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# STRONG LIMIT THEOREMS FOR WEIGHTED QUANTILE PROCESSES

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A thorough description of the almost sure behavior of weighted uniform quantile processes is given. This includes analogues of nearly all known results for weighted uniform empirical processes, such as the James functional law of the iterated logarithm and the Csáki results on the supremum of the standardized empirical process. Subject to the usual regularity conditions, our results extend to the nonuniform quantile process. Also, in the process of obtaining our results, we derive an extension of a theorem of Kiefer, which is likely to be of independent interest.

**1. Introduction and statements of results.** Let $U_1, U_2, \dots,$ be a sequence of independent uniform $(0,1)$ random variables. For each integer $n \geq 1$, let

$$G_n(s) = n^{-1} \sum_{i=1}^{n} 1(U_i \leq s), \qquad 0 \leq s \leq 1,$$

where $1(x \leq y)$ denotes the indicator function, be the empirical distribution function based on the first $n$ of these random variables, and let

$$U_n(s) = U_{k,n}, \qquad (k-1)/n < s \leq k/n, \qquad k = 1, \dots, n,$$

with $U_n(0) = U_{1,n}$, where $U_{1,n} \leq \cdots \leq U_{n,n}$ are the order statistics based on $U_1, \dots, U_n$, be the sample quantile function. We write the uniform empirical process as

$$\alpha_n(s) = n^{1/2}\{G_n(s) - s\}, \qquad 0 \leq s \leq 1,$$

and the uniform quantile process as

$$\beta_n(s) = n^{1/2}\{U_n(s) - s\}, \qquad 0 \leq s \leq 1.$$

We shall use the notation $\tilde{\beta}_n(s)$ to denote the truncated uniform quantile process, which is equal to $\beta_n(s)$ for $1/(n+1) \leq s \leq n/(n+1)$ and defined to be 0 elsewhere.

The purpose of this paper is to provide a complete description of the almost sure behavior of weighted versions of the uniform quantile process. The results that we shall present will be the complete analogues of known results for weighted uniform empirical processes. We begin by stating our results. At the end of this section, we shall discuss related literature on the uniform quantile process and results for the uniform empirical process, which correspond to our

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theorems. Also, some remarks will be made about extensions of our results to the nonuniform quantile process. The proofs will be detailed in the next section.

Our first result is a functional law of the iterated logarithm for the weighted uniform quantile process. Let

$$Q^* = \{q: [0,1] \to [0,\infty): q \text{ is nondecreasing and is strictly positive on } (0,1]\}.$$

Also, let $B[0,1]$ denote the space of bounded real-valued functions defined on $[0,1]$ with the supremum norm and $F[0,1]$ denote the set of absolutely continuous functions $f \in B[0,1]$ such that

$$f(0) = f(1) = 0 \quad \text{and} \quad \int_0^1 (f'(s))^2\, ds \le 1.$$

For any $q \in Q^*$, set $F_q[0,1] = \{f/q\colon f \in F[0,1]\}$.

*Throughout this paper,* $l_n = \log\log(n \vee 3)$ *for* $n \ge 1$.

THEOREM 1. *Let* $q \in Q^*$ *and assume*

$$\lim_{s \downarrow 0} (s \log\log(1/s))^{1/2}/q(s) = \rho \in [0,\infty]. \tag{1.1}$$

*Then with probability* 1 [*wp* 1],

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{0<s\le 1} |\tilde{\beta}_n(s)|/\big(q(s)(2l_n)^{1/2}\big) \\
&\qquad = (2^{-1/2}\rho) \vee \Big(\sup_{0<s\le 1/2} (s(1-s))^{1/2}/q(s)\Big).
\end{aligned} \tag{1.2}$$

*Moreover, when* $\rho = 0$, *then wp* 1 *the sequence*

$$\big\{\tilde{\beta}_n/\big((2l_n)^{1/2}q\big)\big\}_{n=1}^{\infty} \tag{1.3}$$

*is relatively compact in* $B[0,1]$ *with set of limit points equal to* $F_q[0,1]$. *Whereas, if* $\rho > 0$, *wp* 1 *the sequence in* (1.3) *fails to be relatively compact in* $B[0,1]$.

Our next theorem describes the almost sure behavior of the weighted uniform quantile process for a certain subclass of the functions $q \in Q^*$ for which the limit in (1.1) is equal to $\infty$. Here the norming constants depend on the particular weight function and the interval on which the supremum is taken at each stage $n$.

For any $0 < c < \infty$, let $0 < \alpha_c^- < 1 < \alpha_c^+$ be the two solutions of $\lambda - \log\lambda - 1 = c^{-1}$.

THEOREM 2. *Let* $0 \le \nu \le 1/2$ *and* $0 < a_n < 1/2$ *with* $a_n \downarrow 0$.

(I) *If* $na_n/l_n \to 0$ *and* $a_n \ge a/n$, *eventually, for some* $a > 0$, *then wp* 1,

$$\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} n^{1/2} a_n^{1-\nu} |\beta_n(s)|/\big(s^{1-\nu} l_n\big) = 1. \tag{1.4}$$

(II) *If* $na_n/l_n \to c \in (0,\infty)$, *then wp* 1,

$$\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} a_n^{1/2-\nu} |\beta_n(s)|/\big(s^{1-\nu} l_n^{1/2}\big) = \gamma(c,\nu), \tag{1.5}$$

*where* $\gamma(c,1/2) = 2 \vee \{c^{1/2}(\alpha_c^+ - 1)\}$ *and* $\gamma(c,\nu) = c^{1/2}(\alpha_c^+ - 1)$ *if* $0 \le \nu < 1/2$.

(III) *If* $na_n/l_n \uparrow \infty$ *and* $\log\log(1/a_n)/l_n \to c$, *then wp* 1,

$$\begin{aligned}\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} & a_n^{1/2-\nu}|\beta_n(s)|/\left(s^{1-\nu}l_n^{1/2}\right) \\ (1.6) \qquad & = (2(1+c))^{1/2}, \quad \text{if } \nu = 1/2, \\ & = 2^{1/2}, \qquad\qquad \text{if } 0 \le \nu < 1/2.\end{aligned}$$

Let $0 < k_n \le n$, $k_n \uparrow$ and $k_n/n \downarrow 0$. Define two versions of the tail quantile process based on the sequence $\{k_n\}_{n=1}^{\infty}$ to be

$$v_n(s) = (n/k_n)^{1/2}\beta_n(sk_n/n), \qquad 0 \le s \le 1,$$

and

$$\tilde{v}_n(s) = (n/k_n)^{1/2}\tilde{\beta}_n(sk_n/n), \qquad 0 \le s \le 1.$$

Also, let $K[0,1]$ denote the set of absolutely continuous functions $f \in B[0,1]$ such that

$$f(0) = 0 \quad \text{and} \quad \int_0^1 (f'(s))^2\, ds \le 1,$$

and for any $0 < \nu \le 1/2$ set

$$K_\nu[0,1] = \left\{ fI^{-1/2+\nu} \colon f \in K[0,1] \right\},$$

where $I$ denotes the identity function.

The following theorem provides a description of how weighted and unweighted versions of these processes behave almost surely as $n \to \infty$.

THEOREM 3. *Let* $0 < \nu \le 1/2$, $1 \le k_n \le n$, $k_n \uparrow$ *and* $k_n/n \downarrow 0$.

