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A GENERALIZATION OF STOCHASTIC INTEGRATION WITH RESPECT TO SEMIMARTINGALES

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On the real line, there exist σ-finite measures which are not Radon measures, but are nevertheless defined on all bounded intervals

$$\left(\text{e.g. } \frac{1}{x}\sin\frac{1}{x}\,dx, \quad \text{or} \quad \sum_n \frac{(-1)^n}{n}\,\delta_{1/n}\right).$$

Similarly, in stochastic calculus, there exist processes that, though not semimartingales, can be obtained as stochastic integrals of predictable processes with respect to semimartingales. This paper deals with such processes.

0. Introduction; notations. To make stochastic calculations more algebraic, L. Schwartz has recently invented *formal semimartingales*: If H and H' are predictable processes and X and X' are semimartingales, one says that the pairs (H, X) and (H', X') are equivalent if, for some (and thus for any) nonvanishing predictable process K such that KH and KH' are bounded, the stochastic integrals $\int KH\,dX$ and $\int KH'\,dX'$ are equal as processes. Owing to the "associativity property" of stochastic integration, this is an equivalence relation; formal semimartingales are defined as equivalence classes for this relation. The formal semimartingale to which the pair (H, X) belongs is naturally to be understood as the stochastic integral $\int HdX$ (also written $H \cdot X$ throughout this paper). To quote Schwartz [12]: ≪ La possibilité d'écrire $H \cdot X$ sans restriction "libère" complètement des conditions d'intégrabilité, et facilite un grand nombre d'opérations; il n'y a jamais qu'à regarder, à la fin des calculs, si le résultat est une semimartingale formelle ou vraie (un peu comme pour les équations aux dérivées partielles, en utilisant des dérivées-distribution, on ne se préoccupe de la régularité des solutions qu'après les avoir trouvées en tant que distributions). ≫

Given a formal semimartingale $H \cdot X$, it may be interesting to know if it is a true process or not. This paper gives a partial answer to this problem by studying a class of true processes in the space of formal semimartingales. We call them pseudomartingales—pseudosemimartingales would fit better, but enough is enough! It should be emphasized that pseudomartingales are by no means the largest subspace of true processes in the space of formal semimartingales (if ever this is meaningful, which we doubt); see for instance (2.12).

In the first section, we construct and study stochastic integrals $H \cdot X$ under less restrictive assumptions than the usual integrability condition $H \in L(X)$ due to Jacod [7]. In the second section, we define pseudomartingales and show that they are exactly the stochastic integrals $H \cdot X$ constructed in Section 1. The third section is concerned with decompositions of pseudomartingales and the study of particular pseudomartingales (special pseudomartingales, quadratic and decomposable pseudomartingales).

Throughout this paper, we are given a complete probability space $(\Omega, \mathscr{F}, P)$ endowed with a filtration $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ fulfilling the "usual hypotheses" of Dellacherie-Meyer [2]. We use the notations of Dellacherie-Meyer [2] and [3]. In particular, equality between processes means indistinguishability; a function on $[0, \infty)$ is càdlàg if it is right-continuous on $[0, \infty)$ and left-limited on $(0, \infty)$; a process is càdlàg if (almost) all its paths are càdlàg. Given a semimartingale X, the space $L(X)$ of all predictable processes that are integrable

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with respect to X (shortly: X-integrable) has been defined by Jacod [7]; the simplest exposition is Yan [13]; we shall sometimes use the presentation of Chou-Meyer-Stricker [1].

A random set J is called an *optional interval* (resp. *predictable interval*) if it is optional (resp. predictable) and if all its sections $J(\omega)$ are intervals (not necessarily of the same topological kind: some may be open, other closed, etc.). Optional intervals are but a mere generalization of stochastic intervals; their use will make our notations simpler because they are stable under intersections and increasing limits. If J is an optional interval, not only its debut $S = \inf\{t : t \in J\}$, but also its end $T = \sup\{t : t \in J\}$ is a stopping time, for T is the debut of $]\!]S, \infty[\![\cap J^c$.

If f is a càdlàg function on $[0, \infty)$, one can define a simply additive measure df on the ring generated by bounded intervals by

$$df([0, t]) = f(t); \quad df([0, t)) = f(t-)$$

and, for each interval J, a new càdlàg function f^J by $f^J(t) = df(J \cap [0, t])$. If X is a càdlàg process and J an optional interval, a new càdlàg process X^J may be defined path by path: $X_t^J(\omega) = (X(\omega))^{J(\omega)}(t)$. The following properties are easy to check: $X^{J_1 \cap J_2} = (X^{J_1})^{J_2}$; if $J_1 \cup J_2$ is an interval, $XJ_1 \cup J_2 + XJ_1 \cap J_2 = X^{J_1} + X^{J_2}$; if J_n is a sequence of optional intervals, with limit J, then $X^J = \lim_n X^{J_n}$ pointwise (for each ω, the convergence is uniform on compact sets of [0, in)). If X is adapted, so is X^J; if T is a stopping time,

$$X^{[\![0, T]\!]} = X^T; \; X^{[\![0, T[\![} = X^{T-}$$

If X is a semimartingale and J an optional interval, then X^J is exactly the optional integral $I_J \cdot X$; if H is a predictable, X-integrable process, then H is X^J-integrable and $H \cdot (X^J) = (H \cdot X)^J$

A random set A is said to be *thin* if (almost) all its sections $A(\omega)$ are countable (for us, countable will always mean finite or countably infinite).

1. Pseudointegrability w.r.t. a semimartingale. We begin with two lemmas about optional and predictable intervals.

(1.1) LEMMA. *Let $\mathscr{J}$ be a class of optional intervals such that*

(i) *$\mathscr{J}$ contains all graphs of stopping times;*

(ii) *$\mathscr{J}$ is hereditary: any optional interval included in some element of $\mathscr{J}$ is itself in $\mathscr{J}$;*

(iii) *$\mathscr{J}$ is stable under disjoint union: if J_1 and J_2 are in $\mathscr{J}$ with $J_1 \cap J_2 = \phi$ and $J_1 \cup J_2$ an interval, then $J_1 \cup J_2$ is in $\mathscr{J}$;*

(iv) *there exists a sequence in $\mathscr{J}$ covering $[\![0, \infty[\![$.*

Then there exists a sequence of pairwise disjoint predictable intervals in $\mathscr{J}$ covering $[\![0, \infty[\![$.

PROOF. Let (J_n) be a countable covering of $[\![0, \infty[\![$ by elements of $\mathscr{J}$; call S_n the debut of J_n, T_n its end, K_n the interval $]\!]S_n, T_n]\!]$. The predictable interval K_n belongs to $\mathscr{J}$ as the disjoint union of $]\!]S_n, T_n[\![$ (included in J_n) and of a graph of stopping time. Define $K_n^1 = [\![0, S_n]\!]$, $K_n^2 =]\!]T_n, \infty[\![$. The set

$$(K_1 \cup \cdots \cup K_n)^c = \cup_\varepsilon \cap_{i=1}^n K_i^{\varepsilon(i)}$$

(where ε ranges over the finite set $\{1, 2\}^{\{1, \cdots, n\}}$) is a disjoint finite union of predictable intervals; hence $K_{n+1} \cap (\cup_{m<n} K_m)^c$ is the union of a finite family $\mathscr{K}_{n+1}$ of pairwise disjoint predictable intervals, all of them included in K_{n+1} and thus belonging to $\mathscr{J}$. Now $\mathscr{K} = \cup_n \mathscr{K}_n$ is a countable family of pairwise disjoint predictable intervals such that $A = (\cup_{K \in \mathscr{K}} K)^c = (\cup_n]\!]S_n, T_n]\!])^c$ is a predictable set included in $\cup_n [\![S_n]\!]$; by Dellacherie-Meyer [2] page 261, A is the union of a countable family $\mathscr{K}'$ of disjoint graphs of predictable stopping times. The lemma follows by taking the family $\mathscr{K} \cup \mathscr{K}'$. □

The next lemma involves a transfinite induction which constitutes the core of our construction of generalized stochastic integrals.

(1.2) LEMMA. *Let $\mathscr{J}$ be a class of optional (resp. predictable) intervals such that*

(i) *$\mathscr{J}$ is hereditary: any optional (predictable) interval included in some element of $\mathscr{J}$ belongs itself to $\mathscr{J}$;*

(ii) *$\mathscr{J}$ is stable under disjoint unions: if J_1 and J_2 are in $\mathscr{J}$ with $J_1 \cap J_2 = \phi$ and $J_1 \cup J_2$ an interval, then $J_1 \cup J_2$ is also in $\mathscr{J}$;*

(iii) *$\mathscr{J}$ is stable under increasing limits: if (J_n) is an increasing sequence of elements of $\mathscr{J}$, its limit $\cup_n J_n$ is also in $\mathscr{J}$;*

(iv) *there exists in $\mathscr{J}$ a countable covering of $[\![0, \infty[\![$.*

Then $[\![0, \infty[\![$ belongs to $\mathscr{J}$.

PROOF. It needs three steps.

Step 1. If T is a (predictable) stopping time, then its graph $[\![T]\!]$ is in $\mathscr{J}$. Indeed let (J_n) be a countable family in $\mathscr{J}$ covering $[\![0, \infty[\![$. The (predictable) intervals

$$A_n = [\![T]\!] \cap J_n \cap (\cap_{m<n} J_m^c)$$

are in $\mathscr{J}$ because $A_n \subset J_n$; using (ii) one sees by induction that the intervals $\cup_{m<n} A_m$ are in $\mathscr{J}$; (iii) then implies that $[\![T]\!] = \cup_n A_n$ is also in $\mathscr{J}$, which proves the claim.

Now, in the optional case, let $\mathscr{J}'$ be the class of all predictable elements of $\mathscr{J}$. Applying Lemma (1.1) to $\mathscr{J}$, one obtains that $\mathscr{J}'$ also satisfies hypothesis (iv); *thus we may from now on restrict ourselves to the predictable case*, and the optional one will follow by applying the lemma to $\mathscr{J}'$.

