepsilon > 0$ and $M \geqslant \max(1, \alpha\varepsilon^{-1})$. Then Lemma 1 ensures the existence of an admissible sequence $\{n_k\}$ such that (3) holds for some $c > M$ and the series (4) diverges. But $\Sigma_{k=1}^{\infty} P[S_{n_{k-1}} > \varepsilon a_{n_k}] \leqslant \Sigma_{k=1}^{\infty} P[S_{n_{k-1}} > \varepsilon c a_{n_{k-1}}] < \infty$ since $\varepsilon c > \alpha$. Therefore, in view of (6), $\Sigma_{k=1}^{\infty} P[S_{n_k} - S_{n_{k-1}} > (r - \varepsilon)a_{n_k}] = \infty$. Hence $r - \varepsilon \leqslant \beta$ for all $\varepsilon > 0$, proving (ii).

Now let $\eta \equiv \liminf_{n\to\infty} S_n/a_n > -\infty$. If $\eta = +\infty$ then (iii) is trivial, so assume $|\eta| < \infty$. Let $M = \max(1, \alpha\varepsilon^{-1}, |\eta|\varepsilon^{-1})$. By the argument above there is an admissible sequence $\{n_k\}$ obeying (3) with $c > M$ and $\Sigma_{k=1}^{\infty} P[S_{n_k} - S_{n_{k-1}} > (r - \varepsilon)a_{n_k}] = \infty$. By the Borel 0–1 law,

$$\begin{aligned} r - \varepsilon &\leqslant \limsup_{k\to\infty}(S_{n_k} - S_{n_{k-1}})/a_{n_k} \\ &\leqslant \limsup_{k\to\infty} S_{n_k}/a_{n_k} - \liminf_{k\to\infty} S_{n_{k-1}}/a_{n_k} \text{ a.s.} \end{aligned}$$

So $\limsup_{k\to\infty} S_{n_k}/a_{n_k} \geqslant r - \varepsilon + \min(0, \eta)/c \geqslant r - 2\varepsilon$ a.s. for each $r < \alpha$ and $\varepsilon > 0$, so (iii) holds. □

REMARKS. 1. Suppose $P[X_1 = \pm k^k] = A/k^2$ for $k \geqslant 1$, where $A = (2\Sigma_{k=1}^{\infty} k^{-2})^{-1}$ and $X_n \equiv 0$ for $n > 1$; then $S_n = X_1$ for $n \geqslant 1$. Let $a_n = n^{n-1}$. Then, for any $r > 0$ and $n \geqslant r$, $P[S_n > ra_n] \geqslant P[S_n > n^n] = P[X_1 > n^n] = A\Sigma_{k=n+1}^{\infty} k^{-2} > A(n+1)^{-1}$ so $\Sigma_{n=1}^{\infty} P[S_n > ra_n] = \infty$ for each $r > 0$. But the sequence $\{1, 2, 3, \cdots\}$ is admissible so $\alpha = \infty$. But $S_n - S_m = 0$ whenever $n \geqslant m > 1$ so $\beta = 0$. Therefore, part (ii) above may fail when $\alpha = \infty$ and, moreover, the assumption (5) cannot be dropped in part (i).

2. Here is an example to show that equality need not hold in the conclusion of (ii). For $n \geqslant 1$, define $X_{2n-1} = -n$, $X_{2n} = n$, $a_n = n$. Then $S_n = 0$ when n is even and $S_n = -(n+1)/2$ when n is odd; it is easy to see that $\alpha = 0$. Now consider the admissible sequence $\{n_k\}$ defined by $n_k = 2^k$ or $2^k - 1$ accordingly as k is even or odd. But then, if k is odd, $S_{n_{k+1}} - S_{n_k} = X_{2^k} = 2^{k-1} = a_{n_{k+1}}/4$, whence it is evident that $\beta \geqslant \frac{1}{4}$. (In fact, $\beta = \frac{1}{2}$ in this example.)