(I) *If* $k_n^{2\nu}/l_n \to 0$, *then wp* 1,

$$\limsup_{n\to\infty} \sup_{0<s\le 1} k_n^{\nu}|\tilde{v}_n(s)|/\left(s^{1/2-\nu}l_n\right) = 1. \tag{1.7}$$

(II) *If* $k_n^{2\nu}/l_n \to c \in (0,\infty]$, *then wp* 1,

$$\limsup_{n\to\infty} \sup_{0<s\le 1} |\tilde{v}_n(s)|/\left(s^{1/2-\nu}l_n^{1/2}\right) = \tau(c,\nu), \tag{1.8}$$

*where*

$$\begin{aligned}\tau(c,\nu) &= c^{1/2}(\alpha_c^+ - 1), && \nu = 1/2,\ c \in (0,\infty), \\ &= 2^{1/2}, && \nu = 1/2,\ c = \infty, \\ &= c^{-1/2}, && 0 < \nu < 1/2,\ c \in (0,1/2), \\ &= 2^{1/2}, && 0 < \nu < 1/2,\ c \in [1/2,\infty].\end{aligned}$$

(III) *If* $k_n^{2\nu}/l_n \to \infty$, *then wp* 1 *the sequence*

$$\left\{\tilde{v}_n/\left(I^{1/2-\nu}(2l_n)^{1/2}\right)\right\}_{n=1}^{\infty} \tag{1.9}$$

*is relatively compact in* $B[0,1]$ *with set of limit points equal to* $K_\nu[0,1]$.

*Moreover,* (I)–(III) *remain true when* $\tilde{v}_n$ *is replaced by* $v_n$ *in the case* $\nu = 1/2$.

REMARK 1. In parts (I) and (II), when $c < \infty$, statement (1.9) fails to be true.

Our final result dealing with weighted uniform quantile processes concerns strong approximations.

THEOREM 4. (I) *On a rich enough probability space, there exists a sequence of independent Brownian bridges* $B_1, B_2, \dots,$ *and a sequence of independent uniform* (0,1) *random variables* $U_1, U_2, \dots,$ *such that whenever* $q \in Q^*$ *satisfies* (1.1) *with* $\rho = 0$, *then wp* 1,

$$\sup_{0<s\le 1} \left| \tilde{\beta}_n(s) - n^{-1/2} \sum_{i=1}^{n} B_i(s) \right| \Big/ q(s) = o\big(l_n^{1/2}\big). \tag{1.10}$$

(II) *Let* $0 < \nu \le 1/2, 1 \le k_n \le n,\ k_n \uparrow$ *and* $k_n/n \downarrow 0$. *If* $k_n^{2\nu}/l_n \to \infty$, *then on a rich enough probability space there exist a sequence of standard Wiener processes* $W_1, W_2, \dots,$ *and a sequence of independent uniform* (0,1) *random variables* $U_1, U_2, \dots,$ *such that wp* 1,

$$\sup_{0<s\le 1} \left| \tilde{v}_n(s) - k_n^{-1/2} \sum_{i=1}^{n} W_i(sk_n/n) \right| s^{-1/2+\nu} = o\big(l_n^{1/2}\big). \tag{1.11}$$

REMARK 2. Of course, instead of using weight functions in the class $Q^*$, we could have formulated Theorems 1, 2 and 4 in terms of weight functions that agree with members of $Q^*$ on $[0, 1/2]$ and are symmetric about $1/2$. The corresponding statements and proofs of the results for this class of weight functions are, because of symmetry considerations, obvious from our present theorems and therefore left to the reader.

Our last result is both a tool to prove Theorem 3 and an extension of a theorem by Kiefer (1970) concerning the almost sure behavior of the supremum of the Bahadur (1966) process as $n \to \infty$. We use the following notation in the statement of our theorem.

Given a sequence $0 < k_n \le n$, set $a_n^* = (k_n l_n)^{1/2}$, $b_n = \log k_n$, $d_n = 2l_n + b_n$ and $r_n = (a_n^* d_n)^{1/2} n^{-1/2}$. Also, let

$$R_n(k_n) = \sup_{0\le s\le k_n/n} |\alpha_n(s) + \beta_n(s)|.$$

THEOREM 5. *Assume that* $\{k_n\}_{n=1}^{\infty}$ *satisfies*

$$k_n \uparrow \text{ and } k_n/n \downarrow \gamma, \qquad 0 \le \gamma \le 1, \tag{1.12}$$

*and*

$$k_n/l_n \to \infty, \quad \text{as } n \to \infty. \tag{1.13}$$

(I) *If* $\gamma = 0$, *then wp* 1,

$$\limsup_{n \to \infty} r_n^{-1} R_n(k_n) \le 2^{1/4} \tag{1.14}$$

*and if, in addition,*

$$(\log k_n)/l_n \to \infty, \quad \text{as } n \to \infty, \tag{1.15}$$

*then wp* 1,

$$\limsup_{n \to \infty} r_n^{-1} R_n(k_n) = 2^{1/4}. \tag{1.16}$$

(II) *If* $0 < \gamma \le 1$, *then wp* 1,

$$\begin{aligned} \limsup_{n \to \infty} r_n^{-1} R_n(k_n) &= 2^{1/4}(1-\gamma)^{1/4}, \qquad 0 < \gamma \le 1/2, \\ &= 2^{-1/4}\gamma^{-1/4}, \qquad 1/2 < \gamma \le 1. \end{aligned} \tag{1.17}$$

REMARK 3. Subject to regularity conditions, our results can be extended to the nonuniform quantile process as defined in equation (9) on page 640 of Shorack and Wellner (SW) (1986). For instance, if the underlying distribution function $F$ satisfies properties (1)–(3) with $M < 1$ on page 645 of SW (1986), then Theorem 1 with $\rho = 0$, Theorem 2, part (III), Theorem 3, part (II) with $c = \infty$ and part (III), and Theorems 4 and 5 remain true for the nonuniform quantile process. For the sake of brevity, we do not go through the routine details of showing this here.

REMARK 4. Theorem 1 is the analogue for $\tilde{\beta}_n$ of the James (1975) functional law of the iterated logarithm for the weighted uniform empirical process and was first announced in Shorack (1982a). [See also Open Question 1 on page 526 of SW (1986).] The versions for $\alpha_n$ of the various parts of Theorem 2 are to be found in Csáki (1975, 1977), Shorack and Wellner (1978), Mason (1981), Wellner (1978) and Einmahl (1987a). A weaker form of Theorem 2 for the case $\nu = 1/2$ was obtained by Csörgő and Révész (1978). Theorem 2 for the case $\nu = 1/2$ answers Open Question 1 on page 616 of SW (1986). The case when $\nu = 0$ was first given by Wellner (1978). Einmahl and Mason (1988) describe the behavior of the tail empirical process that corresponds to parts (I) and (II) of Theorem 3. Part (III) of this theorem gives the analogues for the tail quantile process of results in Mason (1988). A special case of Theorem 6.1 of Alexander (1982) contains the strong approximation result for the weighted uniform empirical process corresponding to part (I) of Theorem 4. Mason (1988) obtained the strong approximation for the tail empirical process for which Theorem 4, part (II), is the analogue. When $k_n$ is chosen to be $n$, Theorem 5 gives the well-known result by Kiefer (1970) on the Bahadur process. (See Fact 3 in the following discussion.) It also yields an improvement of some results on intermediate order statistics in Watts (1980), specialized to the uniform distribution.

**2. Proofs of the theorems.** For convenient reference later on, we begin by recording a number of facts.

FACT 1 [Kiefer (1972)].