Step 2. Let $\mathscr{E}$ denote the class of all closed optional sets E such that, if U and V are any stopping times with $U \leq V$ and $[\![U, V[\![\cap E = \phi$, then $]\!]U, V]\!] \in \mathscr{J}$. We shall prove that *if E belongs to $\mathscr{E}$, its perfect kernel does also.*

(a) *If (E_n) is a decreasing sequence in $\mathscr{E}$, then $E = \cap_n E_n$ is in $\mathscr{E}$.* Let U and V be stopping times such that $U \leq V$ and $[\![U, V[\![\cap E = \phi$. We want to show that $]\!]U, V]\!]$ belongs to $\mathscr{J}$. Let T_n be the debut of $[\![U, \infty[\![\cap E_n$, and T the debut of $[\![U, \infty[\![\cap E$. Because $[\![U, V[\![\cap E = \phi$, $V \leq T$, and $]\!]U, V]\!]$ is included in $]\!]U, T]\!]$. By (i), it suffices to show that $]\!]U, T]\!]$ is in $\mathscr{J}$. As all E_n are closed, the increasing sequence T_n goes to T. As E_n is in $\mathscr{E}$, $]\!]U, T_n]\!]$ is in $\mathscr{J}$; hence, by (iii), $A = \cup_n]\!]U, T_n]\!]$ is in $\mathscr{J}$. But $]\!]U, T]\!] - A$ is the graph of a predictable stopping time; thus, using Step 1 and (ii), $]\!]U, T]\!]$ is also in $\mathscr{J}$, which establishes the claim.

(b) *If E is a closed optional set and T a stopping time such that $E \cap [\![T]\!] = \phi$ and $E \cup [\![T]\!] \in \mathscr{E}$, then E is also in $\mathscr{E}$.* Indeed, let $U \leq V$ be stopping times with $[\![U, V[\![\cap E = \phi$, and let $T' = (T \vee U) \wedge V$; $T'_n = (T' + 1/n) \wedge V$. Both $]\!]U, T']\!]$ and $]\!]T'_n, V]\!]$ are in $\mathscr{J}$ because neither $[\![U, T'[\![$ nor $[\![T'_n, V[\![$ intersects $E \cup [\![T]\!]$;

$$]\!]T', V]\!] = \lim_n]\!]T'_n, V]\!]$$

is in $\mathscr{J}$ by (iii) and $]\!]U, V]\!] =]\!]U, T']\!] \cup]\!]T', V]\!]$ is in $\mathscr{J}$ by (ii).

(c) *If E is in $\mathscr{E}$, its derived set E' is also in $\mathscr{E}$.* The optional set $E - E'$ is thin. By Dellacherie-Meyer [2] page 261, there exists a sequence (T_n) of stopping times with disjoint graphs such that $E - E' = \cup_n [\![T_n]\!]$. Using (b), one sees by induction that $E_n = E' \cup \cup_{m \geq n} [\![T_m]\!]$ is in $\mathscr{E}$; as E_n is a decreasing sequence with limit E', (a) implies that $E' \in \mathscr{E}$.

(d) Now let $E \in \mathscr{E}$. We are going to prove that the perfect kernel F of E is in $\mathscr{E}$, and Step 2 will be done. By Dellacherie-Meyer [2] page 260, there exists a countable ordinal α such that F is equal to $E^{(\alpha)}$, the derived set of order α of E. Thus it suffices to show by transfinite induction that, if α is any countable ordinal, $E^{(\alpha)}$ belongs to $\mathscr{E}$. We have just seen in (c) that, if $E^{(\alpha)}$ is in $\mathscr{E}$, then its derived set $E^{(\alpha+1)}$ is also in $\mathscr{E}$. It remains to show that, if β is any countable ordinal of the second kind, and if $E^{(\alpha)}$ is in $\mathscr{E}$ for all $\alpha < \beta$, then $E^{(\beta)} = \cap_{\alpha<\beta} E^{(\alpha)}$

is in $\mathscr{E}$. Let $n \mapsto \alpha_n$ be a bijection between N and the ordinals smaller than β; define $\alpha(n) = \max(\alpha_0, \cdots, \alpha_n)$. The sequence $\alpha(n)$ is increasing and $\sup_n \alpha(n) = \beta$. Thus $E^{(\beta)} = \lim_n \downarrow E^{(\alpha(n))}$, and (a) gives the claim.

Step 3. We know by Lemma (1.1) that there exists in $\mathscr{J}$ a sequence (L_n) of pairwise disjoint predictable intervals with $\cup_n L_n = [\![0, \infty[\![$. Let S_n be the debut of L_n, T_n the end of L_n. Define

$$E = \cup_n ([\![S_n]\!] \cup [\![T_n]\!]) = (\cup_n]\!]S_n, T_n[\![)^c.$$

This set is closed and belongs to $\mathscr{E}$ (because if $[\![U, V[\![\cap E = \phi$, $]\!]U, V]\!] \cap L_n = A_n$ is in $\mathscr{J}$; but the A_n are pairwise disjoint and $\cup_{m \le n} A_m$ is an interval; hence $\cup_n A_n =]\!]U, V]\!]$ is in $\mathscr{J}$). Its perfect kernel F also belongs to $\mathscr{E}$ by Step 2. But, as E is thin, F is empty. This proves that $]\!]0, \infty[\![$ is in $\mathscr{J}$, and, as $[\![0]\!]$ is also in $\mathscr{J}$ by Step 1, $[\![0, \infty[\![$ belongs to $\mathscr{J}$: the lemma is established. □

We now come to stochastic integrals.

(1.3) Definition. Let X be a semimartingale, H a predictable process. One says that H is pseudointegrable with respect to X (shortly: X-pseudointegrable) if there exist a càdlàg process Y and a sequence (J_n) of optional intervals covering $[\![0, \infty[\![$ such that, for each n,

$$H \in L(X^{J_n}) \quad \text{and} \quad Y^{J_n} = H \cdot (X^{J_n}).$$

The set of such predictable processes is denoted by $D(X)$.

(1.4) *Unicity of Y. Given a semimartingale X and a process H in D(X), let* $(Y', (J'_n))$ *and* $(Y'', (J''_n))$ *be as in Definition* (1.3). *Then* $Y' = Y''$.

Proof. Let $Z = Y' - Y''$ and let $\mathscr{J}$ be the set of optional intervals J with $Z^J = 0$. We can apply Lemma (1.2) to $\mathscr{J}$ because

(i) $J' \subset J \Rightarrow Z^{J'} = (Z^J)^{J'}$;
(ii) $J_1 \cap J_2 = \phi \Rightarrow Z^{J_1 \cup J_2} = Z^{J_1} + Z^{J_2}$;
(iii) $J_n \uparrow J \Rightarrow Z^J = \lim_n Z^{J_n}$ (pointwise limit);
(iv) if $J_{m,n} = J'_m \cap J''_n$, $\cup_{m,n} J_{m,n} = [\![0, \infty[\![$.

Thus $[\![0, \infty[\![$ is in $\mathscr{J}$, and $Y' = Y''$. □

This makes possible the following definition:

(1.5) Definition. Let X be a semimartingale, H a predictable X-pseudointegrable process. The process Y appearing in Definition (1.3) is called the integral of H with respect to X, and denoted $H \cdot X$.

This definition generates no ambiguity: the class $D(X)$ obviously contains the space $L(X)$ of predictable X-integrable processes, and both integrals agree on $L(X)$. That $D(X)$ may be strictly wider than $L(X)$ is easy to see, even in the deterministic case: Take

$$X_t = t, \qquad H_t = \frac{1}{t} \sin \frac{1}{t} \quad \text{or} \quad X_t = \sum_{n \ge 1/t} \frac{1}{n^2}, \qquad H_{1/n} = (-1)^n n.$$

Properties of the integral.

(1.6) Y is adapted. (Apply Lemma (1.2) to the class of optional intervals J such that Y^J is adapted.)

(1.7) For any stopping time T, $H_T \Delta X_T = \Delta Y_T$. (Equality $H \Delta X = \Delta Y$ holds on each J_n, and thus everywhere.)

The next three properties are easy consequences of Definition (1.3) and of the corresponding properties for usual integrals.

(1.8) Linearity in H. $D(X)$ is a vector space, and $H \mapsto H \cdot X$ is linear.

(1.9) Linearity in X. If H is a given predictable process, the set of all semimartingales X such that $H \in D(X)$ is a vector space, and $X \mapsto H \cdot X$ is linear.

(1.10) Associativity. If X is a semimartingale and H a process in $L(X)$, then a predictable process K belongs to $D(H \cdot X)$ if and only if KH is in $D(X)$; when this holds. $K \cdot (H \cdot X) = (KH) \cdot X$.

Property (1.10) is very important, for it enables us to identify the integral $H \cdot X$ with a formal semimartingale: if (H, X) and (H', X') are two representants of the same formal semimartingale, and if $H \in D(X)$ with $H \cdot X = Y$, then $H' \in D(X')$ with $H' \cdot X' = Y$. It also makes possible, for any formal semimartingale $Z = H \cdot X$, to define the space $D(Z)$ as the set of all predictable processes K such that KH is in $D(X)$. In particular, if, anticipating Section 2, we call integrals such as $H \cdot X$, where $H \in D(X)$, *pseudomartingales*, given any pseudomartingale Y one can define the spaces $L(Y)$ and $D(Y)$, and the corresponding integrals are respectively semimartingales and pseudomartingales.

As was already noted by Schwartz for semimartingales, this associativity property dispenses us from studying integrals $H \cdot X$ where H is a row-vector and X a column-vector in $\mathbb{R}^n$ (these integrals were introduced by Gal'chuk [5] for martingales and extended by Jacod [6] to semimartingales), since such an integral $H \cdot X = \sum_{i=1}^n H^i \cdot X^i$ can always be reduced to the one-dimensional case by choosing a non-vanishing real-valued predictable process K such that, for each i, KH^i is bounded, and noticing that

$$H \cdot X = \frac{1}{K} \cdot \left(\sum (KH^i) \cdot X^i\right).$$

(1.11) If $H \in D(X)$ with $H \cdot X = Y$, and if Q is another probability with $Q \ll P$, then H is X-pseudointegrable for Q, and $H \dot{_P} X$ is a version of $H \dot{_Q} X$. (Immediate from Definition (1.3) and the similar property for usual integrals.) In particular, if $H \in D(X)$ and $H' \in D(X')$, and if A is any event such that, on $[0, \infty) \times A$, $X = X'$ and $H = H'$, then, on the same set, $H \cdot X = H' \cdot X'$.

