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A FUNCTIONAL CENTRAL LIMIT THEOREM FOR WEAKLY DEPENDENT SEQUENCES OF RANDOM VARIABLES

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Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of r.v.'s with $E\,X_n = 0$, $E(\sum_{i=1}^n X_i)^2/n \to \sigma^2 > 0$, $\sup_{n,m}E(\sum_{i=m+1}^{m+n} X_i)^2/n < \infty$. We prove the functional c.l.t. for (X_n) under assumptions on $\alpha_n(k) = \sup\{|P(A\cap B) - P(A)P(B)|: A \in \sigma(X_i\colon 1\le i\le m)$, $B\in\sigma(X_i\colon m+k\le i\le n)$, $1\le m\le n-k\}$ and the asymptotic behaviour of $\|X_n\|_\beta$ for some $\beta\in(2,\infty]$. For the special cases of strongly mixing sequences (X_n) with $\alpha(k) = \sup \alpha_n(k) = O(k^{-a})$ for some $a>1$, or $\alpha(k) = O(b^{-k})$ for some $b>1$, we obtain functions $f_\beta(n)$ such that $\|X_n\|_\beta = o(f_\beta(n))$ for some $\beta\in(2,\infty]$ is sufficient for the functional c.l.t., but the c.l.t. may fail to hold if $\|X_n\|_\beta = O(f_\beta(n))$.

1. Introduction and results. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of r.v.'s on some probability space $(\Omega, \mathscr{A}, P)$, satisfying

$$EX_n = 0,\quad EX_n^2 < \infty \quad\text{for}\quad n\in\mathbb{N}. \tag{1.1}$$

Put $S_n = \sum_{i=1}^n X_i$ for $n\in\mathbb{N}$. In this paper we will make the following assumptions on the variances:

$$ES_n^2/n \to_{n\in\mathbb{N}} \sigma^2 > 0 \quad\text{for some}\quad \sigma>0 \tag{1.2}$$

$$\sup\{E(S_{m+n} - S_m)^2/n : m, n\in\mathbb{N}\} < \infty. \tag{1.3}$$

Consider the space $D = D[0, 1]$ endowed with the Skorokhod topology (see Billingsley, 1968, Section 14) with Borel-σ-algebra $\mathscr{B}$ and define random functions $W_n:\Omega\to D$ by

$$W_n(t) = S_{[nt]}/(\sigma n^{1/2}) \quad\text{for}\quad t\in[0,1],\quad n\in\mathbb{N}. \tag{1.4}$$

W_n is a measurable map from $(\Omega, \mathscr{A})$ into $(D, \mathscr{B})$. If (W_n) is weakly convergent to a standard Brownian motion W on D, then (X_n) is said to satisfy the invariance principle (i.p.). In this paper we will use the mixing coefficients $\alpha_n(k)$, defined by

$$\alpha_n(k) = \sup\{|P(A\cap B) - P(A)P(B)| : A\in\sigma(X_i : 1\le i\le m),$$
$$B\in\sigma(X_i : m+k\le i\le n),\ 1\le m\le n-k\},\quad k\le n-1$$
$$\alpha_n(k) = 0 \quad\text{for}\quad k\ge n.$$

The coefficient of strong mixing introduced by Rosenblatt (1956) then can be written as

$$\alpha(k) = \sup_{n\in\mathbb{N}} \alpha_n(k) \quad\text{for}\quad k\in\mathbb{N}.$$

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Let $\| X \|_\beta$ be defined in the usual way, i.e.

$$\| X \|_\beta = E^{1/\beta} | X |^\beta \qquad \text{for} \quad \beta \in [1, \infty)$$

$$\| X \|_\beta = \operatorname{ess\,sup} | X | \quad \text{for} \quad \beta = \infty.$$

The following theorem is the main result of this paper.

THEOREM. *Let $\beta \in (2, \infty]$ and $\gamma = 2/\beta$ $(2/\infty = 0)$. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in* $[1, \infty)$. *If* (X_n) *satisfies* (1.1), (1.2), (1.3) *and*

$$(1.5) \qquad (\sup_{i \le n} \| X_i \|_\beta^2)(\textstyle\sum_{i \ge a_n} \alpha_n(i)^{1-\gamma} + a_n^{2-\gamma}/n^{1-\gamma}) \to_{n \in \mathbb{N}} 0,$$

then the random functions W_n, $n \in \mathbb{N}$, converge in distribution to the standard Brownian motion W.

The first corollary shows that several known results for strongly mixing sequences are contained in the above theorem.

COROLLARY 1: *Let $\beta \in (2, \infty]$ and $\gamma = 2/\beta$. If (X_n) satisfies* (1.1), (1.2) *and*

$$(1.6) \qquad \textstyle\sum_{i \in \mathbb{N}} \alpha(i)^{1-\gamma} < \infty \quad \textit{and} \quad \limsup_{n \in \mathbb{N}} \| X_n \|_\beta < \infty,$$

then (W_n) converges to W in distribution.

The c.l.t. which follows from Corollary 1 has been proved for the strictly stationary case by Ibragimov (1962), and the general case can be obtained from Theorem 2.1 (A) of Withers (1981). Yoshihara and Oodaira (1972) proved the i.p. for strictly stationary sequences under the assumption (1.6), improving a result of Davydov (1968), but their method does not admit an easy generalization to the non-stationary situation considered in this paper, since the uniform integrability of $(S_{m+n} - S_m)^2/n$, $n, m \in \mathbb{N}$, and of $| X_n |^\beta$, $n \in \mathbb{N}$, if $\beta < \infty$, is crucial for their argument. Using his theory of mixingales, McLeish (1975) obtained an i.p. for the non-stationary case. The above Corollary 1 improves McLeish's result, by weakening and simplifying the mixing condition and dropping the assumption 3.8 (c) of his paper.