(i) *Let* $(n+1)^{-1} \le a_n < 1$ *and* $na_n/l_n \to 0$. *Then wp* 1,

$$\limsup_{n\to\infty} nU_n(a_n)/l_n = 1.$$

(ii) *Let* $0 < a_n < 1$, $na_n/l_n \uparrow \infty$ *and* $a_n \downarrow$. *Then wp* 1,

$$\limsup_{n\to\infty} \pm \beta_n(a_n)/(a_n(1-a_n)l_n)^{1/2} = 2^{1/2}.$$

(iii) *Let* $0 < a_n < 1$, $na_n \uparrow$, $na_n/l_n \to \infty$ *and* $a_n \downarrow 0$. *Then wp* 1,

$$\limsup_{n\to\infty} \pm \alpha_n(a_n)/(a_n(1-a_n)l_n)^{1/2} = 2^{1/2}.$$

*If* $c \in (0, \infty)$, *then wp* 1,

$$\text{(iv)} \qquad \limsup_{n\to\infty} nU_n(cl_n/n)/l_n = c\alpha_c^+$$

*and*

$$\text{(v)} \qquad \liminf_{n\to\infty} nU_n(cl_n/n)/l_n = c\alpha_c^-.$$

FACT 2 [Wellner (1978)].
*If* $c \in (0,1)$, *then wp* 1,

$$\text{(i)} \qquad \limsup_{n\to\infty} \sup_{cl_n/n \le s \le 1} U_n(s)/s = \alpha_c^+$$

*and*

$$\text{(ii)} \qquad \limsup_{n\to\infty} \sup_{cl_n/n \le s \le 1} s/U_n(s) = (\alpha_c^-)^{-1}.$$

FACT 3 [Kiefer (1970)]. *We have wp* 1,

$$\limsup_{n\to\infty} \sup_{0\le s\le 1} n^{1/4}|\alpha_n(s) + \beta_n(s)|/(l_n^{1/2}\log n)^{1/2} = 2^{-1/4}.$$

FACT 4 [Csáki (1977)].

(i) *Let* $0 < a_n \le 1$, $na_n/l_n \to \infty$, $\log\log(1/a_n)/l_n \to c$ *and* $a_n \downarrow$. *Then wp* 1,

$$\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} |\alpha_n(s)|/(sl_n)^{1/2} = (2(1+c))^{1/2}.$$

(ii) *For* $c \in (0, \infty)$, *wp* 1,

$$\limsup_{n\to\infty} \sup_{cl_n/n\le s\le 1} |\alpha_n(s)|/(sl_n)^{1/2} = 2 \vee \{c^{1/2}(\beta_c^+ - 1)\},$$

*where* $\beta_c^+$ *is the root greater than* 1 *of* $\lambda(\log\lambda - 1) + 1 = c^{-1}$.

All these facts can be found in SW (1986).

We shall first prove Theorem 2, then Theorem 1, the upper bound part of Theorem 5, Theorem 3, Theorem 4, and finally the lower bound part of Theorem 5.

PROOF OF THEOREM 2. We shall require a number of lemmas.

LEMMA 1. *For all* $0 < c < d < \infty$, *wp* 1,

$$\limsup_{n\to\infty} \sup_{cl_n/n \le s \le dl_n/n} |\beta_n(s)|/(sl_n)^{1/2} = c^{1/2}(\alpha_c^+ - 1). \tag{2.1}$$

PROOF. Choose any integer $k \ge 1$ and set

$$\varepsilon_k = (d/c)^{1/k} - 1, \qquad c_{j,k} = c(1+\varepsilon_k)^j \quad \text{and} \quad \bar{c}_{j,k} = c_{j,k} l_n/n,$$

for $j = 0, 1, \dots, k$. Notice that for any $j = 0, \dots, k-1$, we have

$$\sup_{\bar{c}_{j,k} \le s \le \bar{c}_{j+1,k}} \beta_n(s)/(sl_n)^{1/2}$$
$$\le \sup_{\bar{c}_{j,k} \le s \le \bar{c}_{j+1,k}} \left\{\beta_n(\bar{c}_{j+1,k}) + n^{1/2}(\bar{c}_{j+1,k} - \bar{c}_{j,k})\right\}/(sl_n)^{1/2},$$

which when $\beta_n(\bar{c}_{j+1,k}) \ge 0$ is

$$\le (1+\varepsilon_k)^{1/2}\beta_n(\bar{c}_{j+1,k})/(\bar{c}_{j+1,k} l_n)^{1/2} + \varepsilon_k d^{1/2}.$$

Applying Fact 1(iv), we obtain for each fixed $0 \le j \le k-1$ and $k \ge 1$, wp 1,

$$\limsup_{n\to\infty} \beta_n(\bar{c}_{j+1,k})/(\bar{c}_{j+1,k} l_n)^{1/2} = c_{j+1,k}^{1/2}\left(\alpha_{c_{j+1,k}}^+ - 1\right).$$

Thus, since

$$\lambda^{1/2}(\alpha_\lambda^+ - 1) \downarrow 1, \quad \text{as } \lambda \uparrow \infty,$$

we see that for each $k \ge 1$, the lim sup in (2.1) is less than or equal to wp 1,

$$(1+\varepsilon_k)^{1/2} c^{1/2}(\alpha_c^+ - 1) + \varepsilon_k d^{1/2}.$$

Observing that $\varepsilon_k \to 0$ as $k \to \infty$, we have wp 1,

$$\limsup_{n\to\infty} \sup_{cl_n/n \le s \le dl_n/n} \beta_n(s)/(sl_n)^{1/2} \le c^{1/2}(\alpha_c^+ - 1). \tag{2.2}$$

Similarly, we can show using Fact 1(v) that wp 1,

$$\limsup_{n\to\infty} \sup_{cl_n/n \le s \le dl_n/n} -\beta_n(s)/(sl_n)^{1/2} \le d^{1/2}(1 - \alpha_d^-). \tag{2.3}$$

Since $c^{1/2}(\alpha_c^+ - 1) > d^{1/2}(1 - \alpha_d^-)$, (2.1) follows from (2.2), (2.3) and Fact 1(iv). □

LEMMA 2. *Let* $0 < a_n < 1/2$ *be such that* $a_n \downarrow 0$, $\log\log(1/a_n)/l_n \to c$ *and* $a_n(nl_n)^{1/2}/\log n \to \infty$ *as* $n \to \infty$. *Then wp* 1,

$$\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} |\beta_n(s)|/(sl_n)^{1/2} = (2(1+c))^{1/2}. \tag{2.4}$$

PROOF. Notice that

$$\sup_{a_n \le s \le 1/2} |\alpha_n(s) + \beta_n(s)|/(sl_n)^{1/2} \le \sup_{0 \le s \le 1} |\alpha_n(s) + \beta_n(s)|/(a_n l_n)^{1/2}.$$

Assertion (2.4) is now an easy consequence of Facts 3 and 4(i). □

LEMMA 3. *For every* $\varepsilon > 0$ *there exists a* $d_0 \in (0, \infty)$ *such that for all* $\delta \in (0, 1/2)$ *and* $d \ge d_0$, *wp* 1,

$$\limsup_{n\to\infty} \sup_{dl_n/n \le s \le n^{-\delta}} |\beta_n(s)|/(sl_n)^{1/2} \le 2(1+\varepsilon). \tag{2.5}$$

PROOF. First note that wp 1,

$$\sup_{0 \le s \le 1} |\beta_n(s) + \alpha_n(U_n(s))| = O(n^{-1/2}). \tag{2.6}$$

Therefore, to establish (2.5), it suffices to prove that wp 1,

$$\limsup_{n\to\infty} \sup_{dl_n/n \le s \le n^{-\delta}} |\alpha_n(U_n(s))|/(sl_n)^{1/2} \le 2(1+\varepsilon), \tag{2.7}$$

for some large enough $d$. The left side of (2.7) is less than or equal to

$$\limsup_{n\to\infty} \sup_{dl_n/n \le s \le n^{-\delta}} |\alpha_n(U_n(s))|/(U_n(s)l_n)^{1/2} \sup_{dl_n/n \le s \le 1} (U_n(s)/s)^{1/2}. \tag{2.8}$$

Now by Fact 2(i), wp 1,

$$\limsup_{n\to\infty} \sup_{dl_n/n \le s \le 1} U_n(s)/s = \alpha_d^+,$$

and by Fact 1(v), wp 1,

$$\liminf_{n\to\infty} nU_n(dl_n/n)/l_n = d\alpha_d^- =: f(d),$$

where $f(d) \uparrow \infty$ as $d \downarrow \infty$. Hence, we see that expression (2.8) is less than or equal to

$$(\alpha_d^+)^{1/2} \limsup_{n\to\infty} \sup_{d'l_n/n \le s \le 1/2} |\alpha_n(s)|/(sl_n)^{1/2},$$

where $d' = 2^{-1}f(d)$, which by Fact 4(ii) is wp 1 equal to

$$(\alpha_d^+)^{1/2}\left\{2 \vee \left((d')^{1/2}(\beta_{d'}^+ - 1)\right)\right\}.$$

Since $\alpha_d^+ \downarrow 1$ and $(d')^{1/2}(\beta_{d'}^+ - 1) \downarrow 2^{1/2}$ as $d' \uparrow \infty$, we have proved (2.5). □

PROOF OF THEOREM 2 FOR $\nu = 1/2$. We begin with part (II). This part follows immediately from Lemma 1, Lemma 2 with $a_n = n^{-1/4}$ and Lemma 3 with $\delta = 1/4$.

