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RANDOM WALKS, CAPACITY AND PERCOLATION ON TREES[1]

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A collection of several different probabilistic processes involving trees is shown to have an unexpected unity. This makes possible a fruitful interplay of these probabilistic processes. The processes are allowed to have arbitrary parameters and the trees are allowed to be arbitrary as well. Our work has five specific aims: First, an exact correspondence between random walks and percolation on trees is proved, extending and sharpening previous work of the author. This is achieved by establishing surprisingly close inequalities between the crossing probabilities of the two processes. Second, we give an equivalent formulation of these inequalities which uses a capacity with respect to a kernel defined by the percolation. This capacitary formulation extends and sharpens work of Fan on random interval coverings. Third, we show how this formulation also applies to generalize work of Evans on random labelling of trees. Fourth, the correspondence between random walks and percolation is used to decide whether certain random walks on random trees are transient or recurrent a.s. In particular, we resolve a conjecture of Griffeath on the necessity of the Nash–Williams criterion. Fifth, for this last purpose, we establish several new basic results on branching processes in varying environments.

1. Introduction. We shall exhibit a simple and useful correspondence between random walks and percolation on arbitrary trees. In the case where the tree is infinite, the random walk is transient if and only if percolation occurs (i.e., there is almost surely an infinite connected component). This is established as a consequence of inequalities relating "crossing probabilities." These probabilities are, on the one hand, the probability that a random walk started at a given vertex hits the boundary of the tree before returning to its starting place (in the infinite case, we mean by this that the walk never returns) and, on the other hand, the probability that the given vertex is connected to the boundary in the corresponding percolation process (by which we mean, in the infinite case, that the component of the vertex is infinite). Our first proof of these inequalities is algebraic and extremely elementary, depending on the well-known relationship between random walks and electrical networks and on recurrence relations for conductances and percolation probabilities arising from the tree structure.

The inequalities relating crossing probabilities can also be expressed by using a capacity with respect to a kernel arising from the percolation probabili-

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ties; this is especially useful for calculating the probability of connection to a subset of the boundary when the tree is infinite. It extends, as well as sharpens, a recent result of Fan ([12], [13]) on random interval coverings. (The connection between such coverings and percolation is explained in [22] and will not be detailed here.)

The capacity which arises above appears more naturally in our second proof of the inequalities between the crossing probabilities. This proof is much more probabilistic than our first one and, although it provides a slightly worse constant, it applies to a useful class of percolation processes wider than the customary edge-independent ones. In particular, we shall apply the result in Section 3 to percolation processes which model random labelling of trees. A random labelling of one tree by another tree is, in brief, an adjacency-preserving random map from the first tree to the second which maps one root to the other; for example, if the second tree is an m-ary tree, then this amounts to choosing a label from $\{1, \dots, m\}$ for every vertex in the first tree. It turns out that such labelling can be modelled by a percolation process on a kind of product tree formed from the two given trees. Thus, by using the results of Section 2, we shall immediately be able to give a close estimate of the distribution of the graph of such a random labelling. This extends and sharpens a recent result of Evans [11], who considered the following special case. For $n, m \geq 2$, assign independently and uniformly to each vertex of an n-ary tree a label in $\{1, \dots, m\}$. Given a set in $\{1, \dots, m\}^{\mathbf{N}}$, Evans gives (implicitly) upper and lower bounds for the probability of finding a non-self-intersecting path beginning at the root of the n-ary tree which is labelled with some sequence drawn from the given set. The bounds of Evans differ by a factor of 16. Besides extending the labelling to a more general setting and answering a broader question, we also reduce the factor of 16 to 4. In addition, we shall show that Evans's method, whose essential ingredient is an elementary case of the Burkholder–Davis–Gundy inequality ([11], [9], Chapter 6, (100.2), [26]), can itself be improved to yield a factor of only 4. Moreover, this improvement can be used as an alternative derivation of one of our main theorems, Theorem 2.4 [although the hypothesis (2.5) would then need to be strengthened].

After establishing the connection between random walks and percolation in Section 2, we see in Section 4 that it reduces the question of deciding the transience or recurrence of certain random walks with drift on random trees to simpler calculations of percolation or survival. The critical value of the drift separating transience from recurrence was already calculated in [22] in the simplest case. Thus, the new aspect here is resolving the question at criticality, as well as extending the generality of our previous results. In this connection, there is a criterion due to Nash–Williams ([27], [14], [24]) which is sufficient for recurrence of reversible random walks. In situations of sufficient regularity, the criterion is necessary as well. With the goal of testing the necessity under random perturbations, Griffeath considered a simple random walk on the following random trees. At time 0, begin with one particle, the root of the tree. At time $n \in \mathbb{Z}^+$, each particle present independently generates two children with probability $(\alpha/n) \wedge 1$ and one child otherwise, where α is a positive constant, and then dies. The corresponding tree is, of course, the associated genealogical tree. Griffeath observed that a "mean" Nash–Williams criterion ensures a.s. recurrence of simple random walk if $\alpha \le 1$; he conjectured a.s. transience if $\alpha > 1$, when the criterion is no longer satisfied. Bramson and Griffeath [5] established transience only for $\alpha > 2$, however. These same results appear independently in [19]. We shall establish the full conjecture in Section 4 and generalize it considerably. We shall also see that with a sufficiently large perturbation, the Nash–Williams criterion is no longer necessary. In order to accomplish these things, we shall establish some new fundamental results on branching processes in varying environments (i.e., with time-inhomogeneous progeny distributions) which are interesting in themselves.

2. Crossing probabilities.

2.1. *Definitions.* We shall use the term *tree* to mean a finite or countable connected graph with a distinguished vertex, 0, called the *root*, and which has no loops or cycles. For any vertex σ, we write $|\sigma|$ for the number of edges on the shortest path from 0 to σ. For vertices σ and τ, write $\sigma \le \tau$ if σ is on the shortest path from 0 to τ; $\sigma < \tau$ if $\sigma \le \tau$ and $\sigma \ne \tau$; and $\sigma \to \tau$ if $\sigma \le \tau$ and $|\tau| = |\sigma| + 1$. We shall use the name of a tree for its vertex set as well. If σ is a vertex of a tree Γ, let Γ^σ denote the subtree formed by the vertices $\{\tau \in \Gamma; \sigma \le \tau\}$ with σ as the root. If $\sigma \ne 0$, then $\overleftarrow{\sigma}$ denotes the unique vertex such that $\overleftarrow{\sigma} \to \sigma$. For $\tau < \sigma$, write $\vec{\tau}(\sigma)$ for the unique vertex satisfying $\tau \to \vec{\tau}(\sigma) \le \sigma$. The edge *preceding* σ, from $\overleftarrow{\sigma}$ to σ, is denoted $e(\sigma)$. We write $\sigma \wedge \tau$ for the vertex farthest from 0 which is less than or equal to both σ and τ. We define the *boundary* $\partial\Gamma$ of a tree Γ as the set of paths beginning at 0 which go through no vertex more than once and which cannot be extended. In the infinite case, we shall also be interested in the *reduced boundary* $\partial'\Gamma$, which is defined to be the subset of $\partial\Gamma$ of infinite paths (if any). Observe that $\partial'\Gamma = \partial\Gamma'$, where Γ' is the *reduced subtree* of Γ whose edges appear in some element of $\partial'\Gamma$. We sometimes identify elements of $\partial\Gamma \setminus \partial'\Gamma$ with their endpoints in Γ. We say that Γ is *locally finite* if the degree of every vertex in Γ is finite, but this is only rarely supposed here.