3. It is noteworthy in connection with (iii) that, as Klass and Teicher (1977) have shown, it is possible to have $\delta = 1$ and yet $\liminf_{n\to\infty} S_n/a_n = -\infty$ a.s., even when $X_1, X_2, \cdots$ are identically distributed with $E|X_1| < \infty$.

The following theorem uncovers additional relationships among α, β and δ under certain modest hypotheses.

THEOREM 2. (i) *Suppose*

$$\lim_{m\to\infty} \mathit{inf}_{m\leqslant n} P[S_n - S_m > -xa_n] > 0 \textit{ for all } x > 0. \tag{7}$$

Then $\beta \geqslant 0$ *and* $\limsup_{n\to\infty} S_n/a_n \leqslant \min(\alpha, \beta)$ a.s.

(ii) *Suppose*

$$\liminf_{m\to\infty} P[S_m > -xa_m] > 0 \text{ for all } x > 0. \tag{8}$$

Then $\alpha \geqslant 0$ *and* $\limsup_{n\to\infty} S_n/a_n \geqslant \beta$ a.s. *Indeed, if* $r > \delta$ *then the series*

$$\Sigma_{k=1}^{\infty} P[S_{j_k} - S_{j_{k-1}} > ra_{j_k}] \tag{9}$$

converges for every integral sequence $0 = j_0 < j_1 < \cdots$.

PROOF. If (7) holds then $\liminf_{k\to\infty} P[S_{n_k} - S_{n_{k-1}} > -xa_{n_k}] > 0$ for every integral sequence $\{n_k\}$ and every $x > 0$, so $\beta \geqslant 0$.

To continue the proof of (i), let $\varepsilon > 0$ and $c > 1$. Define, for $k \geqslant 1$, $n_k = \min\{n | a_n \geqslant ca_{n_{k-1}}\}$, $m_k = n_k - 1$ and $U_k = \max_{n_{k-1} \leqslant n < n_k} S_n$. Note that $a_{m_k} < ca_{n_{k-1}}$. Moreover, by virtue of (7), positive integers $\lambda = \lambda(\varepsilon)$ and $N = N(\varepsilon)$ exist such that

$$\lambda P[S_n - S_m > -\varepsilon a_n] \geqslant 1 \text{ for all } n \geqslant m \geqslant N. \tag{10}$$

There is no loss of generality in assuming that $\min(\alpha, \beta) < \infty$. Suppose first that $\alpha < \infty$. In view of Theorem 1 (ii) it will suffice to show that $\delta \leqslant \alpha$ a.s. If $\alpha \geqslant 0$, let $c = (\alpha + 3\varepsilon)/(\alpha + 2\varepsilon)$. If $\alpha < 0$, let $c = 2$ and assume without loss of generality that $\varepsilon < -\alpha/2$. Then, by (10) and Lemma 2,

$$P[U_k > (\alpha + 2\varepsilon)a_{m_k}] \leqslant \lambda P[S_{m_k} > (\alpha + \varepsilon)a_{m_k}] \equiv \lambda P_k.$$

But, for $k \geqslant 2$, $a_{m_{k+1}} \geqslant a_{n_k} \geqslant ca_{n_{k-1}} \geqslant ca_{m_{k-1}}$, so both $\{m_{2j-1}\}$ and $\{m_{2j}\}$ are admissible. Consequently,

$$\Sigma_{k=1}^{\infty} P[U_k > (\alpha + 2\varepsilon)a_{m_k}] \leqslant \lambda\Sigma_{j=1}^{\infty} P_{2j-1} + \lambda\Sigma_{j=1}^{\infty} P_{2j} < \infty$$

by definition of α; hence $P[U_k > (\alpha + 2\varepsilon)a_{m_k} \text{ i.o.}] = 0$ by the Borel-Cantelli lemma. But then, if $\alpha \geqslant 0$,

$$\begin{aligned} 0 &\leqslant P[S_n/a_n > (\alpha + 3\varepsilon) \text{ i.o.}] \\ &\leqslant P[U_k > c(\alpha + 2\varepsilon)a_{n_{k-1}} \text{ i.o.}] \leqslant P[U_k > (\alpha + 2\varepsilon)a_{m_k} \text{ i.o.}] = 0, \end{aligned}$$

