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THIN BUT UNAVOIDABLE SETS[1]

MAURY BRAMSON AND ARNOLD NEIDHARDT

Courant Institute of Mathematical Sciences

This paper uses randomization to prove the existence of a subset of the circle which is small in the sense that its Lebesque measure is small, and which is large in the sense that a certain stochastic process almost surely visits the set infinitely often.

We are interested here in the following curious problem. Let $\mathbf{X}$ and $\mathbf{Y}$ be independent random variables uniformly distributed on the circle [0, 1]. (1 is identified with 0.) For any $y \in [0, 1]$, let y_n be the nth binary approximation to y (the first n digits of y to the base 2), i.e., $y_n = 2^{-n}\lfloor 2^n y \rfloor$; $\mathbf{X} + \mathbf{Y}_n$ is then a random walk which converges to $\mathbf{X} + \mathbf{Y}$. Also, consider open sets $U \subset [0, 1]$; we denote by $|U|$ the measure of U. We now ask the following: how is $|U|$ related to

$$\mathscr{K}(U) = P(\exists n: \mathbf{X} + \mathbf{Y}_n \in U),$$

the probability of ever hitting U? Specifically, as $|U| \to 0$, need $\mathscr{K}(U) \to 0$ also, and if so, at what rate?

A reasonable initial response is to check the case where U is an interval of (say) length λ. A little calculation shows that $\mathscr{K}(U)$ will be of order of magnitude $-\lambda \log_2 \lambda$; after the first $\log_2 \lambda$ decimal places, $\mathbf{X} + \mathbf{Y}_n$ will most likely either be permanently inside or outside U. If one amends U by allowing the set to consist of m disjoint intervals (say, of equal length), difficulties arise when one attempts to improve appreciably on this bound. For if the intervals are regularly spaced, then the bound will not be improved, whereas if the intervals are irregularly spaced, computation becomes untenable (even for small m). Direct construction of U therefore does not seem a fruitful approach.

We now answer the question: we show that $\mathscr{K}(U) = 1$ is still possible no matter how small $|U|$ is. We proceed indirectly by randomizing U, and by showing the result to be true for most U.

THEOREM. *For any positive length* $\lambda > 0$, *there is an open set* $U \subset [0, 1]$ *with* $|U| \le \lambda$ *and* $\mathscr{K}(U) = 1$.

PROOF. Consider any $\lambda, \delta > 0$. We will first show the existence of an open set V with $|V| \le \lambda$ and $\mathscr{K}(V) \ge 1 - \delta$. V will be the union of L open intervals of length λ/L for some large integer L. Let $c_1, \cdots, c_L$ be the centers of these intervals. Instead of carefully selecting these points, we will show that an "average" choice of these points does sufficiently well, provided L is large enough.

Precisely, once L has been chosen, let $\mathbf{X}, \mathbf{Y}, \mathbf{c}_1, \cdots, \mathbf{c}_L$ be independent random variables uniformly distributed on [0, 1]. Let

$$\mathbf{V} = \cup_{k=1}^{L} \left(\mathbf{c}_k - \frac{\lambda}{2L}, \mathbf{c}_k + \frac{\lambda}{2L}\right).$$

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Note that $\mathbf{V}$ is a random set. Observe that for one of the points $\mathbf{X} + \mathbf{Y}_n$ to land in $\mathbf{V}$, it suffices to have one of the points $\mathbf{c}_k$ land in the union $\mathbf{W}$ of intervals of length λ/L centered at the points $\mathbf{X} + \mathbf{Y}_n$. In order to estimate the measure of this union, let

$$\begin{aligned} N_m(y) &= \text{the number of distinct points in } \{y_0, y_1, \cdots, y_m\} \\ &= 1 + \text{the number of 1's occurring in the first } m \text{ places of the binary expansion of } y. \end{aligned}$$

Then, the union

$$W = W(x, y), = \cup_{n=0}^{\infty} \left(x + y_n - \frac{\lambda}{2L}, x + y_n + \frac{\lambda}{2L} \right)$$

contains at least $N_m(y)$ disjoint intervals of length λ/L, provided $2^{-m} \geq \lambda/L$. Letting $m(L)$ be the largest integer for which $2^{-m} \geq \lambda/L$, we have

$$\begin{aligned} P(\forall n, \mathbf{X} + \mathbf{Y}_n \notin \mathbf{V}) &= P(\mathbf{c}_k \notin \mathbf{W} \text{ for } 1 \leq k \leq L) \\ &= \int_0^1 dx \int_0^1 dy (1 - |W|)^L \\ &\leq \int_0^1 dx \int_0^1 dy (1 - \lambda N_{m(L)}(y)/L)^L \\ &\leq \int_0^1 dx \int_0^1 dy \exp(-\lambda N_{m(L)}(y)). \end{aligned}$$

Now, if we let $L \to \infty$, then $m(L) \to \infty$ and hence $N_{m(L)} \to \infty$ with probability one. By bounded convergence, this implies that the above integral approaches 0.

We have now shown that for fixed $\delta > 0$, for some random set $\mathbf{V}$, with $|\mathbf{V}| \leq \lambda$,

$$P(\forall n, \mathbf{X} + \mathbf{Y}_n \notin \mathbf{V}) \leq \delta.$$

Since $\mathbf{X}$, $\mathbf{Y}_n$, and $\mathbf{V}$ are independent, we may apply Fubini to conclude that the above inequality holds as well for some nonrandom set V. We have thus demonstrated the existence of an open set V with $|V| \leq \lambda$ and $\mathscr{H}(V) \geq 1 - \delta$. To complete the proof, let U_n be an open set with $|U_n| \leq \lambda\, 2^{-n}$ and $\mathscr{H}(U_n) \geq 1 - 2^{-n}$, and set $U = \cup_{n=1}^{\infty} U_n$. The monotonicity of $\mathscr{H}$ shows that U has the desired properties. □

As one would expect, the theorem is easily strengthened to have U visited not just once, but infinitely often. Choose open sets U_n so that $|U_n| \leq 2^{-n}$ and $\mathscr{H}(U_n) = 1$. Let $W_m = \cup_{n=m}^{\infty} U_n$; then $|W_m| \leq 2^{1-m}$ and for each k,

$$\begin{aligned} P(\exists n \geq k, \mathbf{X} + \mathbf{Y}_n \in W_m) &\geq P(\exists n \geq k, \mathbf{X} + \mathbf{Y}_n \in U_l), \quad \text{for} \quad l \geq m, \\ &\geq \mathscr{H}(U_l) - k\,|U_l| \\ &\geq 1 - k \cdot 2^{-l} \to 1 \quad \text{as} \quad l \to \infty. \end{aligned}$$

It follows that for each m, $P(\mathbf{X} + \mathbf{Y}_n \in W_m \text{ i.o. } (n)) = 1$. Thus, we may reformulate the theorem as:

THEOREM′. *For each* $\lambda > 0$, *there is an open set* $U \subset [0, 1]$ *such that* $|U| \leq \lambda$ *and* $P(\mathbf{X} + \mathbf{Y}_n \in U \text{ i.o.}) = 1$.

REMARK. The reader interested in randomly placed arcs may wish to look at [1] and [2].

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NEW YORK UNIVERSITY
COURANT INSTITUTE OF MATHEMATICAL SCIENCES
251 MERCER ST.
NEW YORK, N.Y. 10012

DEPARTMENT OF STATISTICS
AND PROBABILITY
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824