For a (nearest-neighbor) random walk on Γ, we denote the transition probability from σ to τ by $p_{\sigma,\tau}$. For any (bond) percolation process on Γ, we denote the probability of survival of $e(\sigma)$ by p_σ; we suppose that, $\forall\, \sigma \ne 0$, $p_\sigma < 1$ and that, $\forall\, \sigma$, $\Sigma_{\sigma \to \tau} p_\tau < \infty$. For an electrical network with resistors on the edges of Γ, we use C_σ to denote the conductance of the edge $e(\sigma)$.

We assume as known the correspondence between random walks and electrical networks (see, e.g., [10] or [22], Section 4), whereby the transition probabilities of a random walk are defined from an electrical network via the relations

$$p_{\sigma,\tau} = \begin{cases} \dfrac{C_\tau}{C_\sigma + \Sigma_{\sigma \to \psi} C_\psi}, & \text{if } \sigma \to \tau, \\[2ex] \dfrac{C_\sigma}{C_\sigma + \Sigma_{\sigma \to \psi} C_\psi}, & \text{if } \tau \to \sigma, \end{cases}$$

and, conversely, given any positive constant, a_0, conductances are determined from transition probabilities via the relations

$$C_\sigma = a_0 p_{0,\vec{0}(\sigma)} \prod_{0<\tau<\sigma} \frac{p_{\tau,\vec{\tau}(\sigma)}}{p_{\tau,\overleftarrow{\tau}}}.$$

Note that to every random walk on Γ there corresponds a one-parameter family of electrical networks due to the arbitrary scaling constant, a_0. Our new correspondence between electrical networks and percolation will not depend on a scaling constant, but it will depend on the choice of root. Consequently, our correspondence between random walks and percolation will depend on both a scaling constant and the choice of root; more precisely, to every percolation and choice of root there will correspond one random walk, but to every random walk, choice of root and positive scaling constant there will correspond one percolation. We now describe these correspondences.

For normalization purposes, set

$$a_\sigma := \begin{cases} \displaystyle\sum_{\sigma\to\tau} \frac{p_\tau}{1-p_\tau} + \frac{1}{1-p_\sigma}, & \text{if } \sigma \neq 0, \\ \displaystyle\sum_{0\to\tau} \frac{p_\tau}{1-p_\tau}, & \text{if } \sigma = 0. \end{cases}$$

Let us say that a random walk with transition probabilities $\{p_{\sigma,\tau}\}$ and a percolation process with survival probabilities $\{p_\sigma\}$ are *associated* if

$$p_{\sigma,\tau} = \begin{cases} \dfrac{p_\tau}{1-p_\tau} a_\sigma^{-1}, & \text{if } \sigma \to \tau, \\ \dfrac{1}{1-p_\sigma} a_\sigma^{-1}, & \text{if } \tau \to \sigma. \end{cases}$$

Note that such a random walk has a bias toward the root even when all p_σ are equal. Correspondingly, we shall say that an electrical network with conductances $\{C_\sigma\}$ and a percolation process with survival probabilities $\{p_\sigma\}$ are *associated* if

$$C_\sigma = \frac{1}{1-p_\sigma} \prod_{0<\tau\le\sigma} p_\tau.$$

In this case, we find that

$$1 + \sum_{0<\tau\le\sigma} C_\tau^{-1} = \prod_{0<\tau\le\sigma} p_\tau^{-1},$$

whence

$$p_\sigma = \frac{1 + \sum_{0<\tau<\sigma} C_\tau^{-1}}{1 + \sum_{0<\tau\le\sigma} C_\tau^{-1}}.$$

We denote by $0 \to \partial\Gamma$ the event that the random walk started at 0 does not return to 0 before traversing some element of $\partial\Gamma$. By $0 \leftrightarrow \partial\Gamma$ we mean the event that some element of $\partial\Gamma$ remains in the random graph generated by the percolation process. More generally, for $E \subseteq \partial\Gamma$, we write $0 \leftrightarrow E$ for the event that some element of E remains after percolation; and for $\sigma \in \Gamma$, we write $0 \leftrightarrow \sigma$ if 0 and σ lie in the same connected component after percolation. We write $\mathscr{C}(0 \to \partial\Gamma)$ for the effective conductance of the electrical network from 0 to $\partial\Gamma$. (For infinite Γ, this is defined as the limit of conductances over the net of finite subtrees, cf. [22], Section 4.) Similar definitions apply for $\partial'\Gamma$.

We say that a percolation process is *Bernoulli* if the events of survival of the edges are mutually independent.

2.2. *The basic connection.* The fundamental inequalities linking percolation, random walks and electrical networks on trees are as follows in the simplest but most important situation.

THEOREM 2.1. *Let Γ be a tree with mutually associated percolation process, random walk and electrical network. If the percolation is Bernoulli, then*

$$\frac{\mathbb{P}[0 \to \partial\Gamma]}{a_0^{-1} + \mathbb{P}[0 \to \partial\Gamma]} = \frac{\mathscr{C}(0 \to \partial\Gamma)}{1 + \mathscr{C}(0 \to \partial\Gamma)} \le \mathbb{P}[0 \leftrightarrow \partial\Gamma] \le 2\frac{\mathscr{C}(0 \to \partial\Gamma)}{1 + \mathscr{C}(0 \to \partial\Gamma)},$$

which is the same as

$$\frac{\mathbb{P}[0 \leftrightarrow \partial\Gamma]}{2 - \mathbb{P}[0 \leftrightarrow \partial\Gamma]} \le \mathscr{C}(0 \to \partial\Gamma) = a_0\mathbb{P}[0 \to \partial\Gamma] \le \frac{\mathbb{P}[0 \leftrightarrow \partial\Gamma]}{1 - \mathbb{P}[0 \leftrightarrow \partial\Gamma]}.$$

The same inequalities hold with $\partial'\Gamma$ in place of $\partial\Gamma$.

Now if $\Gamma' \neq \varnothing$, then a random walk on Γ is transient iff $\mathbb{P}[0 \to \partial'\Gamma] > 0$. Theorem 2.1 shows that this is equivalent to $\mathbb{P}[0 \leftrightarrow \partial'\Gamma] > 0$, that is, to the occurrence of percolation.