whereas $0 \leqslant P[S_n/a_n > (\alpha + 2\varepsilon) \text{ i.o.}] \leqslant P[U_k > (\alpha + 2\varepsilon)a_{m_k} \text{ i.o.}] = 0$ if $\alpha < 0$. So $\delta \leqslant \alpha + 3\varepsilon$ for every $\varepsilon > 0$ in either case, so $\delta \leqslant \alpha$ as required.

It remains to show that $\delta \leqslant \beta$ when $0 \leqslant \beta < \infty$ but $\alpha = \infty$; there is clearly no harm in assuming $\delta > 0$. A technique similar to one of Martikainen and Petrov (1977) will be employed.

First, define $c = (\beta + 3\varepsilon)/(\beta + 2\varepsilon)$. Then choose an integer $s \geqslant 2$ so large that $(1 + \varepsilon)(1 - c^{-s+1}) > 1$. Fix any $1 \leqslant j \leqslant s$. For $k \geqslant 1$, define $V_k = U_{ks+j}$, $W_k = V_k - S_{m_{(k-1)s+j}}$, $p_k = m_{ks+j}$, and $Q_k = P[W_k > (\beta + 2\varepsilon)a_{p_k}]$. Note that $\{p_k\}$ is admissible; indeed, $a_{p_k} \geqslant a_{n_{ks+j-1}} \geqslant c^{s-1}a_{n_{(k-1)s+j}} \geqslant c^{s-1}a_{p_{k-1}}$ so that

$$a_{p_i} \leqslant c^{-(s-1)(k-i)}a_{p_k} \text{ for each } i \leqslant k. \tag{11}$$

Furthermore, (10) and Lemma 2 imply that $Q_k \leqslant \lambda P[S_{p_k} - S_{p_{k-1}} > (\beta + \varepsilon)a_{p_k}]$. By the definition of β, $\Sigma_{k=1}^{\infty} Q_k < \infty$, so $P[W_k > (\beta + 2\varepsilon)a_{p_k} \text{ i.o.}] = 0$. But this fact, together with (11) and a method of Kruglov (1974), leads to

$$\begin{aligned}\limsup_{k\to\infty}\Sigma_{i=1}^{k}W_i/a_{p_k} &\leqslant \limsup_{k\to\infty}\Sigma_{i=1}^{k}(\beta+2\varepsilon)a_{p_i}/a_{p_k}\\ &\leqslant (\beta+2\varepsilon)\limsup_{k\to\infty}\Sigma_{i=1}^{k}c^{-(s-1)(k-i)}\\ &= (\beta+2\varepsilon)(1-c^{-s+1})^{-1} < (\beta+2\varepsilon)(1+\varepsilon) \text{ a.s.}\end{aligned}$$

But $\Sigma_{i=1}^{k}W_i = \Sigma_{i=1}^{k}V_i - \Sigma_{i=1}^{k}S_{p_{i-1}} = V_k + \Sigma_{i=1}^{k-1}(V_i - S_{p_i}) - S_{p_0} \geqslant V_k - S_{m_j}$, since $V_i \geqslant S_{p_i}$. Therefore, $\limsup_{k\to\infty}V_k/a_{p_k} \leqslant (\beta+2\varepsilon)(1+\varepsilon)$ a.s. Consequently,

$$\begin{aligned}A_j &\equiv \limsup_{k\to\infty}\max_{n_{ks+j-1}\leqslant n<n_{ks+j}}(S_n/a_n)\\ &\leqslant \limsup_{k\to\infty}a_{n_{ks+j-1}}^{-1}V_k\\ &\leqslant \limsup_{k\to\infty}cV_k/a_{p_k} \leqslant c(\beta+2\varepsilon)(1+\varepsilon) = (\beta+3\varepsilon)(1+\varepsilon) \text{ a.s.}\end{aligned}$$

Therefore $\delta \leqslant \max_{1\leqslant j\leqslant s}A_j \leqslant (\beta+3\varepsilon)(1+\varepsilon)$ for every $\varepsilon > 0$, so $\delta \leqslant \beta$, concluding the proof of (i).