The following corollary shows that the moment condition in (1.6) can be considerably relaxed, if one imposes stronger mixing conditions and adds the assumption (1.3).

COROLLARY 2. *Let $\beta \in (2, \infty]$ and $\gamma = 2/\beta$. Assume that (X_n) satisfies* (1.1), (1.2), (1.3) *and one of the following conditions*

$$(1.7) \qquad \begin{gathered} \alpha(k) = O(k^{-a}) \quad \textit{and} \\ \| X_n \|_\beta = o(n^{(1-\gamma)/2-(1-\gamma/2)/(1+a)(1+a)}) \quad \textit{for some} \quad a > 1/(1-\gamma) \end{gathered}$$

$$(1.8) \qquad \begin{gathered} \alpha(k) = O(b^{-k}) \quad \textit{for some} \quad b > 1 \quad \textit{and} \\ \| X_n \|_\beta = o(n^{(1-\gamma)/2}/(\log n)^{1-\gamma/2}) \end{gathered}$$

then (W_n) converges to W in distribution.

The moment conditions in the above corollary cannot be weakened. In Section 3 we will show by examples that the c.l.t. may fail to hold, if the moment condition in (1.7) or (1.8) is weakened, by replacing "o" by "O".

In the following corollary the above theorem is applied to a class of processes which need not satisfy $\alpha(k) \to 0$. Define for $n \in \mathbb{N}$:

$$m_n = \inf\{m \in \mathbb{N} \cup \{0\}: (X_i: 1 \le i \le k) \text{ and } (X_i: k + m < i \le n) \text{ are independent for every } k \in \{1, \cdots n - m - 1\}\}.$$

COROLLARY 3. *Let $\beta \in (2, \infty]$ and $\gamma = 2/\beta$. Assume that (X_n) satisifies* (1.1), (1.2), (1.3) *and*

$$\sup_{i \le n} \| X_i \|_\beta = o(n^{(1-\gamma)/2}/m_n^{1-\gamma/2}). \tag{1.9}$$

Then (W_n) converges to W in distribution.

In general (1.9) cannot be weakened. The processes in Example 1 and 2 of Section 3 satisfy $\sup_{i \le n} \| X_i \|_\beta = O(n^{(1-\gamma)/2}/m_n^{1-\gamma/2})$, but the c.l.t. does not hold in these examples.

If $m = \sup_{n \in \mathbb{N}} m_n < \infty$, then (X_n) is called m-dependent. For such sequences we obtain:

COROLLARY 4. *Let $\beta \in (2, \infty]$ and $\gamma = 2/\beta$. Assume that (X_n) is m-dependent and satisfies* (1.1), (1.2), (1.3)

$$\| X_n \|_\beta = o(n^{(1-\gamma)/2}). \tag{1.10}$$

Then (W_n) converges to W in distribution.

The condition (1.10) cannot be weakened even in the independent case. If $(X_n)_{n \in \mathbb{N}}$ are independent r.v.'s with $P\{X_n = \pm n^{1/2}\} = 1/(2n)$, then as a consequence of the classical Lindeberg Theorem the c.l.t. is not valid, and one has $\| X_n \|_\beta = n^{(1-\gamma)/2}$. In Example 3 of Section 3 we will show that it is impossible to find $\bar{\alpha}(k) > 0$ and $\bar{m}_n \in \mathbb{N} \cup \{0\}$ with $\bar{m}_n \to \infty$ such that every process (X_n) which fulfills (1.1), (1.2), (1.3), (1.10) and $\alpha(k) \le \bar{\alpha}(k)$, $m_n \le \bar{m}_n$ satisfies the c.l.t. Hence the assumption of m-dependence in Corollary 4 cannot be replaced by any weaker assumption of strong mixing and m_n-dependence without strengthening the moment condition.

Under the assumptions of our Theorem one can find $p_n \in \mathbb{N}$ such that $p_n = o(n^{1/2})$ and $\alpha_n(p_n - k) \to_{n \in \mathbb{N}} 0$ for every $k \in N$. Therefore it follows from arguments of Billingsley (1968) that the assertion of our Theorem can be strengthened to "(W_n) is R-mixing to W", i.e. $P(W_n \in B \mid A) \to_{n \in \mathbb{N}} W(B)$ for every W-continuity set $B \in \mathscr{B}$ and every $A \in \mathscr{A}$ with $P(A) > 0$. This extension is useful for the study of weak convergence, if the indices are r.v.'s (see Durrett and Resnick, 1977). The proof of our results is given in Section 2. To prove the c.l.t. we use Bernstein's blocking argument, which is discussed in Lemma 3.1 of Withers (1981). We do not use Theorem 2.1 of Withers, since his moment assumptions are too strong for our purpose. The tightness of (W_n) is also established by a blocking argument. An important tool is Lemma 2.2, which extends an inequality proved by Ottaviani for independent r.v.'s.

The following notations are frequently used:

$[x] = \max\{n \in Z: n \leq x\}$ for $x \in \mathbb{R}$.

$a_n \sim b_n$ means $a_n/b_n \to_{n\in\mathbb{N}} 1$.

For a set $A \subset \Omega$, $I(A)$ is the indicator function of A, i.e. $I(A)(\omega) = 1$ for $\omega \in A$ and $I(A)(\omega) = 0$ for $\omega \notin A$. For a collection $\mathscr{Y}$ of r.v.'s $\sigma(\mathscr{Y})$ denotes the σ-algebra generated by these r.v.'s.