A nice way to express the inequalities of Theorem 2.1 is as follows. Add a vertex x to Γ which is joined to 0 by an edge of conductance 1. Let $0 \to \partial\Gamma, \neg x$ be the event that the random walk on $\Gamma \cup \{x\}$ associated to these conductances and started at 0 does not hit x before traversing some element of $\partial\Gamma$. Then the inequalities become

$$\mathbb{P}[0 \to \partial\Gamma, \neg x] \le \mathbb{P}[0 \leftrightarrow \partial\Gamma] \le 2\mathbb{P}[0 \to \partial\Gamma, \neg x]$$

since

$$\mathbb{P}[0 \to \partial\Gamma, \neg x] = \frac{\mathscr{C}(0 \to \partial\Gamma)}{1 + \mathscr{C}(0 \to \partial\Gamma)},$$

as seen immediately from the fundamental properties of the connection between random walks and electrical networks [10]. The same holds for $\partial'\Gamma$.

We shall use the following little inequalities in the proof of Theorem 2.1.

LEMMA 2.2. *If* $\{x_n\} \subseteq]0,1]$, *then*

$$\sum \frac{1-x_n}{x_n} \le \frac{1-\prod x_n}{\prod x_n} \quad \textit{and} \quad \sum \frac{1-x_n}{1+x_n} \ge \frac{1-\prod x_n}{1+\prod x_n}.$$

In each case, equality holds iff $x_n = 1$ *for all but at most one* n.

PROOF. It is elementary to verify the inequalities when there are only two terms. Induction on the number of terms then leads promptly to the general case. □

PROOF OF THEOREM 2.1. The inequalities for $\partial'\Gamma$ follow from those for $\partial\Gamma$ by using Γ' in place of Γ. Also, by taking limits, it suffices to establish the results for finite trees, so assume that Γ is finite.

The fact that $\mathscr{C}(0 \to \partial\Gamma) = a_0 \mathbb{P}[0 \to \partial\Gamma]$ is the content of [18], Lemma 9-129 (since a_0 is the sum of the conductances of the edges incident to 0).

It remains to prove the inequalities between the conductance and the percolation probability. Let $\mathscr{C}^\Gamma(\sigma)$ be the effective conductance from σ to $\partial\Gamma^\sigma$ in Γ. By the usual series–parallel circuit laws [or [22], (5.3)], we have

$$\mathscr{C}(0 \to \partial\Gamma) = \sum_{|\sigma|=1} \left(C_\sigma^{-1} + \mathscr{C}^\Gamma(\sigma)^{-1}\right)^{-1}.$$

Now

$$|\sigma| = 1 \quad \Rightarrow \quad \mathscr{C}^\Gamma(\sigma) = p_\sigma \mathscr{C}(\sigma \to \partial\Gamma^\sigma),$$

where the latter conductance is defined *not* using the edge conductances in Γ, but via the edge conductances associated to the survival probabilities in Γ^σ, where σ (not 0) is the root. Incorporating this in the formula above, we obtain

$$\begin{aligned}\mathscr{C}(0 \to \partial\Gamma) &= \sum_{|\sigma|=1} \left(C_\sigma^{-1} + p_\sigma^{-1}\mathscr{C}(\sigma \to \partial\Gamma^\sigma)^{-1}\right)^{-1}\\ &= \sum_{|\sigma|=1} \left(p_\sigma^{-1} - 1 + p_\sigma^{-1}\mathscr{C}(\sigma \to \partial\Gamma^\sigma)^{-1}\right)^{-1}\end{aligned}$$

unless $\Gamma = \{0\}$, in which case $\mathscr{C}(0 \to \partial\Gamma) = +\infty$. We also have a recursion formula for the percolation probabilities:

$$\mathbb{P}[0 \leftrightarrow \partial\Gamma] = 1 - \prod_{|\sigma|=1} \left(1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma]\right)$$

unless $\Gamma = \{0\}$, in which case $\mathbb{P}[0 \leftrightarrow \partial\Gamma] = 1$.

Our proof of the theorem will proceed by induction on the cardinality (or height) of Γ, based on these recursion formulas. The theorem is true for $\Gamma = \{0\}$, so given Γ, assume the inequalities are satisfied for the trees Γ^σ

($|\sigma| = 1$). Thus, from

$$\mathscr{C}(\sigma \to \partial\Gamma^\sigma) \leq \frac{\mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma]}{1 - \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma]}, \qquad |\sigma| = 1,$$

we deduce that

$$\begin{aligned}\mathscr{C}(0 \to \partial\Gamma) &\leq \sum_{|\sigma|=1} \Big(p_\sigma^{-1} - 1 + p_\sigma^{-1}\big(\mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma]^{-1} - 1\big)\Big)^{-1} \\ &= \sum_{|\sigma|=1} \frac{1 - (1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma])}{1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma]} \\ &\leq \frac{1 - \prod(1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma])}{\prod(1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma])} \\ &= \frac{\mathbb{P}[0 \leftrightarrow \partial\Gamma]}{1 - \mathbb{P}[0 \leftrightarrow \partial\Gamma]},\end{aligned}$$

having used Lemma 2.2 in the second inequality. In the same way, from

$$\mathscr{C}(\sigma \to \partial\Gamma^\sigma) \geq \frac{\mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma]}{2 - \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma]}, \qquad |\sigma| = 1,$$

we deduce that

$$\begin{aligned}\mathscr{C}(0 \to \partial\Gamma) &\geq \sum_{|\sigma|=1} \Big(p_\sigma^{-1} - 1 + p_\sigma^{-1}\big(2\mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma]^{-1} - 1\big)\Big)^{-1} \\ &= \sum_{|\sigma|=1} \frac{1 - (1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma])}{1 + (1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma])} \\ &\geq \frac{1 - \prod(1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma])}{1 + \prod(1 - p_\sigma \mathbb{P}[\sigma \leftrightarrow \partial\Gamma^\sigma])} \\ &= \frac{\mathbb{P}[0 \leftrightarrow \partial\Gamma]}{2 - \mathbb{P}[0 \leftrightarrow \partial\Gamma]}.\end{aligned}$$

The induction is complete. □

2.3. *Capacity and boundary sets.* We now describe the role of capacity on $\partial\Gamma$. Given distinct points $s, t \in \partial\Gamma$, we write $s \wedge t$ for the vertex of Γ farthest from 0 which is common to both s and t; if $s = t$, then we set $s \wedge t := s$. Under the metric $d(s,t) := e^{-|s \wedge t|}$, $s \neq t$, the boundary is a complete separable metric space. Given an electrical network on Γ, let

$$K(s,t) := 1 + \sum_{0 < \sigma \leq s \wedge t} C_\sigma^{-1}. \tag{2.1}$$

The *potential* of a positive σ-finite Borel measure μ on $\partial\Gamma$ is the function

$$V_\mu(s) := \int_{\partial\Gamma} K(s,t)\, d\mu(t)$$

and the *energy* of μ is the number

$$\mathscr{E}(\mu) := \int_{\partial\Gamma\times\partial\Gamma} K(s,t)\, d(\mu\times\mu)(s,t) = \int_{\partial\Gamma} V_\mu(s)\, d\mu(s).$$

The *capacity* of a subset $E \subseteq \partial\Gamma$ is defined as

$$\operatorname{cap} E := \sup\left\{\mathscr{E}(\mu)^{-1};\ \mu \text{ a Borel probability measure, } \mu(\partial\Gamma \setminus E) = 0\right\}.$$