Next, consider part (I). We must show that wp 1,

$$\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} (na_n)^{1/2} |\beta_n(s)|/(s^{1/2} l_n) = 1. \tag{2.9}$$

First, we establish the lower bound. The left side of (2.9) is wp 1 greater than or equal to

$$\limsup_{n\to\infty} n^{1/2} |\beta_n(a_n)|/l_n = \limsup_{n\to\infty} nU_n(a_n)/l_n = 1,$$

cf. Fact 1(i). Now we consider the upper bound. Since

$$(na_n)^{1/2}/l_n = o(l_n^{-1/2}),$$

it follows from part (II) that it suffices to show that for every $\varepsilon > 0$ there exists a $c \in (0, \infty)$ such that wp 1,

$$\limsup_{n\to\infty} \sup_{a_n \le s \le cl_n/n} (na_n)^{1/2} |\beta_n(s)|/(s^{1/2} l_n) \le 1 + \varepsilon.$$

Since

$$\lim_{n\to\infty} \sup_{a_n \le s \le cl_n/n} (na_n)^{1/2} (ns)^{1/2}/l_n = 0,$$

we must only verify that there exists a $c \in (0, \infty)$ such that wp 1,

$$\limsup_{n\to\infty} nU_n(cl_n/n)/l_n \le 1 + \varepsilon. \tag{2.10}$$

But, by Fact 1(iv), for arbitrary $c \in (0, \infty)$ the left side of (2.10) is equal wp 1 to $c\alpha_c^+$. Noticing that $c\alpha_c^+ \downarrow 1$ as $c \downarrow 0$ completes the proof of part (I).

Finally, we turn to part (III). When $0 \le c < 1$, this part is immediate from Lemma 2. It remains to consider $c = 1$. From Lemma 2, with $a_n = n^{-1/4}$ and Lemma 3 with $\delta = 1/4$, the upper bound follows. The lower bound when $c = 1$ is easily inferred from the already established lower bounds for the case $0 \le c < 1$. This completes the proof of Theorem 2 when $\nu = 1/2$. □

Before we can complete the proof of Theorem 2, we must establish two more lemmas.

LEMMA 4. *Under the conditions of part* (III) *of Theorem* 2, *we have for any* $M \in (1, \infty)$, *wp* 1,

$$\limsup_{n\to\infty} \sup_{a_n \le s \le Ma_n} |\alpha_n(s)|/(sl_n)^{1/2} \le 2^{1/2}. \tag{2.11}$$

The proof is a routine application of Inequalities 2.8 and 2.10 in Einmahl (1987b) and will be omitted; cf. also the proof of Theorem 4.3 there.

LEMMA 5. *Under the conditions of part* (III) *of Theorem* 2, *we have for any* $M \in (1, \infty)$, *wp* 1,

$$\limsup_{n\to\infty} \sup_{a_n \le s \le Ma_n} |\beta_n(s)|/(sl_n)^{1/2} \le 2^{1/2}. \tag{2.12}$$

PROOF. The proof is based on (2.6) and is along the same lines as the proof of Lemma 3. The application of Fact 4(ii) is replaced by an application of Lemma 4. □

PROOF OF THEOREM 2 FOR $0 \le \nu < 1/2$. We first consider the lower bounds. These follow immediately from Fact 1 and

$$\sup_{a_n \le s \le 1/2} |\beta_n(s)| s^{-1+\nu} \ge \beta_n(a_n) a_n^{-1+\nu}.$$

Next, we establish the upper bounds. We begin with part (I). Observe that

$$\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} n^{1/2} a_n^{1-\nu} |\beta_n(s)|/(l_n s^{1-\nu})$$
$$\le \limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} (na_n)^{1/2} |\beta_n(s)|/(l_n s^{1/2}).$$

Applying the theorem for $\nu = 1/2$, see (2.9), finishes the proof of the upper bound for this part. Now, we turn to part (II). Let $c_0$ be the uniquely determined root of the equation

$$c^{1/2}(\alpha_c^+ - 1) = 2.$$

First, suppose that the $c$ in Theorem 2, part II, is less than or equal to $c_0$. For such $c$, $c^{1/2}(\alpha_c^+ - 1) \ge 2$. We have, using the theorem for $\nu = 1/2$ that wp 1,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} a_n^{1/2-\nu} |\beta_n(s)|/(l_n^{1/2} s^{1-\nu}) \\ &\quad \le \limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} |\beta_n(s)|/(sl_n)^{1/2} = c^{1/2}(\alpha_c^+ - 1). \end{aligned} \tag{2.13}$$

The case $c > c_0$ requires more care. Observe that for any $M \in (1, \infty)$ the left side of (2.13) is less than or equal to

$$\limsup_{n\to\infty} \sup_{a_n \le s \le Ma_n} a_n^{1/2-\nu} |\beta_n(s)|/(l_n^{1/2} s^{1-\nu})$$
$$\vee \limsup_{n\to\infty} \sup_{Ma_n \le s \le 1/2} a_n^{1/2-\nu} |\beta_n(s)|/(l_n^{1/2} s^{1-\nu})$$
$$\le \limsup_{n\to\infty} \sup_{a_n \le s \le Ma_n} |\beta_n(s)|/(l_n s)^{1/2}$$
$$\vee \limsup_{n\to\infty} \sup_{Ma_n \le s \le 1/2} M^{-1/2+\nu} |\beta_n(s)|/(l_n s)^{1/2}.$$

Using Lemma 1 and the theorem for $\nu = 1/2$, we see that this last expression is for large enough $M$ less than or equal to wp 1,

$$c^{1/2}(\alpha_c^+ - 1) \vee (2M^{-1/2+\nu}) \le c^{1/2}(\alpha_c^+ - 1).$$

[This trick is due to Alexander (1984).] Note that here and previously, we w.l.o.g. replaced the condition $na_n/l_n \to c$ by $a_n = cl_n/n$.

Finally, we turn to part (III). Proceeding as before, we obtain by applications of Lemma 5 and the theorem for $\nu = 1/2$ that wp 1,

$$\begin{aligned}\limsup_{n\to\infty} \sup_{a_n \le s \le 1/2} & a_n^{1/2-\nu}|\beta_n(s)|/\big(l_n^{1/2}s^{1-\nu}\big) \\ &\le 2^{1/2} \vee M^{-1/2+\nu}(2(1+c))^{1/2}.\end{aligned} \tag{2.14}$$

Again, for large enough $M$, the right side of (2.14) is equal to $2^{1/2}$. This completes the proof of Theorem 2. □

**Proof of Theorem 1.** From Theorem 2, part (I) with $\nu = 1/2$ and $a_n = (n+1)^{-1}$, and Fact 1(i), it is easily seen that wp 1,

$$\limsup_{n\to\infty} \sup_{(n+1)^{-1} \le s \le n^{-1/2}} |\beta_n(s)|/\big(q(s)(2l_n)^{1/2}\big) = 2^{-1/2}\rho. \tag{2.15}$$

Also, from Theorem 2, part (III) with $\nu = 1/2$ and $a_n = n^{-1/2}$, we see, assuming $\rho < \infty$, that for every $\varepsilon > 0$ there exists an $\eta > 0$ such that wp 1,

$$\limsup_{n\to\infty} \sup_{n^{-1/2} \le s \le \eta} |\beta_n(s)|/\big(q(s)(2l_n)^{1/2}\big) < \varepsilon. \tag{2.16}$$

The results in (2.15) and (2.16) are enough to prove (1.3), cf. James (1975).