2. Proofs. The following lemma follows from formula (2.2) of Davydov (1968).

2.1 LEMMA. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of* r.v.'s *with $EX_n = 0$. Let $\beta \in (2, \infty]$ and $\gamma = 2/\beta$. Then for all $i, j \in \{1, \cdots n\}$ with $i \neq j$ the following inequality holds*:

$$|EX_iX_j| \leq 12\alpha_n(|i-j|)^{1-\gamma} \|X_i\|_\beta \|X_j\|_\beta.$$

The proof of the following lemma is omitted, since it is very similar to the proof of Lemma 1.1.6 of Iosifescu, Theodorescu (1969).

2.2 LEMMA. *Let $\eta_1, \cdots \eta_n$ be* r.v.'s. *Put*

$$\alpha = \sup\{|P(A \cap B) - P(A)P(B)|: A \in \sigma(\eta_1, \cdots \eta_k),\ B \in \sigma(\eta_{k+1}, \cdots \eta_n), 1 \leq k \leq n-1\}.$$

Then for every $\varepsilon > 0$:

$$P\{\max_{1\leq r\leq n} |\textstyle\sum_{i=1}^r \eta_i| > \varepsilon\} \leq \frac{P\left\{|\sum_{i=1}^n \eta_i| > \frac{\varepsilon}{2}\right\} + n\alpha}{\min_{1\leq r\leq n-1} P\left\{|\sum_{i=r+1}^n \eta_i| \leq \frac{\varepsilon}{2}\right\}}.$$

PROOF OF THE THEOREM. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of r.v.'s. Assume that (1.1), (1.2), (1.3) are satisfied. Let $\beta \in (2, \infty]$, $\gamma = 2/\beta$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $[1, \infty)$ such that (1.5) is fulfilled. We define a sequence $(b_n)_{n\in\mathbb{N}}$ in $[1, \infty)$ by

$$b_n^{2-\gamma}/n^{1-\gamma} = \textstyle\sum_{i\geq a_n} \alpha_n(i)^{1-\gamma} + a_n^{2-\gamma}/n^{1-\gamma}.$$

Then the following conditions are satisfied:

(2.3) (a) $b_n^{2-\gamma}/n^{1-\gamma} \to_{n\in\mathbb{N}} 0$

(b) $\sum_{i\geq b_n} \alpha_n(i)^{1-\gamma} \leq b_n^{2-\gamma}/n^{1-\gamma}$

(c) $\sup_{i\leq n} \|X_i\|_\beta^2 = o(n^{1-\gamma}/b_n^{2-\gamma})$.

Now we will construct sequences of positive integers $p(n)$, $q(n)$ with the following properties:

(2.4) (a) $\dfrac{q(n)}{p(n)} \longrightarrow 0$

(b) $\dfrac{p(n)}{n} \longrightarrow 0$

(c) $\sum_{i>q(n)} \alpha_n(i)^{1-\gamma} \sup_{i\le n} \| X_i \|_\beta^2 \longrightarrow 0$

(d) $\dfrac{p(n)^{2-\gamma}}{n^{1-\gamma}} \sup_{i\le n} \| X_i \|_\beta^2 \longrightarrow 0$

(e) $\dfrac{n}{p(n)} \alpha_n(q(n)) \longrightarrow 0$

(2.3) (a) and (c) imply that there exists a sequence $r(n)$ of positive integers with $r(n) \longrightarrow \infty$, $r(n)b(n) = o(n)$ and $\sup_{i\le n} \| X_i \|_\beta^2 = o(n^{1-\gamma}/(r(n)^{2-\gamma} b_n^{2-\gamma}))$. Choose $q(n) = 2 \inf\{i \in \mathbb{N}: i \ge b_n\}$ and $p(n) = r(n)q(n)$. Then (2.4) (a)–(d) follow immediately. Since $k \longrightarrow \alpha_n(k)$ is non-increasing, one obtains

$$\left(\frac{n}{p(n)} \alpha_n(q(n))\right)^{1-\gamma} \le \frac{n^{1-\gamma}}{p(n)^{1-\gamma}} \frac{1}{b_n} \sum_{i\ge b_n} \alpha_n(i)^{1-\gamma} \le \frac{b_n^{1-\gamma}}{p(n)^{1-\gamma}} \longrightarrow 0.$$

Now we proceed as in Lemma 3.1 of Withers (1981). With $p = p(n)$, $q = q(n)$ and $k = [n/(p+q)]$ we define

$$\xi_j = \sum \{X_i : j(p+q) + 1 \le i \le j(p+q) + p\} \qquad 0 \le j \le k-1$$

$$\zeta_j = \sum \{X_i : j(p+q) + p + 1 \le i \le (j+1)(p+q)\} \quad 0 \le j \le k-1$$

$$\zeta_k = \sum \{X_i : k(p+q) + 1 \le i \le n\}$$

$$S_n' = \sum_{j=0}^{k-1} \xi_j \qquad S_n'' = \sum_{j=0}^{k} \zeta_j.$$

To prove the c.l.t. it suffices to show that for $n \longrightarrow \infty$

(2.5) (a) $n^{-1} E S_n''^2 \longrightarrow 0$

(b) $n^{-1} \sum_{0\le i<j\le k-1} | E\xi_i \xi_j | \longrightarrow 0$

(c) $k \longrightarrow \infty$

(d) $| E \exp(iuS_n') - \prod_{j=0}^{k-1} E \exp(iu\xi_j) | \longrightarrow_{n\in\mathbb{N}} 0$

uniformly over $u \in \mathbb{R}$.

(e) $n^{-1} \sum_{j=0}^{k-1} E(\xi_j^2 I\{| \xi_j | > \varepsilon\sigma n^{1/2}\}) \longrightarrow_{n\in\mathbb{N}} 0$ for every $\varepsilon > 0$.