We call a nonnegative function θ on the vertices of Γ a *unit flow* if $\theta(0) = 1$ and, $\forall\ \sigma$, $\theta(\sigma) = \sum_{\sigma\to\tau}\theta(\tau)$ if $\exists\ \tau$ with $\sigma \to \tau$. Such functions θ are in 1–1 correspondence with Borel probability measures μ on $\partial\Gamma$ via the equation

$$\theta(\sigma) = \mu(\{s; \sigma \in s\}).$$

The following statements are shown in [22], Section 4: If μ is a probability measure, then

$$V_\mu(s) = 1 + \sum_{0<\sigma\in s} \theta(\sigma) C_\sigma^{-1}$$

and

$$\mathscr{E}(\mu) = 1 + \sum_{0\neq\sigma\in\Gamma} \theta(\sigma)^2 C_\sigma^{-1};$$

the associated random walk is transient iff current flow is possible, which is equivalent to $\operatorname{cap}\partial\Gamma > 0$, in which case unit current flow θ_0 corresponds to harmonic measure μ_0 on $\partial\Gamma$; μ_0 is the unique probability measure on $\partial\Gamma$ of minimum energy; and

$$V_{\mu_0}(s) = \mathscr{E}(\mu_0) = (\operatorname{cap}\partial\Gamma)^{-1} = 1 + \mathscr{C}(0 \to \partial\Gamma)^{-1} \tag{2.2}$$

except for s belonging to a set of capacity 0. Thus, provided $\operatorname{cap}\partial\Gamma > 0$, $V_{\mu_0}(s)$ is 1 plus the sum of the potential drops of θ_0 along the edges of s (for *every* s) and $\mathscr{E}(\mu_0)$ is 1 plus the power loss of θ_0.

Recall that a set is called *analytic* if it is a continuous image of a Borel subset of a complete separable metric space ([1], pages 64–65).

THEOREM 2.3. *Under the hypotheses of Theorem* 2.1, *if* $E \subseteq \partial\Gamma$ *is analytic, then*

$$\operatorname{cap}\partial\Gamma \cdot \mu_0(E) \le \operatorname{cap} E \le \mathbb{P}[0 \leftrightarrow E] \le 2\operatorname{cap} E.$$

PROOF. When $E = \partial\Gamma$, this reduces to Theorem 2.1 by virtue of (2.2). When E is closed, there is a subtree Γ_E of Γ such that $E = \partial\Gamma_E$. (Of course, E is compact if and only if Γ_E is locally finite.) Consequently, the latter two inequalities are again valid. The first inequality follows for analytic (indeed, μ_0-measurable) E by consideration of the restriction of μ_0 to E (when μ_0 exists, i.e., when $\operatorname{cap}\partial\Gamma > 0$).

Now it is not hard to see from the definition that

$$\operatorname{cap} E = \sup\{\operatorname{cap} K;\ K \subseteq E,\ K \text{ compact}\}$$

for all E. Indeed, if $\mu(\partial\Gamma \setminus E) = 0$, then by regularity of μ ([8], Theorem 3.38, page 61), there is an increasing sequence of compact sets $K_n \subseteq E$ such that $\mu(\partial\Gamma \setminus \bigcup K_n) = 0$. By the monotone convergence theorem, we have

$$\mathscr{E}(\mathbf{1}_{K_n} \cdot \mu) \to \mathscr{E}(\mu) \quad \text{as } n \to \infty,$$

whence

$$\liminf_{N \to \infty} \operatorname{cap} K_n \geq \mathscr{E}(\mu)^{-1}.$$

On the other hand, from the Borel–Cantelli lemma and the hypothesis that $\sum_{\sigma \to \tau} p_\tau < \infty$, it follows that the component of 0 after percolation is a.s. locally finite, whence $\{s \in \partial\Gamma;\ 0 \leftrightarrow s\}$ is a.s. compact. It is routine to verify that $0 \leftrightarrow E$ is a measurable event for E compact and then that $\mathbb{P}^*[0 \leftrightarrow E]$ is a Choquet capacity on $\partial\Gamma$, where $\mathbb{P}^*$ denotes outer $\mathbb{P}$-measure. If E is analytic, then $0 \leftrightarrow E$ is analytic with respect to the paving $\{0 \leftrightarrow K;\ K \text{ compact}\}$ ([8], Theorem 3.11, page 43), whence $0 \leftrightarrow E$ is $\mathbb{P}$-measurable ([8], Section 3.3.3, page 58) and

$$\mathbb{P}[0 \leftrightarrow E] = \sup\{\mathbb{P}[0 \leftrightarrow K];\ K \subseteq E,\ K \text{ compact}\}$$

by the Choquet capacitability theorem ([8], Theorem 3.28, page 52). Since cap obeys the same relation, the inequalities for analytic E follow from those for compact E. $\square$

According to this theorem, if $E \subseteq \partial\Gamma$ is analytic and $\mathbb{P}[0 \leftrightarrow E] > 0$, then for every $\alpha > 2/\mathbb{P}[0 \leftrightarrow E]$, E carries a Borel probability measure of energy less than α. The proof implicitly gives such a measure, namely, harmonic measure of random walk on some subtree Γ_K, K being a sufficiently large compact subset of E. However, one might desire to know such a measure in terms of the percolation, rather than as harmonic measure of random walk, especially since the kernel (2.1) has a simple expression directly in terms of the percolation, namely,

$$K(s,t) = \mathbb{P}[0 \leftrightarrow s \wedge t]^{-1}.$$

Later, in discussing Theorem 2.7, we shall see how to get such a measure more explicitly with energy less than α for any $\alpha > 4/\mathbb{P}[0 \leftrightarrow E]$, although an unspecified compact subset K of E still appears.

2.4. *Non-Bernoulli percolation.* The preceding results require only a slight modification for *site* percolation due to a Markov random field (such as the Ising model [21]). This is because the law of the component of 0, given the survival of 0, is the same as that for Bernoulli bond percolation with

$$p_\sigma := \mathbb{P}\left[0 \leftrightarrow \sigma | 0 \leftrightarrow \overleftarrow{\sigma}\right].$$

With this new definition of p_σ used to define $\{p_{\sigma,\tau}\}$, $\{C_\sigma\}$ and cap, the inequalities of Theorems 2.1 and 2.3 remain valid with the percolation probabilities there conditioned on the survival of 0.

Still more general percolation processes have found use (e.g., in [21] and [23]) and continue to elicit attention. In order to relate such processes to random walks, electrical networks and capacity, we need a more flexible method than the algebraic method of Section 2.3. Until now, we have relied on fairly nonprobabilistic means, not having taken even one expectation, although integrals have appeared in defining energy. Our second method is quite a bit more probabilistic than the first and reveals the relationship between these energy integrals and certain expectations. The method enables us to prove the following results. Call a random subtree $\Gamma_0(\omega)$ rooted at 0 a *percolation* on Γ if $\{\omega;\ \sigma \in \Gamma_0(\omega)\}$ is a measurable event for each $\sigma \in \Gamma$. As before, we write $0 \leftrightarrow \sigma$ for $\sigma \in \Gamma_0(\omega)$ and $0 \leftrightarrow E$ for $E \cap \partial\Gamma_0(\omega) \neq \varnothing$.