(1.12) In Definition (1.3), the intervals J_n can be chosen predictable and pairwise disjoint. (Apply Lemma (1.1) to the class of optional intervals J with $H \in L(X^J)$ and $H \cdot X^J = Y^J$.)

(1.13) If $H \in D(X)$ and $H \cdot X = Y$, then for any optional interval J, $H \in D(X^J)$ and $H \cdot X^J = Y^J$. (Obvious from Definition (1.3).)

(1.14) Let X be a semimartingale, H a predictable process, Y a càdlàg process. If there exists a countable family (L_n) of optional intervals covering $[\![0, \infty[\![$ such that $H \in D(X^{L_n})$ and $H \cdot X^{L_n} = Y^{L_n}$, then $H \in D(X)$ and $H \cdot X = Y$. Indeed, for each n, there exists a countable covering $(J_k^n)_{k \in \mathbb{N}}$ of $[\![0, \infty[\![$ by optional intervals such that $H \in L(X^{L_n \cap J_n^k})$, with

$$H \cdot X^{L_n \cap J_n^k} = Y^{L_n \cap J_n^k};$$

(1.14) is proved by noting that $(L_n \cap J_n^k)_{k,n}$ is a countable covering of $[\![0, \infty[\![$. In particular, X-pseudointegrability is prelocal: if $Y = H \cdot X$ on each $[\![0, T_n[\![$ for a sequence T_n of stopping times, $Y = H \cdot X$ on $[\![0, \sup_n T_n[\![$.

(1.15) Proposition. *Let H be a predictable, X-pseudointegrable process, where X is a semimartingale. Then $H \cdot X$ is a semimartingale if and only if H is X-integrable.*

PROOF. The "if" part is well known. For the converse, assume $Y = H\cdot X$ is a semimartingale, and let $\mathcal{J}$ be the class of all predictable intervals J such that $H \in L(X^J)$. It suffices to apply Lemma (1.2) to $\mathcal{J}$. Hypotheses (i) and (ii) are easy to check; (iv) follows from (1.12). To verify (iii), assume $H \in L(X^{J_n})$ for an increasing sequence (J_n) of predictable intervals; we want to show that $H \in L(X^J)$ where $J = \lim_n J_n$. Because Y is a semimartingale, the sequence

$$H\cdot X^{J_n} = I_{J_n}H\cdot X = I_{J_n}\cdot Y$$

converges, for the semimartingale topology (see Mémin [10]) to $I_J\cdot Y$ by the dominated convergence theorem for stochastic integrals. Thus, the sequence of predictable processes $I_{J_n}H$ converges, in the topological space $\mathbb{L}(X)$ of Mémin [10], to some limit. It also converges pointwise to I_JH. Identifying limits, one gets $I_JH \in L(X)$, the desired result. □

No dominated convergence theorem is true for those integrals, and one cannot expect much better than (1.13). This is due to the fact that the order structure on the time axis plays much greater a role here than in usual stochastic integrals. We have lost the globality featured in modern integration theory; but as stochastic integrals are usually understood as processes computed in an adapted way with respect to time, this loss is probably not too serious. The following example shows how important the time structure is in those integrals.

(1.16) *An example where $H \in D(X)$ and $H\cdot X \neq \lim_n(HI_{\{|H|\le n\}}\cdot X)$.* Define sequences $(a_n)_{n\ge1}$ and $(b_n)_{n\ge1}$ by

$$b_{2n} = 3n; \quad b_{4n+1} = 3n + 1; \quad b_{4n+3} = 3n + 2; \quad a_n = \frac{(-1)^n}{nb_n}.$$

As $b_n > n/2$, one has $|a_n| < 2/n^2$ and $X = \sum_n a_nI_{[1/n,\infty[}$ is a semimartingale. Let H be a predictable process such that $H_{1/n} = b_n$. Because $H_{1/n}\Delta X_{1/n} = a_nb_n = (-1)^n/n$, H is in $D(X)$ and $(H\cdot X)_t = \sum_{n\ge1/t}(-1)^n/n$. Nevertheless $L = \lim_n(HI_{\{|H|\le n\}}\cdot X)$ exists, and is not equal to $H\cdot X$ but to $H\cdot X - (1/2)\log 2$. (For $t \ge 1$,

$$H\cdot X_t = -1/1 + 1/2 - 1/3 + 1/4 - \cdots = -\log 2,$$

$$L_t = -1/1 - 1/3 + 1/2 - 1/5 - 1/7 + 1/4 - \cdots = -3/2 \log 2;$$

for $0 < t \le 1$,

$$L_1 - L_t = \sum_{1\le n<1/t} H_{1/n}\Delta X_{1/n} = H\cdot X_1 - H\cdot X_t.)$$

So far, Definition (1.3) enables us to check if a given process Y is, or not, the integral $H\cdot X$, but gives no way of constructing Y when H and X are given. Such a construction is, in fact, hidden inside Lemma (1.2); the rest of this section is devoted to making it explicit.

(1.17) We are given a semimartingale X and a predictable process H. We are going to define inductively, for each ordinal α, two classes $\mathcal{J}_\alpha$ and $\mathcal{J}'_\alpha$ of optional intervals J for which the integral $\mu(J) = H\cdot X^J$ can be defined.

First, define $\mathcal{J}_0$ as the class of all optional intervals J with $H \in L(X^J)$; for $J \in \mathcal{J}_0$, let $\mu(J)$ denote the process $H\cdot X^J$; define $\mathcal{J}'_0 = \mathcal{J}_0$.

Then, suppose $\mathcal{J}_\alpha$ and $\mathcal{J}'_\alpha$ are defined for some ordinal α, and suppose μ is defined as a process-valued mapping on $\mathcal{J}_\alpha \cup \mathcal{J}'_\alpha$. Define $\mathcal{J}_{\alpha+1}$ as the set of all optional intervals J of the form $J_1 \cup J_2$, where J_1 and J_2 are in $\mathcal{J}'_\alpha$ with $J_1 \cap J_2 = \phi$; define $\mu(J) = \mu(J_1) + \mu(J_2)$. Define $\mathcal{J}'_{\alpha+1}$ as the class of all J of the form $\lim_n \uparrow J_n$, with $J_n \in \mathcal{J}_{\alpha+1}$ such that $\lim_n \mu(J_n)$ exists pointwise; for $J \in \mathcal{J}'_{\alpha+1}$, define $\mu(J)$ as this limit.

Last, suppose $\mathcal{J}_\alpha$ and $\mathcal{J}'_\alpha$ are defined for $\alpha < \beta$, where β is an ordinal of the second kind. Define $\mathcal{J}_\beta$ and $\mathcal{J}'_\beta$ as $\cup_{\alpha<\beta}\mathcal{J}_\alpha$.

To make the preceding construction meaningful, we have to show that the definition of μ involves no contradiction. More precisely, we must prove that, if μ is well defined up to level α, then the definition $\mu(J) = \mu(J_1) + \mu(J_2)$ for $J \in \mathcal{J}_{\alpha+1}$ does not depend on the decomposition $J_1 \cup J_2$ of J and agrees with the previously defined $\mu(J)$ if J belongs to some $\mathscr{J}_\beta$ or $\mathscr{J}'_\beta$ with $\beta \le \alpha$; and an analogous statement for the definition $\mu(J) = \lim_n \mu(J_n)$. This can be done in a pedestrian but tedious way. It is much quicker to use what we already know about pseudointegrability: the next lemma makes it obvious.

(1.18) LEMMA. *For each α, $\mathscr{J}_\alpha$ and $\mathscr{J}'_\alpha$ are hereditary, and, if J belongs to $\mathscr{J}_\alpha$ or $\mathscr{J}'_\alpha$, then $H \in D(X^J)$ and $\mu(J) = H \cdot X^J$.*

PROOF. The transfinite induction goes without difficulty, owing to (1.9) (which implies the additivity of μ) and to (1.14) (which implies the stability of μ under increasing limits). □

This construction can be done for any H and X. It leads to a "constructive" characterization of pseudointegrability:

(1.19) THEOREM. *Let X be a semimartingale, H a predictable process. Then $H \in D(X)$ if and only if $[\![0, \infty[\![$ belongs to $\mathscr{J}_\alpha$ for some ordinal α. When this holds, α may be chosen countable.*

PROOF. If $[\![0, \infty[\![$ belongs to $\mathscr{J}_\alpha$, the preceding lemma shows that $H \in D(X)$. Conversely, suppose $H \in D(X)$. We claim that $[\![0, \infty[\![$ belongs to $\mathscr{J}_\alpha$ for some countable α. Define $\mathscr{E}_\alpha$ as the class of all closed optional sets E such that, for U and V stopping times with $U \le V$ and $[\![U, V[\![\cap E = \phi$, $]\!]U, V]\!]$ belongs to $\mathscr{J}_\alpha$. Except that we now have to count the number of times we use properties (ii) and (iii), the proof is very similar to that of Lemma (1.2), so we merely sketch it.

First, similarly to (a), one shows that if (E_n) is any decreasing sequence such that $E_n \in \mathscr{E}_{\alpha_n}$ for each n, then $\bigcap_n E_n$ is in $\mathscr{E}_{(\sup_n \alpha_n)+1}$. Then, as in (b), one checks that if E is a closed set and T a stopping time with $E \cap [\![T]\!] = \phi$ and $E \cup [\![T]\!] \in \mathscr{E}_\alpha$, then E belongs to $\mathscr{E}_{\alpha+1}$.

As in (c) and (d), this implies that, if E is in $\mathscr{E}_\alpha$, its derived set is in $\mathscr{E}_{\alpha+\beta}$ for some countable β and its perfect kernel in $\mathscr{E}_{\alpha+\gamma}$ for some countable γ (β and γ, of course, do not depend only on α).