On account of (2.15) and (2.16) to establish (1.2), it suffices to prove that for every $0 < \eta < 1/2$, wp 1,

$$\limsup_{n\to\infty} \sup_{\eta \le s \le 1} |\beta_n(s)|/\big(q(s)(2l_n)^{1/2}\big) = \sup_{\eta \le s \le 1/2} (s(1-s))^{1/2}/q(s). \tag{2.17}$$

However, this follows immediately from the Finkelstein theorem for the uniform quantile process; see (3) on page 513 of SW (1986).

The statement in (2.15) is sufficient to prove the last line of Theorem 1. □

**Proof of the upper bounds in Theorem 5.** For $a, b > 0$ with $a + b \le 1$, write

$$\omega_n(a; b) = \sup_{\substack{0 \le s \le b \\ 0 \le h \le a}} |\alpha_n(s+h) - \alpha_n(s)|.$$

Let

$$\psi(\lambda) = 2\lambda^{-2}\{(1+\lambda)\log(1+\lambda) - \lambda\}, \qquad 0 < \lambda < \infty.$$

The function $\psi$ has the property that $\psi(\lambda) \uparrow 1$ as $\lambda \downarrow 0$.

We shall require the following two inequalities.

INEQUALITY 1. *Let* $0 < \varepsilon \le 1/2$, $0 < a < 1/4$ *and* $a + b \le 1$. *Then we have for all* $\lambda \ge 0$,

$$P(\omega_n(a; b) \ge \lambda) \le Kba^{-1}\exp\left(-\frac{(1-\varepsilon)\lambda^2}{2a}\psi\left(\frac{\lambda}{n^{1/2}a}\right)\right),$$

*where* $K = K(\varepsilon) \in (0, \infty)$.

INEQUALITY 2. *Let* $\varepsilon \in (0,1)$, $a \in (0,1]$ *and write* $n_j = [(1+\varepsilon/2)^j]$ *for* $j \ge 1$. *Then we have for all* $j \ge 1$ *and* $\lambda > 2(a/\varepsilon)^{1/2}$,

$$P\Big(\max_{n_j < n \le n_{j+1}} \omega_n(a; b) \ge \lambda\Big) \le 2P\big(\omega_{n_{j+1}}(a; b) \ge (1-\varepsilon)\lambda\big).$$

The proofs of these inequalities are very much like those of Inequalities 3.1 and 3.2 in Einmahl and Ruymgaart (1987) and will thus be omitted.

*The reader is advised to recall the notation of Theorem* 5.

LEMMA 6. *Whenever* $\{k_n\}_{n=1}^{\infty}$ *satisfies* (1.12) *and* (1.13), *for all* $0 < c < \infty$, *wp* 1,

$$\limsup_{n\to\infty} r_n^{-1}\omega_n(ca_n^*/n; k_n/n) \le c^{1/2}. \tag{2.18}$$

PROOF. Choose any $0 < \varepsilon < 1$ small to be specified in the following discussion. It suffices to show that $\Sigma P(A_j) < \infty$, where

$$A_j = \Big\{\max_{n_j < n \le n_{j+1}} w_n\big(ca_{n_j}^*/n_j; k_{n_j}/n_j\big) \ge \lambda_j\Big\},$$

$$\lambda_j = n_{j+1}^{-1/2}c^{1/2}(1+2\varepsilon)\big(a_{n_j}^*\big)^{1/2}\big(2l_{n_j} + \log k_{n_j}\big)^{1/2}$$

and $n_j = [(1+\varepsilon/2)^j]$. Now by Inequality 2 for all large $j$, $P(A_j) \le 2P(B_j)$, where

$$B_j = \Big\{\omega_{n_{j+1}}\big(ca_{n_j}^*/n_j; k_{n_j}/n_j\big) \ge (1-\varepsilon)\lambda_j\Big\}.$$

Observe that by (1.13)

$$\begin{aligned}(1-\varepsilon)\lambda_j/\big(n_{j+1}^{1/2}ca_{n_j}^*n_j^{-1}\big)\\ = \big(n_j/n_{j+1}\big)(1-\varepsilon)(1+2\varepsilon)c^{-1/2}\Big(\big(2l_{n_j} + \log k_{n_j}\big)/a_{n_j}^*\Big)^{1/2} \to 0,\\ \text{as } j\to\infty,\end{aligned} \tag{2.19}$$

$$(1-\varepsilon)^2\lambda_j^2/\big(2ca_{n_j}^*n_j^{-1}\big) \sim d_{n_j}(1-\varepsilon)^2(1+2\varepsilon)^2/(2(1+\varepsilon/2)), \quad \text{as } j\to\infty, \tag{2.20}$$

and

$$k_{n_j}/(ca_{n_j}^*) = \big(k_{n_j}/\log\log n_j\big)^{1/2}c^{-1}. \tag{2.21}$$

Applying Inequality 1, we have

$$P(B_j) \le K \frac{c^{-1}k_{n_j}}{a_{n_j}^*} \exp\left( - \frac{n_j(1-\varepsilon)^2\lambda_j^2}{2ca_{n_j}^*} \psi\left( \frac{n_j(1-\varepsilon)\lambda_j}{cn_{j+1}^{1/2}a_{n_j}^*} \right) \right),$$

which by using the fact that $\psi(x) \uparrow 1$ as $x \downarrow 0$ combined with (2.19)–(2.21) is for all large enough $j$ less than

$$Kc^{-1}\left(k_{n_j}/l_{n_j}\right)^{1/2} \exp\left(-2^{-1}(1+\delta)d_{n_j}\right),$$

for some $\delta > 0$ depending on $\varepsilon > 0$ as long as $\varepsilon$ is chosen small enough. This last term is for all sufficiently large $j$ less than $Kc^{-1}(\log n_j)^{-(1+\delta)}$. Since

$$\sum \left(\log n_j\right)^{-(1+\delta)} < \infty,$$

we see that (2.18) follows from the Borel–Cantelli lemma. □

LEMMA 7. *Whenever* $\{k_n\}_{n=1}^{\infty}$ *satisfies* (1.12) *and* (1.13), *wp* 1,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{0\le s\le k_n/n} (n/k_n)^{1/2} |\beta_n(s)| l_n^{-1/2} \\ (2.22) \qquad &= 2^{1/2}(1-\gamma)^{1/2}, \qquad 0 \le \gamma \le 1/2, \\ &= 2^{-1/2}\gamma^{-1/2}, \qquad 1/2 < \gamma \le 1. \end{aligned}$$

PROOF. First, consider the case $\gamma = 0$. From equation (2) on page 584 of SW (1986), we have

$$\begin{aligned} \sup_{0\le s\le k_n/n} |\beta_n(s)| &\le \sup_{0\le s\le k_n/n} |\alpha_n(U_n(s))| + n^{-1/2} \\ (2.23) \qquad &\le \sup_{0\le t\le U_n(k_n/n)} |\alpha_n(t)| + n^{-1/2}. \end{aligned}$$

Using (2.23) in combination with the fact that for all $\lambda > 1$, wp 1,

$$(2.24) \qquad \limsup_{n\to\infty} \sup_{0\le s\le \lambda k_n/n} (n/k_n)^{1/2} |\alpha_n(s)| l_n^{-1/2} = 2^{1/2},$$

cf. Einmahl and Mason (1988), and Fact 1(ii), yields (2.22) for the case $\gamma = 0$.