THEOREM 2.4. *Given a percolation on a tree Γ, define conductances from the relations*

$$\mathbb{P}[0 \leftrightarrow \sigma] = \left(1 + \sum_{0<\tau\le\sigma} C_\tau^{-1}\right)^{-1}, \qquad \sigma \in \Gamma, \tag{2.3}$$

and define cap *using the kernel in* (2.1). *If there is some $M_1 < \infty$ such that, for all $\sigma, \tau \in \Gamma$,*

$$\begin{aligned} &\mathbb{P}[0 \leftrightarrow \sigma \text{ and } 0 \leftrightarrow \tau \mid 0 \leftrightarrow \sigma \wedge \tau] \\ &\qquad \le M_1 \mathbb{P}[0 \leftrightarrow \sigma \mid 0 \leftrightarrow \sigma \wedge \tau]\mathbb{P}[0 \leftrightarrow \tau \mid 0 \leftrightarrow \sigma \wedge \tau], \end{aligned} \tag{2.4}$$

then

$$\frac{1}{M_1}\frac{\mathscr{C}(0 \to \partial\Gamma)}{1 + \mathscr{C}(0 \to \partial\Gamma)} \le \mathbb{P}[0 \leftrightarrow \partial\Gamma]$$

and, for analytic $E \subseteq \partial\Gamma$,

$$M_1^{-1} \operatorname{cap} E \le \mathbb{P}[0 \leftrightarrow E].$$

If there is some $M_2 > 0$ such that, for all $\sigma, \tau \in \Gamma$ and $A \subseteq \Gamma$ with the property that the removal of $\sigma \wedge \tau$ would disconnect τ from every vertex in A,

$$\mathbb{P}[0 \leftrightarrow \tau \mid 0 \leftrightarrow \sigma \text{ and } 0 \not\leftrightarrow A] \ge M_2 \mathbb{P}[0 \leftrightarrow \tau \mid 0 \leftrightarrow \sigma \wedge \tau], \tag{2.5}$$

then

$$\mathbb{P}[0 \leftrightarrow \partial\Gamma] \le \frac{4}{M_2}\frac{\mathscr{C}(0 \to \partial\Gamma)}{1 + \mathscr{C}(0 \to \partial\Gamma)}$$

and, for analytic $E \subseteq \partial\Gamma$,

$$\mathbb{P}[0 \leftrightarrow E] \le 4M_2^{-1} \operatorname{cap} E.$$

REMARK. It will be seen from the proof that in case Γ is well founded (i.e., $\partial'\Gamma = \varnothing$—in particular, if Γ is finite), then it suffices that (2.4) or (2.5) hold only for $\sigma, \tau \in \partial\Gamma$ and $A \subseteq \partial\Gamma$. Thus, in general, it suffices that (2.4) or (2.5) hold for $\sigma, \tau \in \partial\Gamma_n$ and $A \subseteq \partial\Gamma_n$ for some sequence of finite subtrees Γ_n of Γ (rooted at 0) tending to all of Γ.

Percolation processes satisfying (2.4) were termed quasi-Bernoulli in [21] and used therein as well as in [23]. Bernoulli percolation processes, of course, satisfy (2.4) and (2.5) with $M_1 = M_2 = 1$ and with equality.

PROOF OF THEOREM 2.4. By the same methods we used in the proofs of Theorems 2.1 and 2.3, it suffices to assume that Γ is finite and to deduce $M_1^{-1} \operatorname{cap} \partial\Gamma \le \mathbb{P}[0 \leftrightarrow \partial\Gamma]$ from (2.4) and $\mathbb{P}[0 \leftrightarrow \partial\Gamma] \le 4M_2^{-1} \operatorname{cap} \partial\Gamma$ from (2.5). Also, we may as well assume that, $\forall\, s \in \partial\Gamma$, $\mathbb{P}[0 \leftrightarrow s] > 0$.

Define the random variable

$$X := \sum_{s \in \partial\Gamma} \mu_0(s) \mathbf{1}_{0 \leftrightarrow s} \mathbb{P}[0 \leftrightarrow s]^{-1}.$$

Then

$$\mathbb{E}[X^2] = \sum_{s, t \in \partial\Gamma} \mu_0(s)\mu_0(t) \frac{\mathbb{P}[0 \leftrightarrow s \text{ and } 0 \leftrightarrow t]}{\mathbb{P}[0 \leftrightarrow s]\mathbb{P}[0 \leftrightarrow t]}.$$

If (2.4) holds, then

$$\begin{aligned} \mathbb{E}[X^2] &\le M_1 \sum_{s, t \in \partial\Gamma} \mu_0(s)\mu_0(t)\mathbb{P}[0 \leftrightarrow s \wedge t]^{-1} = M_1 \mathscr{E}(\mu_0) \\ &= M_1 (\operatorname{cap} \partial\Gamma)^{-1}. \end{aligned}$$

It follows that

$$\begin{aligned} 1 = \mathbb{E}[X]^2 &= \mathbb{E}[X \mathbf{1}_{0 \leftrightarrow \partial\Gamma}]^2 \le \mathbb{E}[X^2]\mathbb{E}\left[\mathbf{1}^2_{0 \leftrightarrow \partial\Gamma}\right] \\ &\le M_1 (\operatorname{cap} \partial\Gamma)^{-1} \mathbb{P}[0 \leftrightarrow \partial\Gamma], \end{aligned}$$

as desired.

For the other direction, we adapt Shepp's "stopping-time" approach [30]. First, embed Γ in the upper half-plane with 0 at the origin, and order $\partial\Gamma$ clockwise. Denoting this ordering by $\preccurlyeq$, we may express the important feature of this ordering by

$$s_1 \preccurlyeq s_2 \preccurlyeq t_2 \preccurlyeq t_1 \quad \Rightarrow \quad s_1 \wedge t_1 \le s_2 \wedge t_2.$$

Assume that $+\infty$ is a symbol $\notin \partial\Gamma$ and extend the ordering so that every element of $\partial\Gamma$ is $\prec +\infty$. Define two subsets of $\partial\Gamma$ as follows:

$$\partial^+\Gamma := \left\{ s \in \partial\Gamma;\ \sum_{s \preccurlyeq t} \mu_0(t)\mathbb{P}[0 \leftrightarrow s \wedge t]^{-1} \ge \tfrac{1}{2}\mathscr{E}(\mu_0) \right\},$$

$$\partial^-\Gamma := \left\{ s \in \partial\Gamma;\ \sum_{t \prec s} \mu_0(t)\mathbb{P}[0 \leftrightarrow s \wedge t]^{-1} \ge \tfrac{1}{2}\mathscr{E}(\mu_0) \right\}.$$