Now prove (ii). Let $\varepsilon > 0$. By (8), positive numbers $M = M(\varepsilon)$ and $v = v(\varepsilon)$ exist such that

$$P[S_m > -\varepsilon a_m] \geqslant v \text{ for every } m \geqslant M.$$

Clearly $\alpha \geqslant 0$, because the summands in the series (4) are each bounded below by $v(-r) > 0$ when $r < 0$.

Now suppose the series (9) diverges for some r and some sequence $\{j_k\}$; assume $j_1 > M$ without loss of generality. For $k \geqslant 1$, define the events

$$A_k = [S_{j_k} - S_{j_{k-1}} > ra_{j_k}] \text{ and } B_k = [S_{j_{k-1}} > -\varepsilon a_{j_k}].$$

Note that $P(B_k) \geqslant P[S_{j_{k-1}} > -\varepsilon a_{j_{k-1}}] \geqslant v$. Therefore, by Lemma 1 of Baum, Katz and Stratton (1971), $P[A_kB_k \text{ i.o.}] \geqslant v > 0$. Hence, $P[S_{j_k} > (r-\varepsilon)a_{j_k} \text{ i.o.}] \geqslant v$. By Kolmogorov's 0–1 law, $\limsup_{k\to\infty}S_{j_k}/a_{j_k} \geqslant r - \varepsilon$ a.s., which, in turn, implies $\delta \geqslant r - \varepsilon$ for every $\varepsilon > 0$, so $\delta \geqslant r$ as required.

Finally, note that $\delta \geqslant \beta$ trivially if $\beta = -\infty$. If $\beta > -\infty$ then the preceding work shows that $r \leqslant \delta$ whenever $r < \beta$, so $\delta \geqslant \beta$. □

REMARKS. 1. The conditions (7) and (8) are not very stringent. They both hold, for example, if $S_n/a_n \to_P 0$ or if each X_n is symmetrically distributed. More generally, (7) and (8) hold if either

(12) $$\liminf_{n\to\infty}\min_{0\leqslant m\leqslant n}P[S_n - S_m > -xa_n] > 0 \text{ for every } x > 0,$$

or

(13) $$\liminf_{n\to\infty}P[|S_n| \leqslant xa_n] \geqslant p \text{ for some } p > \tfrac{1}{2} \text{ and every } x > 0.$$

Condition (12) was introduced by Martikainen and Petrov (1977), whereas (13) was used by Petrov (1972); an argument on page 1128 of the latter article shows that (13) implies (7) and (8).

2. The proof of (i) in the case where $\alpha < \infty$ is clearly motivated by Kolmogorov's renowned method as modified by Hartman (1941). Part (ii) is a

slight generalization of results of Teicher (1975), Volodin (1977), and Martikainen and Petrov (1977).

The major theorem of this paper follows. It brings Theorems 1 and 2 together to produce necessary and sufficient conditions for (2) and yields the value of δ.

THEOREM 3. *Suppose* (7) *and* (8) *both hold. Then*
(i) $0 \leqslant \limsup_{n\to\infty} S_n/a_n = \beta \leqslant \alpha$ a.s.;
(ii) $\limsup_{n\to\infty} S_n/a_n = \beta = \alpha$ *if* $\alpha < \infty$;
(iii) $\alpha < \infty$ iff $\beta < \infty$ *and* (5) *holds for some* $\varepsilon > 0$.

PROOF. (i) and (ii) are immediate from Theorem 2 and Theorem 1 (ii). Furthermore, if $\alpha < \infty$ then $\beta < \infty$ by part (ii) and (5) holds for any $\varepsilon \geqslant \alpha$; therefore, (iii) is true in light of Theorem 1 (i). □

REMARKS. 1. Theorem 3 (i) is analogous to Theorem 4 of Martikainen and Petrov (1977). The latter result, instead of using admissible sequences, deals with sequences of the form $\{i_k\}$, where i_k is the largest integer such that $a_{i_k} \leqslant c^k$, $c > 1$. Such sequences need not be admissible, but are admissible in the case where $a_n/a_{n-1} \to 1$.