To (a): Using 2.1 and (1.3), one obtains

$$ES_n''^2 \le \sum_{j=0}^{k} E\zeta_j^2 + 2\sum_{1\le i\le n}\sum_{i+p<j\le n}|EX_iX_j|$$
$$\le C(kq+p+q) + 24n\sum_{i>p}\alpha_n(i)^{1-\gamma}\sup_{i\le n}\|X_i\|_\beta^2$$

where C is the constant of (1.3). Now (2.5) (a) follows from (2.4) (a), (b), (c).

To (b): By a similar application of 2.1 follows

$$\sum_{0\le i<j\le k-1}|E\xi_i\xi_j| \le 24n\sum_{i>q}\alpha_n(i)^{1-\gamma}\sup_{i\le n}\|X_i\|_\beta^2.$$

Thus (2.4) (c) implies (2.5) (b).

(c) follows immediately from the definition of k and (2.4)(a), (b). Arguing as in the proof of Theorem 18.4.1 of Ibragimov and Linnik (1971), one obtains (d) from (2.4)(e).

To (e): 1st case: $\beta=\infty$, $\gamma=0$. Then (2.4)(d) implies

$$\sup_{0\le j\le k-1}\|\xi_j\|_\infty = o(n^{1/2}).$$

Hence the sum in (2.5)(e) equals 0 for all sufficiently large n.

2nd case: $\beta\in(2,\infty)$, $\gamma\in(0,1)$. Then we estimate the l.h.s. of (e) by

$$n^{-1}k(\varepsilon\sigma n^{1/2})^{2-\beta}\sup_{0\le j\le k-1}E|\xi_j|^\beta \le n^{-1}k(\varepsilon\sigma n^{1/2})^{2-\beta}p^\beta\sup_{i\le n}\|X_i\|_\beta^\beta$$
$$\sim (\varepsilon\sigma)^{2-\beta}(p^{2-\gamma}n^{\gamma-1}\sup_{i\le n}\|X_i\|_\beta^2)^{\beta/2}$$

which goes to 0 according to (2.4)(d).

All points of (2.5) are proved. Hence we have proved the c.l.t. under the assumptions of our Theorem. Using (1.2) we obtain

(2.6) $\qquad W_n(t)\to_{n\in\mathbb{N}} W(t)$ in distribution for every $t\in[0,1]$.

Let $0\le t_i<\cdots<t_k\le 1$ be given. We wish to show:

(2.7) $\qquad (W_n(t_1),\cdots,W_n(t_k))\to_{n\in\mathbb{N}}(W(t_1),\cdots,W(t_k))$ in distribution.

It follows from Prohorov's classical characterization of tightness that (2.6) implies the tightness of $(W_n(t_1),\cdots,W_n(t_k))_{n\in\mathbb{N}}$. Let $Q\,|\,\mathscr{B}(\mathbb{R}^{\{t_i,\cdots,t_k\}})$ be the weak limit distribution of some subsequence of $(W_n(t_1),\cdots,W_n(t_k))$. According to (2.6) the marginal distributions $Q\pi_{t_i}^{-1}$ are normal with variance t_i. Take $r_n=(2+a_n)/n$, where (a_n) is the sequence in the assumption of the Theorem. Then $r_n\to 0$ and $\alpha_n([nr_n]-1)\to 0$. Using (1.3) one obtains $E(W_n(t_i+r_n)-W_n(t_i))^2\to_{n\in\mathbb{N}}0$. Therefore $Q(\pi_{t_1},\pi_{t_2}-\pi_{t_1},\cdots,\pi_{t_k}-\pi_{t_{k-1}})^{-1}$ is the weak limit distribution of some subsequence of $(W_n(t_1),W_n(t_2)-W_n(t_1+r_n),\cdots,W_n(t_k)-W_n(t_{k-1}+r_n))$. Now $\alpha_n([nr_n]-1)\to 0$ implies that $\pi_{t_1},\pi_{t_2}-\pi_{t_1},\cdots,\pi_{t_k}-\pi_{t_{k-1}}$ are independent under Q. Hence Q is the distribution of $(W(t_1),\cdots,W(t_k))$. This argument proves (2.7).

Finally, we have to prove the tightness of the sequence (W_n). According to Billingsley (1968) Theorem 15.5, it suffices to show for every $\varepsilon>0$:

(2.8) $\qquad \lim_{\delta\downarrow 0}\limsup_{n\in\mathbb{N}}P\{w(W_n,\delta)>\varepsilon\}=0.$

Let $\varepsilon>0$ be given. For $\delta>0$ and $n\in\mathbb{N}$ holds

$$P\{w(W_n,\delta)>\varepsilon\}\le\sum_{a=0}^{[1/\delta]}P\{\max_{[na\delta]<r\le[n(a+1)\delta]}|S_r-S_{[na\delta]}|>\tfrac{1}{3}\,\varepsilon\sigma n^{1/2}\}.$$

For the next step of the proof let $a \in \{0, \cdots, [1/\delta]\}$ be fixed. Let $p = p(n)$, $q = q(n)$ be sequences of positive integers satisfying (2.4). Put $m = m(n) = \max\{i \in \mathbb{N}: (p+q)i \le [n(a+1)\delta] - [na\delta]\}$. For $j \in \{0, \cdots, m-1\}$ define

$$\varphi_j = \sum \{X_i : [na\delta] + j(p+q) + 1 \le i \le [na\delta] + j(p+q) + p\}$$

$$\psi_j = \sum \{X_i : [na\delta] + j(p+q) + p + 1 \le i \le [na\delta] + (j+1)(p+q)\}.$$