Then $\partial\Gamma = \partial^+\Gamma \cup \partial^-\Gamma$ since $V_{\mu_0}(s) = \mathscr{E}(\mu_0)$ for all $s \in \partial\Gamma$. Define a random variable

$$s^+(\omega) := \begin{cases} \min \partial^+\Gamma \cap \Gamma_0(\omega), & \text{if } \partial^+\Gamma \cap \Gamma_0(\omega) \neq \varnothing, \\ +\infty, & \text{otherwise,} \end{cases}$$

which has properties similar to a stopping time. Now

$$1 = \mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|s^+]] \geq \mathbb{E}\Bigg[\sum_{s^+ \preccurlyeq s} \mu_0(s)\mathbb{P}[0 \leftrightarrow s|s^+]\mathbb{P}[0 \leftrightarrow s]^{-1}\Bigg].$$

Let $A_t := \{s \in \partial\Gamma;\ s \prec t\}$. Then

$$\mathbb{P}[0 \leftrightarrow s|s^+] = \sum_{t \in \partial\Gamma} \mathbb{P}[0 \leftrightarrow s|0 \leftrightarrow t \text{ and } 0 \not\leftrightarrow A_t]\mathbf{1}_{s^+=t},$$

whence if (2.5) holds, then

$$\begin{aligned} 1 &\geq M_2\mathbb{E}\Bigg[\sum_{s^+ \preccurlyeq s} \mu_0(s)\frac{\mathbb{P}[0 \leftrightarrow s|0 \leftrightarrow s \wedge s^+]}{\mathbb{P}[0 \leftrightarrow s]}\Bigg] \\ &= M_2\mathbb{E}\Bigg[\sum_{s^+ \preccurlyeq s} \mu_0(s)\mathbb{P}[0 \leftrightarrow s \wedge s^+]^{-1}\Bigg]. \end{aligned}$$

Since $s^+ \in \partial^+\Gamma$ when $0 \leftrightarrow \partial^+\Gamma$, it follows that

$$1 \geq M_2\mathbb{E}\big[\tfrac{1}{2}\mathscr{E}(\mu_0)\mathbf{1}_{0 \leftrightarrow \partial^+\Gamma}\big] = \tfrac{1}{2}M_2\mathscr{E}(\mu_0)\mathbb{P}[0 \leftrightarrow \partial^+\Gamma].$$

Similarly, $1 \geq \frac{1}{2}M_2\mathscr{E}(\mu_0)\mathbb{P}[0 \leftrightarrow \partial^-\Gamma]$. Addition of these inequalities yields

$$\begin{aligned} \mathbb{P}[0 \leftrightarrow \partial\Gamma] &\leq \mathbb{P}[0 \leftrightarrow \partial^+\Gamma] + \mathbb{P}[0 \leftrightarrow \partial^-\Gamma] \\ &\leq 4M_2^{-1}\mathscr{E}(\mu_0)^{-1} = 4M_2^{-1}\operatorname{cap}\partial\Gamma, \end{aligned}$$

as desired. □

It was observed in [22] that when Γ is infinite and locally finite and p_σ is constant, then the expected size of the component of 0 is finite iff the associated random walk is positive recurrent. However, only half of this equivalence remains true in general.

PROPOSITION 2.5. *Assume* (2.3) *for a percolation on a tree* Γ. *Then* $\mathbb{E}[\operatorname{card}\Gamma_0(\omega)] < \infty$ *if the associated random walk is positive recurrent, but not conversely.*

PROOF. We have the estimate

$$\begin{aligned} \mathbb{E}[\operatorname{card}\Gamma_0(\omega)] &= \mathbb{E}\Bigg[\sum_{\sigma \in \Gamma} \mathbf{1}_{0 \leftrightarrow \sigma}\Bigg] = \sum_{\sigma \in \Gamma} \mathbb{P}[0 \leftrightarrow \sigma] \\ &= 1 + \sum_{0 \neq \sigma \in \Gamma} \Bigg(1 + \sum_{0 < \tau \leq \sigma} C_\tau^{-1}\Bigg)^{-1} \leq 1 + \sum_{0 \neq \sigma \in \Gamma} C_\sigma. \end{aligned}$$

Since the walk is positive recurrent iff $\sum C_\sigma < \infty$ ([18], Proposition 9–131), positive recurrence entails the finiteness of $\mathbb{E}[\operatorname{card}\Gamma_0(\omega)]$. On the other hand, if

$\Gamma = \mathbf{N}$ with $C_n = 1/n$, then the walk is null recurrent while the expected size of $\Gamma_0(\omega)$ is finite. $\square$

It would be nice to have a direct probabilistic proof of the relationship between random walks and percolation—one that does not use the medium of electrical networks or its equivalents. Unfortunately, we were not successful in finding such a proof.

2.5. *General percolation and capacity.* As we shall see in Section 3, it is quite useful to have a version of Theorem 2.4 which relates even more general percolation to capacity. Here, conductances (and random walks) may no longer play a role. Furthermore, the capacity may no longer arise from a kernel on $\partial\Gamma$. Rather, the capacity will be defined through a limiting energy which arises from a kernel on Γ, not on $\partial\Gamma$. Therefore, we must redevelop some of our earlier theory of capacity on trees ([22], Section 4), sketched in Section 2.3, to this more general setting.

Given a tree Γ, define the *height* of Γ as

$$\operatorname{ht}\Gamma := \sup\{|\sigma|;\, \sigma \in \Gamma\}.$$

For $n \geq 0$, write

$$\Pi_n := \{\sigma \in \Gamma;\, |\sigma| = n \text{ or } (|\sigma| < n \text{ and } \sigma \in \partial\Gamma)\}.$$

For a general percolation as defined in Section 2.4, write $P_\sigma := \mathbb{P}[\sigma \in \Gamma_0(\omega)]$ and

$$K(\sigma,\tau) := \begin{cases} \dfrac{\mathbb{P}[\sigma,\tau \in \Gamma_0(\omega)]}{P_\sigma P_\tau}, & \text{if } P_\sigma P_\tau \neq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{2.6}$$

Let θ be a real-valued function on Γ such that, $\forall\, \sigma \in \Gamma \setminus \partial\Gamma$, $\theta(\sigma) = \sum_{\sigma \to \tau} \theta(\tau)$. If

$$\forall\, \sigma \in \Gamma, \qquad P_\sigma = 0 \quad \Rightarrow \quad \theta(\sigma) = 0,$$

then we say that θ is *subordinate* to the percolation. Set

$$X_n^\theta(\omega) := \sum_{\substack{\sigma \in \Pi_n \\ P_\sigma \neq 0}} \frac{\theta(\sigma)}{P_\sigma} \mathbf{1}_{0 \leftrightarrow \sigma}(\omega)$$

and

$$\mathscr{E}_n(\theta) := \mathbb{E}\Big[\big(X_n^\theta\big)^2\Big] = \sum_{\sigma,\tau \in \Pi_n} K(\sigma,\tau)\theta(\sigma)\theta(\tau).$$

Now $\mathscr{E}_n \geq 0$ and, from simple algebra,

$$\mathscr{E}_n\left(\frac{\theta_1 + \theta_2}{2}\right) + \mathscr{E}_n\left(\frac{\theta_1 - \theta_2}{2}\right) = \frac{\mathscr{E}_n(\theta_1) + \mathscr{E}_n(\theta_2)}{2} \tag{2.7}$$