2. Petrov (1974) and Martikainen and Petrov (1977) have shown that it may be sufficient in some cases to consider only subscript sequences of the form $i_k = [c^k]$, the integral part of c^k, where $c > 1$, rather than admissible sequences. Their results apply in the special case where $\lim_{k\to\infty} a_{i_k}/a_{i_{k-1}}$ exists and is finite for every $c > 1$; this condition was introduced by Baum, Katz and Stratton (1971). While such a sequence $\{i_k\}$ need not be admissible, it must satisfy $a_n/a_{n-1} \to 1$.

The next theorem generalizes a result of Petrov (1972) and, therefore, some of Petrov's earlier results.

THEOREM 4. *Suppose* (7) *and* (8) *hold. Define the constant*

$$\gamma \equiv \inf\left\{ r \middle| \Sigma_{k=1}^{\infty} (\log a_{n_k})^{-r^2} < \infty \textit{ for all admissible } \{n_k\} \right\}.$$

Let ξ, ν *and* η *satisfy* $\xi \geqslant \gamma$, $\nu \leqslant \gamma$ *and* $\eta > 0$.
(i) *Suppose that, for all* $x \in (\xi, \xi + \eta)$,

$$\limsup_{n\to\infty} (\log a_n)^{(1-\varepsilon)x^2} P[S_n > x a_n] < \infty \textit{ for all } \varepsilon > 0. \tag{14}$$

Then $\limsup_{n\to\infty} S_n/a_n \leqslant \xi$ a.s. *and* $\gamma \leqslant 1$.
(ii) *If, in addition,*

$$\liminf_{n\to\infty} (\log a_n)^{(1+\varepsilon)x^2} P[S_n > x a_n] > 0 \tag{15}$$

for all $\varepsilon > 0$ *and all* $x \in (\nu - \eta, \nu)$, *then* $\limsup_{n\to\infty} S_n/a_n \geqslant \nu$ a.s.

PROOF. Note that $\gamma \leqslant 1$, since $a_{n_k} \geqslant c^{k-1} a_{n_1}$ for $k \geqslant 1$, by (3).

By dint of Theorem 3 (ii), it will suffice in (i) to show that $\alpha < \infty$ and $\alpha \leqslant \xi$. To this end, let $\xi < x < \xi + \eta$. Pick $\varepsilon > 0$ so that $(1 - \varepsilon)x^2 > \gamma^2$. By (14), $P[S_{n_k} > x a_{n_k}] = O((\log a_{n_k})^{-(1-\varepsilon)x^2})$ for every admissible sequence $\{n_k\}$. Consequently, $\alpha < \infty$ and $x \geqslant \alpha$ in light of the definition of γ, so $\xi \geqslant \alpha$ as required.

For part (ii), let $\nu - \eta < x < \nu$ and select $\varepsilon > 0$ so that $y \equiv (1 + \varepsilon)x^2 < \gamma^2$. Then there is an admissible sequence $\{n_k\}$ such that $\Sigma_{k=1}^{\infty}(\log a_{n_k})^{-y} = \infty$. With the aid of (15), this implies $\Sigma_{k=1}^{\infty} P[S_{n_k} > xa_{n_k}] = \infty$. So $x < \alpha$; i.e., $\nu \leqslant \alpha < \infty$. Part (ii) now follows from Theorem 3 (ii). □

REMARK. Theorem 1 of Petrov (1972) assumes that (13) holds, $\xi = \nu = 1$, $a_n/a_{n-1} \to 1$ (so $\gamma = 1$) and, in part (ii), that (14) holds with $|S_n|$ in place of S_n.