Then we have

$$\begin{aligned} &P\{\max_{[na\delta]<r\le[n(a+1)\delta]} | S_r - S_{[na\delta]} | > \tfrac{1}{3}\, \varepsilon\sigma n^{1/2}\} \\ &\qquad \le (m+1)\max_{0\le b\le n-(p+q)} P\{\max_{1\le r\le p+q} | S_{b+r} - S_b | > \tfrac{1}{9}\varepsilon\sigma n^{1/2}\} \\ &\qquad\quad + P\{\max_{0\le r\le m-1} | \textstyle\sum_{j=0}^{r} \varphi_j | > \tfrac{1}{9}\varepsilon\sigma n^{1/2}\} \\ &\qquad\quad + P\{\max_{0\le r\le m-1} | \textstyle\sum_{j=0}^{r} \psi_j | > \tfrac{1}{9}\varepsilon\sigma n^{1/2}\} = I + II + III. \end{aligned}$$

To estimate I we distinguish two cases.

1st case: $\beta = \infty$, $\gamma = 0$. Then we have in I

$$\text{ess sup} | S_{b+r} - S_b | \le (p+q)\sup_{i\le n} \| X_i \|_\infty = o(n^{1/2})$$

according to (2.4) (a), (d). Hence I equals 0 for all sufficiently large n.

2nd case: $\beta \in (2, \infty)$ $\gamma \in (0, 1)$. Let $b \in \{0, \cdots, n-(p+q)\}$. Then we have

$$\begin{aligned} P\{\max_{1\le r\le p+q} | S_{b+r} - S_b | > \tfrac{1}{9}\varepsilon\sigma n^{1/2}\} &\le P\{\textstyle\sum_{i=1}^{p+q} | X_{b+i} | > \tfrac{1}{9}\varepsilon\sigma n^{1/2}\} \\ &\le (\varepsilon\sigma/9)^{-\beta} n^{-\beta/2} (p+q)^\beta \sup_{i\le n} \| X_i \|_\beta^\beta. \end{aligned}$$

Using the definition of m and (2.4) (a), (d), we obtain

$$(m+1)n^{-\beta/2}(p+q)^\beta \sup_{i\le n} \| X_i \|_\beta^\beta \sim \delta(p^{2-\gamma} n^{\gamma-1} \sup_{i\le n} \| X_i \|_\beta^2)^{\beta/2} \to_{n\in\mathbb{N}} 0.$$

Hence I goes to 0 for $n \to \infty$.

Now we will estimate II and III. Using 2.1 and (1.3), we obtain:

$$\begin{aligned} &\max_{J\subset(0,\cdots,m-1)} E(\textstyle\sum_{j\in J} \psi_j)^2/(\sigma^2 n) \\ &\qquad \le \textstyle\sum_{0\le j\le m-1} E\psi_j^2/(\sigma^2 n) + 2 \sum_{1\le i\le n} \sum_{i+p\le j\le n} | EX_iX_j |/(\sigma^2 n) \\ &\qquad \le Cmq/(\sigma^2 n) + 24\sigma^{-2} \textstyle\sum_{i>p} \alpha_n(i)^{1-\gamma} \sup_{i\le n} \| X_i \|_\beta^2 \to_{n\in\mathbb{N}} 0. \end{aligned}$$

Similarly:

$$\begin{aligned} &\max_{J\subset\{0,\cdots,m-1\}} E(\textstyle\sum_{j\in J} \varphi_j)^2/(\sigma^2 n) \\ &\qquad \le \textstyle\sum_{0\le j\le m-1} E\varphi_j^2/(\sigma^2 n) + 2 \sum_{1\le i\le n} \sum_{i+q<j\le n} | EX_iX_j |/(\sigma^2 n) \\ &\qquad \le Cmp/(\sigma^2 n) + 24\sigma^{-2} \textstyle\sum_{i>q} \alpha_n(i)^{1-\gamma} \sup_{i\le n} \| X_i \|_\beta^2. \end{aligned}$$

Using (2.4) and the definition of m, we obtain that the last expression converges to $C\sigma^{-2}\delta$ for $n \to \infty$, where C is the constant of (1.3). We can choose $\delta_0(\varepsilon) > 0$ such that

$$(18/\varepsilon)^2 C\sigma^{-2}\delta_0(\varepsilon) < \tfrac{1}{2}.$$

From now on we will assume $\delta < \delta_0(\varepsilon)$. Then we obtain by an application of

Tschebyscheff's inequality

$$\min_{0\le r\le m-2} P\{|\textstyle\sum_{i=r+1}^{m-1} \varphi_i| \le 1/18\ \varepsilon\sigma n^{1/2}\} \ge 1/2$$

for all sufficiently large n. Now we apply Lemma 2.2 and obtain:

$$P\{\max_{0\le r\le m-1} |\textstyle\sum_{j=0}^{r} \varphi_j| > 1/9\ \varepsilon\sigma n^{1/2}\}$$

$$\le 2P\{|\textstyle\sum_{j=0}^{m-1} \varphi_j| > 1/18\ \varepsilon\sigma n^{1/2}\} + 2m\alpha_n(q+1).$$