[which amounts to the parallelogram law for the inner-product seminorm

$\mathscr{E}_n(\cdot)^{1/2}]$. In particular, $\mathscr{E}_n$ is convex. Write $\mathscr{F}_n$ for the σ-field generated by the events $\{0 \leftrightarrow \sigma;\ |\sigma| \le n\}$. If

$$(2.8) \qquad \forall\, \sigma \neq 0, \qquad \mathbb{P}\big[0 \leftrightarrow \sigma \,|\, \mathscr{F}_{|\overleftarrow{\sigma}|}\big] = \mathbb{P}\big[0 \leftrightarrow \sigma \,|\, 0 \leftrightarrow \overleftarrow{\sigma}\big] \mathbf{1}_{0 \leftrightarrow \overleftarrow{\sigma}},$$

then $\{(X_n^\theta, \mathscr{F}_n)\}$ is a martingale, so that by Jensen's inequality we have

$$(2.9) \qquad \mathscr{E}_n(\theta) \le \mathscr{E}_{n+1}(\theta).$$

In such a situation, we may define

$$\mathscr{E}(\theta) := \lim_{n \to \infty} \mathscr{E}_n(\theta).$$

Let $\mathscr{U}$ be the convex set of unit flows on Γ which are subordinate to the percolation.

PROPOSITION 2.6. *If a percolation on a locally finite tree Γ satisfies* (2.8), *then*

$$\inf_{\theta \in \mathscr{U}} \mathscr{E}(\theta) = \lim_{n \to \infty} \inf_{\theta \in \mathscr{U}} \mathscr{E}_n(\theta).$$

If this number is finite, then the infima are minima and there is a unique $\theta \in \mathscr{U}$ minimizing $\mathscr{E}$.

PROOF. Give $\mathscr{U}$ the inherited topology from the product space ${}^{\Gamma}\mathbb{R}$. Then $\mathscr{E}_n$ is continuous on $\mathscr{U}$. Since Γ is locally finite, $\mathscr{U}$ is compact, whence $\mathscr{E}_n$ has a minimum on $\mathscr{U}$ unless $\mathscr{U}$ is empty. The asserted equality is now clear from (2.9), as is the fact that the infima are minima when finite. Lastly, if both θ and θ' minimize $\mathscr{E}$ on $\mathscr{U}$, then

$$\min \mathscr{E} = \mathscr{E}\left(\frac{\theta + \theta'}{2}\right) + \mathscr{E}\left(\frac{\theta - \theta'}{2}\right)$$

by virtue of (2.7), which implies that $\mathscr{E}((\theta - \theta')/2) = 0$. From (2.9), we may conclude that $\mathscr{E}_n((\theta - \theta')/2) = 0$ for all n, whence $X_n^{(\theta-\theta')/2} = 0$ a.s. for all n, so $\theta = \theta'$. □

If (2.8) is satisfied, μ is a Borel probability measure on $\partial\Gamma$, θ is the unit flow corresponding to μ, and θ is subordinate to the percolation, then we say that μ is *subordinate* and we define $\mathscr{E}(\mu) := \mathscr{E}(\theta)$. Finally, for $E \subseteq \partial\Gamma$, set

$$(2.10) \quad \operatorname{cap} E := \sup\left\{\mathscr{E}(\mu)^{-1};\ \mu \text{ subordinate and } \mu(\partial\Gamma \setminus E) = 0\right\} \vee 0.$$

We now have enough theory to state and prove our most general results on percolation.

THEOREM 2.7. *Let Γ be a tree of finite height. Given a percolation on Γ, define* cap *as in* (2.10) *using the kernel* (2.6). *For $E \subseteq \partial\Gamma$, we have*

$$(2.11) \qquad \operatorname{cap} E \le \mathbb{P}[0 \leftrightarrow E].$$

If there is some $M > 0$ such that, for all $\sigma, \tau \in \partial\Gamma$ and $A \subseteq \partial\Gamma$ with the *property that the removal of* $\sigma \wedge \tau$ *would disconnect* τ *from every vertex in* A,

$$\mathbb{P}[0 \leftrightarrow \tau | 0 \leftrightarrow \sigma \text{ and } 0 \not\leftrightarrow A] \geq M \mathbb{P}[0 \leftrightarrow \tau | 0 \leftrightarrow \sigma], \tag{2.12}$$

then, *for all* $E \subseteq \partial\Gamma$,

$$\mathbb{P}[0 \leftrightarrow E] \leq 4M^{-1} \operatorname{cap} E. \tag{2.13}$$

The proof is exactly as for Theorem 2.4. Using reasoning similar to that used in proving Theorems 2.1 and 2.3, we may deduce the following corollary of Theorem 2.7 which lifts the restriction that the height of Γ be finite.

COROLLARY 2.8. *Given a percolation on a locally finite tree which satisfies* (2.8), *define* cap *as in* (2.10). *For any analytic set* $E \subseteq \partial\Gamma$, *we have* (2.11). *If* (2.12) *holds for all* $\sigma, \tau \in \Gamma$ *and* $A \subseteq \Gamma$ *with the property that the removal of* $\sigma \wedge \tau$ *would disconnect* τ *from* A, *then*, *for analytic* $E \subseteq \partial\Gamma$, *we have* (2.13).

It is interesting that the method Evans uses in his study of random labelling [11] can also be used to prove Theorem 2.7, although it requires a stricter hypothesis than (2.12). On the other hand, Evans's method has an advantage in that it gives a measure of small energy directly in terms of the percolation. Finally, as mentioned in the introduction, the constant arising in his method can be improved from 16 to 4. To illustrate all of this, we shall sketch how the method gives Theorem 2.7 in the special case of Bernoulli percolation.

The proof of (2.11) is no different than that of Theorem 2.4. To prove (2.13), we order the vertices of $\partial\Gamma$ (assumed finite) as in the proof of Theorem 2.4. Define

$$s^*(\omega) := \begin{cases} \min \partial\Gamma \cap \Gamma_0(\omega), & \text{if } \partial\Gamma \cap \Gamma_0(\omega) \neq \varnothing, \\ +\infty, & \text{otherwise}. \end{cases}$$

Let $\mathscr{A}_s$ be the σ-field generated by the events $\{0 \leftrightarrow t;\ t \succcurlyeq s\}$. We claim that the probability measure

$$\mu(s) := \mathbb{P}[s^* = s] / \mathbb{P}[0 \leftrightarrow \partial\Gamma]$$

satisfies

$$\mathbb{P}[0 \leftrightarrow \partial\Gamma] \leq 4\mathscr{E}(\mu)^{-1} \leq 4 \operatorname{cap} \partial\Gamma.$$

Indeed, the Burkholder–Davis–Gundy inequality ([11], [9], Chapter 6, (100.2), [26]) yields

$$\begin{aligned} 4\mathbb{P}[0 \leftrightarrow \partial\Gamma] &= 4\mathbb{E}\left[\left(\sum_{s \in \partial\Gamma} \mathbf{1}_{s^* = s}\right)^2\right] \geq \mathbb{E}\left[\left(\sum_{s \in \partial\Gamma} \mathbb{E}[\mathbf{1}_{s^* = s} | \mathscr{A}_s]\right)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{s \in \partial\Gamma} \mathbb{P}[s^* = s | 0 \leftrightarrow s] \mathbf{1}_{0 \leftrightarrow s}\right)^2\right] \\ &= \mathbb{E}\left[\left(\sum_{s \in \partial\Gamma} \mathbb{P}[0 \leftrightarrow \partial\Gamma] \mu(s) \mathbf{1}_{0 \leftrightarrow s} \mathbb{P}[0 \leftrightarrow s]^{-1}\right)^2\right] \\ &= \mathbb{P}[0 \leftrightarrow \partial\Gamma]^2 \mathscr{E}(\mu). \end{aligned}$$

This gives (2.13) for $E = \partial\Gamma$ and hence for all E.