When $0 < \gamma \le 1$, it can be inferred from Finkelstein (1971) that wp 1,

$$\begin{aligned} &\lim_{\lambda\downarrow 1} \limsup_{n\to\infty} \sup_{0\le s\le \lambda k_n/n} |\alpha_n(s)| l_n^{-1/2} \\ (2.25) \qquad &= (2\gamma(1-\gamma))^{1/2}, \qquad 0 < \gamma \le 1/2, \\ &= 2^{-1/2}, \qquad 1/2 < \gamma \le 1. \end{aligned}$$

Also, by Fact 1(ii), wp 1,

$$\begin{aligned} (2.26) \qquad \limsup_{n\to\infty} \beta_n(k_n/n \wedge 2^{-1}) l_n^{-1/2} &= (2\gamma(1-\gamma))^{1/2}, \qquad 0 < \gamma \le 1/2, \\ &= 2^{-1/2}, \qquad 1/2 < \gamma \le 1. \end{aligned}$$

Thus (2.22) follows from Fact 1(ii), (2.25) and (2.26) in this case. □

Set for $0 < k_n \le n$,

$$R_n^*(k_n) = \sup_{0 \le s \le k_n/n} |\alpha_n(s) - \alpha_n(U_n(s))|.$$

LEMMA 8. *Whenever* $\{k_n\}_{n=1}^{\infty}$ *satisfies* (1.12) *and* (1.13), *wp* 1,

$$\begin{aligned} \limsup_{n\to\infty} r_n^{-1} R_n^*(k_n) &\le 2^{1/4}(1-\gamma)^{1/4}, \qquad 0 \le \gamma \le 1/2, \\ &\le 2^{-1/4}\gamma^{-1/4}, \qquad 1/2 < \gamma \le 1, \end{aligned} \tag{2.27}$$

PROOF. Choose any $\varepsilon > 0$. By Lemma 7, for almost every $\omega$ there exists an $N_\omega$ such that for all $n \ge N_\omega$,

$$\sup_{0 \le s \le k_n/n} |U_n(s) - s| \le c a_n^*/n,$$

where

$$\begin{aligned} c &= (1+\varepsilon)2^{1/2}(1-\gamma)^{1/2}, \qquad 0 \le \gamma \le 1/2, \\ &= (1+\varepsilon)2^{-1/2}\gamma^{-1/2}, \qquad 1/2 < \gamma \le 1. \end{aligned}$$

Assertion (2.27) is now an immediate consequence of Lemma 6. □

Since by (2.6), $R_n(k_n) = R_n^*(k_n) + O(n^{-1/2})$, this last lemma completes the proof of the upper bounds in Theorem 5.

PROOF OF THEOREM 3. First, we assume $\nu = 1/2$. Consider part (I). In this case, by Fact 1(i), wp 1,

$$\limsup_{n\to\infty} \sup_{0 \le s \le 1} k_n^{1/2} |\tilde{v}_n(s)|/l_n = \limsup_{n\to\infty} nU_n(k_n/n)/l_n = 1.$$

Next, we prove part (II) when $\nu = 1/2$. The appropriate lower bound results for the lim sup follow from Fact 1(ii) and (iv). Now we turn to the upper bound part of the proof of (II), and first assume that $c < \infty$. Choose any $0 < \delta < 1$. Note that

$$\limsup_{n\to\infty} \sup_{0 \le s \le \delta} |\tilde{v}_n(s)| l_n^{-1/2} \le \limsup_{n\to\infty} \left(nU_n(\delta k_n/n) + \delta k_n\right)/(k_n l_n)^{1/2},$$

which by Fact 1(iv) is wp 1 less than or equal to $\delta c^{1/2}(1 + \alpha_{\delta c}^+)$. Since $\lambda \alpha_\lambda^+ \downarrow 1$ as $\lambda \downarrow 0$ and $\lambda^{1/2}(\alpha_\lambda^+ - 1) > \lambda^{-1/2}$, we see that for all $\delta > 0$ small enough $\delta c^{1/2}(1 + \alpha_{\delta c}^+) < c^{-1/2} < c^{1/2}(\alpha_c^+ - 1)$. Hence, to finish the proof of the upper bound result when $c < \infty$, it suffices to prove that for all $0 < \delta < 1$, wp 1,

$$\limsup_{n\to\infty} \sup_{\delta \le s \le 1} |\tilde{v}_n(s)| l_n^{-1/2} \le c^{1/2}(\alpha_c^+ - 1). \tag{2.28}$$

We follow the steps of the proof of Lemma 1. Write $\varepsilon_k = \delta^{-1/k} - 1$ and

$c_{j,k} = \delta c(1+\varepsilon_k)^j$. It is clear that for any $j = 0, 1, \dots, k-1$, $k \geq 1$,

$$\sup_{c_{j,k} \leq s \leq c_{j+1,k}} \tilde{v}_n(s) l_n^{-1/2} \leq \tilde{v}_n(c_{j+1,k}) l_n^{-1/2} + c^{-1/2} \varepsilon_k c_{j,k}.$$

Applying Fact 1(iv), we have wp 1,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{c_{j,k} \leq s \leq c_{j+1,k}} \tilde{v}_n(s) l_n^{-1/2} \\ (2.29) \qquad &\leq \left(c_{j+1,k}/c\right)^{1/2} c_{j+1,k}^{1/2} \left(\alpha_{c_{j+1,k}} - 1\right) + \varepsilon_k c^{1/2} \\ &\leq c^{1/2}(\alpha_c^+ - 1) + \varepsilon_k c^{1/2}. \end{aligned}$$

It can be shown along similar lines that wp 1,

$$(2.30) \qquad \limsup_{n\to\infty} \sup_{c_{j,k} \leq s \leq c_{j+1,k}} -\tilde{v}_n(s) l_n^{-1/2} \leq 1 + \varepsilon_k c^{1/2}.$$

Combining (2.29) and (2.30) with the fact that $\varepsilon_k \to 0$ as $k \to \infty$ completes the proof of part (II) when $c < \infty$ and $\nu = 1/2$. The case when $c = \infty$ is already proven in Lemma 7. This completes the proof of part (II) when $\nu = 1/2$.

The proof of part (III) when $\nu = 1/2$ is a direct consequence of Lemma 7 and Corollary 2 of Mason (1988). (Obviously, the arguments just given work with $\tilde{v}_n$ replaced by $v_n$ when $\nu = 1/2$.)

For the case when $0 < \nu < 1/2$, we need a number of lemmas.

LEMMA 9. *Let* $0 < \nu < 1/2$. *Then we have wp* 1,

$$(2.31) \qquad \limsup_{n\to\infty} \sup_{(n+1)^{-1} \leq s \leq l_n/n} n^{\nu} |\beta_n(s)| / \left(s^{1/2-\nu} l_n\right) = 1.$$

PROOF. The lower bound is immediate from Fact 1(i). For the upper bound, since

$$\lim_{n\to\infty} \sup_{(n+1)^{-1} \leq s \leq l_n/n} n^{\nu} n^{1/2} s / \left(s^{1/2-\nu} l_n\right) = 0,$$

it suffices to prove that wp 1,

$$(2.32) \qquad \limsup_{n\to\infty} \sup_{(n+1)^{-1} \leq s \leq l_n/n} n^{1/2+\nu} U_n(s) / \left(s^{1/2-\nu} l_n\right) \leq 1.$$