Now, the set E in Step 3 of the proof of Lemma (1.2) belongs to $\mathscr{E}_0$, thus its perfect kernel, ϕ, is in $\mathscr{E}_\gamma$ for some countable γ. Hence $]\!]0, \infty[\![$ is in $\mathscr{J}_\gamma$, and $[\![0, \infty[\![$ in $\mathscr{J}_{\gamma+1}$, which proves the claim. □

(1.20) REMARK. In the construction of $\mathscr{J}'_{\alpha+1}$ from $\mathscr{J}_{\alpha+1}$, we required the sequence (J_n) to be increasing. Facts (1.18) and (1.19) remain true if we drop this requirement. Indeed, if J_n is a sequence of optional intervals with limit J, if H is in $D(X^{J_n})$ for all n, and if $\lim_n H \cdot X^{J_n}$ exists, then H is in $D(X^J)$ and $H \cdot X^J$ is equal to this limit. This follows at once from (1.14) and from the inclusion $J \subset \bigcup_n J_n$.

2. Pseudomartingales. In this section, we shall make use of the notion of "semimartingale in an open set" studied by Meyer and Stricker [11] after an idea of Schwartz. We first recall quickly a few facts from [11].

Given a process X and an open random set A (for us, A will always be optional), we say that X is a semimartingale in the open set A if there exist a countable covering of A by optional open sets A^n and a sequence of semimartingales X^n such that, for each n, $X = X^n$ on A^n. The next two facts are proved by Meyer-Stricker:

(2.1) If $(A_\iota)_{\iota \in I}$ is a (not necessarily countable) family of open sets, and if X is a semimartingale in each A_ι, then X is a semimartingale in the essential union ess $\bigcup_{\iota \in I} A_\iota$. This implies that, given a process X, there exists a largest open set A (up to evanescent sets) such that X is a semimartingale in A; if X is optional, so is A.

(2.2) If T is a stopping time, and if an adapted process X is a semimartingale in the open set $[\![0, T[\![$, then there exists an increasing sequence of stopping times T_n, with limit T, such that each X^{T_n} is a semimartingale.

A consequence of (2.2) is the following translation into our notations of a particular case of Theorem 4 of Meyer-Stricker:

(2.3) LEMMA. *Let Y be a càdlàg adapted process and O an optional open set. If Y is a semimartingale in O, there exists a sequence (L_n) of pairwise disjoint predictable intervals covering O such that each Y^{L_n} is a semimartingale.*

PROOF. For $s \geq 0$, the process $\tilde{Y} = (Y_t)_{t\geq s}$ is a semimartingale (for the filtration $(\mathscr{F}_t)_{t\geq s}$) in the open set $[\![s, \infty[\![\cap O$ (open in $[\![s, \infty[\![$). Let T^s be the debut of $[\![s, \infty[\![\cap O^c$; $\tilde{Y}$ is a semimartingale in $[\![s, T^s[\![$. By (2.2), there exists an increasing sequence (T^s_n) of stopping times such that $T^s_n \geq s$, $\lim_n T^s_n = T^s$ and $Y^{[\![s, T^s_n]\!]}$ is a semimartingale. The family $([\![s, T^s_n]\!])_{s,n}$, where s ranges over the rational numbers, can be ordered to get a sequence $(J_k)_{k\in\mathbb{N}}$ of predictable intervals covering O and such that each Y^{J_k} is a semimartingale. The J_k are not disjoint, but noticing that $J_k \cap (\bigcup_{i<k} J_i^c)$ is a finite union of disjoint predictable intervals, it is easy to replace them by a disjoint sequence (L_n) as in Lemma (1.1). □

We are now ready to define pseudomartingales:

(2.4) DEFINITION. A process Y is a pseudomartingale if it is càdlàg, adapted, and if there exists a thin closed set E such that Y is a semimartingale in E^c.

Using (2.1) it can be remarked that a càdlàg adapted process Y is a pseudomartingale if and only if the smallest closed set E such that Y is a semimartingale in E^c is thin. Pseudomartingales are a vector space.

The next two theorems show the relationship between pseudomartingales and pseudointegrability.

(2.5) THEOREM. *For a càdlàg adapted process Y, the following three conditions are equivalent*:

(i) *Y is a pseudomartingale*;
(ii) *there exists a sequence (J_n) of optional intervals covering $[\![0, \infty[\![$ such that each Y^{J_n} is a semimartingale*;
(iii) *there exists a semimartingale X and a predictable X-pseudointegrable process H with $Y = H \cdot X$.*

When they hold, the J_n in (ii) *may be chosen pairwise disjoint and predictable, and the process H in* (iii) *positive and bounded away from zero.*

PROOF. Denote by (ii′) the statement (ii) with J_n predictable and pairwise disjoint, and by (iii′) the statement (iii) with $H \geq 1$. (i) ⇒ (ii′). Let E be a thin optional closed set such that Y is a semimartingale in E^c. According to Lemma (2.3), there exists a covering (L_n) of E^c by pairwise disjoint predictable intervals such that Y^{L_n} are semimartingales. Now $(\bigcup_n L_n)^c$ is a thin predictable set, thus the union of a sequence (G_n) of pairwise disjoint predictable graphs. As Y is càdlàg and adapted, each Y^{G_n} is a semimartingale. Reordering the G_n and L_n into a sequence (J_n) gives the result. (ii′) ⇒ (iii′). Let (α_n) be a sequence in $(0, 1]$ such that

$$d(\alpha_n Y^{J_n}, 0) < 2^{-n},$$

where d is a complete, translation-invariant distance on semimartingales, compatible with the topology of semimartingales (see Dellacherie-Meyer [3] page 315; if he prefers, the reader may stop all processes at a fixed time t, choose a new probability for which each Y^{J_n} is in the Banach space $\mathscr{S}^1$ of semimartingales, and use the $\mathscr{S}^1$-norm instead of d). Such a sequence (α_n) exists because the space of semimartingales is a topological vector space.

The partial sums of the series $\sum_n \alpha Y^{J_n}$ are a Cauchy sequence for d; hence the series has a sum which is a semimartingale X verifying $X^{J_n} = \alpha_n Y^{J_n}$. Define $H = \sum_n (1/\alpha_n) I_{J_n}$; H is a predictable process with $H \geq 1$, and for each n, H is X^{J_n}-integrable (because H is bounded on J_n) with $H \cdot X^{J_n} = Y^{J_n}$. Thus, by Definition (1.3), H is X-pseudointegrable and $H \cdot X = Y$. (iii) $\Rightarrow$ (ii) follows immediately from Definition (1.3). (ii) $\Rightarrow$ (i). Let $0 = \cup_n \mathring{J}_n$, where $\mathring{J}_n$ is the interior of J_n. Then Y is a semimartingale in each $\mathring{J}_n$, thus in 0, and 0^c is included in $\cup_n \partial J_n$, hence thin. □

(2.6) REMARK. Another variation on assertion (ii) will be useful in the sequel: if Y is a pseudomartingale, there exist a sequence (s_n) of deterministic times and a sequence (S_n) of stopping times such that each $Y^{\rrbracket s_n, S_n \rrbracket}$ is a semimartingale and $(\cup_n \rrbracket s_n, S_n \rrbracket)^c$ is thin. (This follows from a standard argument in the theory of semimartingales in open sets, already used in Lemma (2.3): for each rational s, there exists by (2.2) a sequence (T_n^s) of stopping times such that $Y^{\rrbracket s, T_n^s \rrbracket}$ is a semimartingale and $\sup_n T_n^s$ is the debut of $\llbracket s, \infty \llbracket \cap 0^c$.)

We already saw, in the remark following (1.10), that, given a pseudomartingale Y, one can define the spaces $L(Y)$ of predictable Y-integrable processes and $D(Y)$ of predictable Y-pseudointegrable processes. It is easy to check that properties (1.6) to (1.15) still hold when the semimartingale X is replaced by a pseudomartingale.

(2.7) THEOREM. *Let X be a pseudomartingale, H a predictable process, Y a process. Then H is in $D(X)$ with $H \cdot X = Y$ if and only if*
(i) *Y is a pseudomartingale, and*
(ii) *for all optional (resp. predictable) intervals J, H is X^J-integrable if and only if Y^J is a semimartingale, and, for every such J, $Y^J = H \cdot (X^J)$.*

PROOF. Replacing X by X^L, where L is a suitable predictable interval, and using (1.14), we may suppose X is a semimartingale.

If H is X-pseudointegrable with $H \cdot X = Y$, we already know that Y is a pseudomartingale; if J is any optional interval, we have seen in (1.13) that $H \in D(X^J)$ and $H \cdot X^J = Y^J$; if Y^J is a semimartingale, (1.15) implies that H is X^J-integrable.

Conversely, suppose that (i) and the predictable form of (ii) hold. By Theorem (2.5), there exists a sequence (L_N) of predictable intervals covering $\llbracket 0, \infty \llbracket$, each Y^{L_n} being a semimartingale. Then $H \in D(X)$ and $H \cdot X = Y$ by Definition (1.3). □

In the rest of this section, we are interested in the behaviour of "pseudointegrals" under a change of filtration and in the Jacod-Meyer convexity property for pseudointegrability. The proofs rest on similar properties for semimartingales in open sets that we lazily borrow from Meyer-Stricker [11].