An admissible sequence $\{n_k\}$ which also obeys

$$\limsup_{k\to\infty} a_{n_k}/a_{n_{k-1}} < \infty \tag{16}$$

will be called a *controlled admissible sequence*. It is readily verified that such sequences exist iff

$$\limsup_{n\to\infty} a_n/a_{n-1} < \infty. \tag{17}$$

Define the (possibly infinite) constants α' and β' by using the phrase "controlled admissible" instead of "admissible" in the definitions of α and β respectively. Under (17), it can be shown directly from the definitions that $\alpha' = \alpha$ and $\beta' = \beta$. So, if (17) holds, Theorem 3 remains valid with α' and β' in place of α and β respectively throughout. Therefore, the value of δ can be determined by focussing only on controlled sequences when (17) holds. The next result will prove that the finitude of δ is assured if $\Sigma_{k=1}^{\infty} P[S_{n_k} - S_{n_{k-1}} > ra_{n_k}]$ converges for some r and *some* controlled admissible sequence $\{n_k\}$.

THEOREM 5. *Suppose* (7), (8) *and* (17) *all hold. Then* $\limsup_{n\to\infty} S_n/a_n < \infty$ a.s. iff

$$\Sigma_{k=1}^{\infty} P\left[S_{n_k} - S_{n_{k-1}} > r_0 a_{n_k}\right] < \infty \tag{18}$$

for some number r_0 *and some admissible sequence satisfying* (16).

PROOF. The "only if" part is clear from Theorem 3(i). For the converse, suppose (18) holds for some r_0 and *some* controlled admissible sequence $\{n_k\}$. Choose $L > \limsup_{k\to\infty} a_{n_k}/a_{n_{k-1}}$. Notice that $r_0 \geqslant 0$ because of (7) and (18). Also, from (3), $a_{n_k} \geqslant c^{k-j} a_{n_j}$ for every $j \leqslant k$. So, using (18) and the Borel-Cantelli lemma,

$$\begin{aligned}\limsup_{k\to\infty} S_{n_{k-1}}/a_{n_k} &= \limsup_{k\to\infty} \Sigma_{j=1}^{k-1}\left(S_{n_j} - S_{n_{j-1}}\right)/a_{n_k} \\ &\leqslant \limsup_{k\to\infty} r_0 \Sigma_{j=1}^{k-1} c^{j-k} = r_0/(c-1) \text{ a.s.}\end{aligned}$$

For $k \geqslant 1$, define $M_k = \max_{n_{k-1} < n \leqslant n_k} S_n$. Then, using (10) and Lemma 2,

$$R_k \equiv P\left[M_k - S_{n_{k-1}} > (r_0 + \varepsilon)a_{n_k}\right] \leqslant \lambda(\varepsilon) P\left[S_{n_k} - S_{n_{k-1}} > r_0 a_{n_k}\right]$$

for every $\varepsilon > 0$, so $\Sigma_{k=1}^{\infty} R_k < \infty$.

Note that $\delta \geqslant 0$ by Theorem 3(i); there is no loss of generality in assuming that $\delta > 0$. Consequently,

$$\begin{aligned}\delta &\leqslant \limsup_{k\to\infty} M_k/a_{n_{k-1}} \leqslant L \limsup_{k\to\infty} M_k/a_{n_k} \\ &\leqslant L \limsup_{k\to\infty}\left(M_k - S_{n_{k-1}}\right)/a_{n_k} + L \limsup_{k\to\infty} S_{n_{k-1}}/a_{n_k} \\ &\leqslant L(r_0 + \varepsilon) + Lr_0/(c-1) < \infty, \text{ as desired.}\end{aligned}$$

□

It will now be shown that Theorem 4 of Kruglov (1974) and the a.s. stability criterion of Loève ((1963), page 252) are consequences of Theorem 5.