(2.4) implies $m\alpha_n(q+1) \to_{n\in\mathbb{N}} 0$. Since

$$\|\textstyle\sum_{j=0}^{m-1} \psi_j/(\sigma\, n^{1/2})\|_2 \to_{n\in\mathbb{N}} 0$$

$$\|(S_{[n(a+1)\delta]} - S_{[na\delta]} - \textstyle\sum_{j=0}^{m-1} (\varphi_j + \psi_j))/(\sigma\, n^{1/2})\|_2 \to_{n\in\mathbb{N}} 0$$

we obtain:

$$\limsup_{n\in\mathbb{N}} P\{\max_{0\le r\le m-1} |\textstyle\sum_{j=0}^{r} \varphi_j| > 1/9\ \varepsilon\sigma n^{1/2}\}$$

$$\le 2 \limsup_{n\in\mathbb{N}} P\{|S_{[n(a+1)]} - S_{[na\delta]}| > 1/19\,\varepsilon\sigma n^{1/2}\}.$$

Another application of Tschebyscheff's inequality and Lemma 2.2 yields

$$\limsup_{n\in\mathbb{N}} P\{\max_{0\le r\le m-1} |\textstyle\sum_{j=0}^{r} \psi_j| > 1/9\ \varepsilon\sigma n^{1/2}\}$$

$$\le 2 \limsup_{n\in\mathbb{N}} P\{|\textstyle\sum_{j=0}^{m-1} \psi_j| > 1/18\,\varepsilon\sigma n^{1/2}\} + 2 \limsup_{n\in\mathbb{N}} m\alpha_n(p+1) = 0.$$

Summing up our results for $a = 0, \cdots, [1/\delta]$, we obtain:

$$\limsup_{n\in\mathbb{N}} P\{w(W_n, \delta) > \varepsilon\} \le 2 \sum_{a=0}^{[1/\delta]} \limsup_{n\in\mathbb{N}} P\left\{\frac{|S_{[n(a+1)\delta]} - S_{[na\delta]}|}{\sigma n^{1/2}} > \frac{\varepsilon}{19}\right\}$$

$$\le 2\left(\frac{1}{\delta} + 1\right) N(0, \delta)\left\{x \in \mathbb{R}: |x| > \frac{\varepsilon}{19}\right\} \to_{\delta\downarrow 0} 0.$$

Here we have used the weak convergence of $W_n((a+1)\delta) - W_n(a\delta)$ to $N(0, \delta)$, the normal distribution with mean 0 and variance δ. Now the proof of (2.8) is complete, and this was the last step of the proof of our Theorem.

PROOF OF COROLLARY 1. Since finitely many X_n may be truncated, without affecting the asymptotic behaviour of (W_n), we may assume w.l.g. $\sup_{n\in\mathbb{N}} \|X_n\|_\beta < \infty$. Then (1.6) implies (1.5) for every sequence (a_n) with $a_n \to \infty$ and $a_n^{2-\gamma}/n^{1-\gamma} \to 0$. Using Lemma 2.1 one obtains

$$E(S_{m+n} - S_m)^2 \le n \sup_{i\in\mathbb{N}} \|X_i\|_2^2 + 24n \textstyle\sum_{i\in\mathbb{N}} \alpha(i)^{1-\gamma} \sup_{i\in\mathbb{N}} \|X_i\|_\beta^2.$$

Hence (1.3) follows from the assumptions of Corollary 1.

PROOF OF COROLLARY 2. W.l.g. we may assume $\|X_n\|_\beta < \infty$ for all $n \in \mathbb{N}$.

Then the moment conditions in (1.7), respectively (1.8), imply

$$\sup_{i\le n} \| X_i \|_\beta = o(n^{((1-\gamma)/2)(a/(a+1))-(1/2)(1/(a+1))})$$

respectively

$$\sup_{i\le n} \| X_i \|_\beta = o(n^{(1-\gamma)/2}/(\log n)^{1-\gamma/2}).$$

If (1.7) is fulfilled, then apply the Theorem with $a_n = n^{1/(a+1)}$. If (1.8) is fulfilled, then take $a_n = \max(1, \log n/\log b)$.

Corollary 3 follows by an application of the Theorem with $a_n = m_n + 1$. To prove Corollary one may assume w.l.g. $\| X_n \|_\beta < \infty$ for all $n \in \mathbb{N}$. Then one has $\sup_{i\le n} \| X_i \|_\beta = o(n^{(1-\gamma)/2})$ and the assertion follows as a special case of Corollary 3.

3. Examples. In Lemma 3.1 the common structure of the examples in this Section is discussed.

3.1 LEMMA. *Let $g: \mathbb{N} \to \mathbb{N}$ have the following properties*
(i) *g is non-decreasing*
(ii) *$g(n) \to \infty$ for $n \to \infty$*
(iii) *$g(2n) \le Mg(n)$ for all $n \in \mathbb{N}$ with some constant M.*
Denote $g^{-1}(n) = \min\{i \in \mathbb{N}: g(i) \ge n\}$, $n \in \mathbb{N}$, $G(n) = \sum_{i=1}^n g(i)$, $n \in \mathbb{N} \cup \{0\}$, $G^{-1}(n) = \min\{i \in \mathbb{N}: G(i) \ge n\}$, $n \in \mathbb{N}$. Let $(Y_i)_{i\in\mathbb{N}}$ be an independent sequence of r.v.'s *with*

$$P\left\{ Y_i = \pm \frac{G(i)^{1/2}}{g(i)} \right\} = \frac{1}{2}\frac{g(i)}{G(i)}, \quad P\{Y_i = 0\} = 1 - \frac{g(i)}{G(i)}$$

for $i \in \mathbb{N}$. Consider the process $(X_n)_{n\in\mathbb{N}}$, defined by $X_n = Y_i$ for $G(i-1) < n \le G(i)$. Then (X_n) has the following properties:

(3.2) *$EX_n = 0$, $EX_n^2 < \infty$ for $n \in \mathbb{N}$*
(3.3) *$ES_n^2/n \to_{n\in\mathbb{N}} 1$*
(3.4) *$E(S_{m+n} - S_m)^2 \le n$ for $n, m \in \mathbb{N}$*
(3.5) *$\alpha(k) \le \sup\{g(i)/G(i): g(i) \ge k + 1\}$ for $k \in \mathbb{N}$*
(3.6) *$m_n \le g(G^{-1}(n)) - 1$ for $n \in \mathbb{N}$*
(3.7) *For $\beta \in (2, \infty]$, $\gamma = 2/\beta$ holds*

$$\| X_n \|_\beta = (G(G^{-1}(n))^{(1-\gamma)/2}/(g(G^{-1}(n))^{1-\gamma/2} \quad \textit{for} \quad n \in \mathbb{N}$$

(3.8) *(X_n) does not satisfy the* c.l.t.