Suppose we have another filtration $\mathbb{G} = (\mathscr{G}_t)_{t \leq 0}$ on $(\Omega, \mathscr{F}, P)$, satisfying the usual conditions and larger than $\mathbb{F}$ (i.e. $\forall t \mathscr{F}_t \subset \mathscr{G}_t \subset \mathscr{F}$); this implies that every $\mathbb{F}$-predictable process is $\mathbb{G}$-predictable. Wherever necessary, we shall use such notations as $D_{\mathbb{F}}(X)$ or $H \overset{\cdot}{_{\mathbb{F}}} X$. We shall start with two lemmas belonging to semimartingale theory (the prefix "pseudo" does not appear):

(2.8) LEMMA. *Let S be a $\mathbb{G}$-stopping time, X an $\mathbb{F}$-semimartingale such that X^S is a $\mathbb{G}$-semimartingale, and H a predictable process in both $L_{\mathbb{F}}(X)$ and $L_{\mathbb{G}}(X^S)$. The integrals $H \overset{\cdot}{_{\mathbb{F}}} X$ and $H \overset{\cdot}{_{\mathbb{G}}} X^S$ are equal on $\llbracket 0, S \rrbracket$.*

PROOF. The result is obvious if $H = I_{]s,t] \times A}$, where A is in $\mathscr{F}_s$. It can be extended to all bounded, $\mathbb{F}$-predictable processes H by the monotone class theorem. In the general case, $H^n = HI_{\{|H| \leq n\}}$ is a sequence of bounded, $\mathbb{F}$-predictable processes such that $H \overset{\cdot}{_{\mathbb{F}}} X = \lim_n H^n \overset{\cdot}{_{\mathbb{F}}} X$ and $H \overset{\cdot}{_{\mathbb{G}}} X^S = \lim_n H^n \overset{\cdot}{_{\mathbb{G}}} X^S$, where the limits are to be taken in the topology of compact convergence in probability (see Dellacherie-Meyer [3]); this proves the lemma. □

(2.9) LEMMA. *Let S be a $\mathbb{G}$-stopping time. There exists a sequence (T_n) of $\mathbb{F}$-stopping times, such that $\sup_n T_n \geq S$, with the following property: If an $\mathbb{F}$-semimartingale X is such that X^S is a $\mathbb{G}$-semimartingale, and if an $\mathbb{F}$-predictable process H belongs to $L_{\mathbb{G}}(X^S)$, then H is in $L_{\mathbb{F}}(X^{T_n})$ for all n.*

PROOF. Let Q stand for the $\mathbb{F}$-optional projection of $I_{[\![0,S[\![}$; define

$$T_n = \inf\left\{t: Q_t \leq \frac{1}{n}\right\}; \qquad T = \inf\{t: Q_t = 0\}.$$

Because $P[S > T \mid \mathscr{F}_T] = Q_T = 0$, $S \leq T$ a.s.; as Q is a positive supermartingale, Q sticks at zero when it hits it, and $T = \lim_n T_n$ (see Dellacherie-Meyer [3], page 86). Thus $\lim_n T_n \geq S$.

If A is any $\mathbb{F}$-optional right-continuous increasing process, one has

$$\frac{1}{n} E[A_{T_n-}] \leq E\left[\int_0^{T_n-} Q_t\, dA_t\right] \leq E\left[\int_0^\infty I_{[\![0,S[\![}(t)\, dA_t\right] \leq E[A_S].$$

The topology of the $\mathbb{F}$-semimartingales may be defined with the quasinorm

$$\|X\|_{\mathscr{S}(\mathbb{F})} = \textstyle\sum_m 2^{-m} \sup_K E[1 \wedge sup_{t\leq m} |K\cdot X|_t],$$

where K ranges over all $\mathbb{F}$-predictable processes with $|K| \leq 1$; similarly for $\mathbb{G}$. If X is an $\mathbb{F}$-semimartingale such that X^S is a $\mathbb{G}$-semimartingale, the preceding inequality applied to

$$A_t = 1 \wedge \sup_{s\leq t\wedge m} |K\cdot X|_s$$

yields

$$\|X^{T_n-}\|_{\mathscr{S}(\mathbb{F})} \leq n\|X^S\|_{\mathscr{S}(\mathbb{G})}.$$

Now if H is an $\mathbb{F}$-predictable process in $L_{\mathbb{G}}(X^S)$, the sequence $H^k\cdot X^S$ (where H^k stands for $HI_{\{|H|\leq k\}}$) is a Cauchy sequence for $\|\ \ \|_{\mathscr{S}(\mathbb{G})}$, hence the sequence $H^k\cdot X^{T_n-}$ is a Cauchy sequence for $\|\ \ \|_{\mathscr{S}(\mathbb{F})}$; this implies $H \in L_{\mathbb{F}}(X^{T_n-})$ (see Chou-Meyer-Stricker [1]), whence $H \in L_{\mathbb{F}}(X^{T_n})$. □

The filtration $\mathbb{G}$ still being larger than $\mathbb{F}$, we now study how pseudointegrability for $\mathbb{G}$ implies pseudointegrability for $\mathbb{F}$.

(2.10) THEOREM.
(a) *Every $\mathbb{F}$-adapted $\mathbb{G}$-pseudomartingale is an $\mathbb{F}$-pseudomartingale.*
(b) *Let X be a pseudomartingale for both filtrations, and H an $\mathbb{F}$-predictable process in $D_{\mathbb{G}}(X)$. Then H belongs to $D_{\mathbb{F}}(X)$, and $H \dot{_\mathbb{G}} X = H \dot{_\mathbb{F}} X$.*

PROOF. (a) This is a corollary of Theorem 12 of Meyer-Stricker [11] (if a càdlàg $\mathbb{F}$-adapted process is a $\mathbb{G}$-semimartingale in some open set, it is an $\mathbb{F}$-semimartingale in some larger $\mathbb{F}$-optional open set).

(b) There exists a countable covering (L_k) of $[\![0, \infty[\![$ by $\mathbb{F}$-optional intervals such that X^{L_k} is an $\mathbb{F}$-semimartingale. It suffices to prove the theorem for each X^{L_k}, thus we may suppose that X is an $\mathbb{F}$-semimartingale. By (2.6), there exists a sequence of intervals $J_k =]\!]s_k, S_k]\!]$ such that s_k are constant times, S_k are $\mathbb{G}$-stopping times, X^{J_k} and $(H \dot{_\mathbb{G}} X)^{J_k}$ are $\mathbb{G}$-semimartingales, and $(\cup_k J_k)^c$ is thin. Let $Y = H \dot{_\mathbb{G}} X$. Using Lemma (2.9), construct a family T_k^n of $\mathbb{F}$-stopping times such that $\sup_n T_k^n \geq S_k$ and $H \in L_{\mathbb{F}}(X^{I_k^n})$, where $I_k^n =]\!]s_k, T_k^n]\!]$. We are going to demonstrate that $H \dot{_\mathbb{F}} X^{I_k^n} = Y^{I_k^n}$; because the complementary of $\cup_{k,n} I_k^n$ is thin, and because $\Delta Y = H\Delta X$, this will prove the theorem.

So, restricting ourselves to I_k^n, we may suppose that $H \in L_{\mathbb{F}}(X)$ and we want to show that $H \dot{_\mathbb{F}} X = Y$. Lemma (2.8) implies that $Y - H \dot{_\mathbb{F}} X$ is constant on each interval $]\!]s_i, T_i^j \wedge S_i]\!]$, thus on $]\!]s_i, S_i[\![$. As these intervals cover $[\![0, \infty[\![$ but a thin closed set, and as $Y - H \dot{_\mathbb{F}} X$ is continuous (because $\Delta Y = H\Delta X$) and null at time zero, $Y = H \dot{_\mathbb{F}} X$. □

By what we have just seen, when the filtration is replaced by a smaller one, pseudointegrals behave as nicely as ordinary integrals. We now turn to the converse problem: enlarging the filtration. We are going to see that, in this case also, pseudomartingales behave like semimartingales: not better, but not worse. The filtration $\mathbb{G}$ being, as always, larger than $\mathbb{F}$, we try to deduce pseudointegrability in $\mathbb{G}$ from pseudointegrability in $\mathbb{F}$.

(2.11) THEOREM. *Suppose that, for $\mathbb{F}$, X is a pseudomartingale and H a predictable X-pseudointegrable process; suppose that, for $\mathbb{G}$, X and $H \dot{_{\mathbb{F}}} X$ are still pseudomartingales. Then H belongs to $D_{\mathbb{G}}(X)$ and $H \dot{_{\mathbb{G}}} X = H \dot{_{\mathbb{F}}} X$.*

PROOF. There exist $\mathbb{F}$-optional intervals L_n covering $[\![0, \infty[\![$ such that X^{L_n} is a semimartingale and $H \in L_{\mathbb{F}}(X^{L_n})$. Restricting ourselves to such an interval, we may suppose that X is an $\mathbb{F}$-semimartingale and $H \in L_{\mathbb{F}}(X)$. Put $Y = H \dot{_{\mathbb{F}}} X$. As X and Y are pseudomartingales, there exists by (2.6) a sequence of $\mathbb{G}$-optional intervals $J_n =]\!]s_n, S_n]\!]$ (where s_n are constant times) such that $(\cup_n J_n)^c$ is thin and X^{J_n} and Y^{J_n} are $\mathbb{G}$-semimartingales. Let $K = (1 + |H|)^{-1}$, so that K and KH are bounded and K never vanishes. The associativity property for stochastic integrals implies $K \dot{_{\mathbb{F}}} Y = KH \dot{_{\mathbb{F}}} X$, whence $(K \dot{_{\mathbb{F}}} Y)^{J_n} = (KH \dot{_{\mathbb{F}}} X)^{J_n}$. Applying Lemma (2.8) to the translated filtrations $(\mathscr{F}_t)_{t \geq s_k}$ and $(\mathscr{G}_t)_{t \geq s_k}$ now yields $K \dot{_{\mathbb{G}}} Y^{J_n} = KH \dot{_{\mathbb{G}}} X^{J_n}$. It is possible to integrate K^{-1} on the left-hand side, hence also on the right-hand side, and, using associativity again, one finds that H is in $L_{\mathbb{G}}(X^{J_n})$ with $H \dot{_{\mathbb{G}}} X^{J_n} = Y^{J_n}$. Because $(\cup_n J_n)^c$ is thin and $H \Delta X = \Delta Y$, this implies $H \in D_{\mathbb{G}}(X)$ and $H \dot{_{\mathbb{G}}} X = Y$. □

In this theorem, the assumption that $H \dot{_{\mathbb{F}}} X$ is a $\mathbb{G}$-pseudomartingale is not a consequence of the other hypotheses. A similar phenomenon is known to occur for semimartingales, as shown by an example due to Jeulin-Yor [8], where the processes are continuous. We give below an example of a semimartingale A (for both filtrations $\mathbb{F}$ and $\mathbb{G}$) and a process H in $L_{\mathbb{F}}(A)$ but not in $D_{\mathbb{G}}(A)$: the $\mathbb{F}$-semimartingale $H \dot{_{\mathbb{F}}} A$ is not a $\mathbb{G}$-pseudomartingale. This example involves jumps; there probably exist similar examples in the continuous case but we do not know any.