By Fact 1(i) again, wp 1,

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{(n+1)^{-1} \leq s \leq l_n^{1/2}/n} n^{1/2+\nu} U_n(s) / \left(s^{1/2-\nu} l_n\right) \\ (2.33) \qquad &\leq \limsup_{n\to\infty} n U_n\left(l_n^{1/2}/n\right) / l_n = 1, \end{aligned}$$

and by Fact 1(iv)

$$\begin{aligned} &\limsup_{n\to\infty} \sup_{l_n^{1/2}/n \leq s \leq l_n/n} n^{1/2+\nu} U_n(s) / \left(s^{1/2-\nu} l_n\right) \\ (2.34) \qquad &\leq \limsup_{n\to\infty} n U_n(l_n/n) l_n^{\nu/2-5/4} = 0. \end{aligned}$$

Combining (2.33) and (2.34) yields (2.32). □

LEMMA 10. *Let* $0 < \nu < 1/2$, $1 \le k_n \le n$, $k_n \uparrow$, $k_n/n \downarrow 0$ *and* $k_n^{2\nu}/l_n \to c \in (0, \infty]$. *Then wp* 1,

$$\limsup_{n\to\infty} \sup_{l_n/n \le s \le k_n/n} (n/k_n)^\nu |\alpha_n(s)|/(s^{1/2-\nu} l_n^{1/2}) = 2^{1/2}. \tag{2.35}$$

PROOF. The lower bound is a consequence of Fact 1(iii). Next, set $m_n = k_n/\log k_n$. Observe that

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{l_n/n \le s \le m_n/n} (n/k_n)^\nu |\alpha_n(s)|/(s^{1/2-\nu} l_n^{1/2}) \\
&\quad \le \limsup_{n\to\infty} \sup_{l_n/n \le s \le m_n/n} (m_n/k_n)^\nu |\alpha_n(s)|/((s l_n)^{1/2}) \\
&\quad \le (\log k_n)^{-\nu} \limsup_{n\to\infty} \sup_{l_n/n \le s \le 1/2} |\alpha_n(s)|/(s l_n)^{1/2}.
\end{aligned} \tag{2.36}$$

From Fact 4(ii), we see that this last term is equal to 0 wp 1.

Similarly, we have

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{m_n/n \le s \le k_n/n} (n/k_n)^\nu |\alpha_n(s)|/(s^{1/2-\nu} l_n^{1/2}) \\
&\quad \le \limsup_{n\to\infty} \sup_{m_n/n \le s \le k_n/n} |\alpha_n(s)|/(s l_n)^{1/2}.
\end{aligned} \tag{2.37}$$

A straightforward application of Inequalities 2.8 and 2.10 in Einmahl (1987b) shows that the right side of inequality (2.37) is less than or equal to $2^{1/2}$, wp 1. □

LEMMA 11. *Under the conditions of Lemma* 10, *wp* 1,

$$\sup_{l_n/n \le s \le k_n/n} (n/k_n)^\nu |\alpha_n(s) + \beta_n(s)|/(s^{1/2-\nu} l_n^{1/2}) = o(1). \tag{2.38}$$

PROOF. For the proof, we will require the already proven upper bounds in Theorem 5. First, assume $1/4 < \nu < 1/2$. Theorem 5 and the last stated assumption on $k_n$ in Lemma 10 yield

$$\begin{aligned}
&\sup_{l_n/n \le s \le k_n/n} (n/k_n)^\nu |\alpha_n(s) + \beta_n(s)|/(s^{1/2-\nu} l_n^{1/2}) \\
&\quad \le n^{1/2} k_n^{-\nu} l_n^{-1+\nu} R_n(k_n) \\
&\quad = O\Big(k_n^{-\nu+1/4} l_n^{-3/4+\nu} (\log(k_n \log n))^{1/2}\Big) = o(1).
\end{aligned}$$

The proof for the case $0 < \nu \le 1/4$ is more intricate. Observe that it suffices to show that (2.38) holds when the supremum is taken over $[c l_n/n, k_n^{3\nu}/n]$ or $[k_n^\alpha/n, k_n^{\alpha+\nu}/n]$ for arbitrary $\alpha \in [3\nu, 1-\nu]$, since the interval $[l_n/n, k_n/n]$ can be written as the finite union of intervals of this form.

First, consider $[l_n/n, k_n^{3\nu}/n]$. From Theorem 5, we have wp 1,

$$\sup_{l_n/n \le s \le k_n^{3\nu}/n} (n/k_n)^{\nu} |\alpha_n(s) + \beta_n(s)| / \left(s^{1/2-\nu} l_n^{1/2}\right)$$

$$\le n^{1/2} k_n^{-\nu} l_n^{-1+\nu} R_n\left(k_n^{3\nu}\right) = O\left(\left(\log(k_n \log n)\right)^{1/2} k_n^{-\nu/4} l_n^{-3/4+\nu}\right) = o(1).$$

For the interval $[k_n^{\alpha}/n, k_n^{\alpha+\nu}/n]$ for $3\nu \le \alpha \le 1 - \nu$, we get wp 1,

$$\sup_{k_n^{\alpha}/n \le s \le k_n^{\alpha+\nu}/n} (n/k_n)^{\nu} |\alpha_n(s) + \beta_n(s)| / \left(s^{1/2-\nu} l_n^{1/2}\right)$$

$$\le n^{1/2} k_n^{-\nu-\alpha(1/2-\nu)} l_n^{-1/2} R_n\left(k_n^{\alpha+\nu}\right)$$

$$= O\left(\left(\log(k_n \log n)\right)^{1/2} l_n^{-1/4} k_n^{-\gamma}\right),$$

where $\gamma = \nu + \alpha(1/2 - \nu) - 1/4(\alpha + \nu) \ge (3/2)\nu(1 - 2\nu)$. Routine bounds verify that this last term is $o(1)$. This completes the proof of the lemma. □

We are now prepared to finish the proof of Theorem 3 for the case $0 < \nu < 1/2$. Consider part (I). The lower bound again follows from Fact 1(i). For the upper bound define $h_n = k_n \vee l_n$ and observe that $h_n^{2\nu}/l_n \to 0$. By Lemma 9, it is sufficient to prove that wp 1,

$$\lim_{n\to\infty} \sup_{l_n/n \le s \le h_n/n} n^{\nu} |\beta_n(s)| / \left(s^{1/2-\nu} l_n\right) = 0. \tag{2.39}$$

The left side of (2.39) is less than or equal to

$$\lim_{n\to\infty} \left(h_n^{2\nu}/l_n\right)^{1/2} \sup_{l_n/n \le s \le 1/2} |\beta_n(s)| / (s l_n)^{1/2}. \tag{2.40}$$

From Theorem 2, part (II), we conclude that the limit in (2.40) is 0, wp 1.

Part (II) follows from Lemmas 9–11.

Finally, consider part (III). From part (II), with $c = \infty$, it is apparent that for $0 < \delta < 1$, wp 1,

$$\limsup_{n\to\infty} \sup_{0 \le s \le \delta} |\tilde{v}_n(s)| / \left(s^{1/2-\nu} l_n^{1/2}\right) = 2^{1/2} \delta^{\nu}.$$

This, in combination with the already proven part (III) for the case $\nu = 1/2$, completes the proof of part (III) for $0 < \nu < 1/2$. □

**Proof of Theorem 4.** For the proof of part (I), we assume that we are on a rich enough probability space such that statement (1.10) holds with $q = 1$. [Such probability spaces exist, cf. Csörgő and Révész (1981).] Now choose any $q \in Q^*$. Clearly, to complete the proof, it suffices to show that wp 1,

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} \sup_{0 < s \le \delta} |\tilde{\beta}_n(s)| / \left(q(s) l_n^{1/2}\right) = 0 \tag{2.41}$$

and

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sup_{0 < s \le \delta} \left| n^{-1/2} \sum_{i=1}^{n} B_i(s) \right| / \left( q(s) l_n^{1/2} \right) = 0. \tag{2.42}$$