PROOF. (3.2) is obvious. To (3.3): (i), (ii), (iii) imply $g(n)/G(n) \to 0$. Hence

$$\frac{g(G^{-1}(n))}{n} \le \frac{g(G^{-1}(n))}{G(G^{-1}(n) - 1)} = \frac{g(G^{-1}(n))}{G(G^{-1}(n)) - g(G^{-1}(n))} \to_{n\in\mathbb{N}} 0.$$

Let $n \in \mathbb{N}$, $i = G^{-1}(n)$. Then $G(i-1) < n \le G(i)$ and one has:

$$ES_n^2 = \sum_{\nu=1}^{i-1} g(\nu)^2 EY_\nu^2 + (n - G(i-1))^2 EY_i^2$$
$$= G(i-1) + (n - G(i-1))^2/g(i)$$
$$|ES_n^2/n - 1| = |(G(i-1) - n)/n + (n - G(i-1))^2/(ng(i))|$$
$$\le g(i)/n \to_{n\in\mathbb{N}} 0.$$

To (3.4): Let $m, n \in \mathbb{N}$ and $G(i-1) < m+1 \le G(i)$, $G(i+t) < m+n \le G(i+t+1)$ for some $i \in \mathbb{N}$, $t \in \mathbb{N} \cup \{0\}$. Then one has:

$$E(S_{m+n} - S_m)^2$$
$$= (G(i) - m)^2 EY_i^2 + \sum_{\nu=i+1}^{i+t} g(\nu)^2 EY_\nu^2 + (m+n-G(i+t))^2 EY_{i+t+1}^2$$
$$= (G(i)-m)^2/g(i) + \sum_{\nu=i+1}^{i+t} g(\nu) + (m+n-G(i+t))^2/g(i+t+1)$$
$$\le G(i) - m + \sum_{\nu=i+1}^{i+t} g(\nu) + m + n - G(i+t) = n.$$

If $G(i-1) < m+1 \le m+n \le G(i)$, the assertion follows similarly.

To (3.5): $(X_1, \cdots, X_n)$ and $(X_{n+k}, X_{n+k+1}, \cdots)$ are independent, unless there exists $i \in \mathbb{N}$ with $G(i-1) < n$, $n+k \le G(i)$. In the latter case it follows from Lemma 8 of Bradley (1981) that

$$\sup\{|P(A \cap B) - P(A)P(B)|: A \in \sigma(X_1, \cdots, X_n), B \in \sigma(X_{n+k}, X_{n+k+1}, \cdots)\}$$
$$\le \sup\{|P(A \cap B) - P(A)P(B)|: A, B \in \sigma(Y_i)\}.$$

Since $P|\sigma(Y_i)$ has an atom with measure $1 - g(i)/G(i)$, it is easy to check that $|P(A \cap B) - P(A)P(B)| \le g(i)/G(i)$ for $A, B \in \sigma(Y_i)$. Hence $\alpha(k) \le \sup\{g(i)/G(i): g(i) \ge k+1\}$.

To (3.6): $m_n \le \max\{g(i) - 1: G(i-1) < n\} = g(G^{-1}(n)) - 1$.

(3.7) follows easily from the definition of X_n.

To (3.8): Consider $Z_n = \sum_{\nu=G(n-1)+1}^{G(n)} X_\nu = g(n) Y_n$. Then $(Z_n)_{n\in\mathbb{N}}$ is an independent sequence with $EZ_n = 0$, $EZ_n^2 = g(n)$, $E(\sum_{i=1}^n Z_i)^2 = G(n)$. It suffices to show that $\sum_{i=1}^n Z_i/G(n)^{1/2}$ is not asymptotically normal. Since $\max_{1\le i\le n} g(i)/G(n) = g(n)/G(n) \to 0$, the Lindeberg condition is necessary for asymptotic normality. Hence it suffices to find $\varepsilon > 0$ such that

$$G(n)^{-1} \sum_{i=1}^n E(Z_i^2 I\{|Z_i| \ge \varepsilon G(n)^{1/2}\}) \not\to 0,$$

i.e.

$$G(n)^{-1} \sum \{g(i): 1 \le i \le n,\ G(i)^{1/2} \ge \varepsilon G(n)^{1/2}\} \not\to 0.$$

The assumption (iii) implies $G(2n) \le 2M\, G(n)$ for $n \in \mathbb{N}$. Put $\varepsilon = (2M)^{-1/2}$. Then one obtains

$$G(2n)^{-1} \sum \{g(i): 1 \le i \le 2n,\ G(i)^{1/2} \ge (2M)^{-1/2} G(2n)^{1/2}\}$$
$$\ge G(2n)^{-1} \sum \{g(i): n \le i \le 2n\} \ge 1/2,$$

which proves our claim.