(2.12) EXAMPLE. For $n \geq 1$ and k odd with $0 < k < 2^n$, let $A^{n,k} = \varepsilon_{n,k} I_{[\![k2^{-n}, \infty[\![}$, where $\varepsilon_{n,k}$ are i.i.d. random variables with $P(\varepsilon_{n,k} = +1) = P(\varepsilon_{n,k} = -1) = ½$. For their natural filtration $\mathbb{F}$, $A = \sum_{n,k} 4^{-n} A^{n,k}$ and $M = \sum_{n,k} 2^{-n} A^{n,k}$ are square integrable martingales; moreover $M = H \cdot A$, where $H_{k2^{-n}} = 2^n$. For the constant filtration $\mathbb{G}$, where $\mathscr{G}_t = \mathscr{F}_1$ $\forall t$, A is still a semimartingale, because it has finite variation. But M has infinite variation on every non-empty open subset of $[0, 1]$; this implies that there exist no non-empty open subinterval of $[\![0, 1]\!]$ on which M is a $\mathbb{G}$-semimartingale, and M cannot be a $\mathbb{G}$-pseudomartingale.

To conclude this section, we now give the "pseudo" version of the Jacod-Meyer convexity property. We are given the space $(\Omega, \mathscr{F}, P)$, the filtration $\mathbb{F}$, and a (finite or infinite) sequence (P_n) of probabilities absolutely continuous with respect to P, such that the events $\Omega_n = \{dP_n/dP > 0\}$ (defined up to a P-null set) cover Ω. Let $\mathbb{F}_n$ denote the filtration $\mathbb{F}$ completed for P_n. According to (1.11), pseudointegration is not affected when $(\mathbb{F}, P)$ is replaced by $(\mathbb{F}_n, P_n)$. Here is a converse result.

(2.13) THEOREM.
(a) *If an $\mathbb{F}$-adapted process is an $(\mathbb{F}_n, P_n)$-pseudomartingale for each n, it is also an $(\mathbb{F}, P)$-pseudomartingale.*
(b) *Let, for $(\mathbb{F}, P)$, X be a pseudomartingale and H a predictable process. If, for each n, H is X-pseudointegrable under $(\mathbb{F}_n, P_n)$, then it is also X-pseudointegrable under $(\mathbb{F}, P)$.*

PROOF. (a) For each n, there exists a thin closed set E_n included in $\mathbb{R}_+ \times \Omega_n$ such that the given process Y is an $(\mathbb{F}_n, P_n)$ semimartingale in E_n^c. By Theorem 10 of Meyer-Stricker

[11], Y is an $(\mathbb{F}, P)$ semimartingale in the (non $\mathbb{F}$-optional) open set $O_n = E_n^c \cap (\mathbb{R}_+ \times \Omega_n)$, thus in the union $O = \cup_n O_n$; but O^c is thin because $O^c = \cup_n E_n$; so Y is an $(\mathbb{F}, P)$-pseudomartingale.

(b) Replace Ω_n by $\Omega'_n = \Omega_n \cap (\cup_{m<n}\Omega_m)^c$, define P'_n as the probability P_n conditioned by Ω'_n if $P_n(\Omega'_n) > 0$ and replace P_n by P'_n; drop the P_n for which $P_n(\Omega'_n) = 0$. This gives a new (perhaps shorter) sequence $(\mathbb{F}'_n, P'_n)$ which satisfies the hypothesis of the theorem and such that the events Ω'_n are pairwise disjoint. Hence we may suppose that the sets Ω_n are pairwise disjoint. Denoting the integral $H \cdot X$ computed under $(\mathbb{F}_n, P_n)$ by Y^n, there exists a càdlàg process Y such that, on $J_n = \mathbb{R}_+ \times \Omega_n$, $Y = Y^n$. Let $\mathbb{G}$ be the smallest enlargement of $\mathbb{F}$ such that $\mathscr{G}_0$ contains all the events Ω_n. As those events are disjoint, the P_n-completion of $\mathbb{G}$ is $\mathbb{G}_n = \mathbb{F}_n$. The process X is $\mathbb{G}$-adapted, and it is a $(\mathbb{G}_n, P_n)$-pseudomartingale for every n; by (a) X is a $(\mathbb{G}, P)$-pseudomartingale.

For fixed n, let L_k^n be a sequence of $(\mathbb{F}_n, P_n)$-optional intervals covering $[\![0, \infty[\![$ such that H is $X^{L_k^n}$-integrable with $H \cdot X^{L_k^n} = Y^{L_k^n}$ up to P_n-indistinguishability. The intervals $J_k^n = L_k^n \cap (\mathbb{R}_+ \times \Omega_n)$ are $\mathbb{G}_n$-optional, and H is in $L_{\mathbb{G}}(X^{J_k^n})$ up to P-indistinguishability. As $\cup_{n,k} J_k^n = [\![0, \infty[\![$, this implies that H is X-pseudointegrable for $(\mathbb{G}, P)$, with integral Y. By (2.10.b), H is also X-pseudointegrable for $(\mathbb{F}, P)$ with integral Y. □

3. Quadratic and decomposable pseudomartingales. In this section we shall attempt, given a pseudomartingale $Y = H \cdot X$, where X is a semimartingale, to split X into parts that make the integral easier to define or to study. In other words, we are looking for decompositions of pseudomartingales into simpler processes. The main result is (3.10): A pseudomartingale is the sum of a local martingale and a process that can be studied path by path if and only if it has a continuous martingale part.

Our first step will be to define $H \cdot X$, when possible, path by path. All that has been done so far can also be done in a deterministic setting, where we deal with functions on $[0, \infty)$ instead of processes, where càdlàg functions with bounded variation on compact sets replace semimartingales, and Stieltjes integrals replace stochastic integrals. In this frame, a measurable function h is naturally said to be pseudointegrable with respect to a càdlàg function a with bounded variation on compact sets if there exist a càdlàg f and a countable covering of $[0, \infty)$ by intervals J_n, such that, for each n, h is a^{J_n}-integrable (in the Stieltjes sense), and $f^{J_n} = h \cdot a^{J_n}$. Linearity, associativity, etc. ... remain true for those integrals (just apply the corresponding probabilistic properties to the case when P is a Dirac mass). There is no ambiguity in writing $D(a)$ for the space of all measurable a-pseudointegrable functions. Functions such as $h \cdot a$, where h is in $D(a)$, will be called *primitives of measures*; they are the deterministic counterpart of pseudomartingales. By (2.5), a primitive of measure can also be characterized by the existence of a countable covering of $[0, \infty)$ by intervals on which it has finite variation.

(3.1) Definition.

(a) A process is a primitive of measure if it is càdlàg, adapted and if (almost) all its paths are primitives of measure as functions.

(b) The process A being a primitive of measure, a process H is pathwise pseudointegrable with respect to A if, for (almost) all $\omega \in \Omega$, the function $H(\omega)$ is in $D(A(\omega))$.

The space of all *predictable* pathwise A-pseudointegrable processes will be denoted by $D_{\text{path}}(A)$. It is included in $D(A)$:

(3.2) Proposition. *Let A be a primitive of measure.*

(a) *A is a pseudomartingale. More precisely, there exist a predictable process $H \geq 1$, a process B with finite variation and a countable covering of $[\![0, \infty[\![$ by pairwise disjoint predictable intervals J_n such that $H \cdot B^{J_n}$ exists in the Stieltjes sense and is equal to A^{J_n}.*

(b) *If K is a predictable, pathwise A-pseudointegrable process, K is A-pseudointegrable, and both integrals ($K \cdot A$ and the integral computed path by path) agree.*

PROOF. (a) The proof uses the fact that the set $\mathscr{V}$ of all processes with finite variation is a complete t.v.s. for the quasinorm

$$\|X\|_{\mathscr{V}} = \sum_n 2^{-n} E\left[1 \wedge \int_0^n |dX_s|\right].$$

For $s > 0$, define stopping times T_n by

$$T_n^s = \inf\left\{t \geq s: \int_{(s,t]} |dA_u| \geq n\right\};$$

let $L_n^s =]\!]s, T_n^s[\![$ and $0 = \cup_{s,n} L_n^s$, where s ranges over the rational numbers. Each $A^{L_n^s}$ is a process with finite variation. Using the fact that 0^c is thin and that $A^{[\![T]\!]}$ is in $\mathscr{V}$ for each stopping time T, Lemma (1.1) gives a sequence (J_n) of pairwise disjoint predictable intervals such that each A^{J_n} is in $\mathscr{V}$. Now choose $\alpha_n \in (0, 1]$ such that $\|\alpha_n A^{J_n}\|_{\mathscr{V}} \leq 2^{-n}$, define

$$B = \sum_n \alpha_n A^{J_n}; \quad H = \sum_n \frac{1}{\alpha_n} I_{J_n},$$

and the required property is easy to check.

(b) Using the stopping times

$$T_n^s = \inf\left\{t \geq s: \int_{(s,t]} (1 + |K_u|)\,|dA_u| \geq n\right\},$$

it is easy to construct countably many optional intervals J_n such that A^{J_m} is in $\mathscr{V}$, $K \cdot A^{J_m}$ exists as a Stieltjes integral and is equal to Y^{J_m}, and $(\cup_m J_m)^c$ is thin. As $\Delta Y = K\Delta A$, this proves the proposition. □

So $D_{\text{path}}(A)$ is included in $D(A)$. What about the converse? In Example (2.12), H is in $D(A)$ (and even in $L(A)$), but, as $H \cdot A$ has infinite variation on every non-empty open set in (0, 1), $H \cdot A$ is not a primitive of measure and H cannot belong to $D_{\text{path}}(A)$: the converse is not always true. But it holds when A is predictable.