Statement (2.41) is immediate from (1.3) of Theorem 1. To prove (2.42), we use the fact that for any $0 < \delta < \infty$, wp 1,

$$\limsup_{n \to \infty} \sup_{0 < s \le \delta} \left| \sum_{i=1}^{n} B_i(s) \right| / (4sn \log\log(n/s))^{1/2} \le 1, \tag{2.43}$$

cf. Corollary 1.15.2 on page 81 of Csörgő and Révész (1981). Hence to establish (2.42), it is enough to verify that

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sup_{0 < s \le \delta} (\log\log(n/s))^{1/2} / (\log\log(1/s) l_n)^{1/2} = 0. \tag{2.44}$$

Observe that since $n/s \le s^{-2}$ for $0 < s < 1/n$,

$$\sup_{0 < s \le 1/n} (\log\log(n/s))^{1/2} / (\log\log(1/s) l_n)^{1/2}$$
$$\le \sup_{0 < s \le 1/n} (\log(2\log(1/s)))^{1/2} / (\log\log(1/s) l_n)^{1/2},$$

which converges to 0 as $n \to \infty$. Also, for $1/n \le s \le \delta \le 1/3$,

$$\limsup_{n \to \infty} \sup_{1/n \le s \le \delta} (\log\log(n/s))^{1/2} / (\log\log(1/s) l_n)^{1/2}$$
$$\le \lim_{n \to \infty} (\log(2\log n))^{1/2} / (\log\log(1/\delta) l_n)^{1/2} = (\log\log(1/\delta))^{-1/2}.$$

Since the right side of this inequality goes to 0 as $\delta \downarrow 0$, we see that we have proven (2.44). This finishes the proof of part (I).

The proof of part (II) proceeds very much like that of part (I). Briefly, it follows directly from Theorem 1 of Mason (1988), Theorem 5, Lemma 7 of Mason (1988) and Theorem 3, part (III). □

PROOF OF THE LOWER BOUNDS IN THEOREM 5. In order to finish the proof of Theorem 5, we need one more lemma. For $a, b > 0$ with $a + b \le 1$ and $0 < \varepsilon < 1$, set

$$\tilde{\omega}_n(a, b, \varepsilon) = \sup_{(1-\varepsilon)b \le s \le b} |\alpha_n(s + a) - \alpha_n(s)|.$$

LEMMA 12. *Whenever* $\{k_n\}_{n=1}^{\infty}$ *satisfies* (1.15), *for all* $0 < \varepsilon < 1$ *and* $0 < c < \infty$, *wp* 1,

$$\lim_{n \to \infty} s_n^{-1} \tilde{\omega}_n(c a_n^*/n, k_n/n, \varepsilon) = c^{1/2}, \tag{2.45}$$

*where* $s_n = (a_n^* b_n/n)^{1/2}$.

PROOF. Since $\omega_n(ca_n^*/n; k_n/n) \ge \tilde{\omega}_n(ca_n^*/n, k_n/n, \varepsilon)$, by Lemma 6 to prove (2.45), it is sufficient to show that wp 1,

$$\liminf_{n\to\infty} s_n^{-1}\tilde{\omega}_n(ca_n^*/n, k_n/n, \varepsilon) \ge c^{1/2}. \tag{2.46}$$

Choose any $0 < \delta < 1$ and set $m_n = ca_n^*$.

$$\begin{aligned} &P\Big(\tilde{\omega}_n(ca_n^*/n, k_n/n, \varepsilon) \le ((1-\delta)c)^{1/2}s_n\Big) \\ &\quad \le P\Big(\max_{(1-\varepsilon)k_n/m_n \le j \le k_n/m_n} |\alpha_n((j+1)m_n/n) - \alpha_n(jm_n/n)| \\ &\qquad\qquad \le ((1-\delta)c)^{1/2}s_n\Big), \end{aligned}$$

which by Mallows (1968) is less than or equal to

$$\Big(P\big(\alpha_n(m_n/n) \le ((1-\delta)c)^{1/2}s_n\big)\Big)^{M_n},$$

where $M_n = \varepsilon k_n/(2m_n)$. By an application of an inequality due to Kolmogorov, cf. Einmahl (1987b), page 70, this last term is for large $n$ less than or equal to

$$\begin{aligned} &\{1 - \exp(-(1+\delta)(1-\delta)b_n/(2(1-ca_n^*/n)))\}^{M_n} \\ &\quad \le \exp\Big(-\varepsilon k_n^{\delta^2/3}/\big(2cl_n^{1/2}\big)\Big). \end{aligned}$$

The Borel–Cantelli lemma and the arbitrary choice of $0 < \delta < 1$ complete the proof of (2.46). □

We are now prepared to prove statements (1.16) and (1.17). It is enough to show that they hold with $R_n(k_n)$ replaced by $R_n^*(k_n)$. First, consider the case $\gamma = 0$. We shall adapt the methods of Shorack (1982b). Choose any $0 < \varepsilon < 1$ and set

$$\begin{aligned} h_\varepsilon(s) &= 1 - \varepsilon, && 1 - \varepsilon < s \le 1, \\ &= s, && 0 \le s \le 1 - \varepsilon. \end{aligned}$$

Notice that $h_\varepsilon \in K[0,1]$. Lemma 12 gives that wp 1,

$$\begin{aligned} &\lim_{n\to\infty} \sup_{1-\varepsilon\le s\le 1} s_n^{-1}\big|\alpha_n(sk_n/n) - \alpha_n\big(sk_n/n + (2l_nk_n)^{1/2}h_\varepsilon(s)/n\big)\big| \\ &\quad = 2^{1/4}(1-\varepsilon)^{1/2}. \end{aligned} \tag{2.47}$$

Let $g_n$ be any sequence of functions such that $\sup_{0\le s\le 1}|g_n(s)| \to 0$ as $n \to \infty$. Then, since $s_n/r_n \to 1$, we obtain by Lemma 6 that wp 1,

$$\begin{aligned} \lim_{n\to\infty} \sup_{1-\varepsilon\le s\le 1} s_n^{-1}\Big|&\alpha_n\big(sk_n/n + (2l_nk_n)^{1/2}(h_\varepsilon(s) + g_n(s))/n\big) \\ &- \alpha_n\big(sk_n/n + (2l_nk_n)^{1/2}h_\varepsilon(s)/n\big)\Big| = 0, \end{aligned}$$

which by (2.47) gives that wp 1,

$$
\begin{aligned}
&\lim_{n\to\infty} \sup_{1-\varepsilon\le s\le 1} s_n^{-1}\Big|\alpha_n(sk_n/n) \\
(2.48)\qquad &\qquad\qquad -\alpha_n\big(sk_n/n + (2l_nk_n)^{1/2}(h_\varepsilon(s) + g_n(s))/n\big)\Big| \\
&= 2^{1/4}(1-\varepsilon)^{1/2}.
\end{aligned}
$$

Now by Theorem 3, part III, with $\nu = 1/2$, for almost every $\omega$ there exists a subsequence $m_\omega$ such that

$$g_m(s) = m^{1/2}\beta_m(sk_m/m)/(2k_ml_m)^{1/2} - h_\varepsilon(s), \qquad 0 \le s \le 1,$$

satisfies $\sup_{0\le s\le 1}|g_m(s)| \to 0$ as $m \to \infty$, along the subsequence $m_\omega$. Thus since

$$U_m(s) = s + (2l_mk_m)^{1/2}\{h_\varepsilon(s) + g_m(s)\}/m,$$

(2.48) implies by the arbitrary choice of $0 < \varepsilon < 1$, that wp 1,

$$\limsup_{n\to\infty} r_n^{-1}R_n^*(k_n) = \limsup_{n\to\infty} s_n^{-1}R_n^*(k_n) \ge 2^{1/4}.$$

Therefore, by Lemma 8, we have (1.16).

The case when $0 < \gamma \le 1$, follows along the same lines; however, instead of using Theorem 3, we apply Finkelstein's theorem for the uniform quantile process, cf. Shorack (1982b). This completes the proof of Theorem 5. □

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