COROLLARY 1. *Suppose* (17) *holds. Let* $\{n_k\}$ *be an admissible sequence satisfying* (16). *Then*

$$\limsup_{n\to\infty}|S_n - \mathrm{med}(S_n)|/a_n < \infty \text{ a.s.} \tag{19}$$

iff, *for some* $\varepsilon > 0$,

$$\Sigma_{k=1}^{\infty}P\left[|S_{n_k} - S_{n_{k-1}} - \mathrm{med}(S_{n_k} - S_{n_{k-1}})| > \varepsilon a_{n_k}\right] < \infty. \tag{20}$$

("med(X)" *denotes a median of* X).

Moreover,

$$(S_n - \mathrm{med}(S_n))/a_n \to 0 \text{ a.s. } as\ n \to \infty \tag{21}$$

iff (20) *holds for every* $\varepsilon > 0$.

PROOF. In view of the symmetrization lemma (see Loève (1963), page 247), the summands X_n may be assumed to have symmetric distributions. Without loss of generality, therefore, (7) and (8) hold, and the medians in (19), (20) and (21) may be ignored. The corollary now follows readily from Theorem 5. $\square$

REMARKS. 1. New conditions equivalent to (21), without the assumption (17), have been presented by Volodin and Nagaev (1977). Their result involves sequences $\{i_k\}$ similar to those of Martikainen and Petrov (1977), rather than admissible sequences.

2. A result of Steiger (1973) also follows from Theorem 5.

4. On the law of the iterated logarithm. Throughout this section, let $\{X_n\}$ be independent rv with $E(X_n) = 0$ and $E(X_n^2) < \infty$ for $n \geqslant 1$. Define $S_n = \Sigma_{j=1}^{n}X_j$, $s_n^2 = E(S_n^2)$ and $t_n = (2 \log \log s_n^2)^{\frac{1}{2}}$. Assume $s_n \to \infty$ as $n \to \infty$.

Suppose, first, that $a_n \equiv s_n t_n$, $n \geqslant 1$. Then $S_n/a_n \to_p 0$ by Chebychev's inequality, so (7) and (8) hold. Moreover, a sequence $\{n_k\}$ is admissible (cf. (3)) provided

$$\liminf_{k\to\infty} s_{n_k}t_{n_k}/\left(s_{n_{k-1}}t_{n_{k-1}}\right) > 1. \tag{22}$$

But, in most proofs of laws of the iterated logarithm, it is preferable to deal with sequences $\{n_k\}$ for which

$$\liminf_{k\to\infty} s_{n_k}/s_{n_{k-1}} > 1. \tag{23}$$

Fortunately, it isn't hard to show that (23) holds for a sequence $\{n_k\}$ if and only if (22) holds too. The following result is now immediate from Theorem 3.

THEOREM 6. (i) $\limsup_{n\to\infty} S_n/(s_n t_n) = \beta^*$ a.s., *where* β^* *is the infimum of those numbers* r *for which* $\Sigma_{k=1}^{\infty}P[S_{n_k} - S_{n_{k-1}} > rs_{n_k}t_{n_k}] < \infty$ *for every sequence* $\{n_k\}$ *obeying* (23).

(ii) *If a number* r *exists such that* (A) $\Sigma_{k=1}^{\infty}P[S_{n_k} > rs_{n_k}t_{n_k}] < \infty$ *for every sequence* $\{n_k\}$ *obeying* (23), *then* $\limsup_{n\to\infty} S_n/(s_n t_n) = \alpha^*$ a.s., *where* α^* *is the infimum of the set of numbers* r *for which condition* (A) *holds.*

REMARK. Weiss (1959) concocted a family of counterexamples for which $\limsup_{n\to\infty} S_n/(s_n t_n) > 1$ a.s., even though the X_n's are symmetric and bounded. It follows from Weiss' Lemma 4 that $\alpha^* \leqslant 2$ in her examples. In view of Theorem 6, then, $1 < \alpha^* \leqslant 2$.

In the final theorem, a strengthened version of Theorem 1 of Tomkins (1972) will be deduced from Theorem 4.