(3.3) PROPOSITION. *If A is a predictable primitive of measure, $D_{\text{path}}(A) = D(A)$.*

(3.4) LEMMA. *If X is a predictable pseudomartingale and if H is X-pseudointegrable, then $H \cdot X$ is predictable.*

This lemma is an obvious consequence of the following characterization of predictable processes (Dellacherie-Meyer [3]): A càdlàg adapted process Z is predictable if and only if $\Delta Z_T = 0$ for every totally inaccessible stopping time T and ΔZ_T is $\mathscr{F}_{T-}$-measurable for every predictable stopping time T. □

PROOF OF PROPOSITION (3.3). Let A be a predictable primitive of measure and H an element of $D(A)$. By the preceding lemma, $H \cdot A$ is predictable. There exists a countable covering of $[\![0, \infty[\![$ by predictable intervals J_n such that A^{J_n} is a semimartingale and $H \in L(A^{J_n})$. Both semimartingales A^{J_n} and $H \cdot A^{J_n}$ are predictable, thus special. Theorem 1 of Yan [13] shows that the integral $H \cdot A^{J_n}$ is pathwise computable. Hence H is in $D_{\text{path}}(A)$. □

The next natural question is the behaviour of integrals $H \cdot M$, where M is a local martingale. We need to define a restricted class of pseudomartingales:

(3.5) DEFINITION. A pseudomartingale Y is special if there exists a local martingale M such that $Y - M$ is predictable.

Of course, a semimartingale is special as a semimartingale if and only if it is special as a pseudomartingale: this definition creates no ambiguity. As a matter of fact, Schwartz has extended the definition of specialness to formal semimartingales.

(3.6) THEOREM. *Let M be a local martingale, H a predictable, M-pseudointegrable process. If the integral $H\cdot M$ is special, it is a local martingale, and H is M-integrable.*

PROOF. Let $H\cdot M = N + Z$, where N is a local martingale and Z a predictable process; let $\mathcal{J}$ denote the class of all predictable intervals J such that $H \in L(M^J)$ and $H\cdot M^J$ is a local martingale. We shall apply Lemma (1.2) to $\mathcal{J}$. Hypotheses (i) and (ii) are obviously fulfilled, (iv) is a consequence of (1.12). It remains to show that, if J_n is an increasing sequence in $\mathcal{J}$, its limit J also belongs to $\mathcal{J}$.

We have

$$H\cdot M^J = N^J + Z^J; \quad H\cdot M^{J_n} = N^{J_n} + Z^{J_n}.$$

Because each $H\cdot M^{J_n}$ is a local martingale, so is Z^{J_n}. We want to show that Z^J is also a local martingale. Since Z is predictable, the increasing process $Z_t^{*2} = \sup_{s\leq t} Z_s^2$ is locally integrable; thus, by stopping, we may assume that Z_∞ exists and Z_∞^* is in L^2. Now Z^{J_n} is a square integrable martingale such that, when n goes to infinity, $Z_\infty^{J_n}$ converges to Z_∞^J, the convergence being dominated by 2 Z_∞^*; thus Z^{J_n} tends to a limit in the space of square integrable martingales. As Z^{J_n} goes to Z^J pointwise, identifying limits shows that Z^J is a martingale. □

When it is not special, the integral $H\cdot M$ need not be a semimartingale. Here is an example, borrowed from Lepingle-Mémin [9] via Meyer-Stricker [11]:

Let (t_n) be a strictly monotone convergent sequence in $[0, \infty)$. Define a process Y, constant but for independent jumps at times t_n, by

$$\Delta Y_{t_n} = \begin{cases} (-1)^{n+1}\left(n - \dfrac{1}{n}\right) \text{ with probability } \dfrac{1}{n^2} \\[2ex] (-1)^n \dfrac{1}{n} \text{ with probability } 1 - \dfrac{1}{n^2} \end{cases}$$

With probability one, all but finitely many jumps have the value $(-1)^n 1/n$, and the series $\sum_n \Delta Y_{t_n}$ converges; thus Y is a pseudomartingale. As $E[\Delta Y_{t_n}] = 0$, it is easy to write Y as an integral with respect to a square integrable martingale. But Y is not a semimartingale, for if H is a predictable process with $H_{t_n} = (-1)^n$, the integral $H\cdot Y$ does not exist (it should have a jump of size $1/n$ at time t_n for all but finitely many n).

Given a pseudomartingale Y (or more generally a formal semimartingale), it is easy to see that its "continuous local martingale" part Y^c and its quadratic variation $[Y, Y]$ are well defined as formal semimartingales: writing $Y = H\cdot X$, where X is a semimartingale, one verifies that neither $H\cdot X^c$ nor $H^2\cdot[X, X]$ depend on the choice of H and X. If $[Y, Y]$ is a pseudomartingale (i.e. if $H^2 \in D([X, X])$), it is an increasing process and $H^2\cdot[X, X]$ is a Stieltjes integral. Such pseudomartingales Y will be called *quadratic* pseudomartingales; we shall come back to them later. If Y^c is a pseudomartingale, by (3.6) Y^c is a local martingale and $H \in L(X^c)$. When this happens, we shall say that Y is a *decomposable* pseudomartingale because of (3.10) below. To avoid the statement "a pseudomartingale is decomposable if and only if it has a decomposition," this name will be used only after (3.10); temporarily we shall merely say that "Y^c exists." There are pseudomartingales Y (even continuous ones) such that Y^c does not exist.

(3.7) EXAMPLE. We need three ingredients: a continuous local martingale M on $[0, 1981)$ with explosion at time 1981, so that $[M, M]_{1981} = +\infty$ (e.g.

$$M_t = \int_0^t \frac{1}{1981 - s}\, dB_s,$$

where B is a brownian motion), a diffeomorphism f between $[0, 1981)$ and $[1, \infty)$$\left(\text{e.g. } f(t) = -\frac{1}{1981 - t}\right)$, and a bounded difference of two convex functions, ϕ, such that $|\phi'_{\text{left}}(x)| = 1$ for all x (e.g. $\phi(x)$ is the distance from x to the nearest even integer). Define

$$Y_t = \begin{cases} \frac{1}{f(t)}\phi(f(t)M_t) & \text{for} \quad 0 \le t < 1981 \\ 0 & \text{for} \quad t \ge 1981. \end{cases}$$

Because $|Y_t| \le 1/f(t)\, \|\phi\|_\infty$ for $t < 1981$, Y is continuous; being a semimartingale on each interval $[0, 1981 - \varepsilon]$, Y is a pseudomartingale.

The Tanaka-Meyer formula shows that

$$N_t = \int_0^t \phi'_{\text{left}}(f(s)M_s)\,dM_s$$

is the martingale part of the semimartingale Y on each interval $[0, 1981 - \varepsilon]$. For $t < 1981$, $[N, N]_t = [M, M]_t$; thus N explodes at time 1981, and there is no pseudomartingale agreeing with N on $[0, 1981)$; hence Y^c is not a pseudomartingale.

We arrive at the decomposition theorem. The proof uses the space $\mathscr{H}^1$ of martingales M such that $[M, M]_\infty^{1/2}$ is integrable; the reader who likes hieroglyphics can decipher [4] and use the topology of local martingales instead; he will save a localisation in the proof of the theorem.

(3.8) LEMMA. *Let M be a martingale in $\mathscr{H}^1$ with $M^c = 0$ and H a predictable process M-integrable in $\mathscr{H}^1$. There exists a sequence of martingales with finite variation (M^n) in $\mathscr{H}^1$, such that each $H\cdot M^n$ exists in the Stieltjes sense, and* $\lim_n M^n = M$, $\lim_n H\cdot M^n = H\cdot M$ *in $\mathscr{H}^1$.*

PROOF. Let (T_k) be a sequence of stopping times with disjoint graphs that exhausts all jumps of M; define

$$A^n = \textstyle\sum_{k\le n} \Delta M_{T_k} I_{[\![T_k,\infty[\![}.$$

Compensating A^n yields a martingale M^n in $\mathscr{H}^1$ with finite variation; as $[M^n, M^n] \le [M, M]$, one has

$$(H^2\cdot[M^n, M^n])^{1/2} \le (H^2\cdot[M, M]^{1/2}),$$

and H is M^n-integrable in $\mathscr{H}^1$. But $H\cdot M^n - H\cdot A^n$ is predictable; hence $H\cdot M^n$ has finite variation and this integral exists in the Stieltjes sense. Last, M^n goes to M in $\mathscr{H}^1$ for

$$[M - M^n, M - M^n]_\infty^{1/2} = (\textstyle\sum_{k>n} \Delta M_{T_k}^2)^{1/2}$$

goes to zero in L^1 (convergence dominated by $[M, M]_\infty^{1/2}$); similarly, $H\cdot M^n$ goes to $H\cdot M$ in $\mathscr{H}^1$. □

(3.9) THEOREM. *Let $Y = H\cdot X$, where X is a semimartingale and H a predictable X-pseudointegrable process. Then Y^c exists if and only if $X = M + A$, where M is a local martingale, H is M-integrable in the sense of local martingales, A is a primitive of measure and H is in $D_{\text{path}}(A)$.*

PROOF. If such a decomposition exists, $H\cdot M^c$ exists; let J_n be a sequence of intervals covering $[\![0, \infty[\![$ with A^{J_n} in $\mathscr{V}$ and $H \in L(A^{J_n})$; as $(A^{J_n})^c = 0$, $(H\cdot A^{J_n})^c = 0$, whence $(H\cdot A)^c$ exists (and is zero). Thus Y^c exists.

Conversely, suppose Y^c exists. We already know that H is X^c-integrable, where X^c is the continuous local martingale part of X; it remains to find a decomposition of $X - X^c$. Thus we may suppose $X^c = Y^c = 0$. There exist disjoint predictable intervals J_n such that $H \in L(X^{J_n})$; hence there exists for each n a decomposition $N^n + B^n$ of X^{J_n} such that

$H \cdot N^n$ exists in the sense of local martingales and $H \cdot B^n$ in the Stieltjes sense. We may suppose

$$N^n = I_{J_n} \cdot N^n; \quad B_n = I_{J_n} \cdot B_n.$$

As the result we want to prove is local, we may, by stopping, suppose N^n and $H \cdot N^n$ in $\mathscr{H}^1$. The preceding lemma gives a sequence $N^{n,k}$ of martingales with bounded variation such that $H \cdot N^{n,k}$ exists in the Stieltjes sense and, in $\mathscr{H}^1$,

$$\lim_k N^{n,k} = N^n; \quad \lim_k H \cdot N^{n,k} = H \cdot N^n.$$

For each n, chose $k(n)$ such that

$$\| N^{n,k(n)} - N^n \|_{\mathscr{H}^1} + \| H \cdot N^{n,k(n)} - H \cdot N^n \|_{\mathscr{H}^1} \le 2^{-n};$$

define $M = \sum_n (N^n - N^{n,k(n)})$, where the series converges in $\mathscr{H}^1$. One has $H \cdot M = \sum_n H \cdot (N^n - N^{n,k(n)})$ and H is M-integrable in the martingale sense. Putting $A = X - M$,

$$A^{J_n} = (N^n + B^n) - (N^n - N^{n,k(n)}) = B^n + N^{n,k(n)}$$

shows that A^{J_n} has finite variation and $H \cdot A^{J_n}$ exists in the Stieltjes sense, whence A is a primitive of measure and $H \in D_{\text{path}}(A)$. □

(3.10) Corollary. *A pseudomartingale Y can be decomposed into a local martingale and a primitive of measure if and only if Y^c exists.*

The next two theorems treat the canonical decomposition of a special decomposable pseudomartingale.