EXAMPLE 1. For every $a > 1$ there exists a process $(X_n)_{n\in\mathbb{N}}$ such that (1.1), (1.2), (1.3) are fulfilled, and

$$\alpha(k) = O(k^{-a}), \quad m_n = O(n^{1/(a+1)})$$

$$\| X_n \|_\beta = O(n^{(1-\gamma)/2-(1-\gamma/2)/(a+1)}) \quad \text{for all} \quad \beta \in (2, \infty], \quad \gamma = 2/\beta$$

but (X_n) does not satisfy the c.l.t.

PROOF. Define $g(n) = [n^{1/a}]$. Then (i), (ii), (iii) of Lemma 3.1 hold, and one has the following asymptotic relations:

$$g(n) \sim n^{1/a}, \quad g^{-1}(n) \sim n^a$$

$$G(n) \sim \frac{a}{a+1} n^{(a+1)/a}, \quad G^{-1}(n) \sim \left(\frac{a+1}{a}\right)^{a/(a+1)} n^{a/(a+1)}$$

$$\frac{g(n)}{G(n)} \sim \frac{a+1}{a} n^{-1}$$

$$\sup\left\{\frac{g(i)}{G(i)} : g(i) \geq k+1\right\} \sim \frac{a+1}{a} \frac{1}{g^{-1}(k+1)} \sim \frac{a+1}{a} k^{-a}$$

$$g(G^{-1}(n)) \sim \left(\frac{a+1}{a}\right)^{1/(a+1)} n^{1/(a+1)}$$

$$G(G^{-1}(n)) \sim n.$$

Using these relations, one obtains the assertion from Lemma 3.1.

EXAMPLE 2. There exists a process $(X_n)_{n\in\mathbb{N}}$, such that (1.1), (1.2), (1.3) are fulfilled, and

$$\alpha(k) = O(e^{-k}), \quad m_n = O(\log n)$$

$$\| X_n \|_\beta = O(n^{(1-\gamma)/2}/(\log n)^{1-\gamma/2}) \quad \text{for all} \quad \beta \in (2, \infty], \quad \gamma = 2/\beta$$

but (X_n) does not satisfy the c.l.t.

PROOF. Define $g(n) = 1 + [\log n]$. Then (i), (ii), (iii) of Lemma 3.1 hold, and one has the following asymptotic relations:

$$g(n) \sim \log n, \quad g^{-1}(n) \sim e^{n-1}$$

$$G(n) \sim n \log n, \quad G^{-1}(n) \sim n/\log n$$

$$g(n)/G(n) \sim n^{-1}$$

$$\sup\{g(i)/G(i): g(i) \geq k+1\} \sim 1/g^{-1}(k+1) \sim e^{-k}$$

$$g(G^{-1}(n)) \sim \log n, \quad G(G^{-1}(n)) \sim n.$$

The assertion now follows from 3.1.

EXAMPLE 3. Let $\bar{\alpha}(k) > 0$, $k \in \mathbb{N}$, and $\bar{m}_n \in \mathbb{N}$ with $\bar{m}_n \to \infty$ for $n \to \infty$, be given. Then there exists a process $(X_n)_{n\in\mathbb{N}}$ such that (1.1), (1.2), (1.3) are fulfilled, and

$$\alpha(k) \le \bar{\alpha}(k) \quad \text{for all} \quad k \in \mathbb{N}, \quad m_n \le \bar{m}_n \quad \text{for all} \quad n \in \mathbb{N}$$

$$\| X_n \|_\beta = o(n^{(1-\gamma)/2}) \quad \text{for all} \quad \beta \in (2, \infty], \quad \gamma = 2/\beta$$

but (X_n) does not satisfy the c.l.t.

PROOF. W.l.g. we assume $\bar{\alpha}(k+1) \le \bar{\alpha}(k)$ and $\bar{m}_n \le \bar{m}_{n+1}$ for all $k, n \in \mathbb{N}$. Define $g(1) = 1$ and inductively: $g(n+1) = g(n) + 1$, if

$$g(n) + 1 \le 2\, g([(n+1)/2])$$

and $g(n) + 1 \le \bar{m}_{n+1} + 1$ and $g(n) + 1 \le \bar{\alpha}(g(n))n$, and $g(n+1) = g(n)$ otherwise. Then $g(n)$ is non-decreasing and $g(n) \to \infty$. It is easy to prove by induction: $g(n) \le 2\, g([n/2])$, $g(n) \le \bar{m}_n + 1$ for all $n \in \mathbb{N}$, and $g(n) \le \bar{\alpha}(k)n$ for all $n \in \mathbb{N}$ and $k < g(n)$. Let $(X_n)_{n\in\mathbb{N}}$ be the process of Lemma 3.1. Then (1.1), (1.2), (1.3) are fulfilled, and

$$\alpha(k) \le \sup\{g(i)/i : g(i) \ge k+1\} \le \bar{\alpha}(k) \quad \text{for} \quad k \in \mathbb{N}$$

$$m_n \le g(G^{-1}(n)) - 1 \le g(n) - 1 \le \bar{m}_n \quad \text{for} \quad n \in \mathbb{N}$$

$$\begin{aligned} \| X_n \|_\beta &= (G(G^{-1}(n) - 1) + g(G^{-1}(n)))^{(1-\gamma)/2} / (g(G^{-1}(n)))^{1-\gamma/2} \\ &\le (n + o(n))^{(1-\gamma)/2} / (g(G^{-1}(n)))^{1-\gamma/2} = o(n^{(1-\gamma)/2}) \\ &\quad \text{for} \quad \beta \in (2, \infty], \quad \gamma = 2/\beta. \end{aligned}$$

In the last line we have used the relation $g(G^{-1}(n)) = o(n)$, which has been established in the proof of (3.3). Finally we obtain from Lemma 3.1 that (X_n) does not satisfy the c.l.t.

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