THEOREM 7. (i) *Let* $\{B_n\}$ *and* $\{c_n\}$ *be positive real sequences satisfying* $B_n \uparrow \infty$, $c_n^2 \log\log B_n \to 0$ *as* $n \to \infty$, *for all large n*, $E\exp\{tB_n^{-1}S_n\} \leqslant \exp\{(t^2/2)(1 + |t|c_n/2)$ *whenever* $|t|c_n \leqslant 1$. *Then*

$$\limsup_{n\to\infty} S_n/\left(2B_n^2 \log\log B_n^2\right)^{\frac{1}{2}} \leqslant \gamma^* \leqslant 1 \text{ a.s.} \tag{24}$$

where γ^* *is the infimum of the set of numbers r satisfying* $\sum_{k=1}^{\infty}(\log B_{n_k})^{-r^2} < \infty$ *for every integral sequence* $\{n_k\}$ *obeying*

$$\liminf_{k\to\infty} B_{n_k}/B_{n_{k-1}} > 1. \tag{25}$$

(ii) *If, in addition,* $E\exp\{tB_n^{-1}S_n\} \geqslant \exp\{(t^2/2)(1 - tc_n)\}$ *for all large n and all* $0 \leqslant t \leqslant c_n^{-1}$, *then*

$$\limsup_{n\to\infty} S_n/\left(2B_n^2 \log\log B_n^2\right)^{\frac{1}{2}} = \gamma^* \text{ a.s.} \tag{26}$$

PROOF. Let $a_n^2 = 2B_n^2 \log\log B_n^2$ and $u_n = a_n/B_n$. Let $x > 0$ and $\varepsilon > 0$. If n is so large that $xu_nc_n < 2\varepsilon$ then

$$P[|S_n| > xa_n] \leqslant 2\exp\{-(x^2u_n^2/2)(1 - xu_nc_n/2)\} \leqslant 2(\log B_n^2)^{-(1-\varepsilon)x^2}$$

by Lemma 1(i) of Tomkins (1972); therefore, (14) is true and $S_n/a_n \to_p 0$, so (7) and (8) hold. Note, too, that (3) holds for some $c > 1$ iff (25) holds, by the same argument as that preceding Theorem 6. Since $2\log a_n/\log B_n^2 \to 1$, it is clear that $\gamma^* = \gamma$, where γ is the constant defined in Theorem 4. Consequently part (i) follows from Theorem 4(i), using $\xi = \gamma$.

Similarly, the assumptions in (ii), with the aid of Lemma 1(ii) of Tomkins (1972), ensure the applicability of Theorem 4 (ii) with $\nu = \gamma$. □

REMARKS. 1. Following Hartman (1941), it is easy to check that $\gamma^* = 1$ if $\limsup_{n\to\infty} B_n/B_{n-1} < \infty$. Hence, Theorem 1 of Tomkins (1972), which contains the additional hypothesis $B_n/B_{n-1} \to 1$, is contained in Theorem 7.

2. A similar application of Theorem 4 allows one to eliminate the hypothesis $EX_n^2 = o(s_n^2)$ in Theorem 1 of Tomkins (1971), provided the conclusions (24) and (26) respectively, with $B_n = s_n$, replace (1) and (2) of the paper just cited.

3. Since the hypotheses of Theorem 7 are satisfied for any sequence of normally distributed rv, Theorem 7 extends Hartman's theorem (1941) to a wide class of rv. The same comments apply to the result alluded to in the preceding remark.

4. Marcinkiewicz and Zygmund (1937) discovered, in the Bernoulli case, an interesting phenomenon, one which, as Feller (1969a) noted, may contradict one's intuition: roughly speaking, the faster $\{s_n\}$ diverges, the smaller the value of

lim $\sup_{n\to\infty} S_n/(s_n t_n)$ becomes (cf. also Feller (1969b)). In view of Remark 3 above, this behavior is common to a wide class of rv which obey the central limit theorem. For example, if the assumptions in Theorem 7 hold with $B_n^2 = s_n^2 = n^n$ then $\alpha = \beta = \gamma = \gamma^* = 1$, but $\gamma^* = 0$ if $s_n^2 = n^{n^n}$.

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