(3.11) Theorem. *A pseudomartingale can be decomposed into a local martingale M and a predictable primitive of measure if and only if it is special and decomposable. If one requires $M_0 = 0$, this decomposition is unique.*

Proof. The necessity is obvious. For sufficiency, suppose a pseudomartingale Y is both special and decomposable. As Y is special, $Y = N + Z$, where N is a local martingale and Z a predictable process; define $M = N + Y^c - N^c$, $A = Y - M$, so that $Y = M + A$ with $M^c = Y^c$ and A predictable. It suffices to prove that A is a primitive of measure. As Y is decomposable, $Y = L + B$, where L is a local martingale with $L^c = Y^c$ and B a primitive of measure. There exist predictable intervals J_n covering $[\![0, \infty[\![$ such that B^{J_n} has finite variation. Thus $A^{J_n} = B^{J_n} + (L - M)^{J_n}$ is a predictable semimartingale, with $(A^{J_n})^c = 0$. This implies that A^{J_n} has finite variation, and A is a primitive of measure.

As for unicity, if a local martingale N with $N_0 = 0$ is also a predictable primitive of measure, there exist predictable intervals J_n covering $[\![0, \infty[\![$ such that N^{J_n} is a predictable local martingale, null at time zero, with finite variation. Hence $N^{J_n} = 0$, and $N = 0$. □

(3.12) Proposition. *Let $X = M + A$ and $Y = N + B$ be two special decomposable pseudomartingales, with their canonical decompositions; let H be in $D(X)$ with $H \cdot X = Y$. Then H is in $L(M)$ with $H \cdot M = N$ and in $D_{\text{path}}(A)$ with $H \cdot A = B$.*

Proof. Choose predictable intervals J_n covering $[\![0, \infty[\![$ such that A^{J_n} and B^{J_n} have finite variation. The semimartingales X^{J_n} and Y^{J_n} are special, with canonical decompositions $M^{J_n} + A^{J_n}$ and $N^{J_n} + B^{J_n}$; moreover $H \in D(X^{J_n})$ with $H \cdot X^{J_n} = Y^{J_n}$, which implies, by (1.15), that $H \in L(X^{J_n})$. Now Theorem 1 of Yan [13] implies that

$$H \in L(M^{J_n}) \quad \text{with} \quad H \cdot M^{J_n} = N^{J_n}$$

$$H \in L(A^{J_n}) \quad \text{with} \quad H \cdot A^{J_n} = B^{J_n}.$$

So $H \in D(M)$ with $H \cdot M = N$, and $H \in D(A)$ with $H \cdot A = B$. By (1.15), $H \in L(M)$ and by (3.3) $H \in D_{\text{path}}(A)$. □

We now turn to quadratic pseudomartingales. We shall show that they are decomposable, and that they give rise to a nice characterization of specialness and to a more accurate version of (3.9).

Recall that a pseudomartingale Y is quadratic if, when written $H\cdot X$ with X semimartingale and $H \in D(X)$, H^2 is $[X, X]$-pseudointegrable (hence also integrable in the Stieltjes sense); this does not depend on the choice of H and X.

(3.13) PROPOSITION. (a) *Quadratic pseudomartingales are a vector space, in which*

$$[Y+Z, Y+Z]^{1/2} \leq [Y, Y]^{1/2} + [Z, Z]^{1/2}$$

holds.
(b) *Every quadratic pseudomartingale is decomposable.*

REMARK. The Kunita-Watanabe inequality also holds for quadratic pseudomartingales; we won't need it.

PROOF. (a) Write $Y = H\cdot U$ and $Z = K\cdot V$, with U and V semimartingales, and approximate H and K by bounded predictable processes H^n and K^n. The desired inequality is true for $Y^n = H^n\cdot U$ and $Z^n = K^n\cdot V$; taking limits, it holds for Y and Z.

(b) Write $Y = H\cdot X$, where X is a semimartingale. As $H^2\cdot[X, X]$ exists, so does $H^2\cdot\langle X^c, X^c\rangle$, and the integral $H\cdot(X^c)$ exists in the sense of local martingales. □

Of course, the converse is not true. Even in the deterministic case, though all pseudomartingales are decomposable, some are not quadratic, for the convergence of the series $\sum x_n$ dose not imply $\sum x_n^2 < \infty$.

(3.14) PROPOSITION. *A quadratic pseudomartingale Y is special if and only if the increasing process $[Y, Y]^{1/2}$ is locally integrable.*

PROOF. If Y is special, then $Y = M + A$ where M is a local martingale and A a predictable primitive of measure. As $[M, M]^{1/2}$ is locally integrable, it suffices to show that $[A, A]^{1/2}$ has the same property. But, because A is predictable, so are $[A, A]$ (see (3.4)) and $[A, A]^{1/2}$; this gives the result.

Conversely, suppose $[Y, Y]^{1/2}$ is locally integrable; we want to show that Y is special. As specialness is a local property, we may suppose $[Y, Y]_\infty^{1/2}$ is integrable. Let (T_n) be a sequence of stopping times with disjoint graphs that exhausts the jumps of Y; define

$$A^n = \textstyle\sum_{k\leq n} \Delta Y_{T_k} I_{[\![T_k,\infty[\![}.$$

As each ΔY_{T_k} belongs to L^1, A^n is integrable. Compensating A^n yields a martingale M^n such that (Dellacherie-Meyer [3] page 307)

$$E[[M^n, M^n]_\infty^{1/2}] \leq 3E[[A^n, A^n]_\infty^{1/2}].$$

Now $[A^n, A^n]_\infty = \sum_{k\leq n} \Delta Y_{T_k}^2 \leq [Y, Y]_\infty$ implies that the sequence (M^n) is bounded in the space $\mathscr{H}^1$. As the increments $M^{n+1} - M^n$ are orthogonal in the sense of Lemma 3 of Meyer-Stricker [11], the sequence (M^n) converges, in $\mathscr{H}^1$, to a limit M. If T is a totally inaccessible stopping time,

$$\Delta M_T = \lim_n \Delta A_T^n = \Delta Y_T;$$

if T is predictable,

$$\Delta M_T = \lim_n (\Delta A_T^n - E[\Delta A_T^n \mid \mathscr{F}_{T-}]) = \Delta Y_T - E[\Delta Y_T \mid \mathscr{F}_{T-}].$$

Putting $Z = Y - M$, this shows that $\Delta Z_T = 0$ for totally inaccessible T and $\Delta Z_T = E[\Delta Z_T \mid \mathscr{F}_{T-}]$ for predictable T. Hence Z is predictable and Y special.

A consequence of this proposition is the possibility of making special a given quadratic pseudomartingale by a suitable change of probability. For a nonquadratic pseudomartingale, this is false in general, even in the decomposable case. Here is an example of a (nonquadratic) bounded pseudomartingale Y, with $Y^c = 0$, that is not the sum of a semimartingale and a predictable process (hence no change of probability can make Y special).

Let (T_n) be a sequence of independent random variables with uniform laws on $(2^{-n-1}, 2^{-n}]$ respectively. The process

$$Y = \sum_n (-1)^n\, n^{-1/2} I_{\llbracket T_n,\infty\llbracket}$$

is a pseudomartingale for its natural filtration (it is even a primitive of measure); the stopping times T_n are totally inaccessible. If $Y = X + Z$ where Z is a predictable pseudomartingale, then $\Delta X_{T_n} = (-1)^n n^{-1/2}$, whence

$$[X, X]_1 \geq \sum_n \Delta X^2_{T_n} = +\infty,$$

and X cannot be a semimartingale.

The next proposition shows that, for quadratic pseudomartingales, a slightly stronger version of Theorem (3.9) holds: the primitive of measure A can be chosen with finite variation. Though this proposition is stated only for quadratic pseudomartingales, it also holds, with the same proof, for pseudomartingales that can be made special by an equivalent change of probability, i.e. pseudomartingales of the form $X + Z$, where X is a semimartingale and Z a predictable process.

(3.15) PROPOSITION. *Let X be a semimartingale, H a predictable X-pseudointegrable process such that $H \cdot X$ is quadratic. Then X can be decomposed into a local martingale M and a process A with finite variation such that H is M-integrable in the sense of local martingales and pathwise A-pseudointegrable.*

PROOF. Let $Y = H \cdot X$. There exists a probability Q equivalent to P that makes $[X, X]^{1/2}$ and $[Y, Y]^{1/2}$ locally integrable. For Q, X and Y are special; let $N + B$ be the canonical decomposition of X. Proposition (3.12) shows that H is in $L(N)$ and in $D_{\text{path}}(B)$; though this is proved using Q, it also holds under P. Now split N into $M + C$, where M and $H \cdot M$ are local martingales (for P) and C has finite variation and $H \cdot C$ is a Stieltjes integral. The process H is pathwise pseudointegrable with respect to B and C, thus also to $A = B + C$. □

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1) The standard reference on Schwartz'formal semimartingales is:

SCHWARTZ, L. (1981). Les semimartingales formelles. Séminaire de Probabilités XV. *Lecture Notes in Math.* **850**. Springer, Berlin.

2) Proof of Lemma (1.2) has been shortened and question preceding (2.12) has been answered by

ABOULAICH, R. (1982). Intégrales stochastiques généralisées. To appear in Séminaire de Probabilités XVI. *Lecture Notes in Math.* Springer, Berlin.

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