3. Random labelling. We shall apply Corollary 2.8 to a general scheme of randomly labelling trees. Suppose that Γ and T are two trees and that Δ is not an element of T. A *random labelling* of Γ by T is a Markov random field $L: \Gamma \to T \cup \{\Delta\}$ such that

$$L(0) = 0,$$

$$(\sigma \to \tau \text{ and } L(\sigma) \neq \Delta) \quad \Rightarrow \quad (L(\sigma) \to L(\tau) \vee L(\tau) = \Delta)$$

and

$$(\sigma \to \tau \text{ and } L(\sigma) = \Delta) \quad \Rightarrow \quad L(\tau) = \Delta.$$

We may think of Δ as the empty label. Thus, the vertices of Γ are labelled with vertices of T at the same distance from the root in such a way that an element of $\partial\Gamma$ is labelled with an element of ∂T (or possibly with a vertex of T), except that when $L(\sigma) = \Delta$, we consider σ, as well as all paths of $\partial\Gamma$ through σ, to be unlabelled. In the obvious way, L induces a map from $\partial\Gamma$ to $\partial T \cup T \cup \{\Delta\}$, which we also call L. Because of the Markov property, the distribution of L is determined by the numbers $\{\mathbb{P}[L(\sigma) = x];\ \sigma \in \Gamma,\ x \in T,\ |\sigma| = |x|\}$.

In the special case where ∂T and $\partial' T$ are singletons, we may interpret the labelling as a percolation, where a random subtree of Γ is formed from those vertices σ for which $L(\sigma) \neq \Delta$. Another special case is where T is an n-ary tree for some n. Here, the random labelling can be thought of as simply choosing one (or no) letter from a size-n alphabet for each vertex of Γ other than the root. (Our setup only allows one letter for the root. If we wish to allow n letters for the root, then we may use our current setup by simply considering a new tree, namely, Γ with a new vertex adjoined to the root and deemed to be the new root.) This is the type of labelling studied by Evans [11], although he considers only the case where Γ is an m-ary tree for some $m \geq 2$ and L satisfies $\mathbb{P}[L(\sigma) = x] = n^{-|x|}$ for $|\sigma| = |x|$ (so that $L \neq \Delta$ a.s.). Given a set $E \subseteq \partial T$, Evans estimates $\mathbb{P}[L^{-1}(E) \neq \varnothing]$.

We shall estimate more general probabilities related to the graph of L, for which purpose we introduce the *product tree* $\Gamma \cdot T$. This is defined to have vertex set $\{(\sigma, x);\ \sigma \in \Gamma,\ x \in T,\ |\sigma| = |x|\}$, root $(0, 0)$ and successor relation

$$(\sigma, x) \to (\tau, y) \quad \Leftrightarrow \quad \sigma \to \tau \text{ and } x \to y.$$

The graph of L on Γ may be regarded as the random subtree (or percolation in the sense of Section 2.4)

$$(\Gamma \cdot T)_0 := \{(\sigma, x) \in \Gamma \cdot T;\ L(\sigma) = x\}.$$

For this percolation on $\Gamma \cdot T$, the kernel of (2.6) becomes

$$K((\sigma, x), (\tau, y))$$

$$= \begin{cases} \mathbb{P}[L(\sigma \wedge \tau) \leq x \wedge y]^{-1}, & \text{if } |\sigma \wedge \tau| \leq |x \wedge y| \text{ and} \\ & \quad \mathbb{P}[L(\sigma) = x]\mathbb{P}[L(\tau) = y] > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{3.1}$$

The condition (2.8) is evidently satisfied, as is (2.12) with $M = 1$ and with equality, whence Corollary 2.8 gives the following estimates of the distribution of the graph of L.

THEOREM 3.1. *Let L be a random labelling of a locally finite tree Γ by a locally finite tree, T. For analytic sets $E \subseteq \partial(\Gamma \cdot T)$,*

$$\operatorname{cap} E \le \mathbb{P}[\exists\, (s, w) \in E,\, L(s) = w] \le 4 \operatorname{cap} E,$$

where cap *is as defined in* (2.10) *using* (3.1).

We call Γ [respectively, L] *spherically symmetric* if the number of edges incident to σ [respectively, the law of $L(\sigma)$] is a function only of $|\sigma|$. If Γ and L are both spherically symmetric, then it turns out that $\operatorname{cap}(\partial\Gamma \times E)$ can be expressed as a capacity of E in ∂T. We proceed to carry out this reduction in order to show how Theorem 3.1 incorporates Evans's results [11]. Assume that both Γ and T are locally finite. Define

$$\nu(x) := \mathbb{P}[L(\sigma) = x], \qquad x \in T,\, |\sigma| = |x|. \tag{3.2}$$

We may, by taking a smaller tree if necessary, assume that $\forall\, x \in T$, $\nu(x) \neq 0$. Let

$$M_k := \operatorname{card}\{\sigma \in \Gamma;\, |\sigma| = k\}.$$

Fix an integer $n \le \operatorname{ht} \Gamma$. If the unit flow θ'_0 on $\Gamma \cdot T$ minimizes $\mathscr{E}$ on $\mathscr{U}$ (see Section 2.5), then $\theta'_0(\sigma, x)$ depends only on x in light of our assumed spherical symmetry and the convexity of $\mathscr{E}$. Thus, writing $\theta_0(x) := \sum_{|\sigma|=|x|}\theta'_0(\sigma, x)$, we obtain that

$$\begin{aligned}
\mathscr{E}_n(\theta'_0) &= \sum_{\substack{|x|=|y|=n\\ 0\le z\le x\wedge y}} \nu(z)^{-1} \sum_{\substack{|\sigma|=|\tau|=n\\ |\sigma\wedge\tau|=|z|}} \theta'_0(\sigma, x)\theta'_0(\tau, y)\\
&= \sum_{\substack{|x|=|y|=n\\ 0\le z\le x\wedge y}} \nu(z)^{-1}\theta_0(x)\theta_0(y) \sum_{\substack{|\sigma|=|\tau|=n\\ |\sigma\wedge\tau|=|z|}} M_n^{-2}.
\end{aligned}$$

If we set

$$f(k, n) := \sum_{\substack{|\sigma|=|\tau|=n\\ |\sigma\wedge\tau|=k}} M_n^{-2},$$

then an easy calculation yields that

$$f(k, n) = \begin{cases} M_k^{-1} - M_{k+1}^{-1}, & \text{if } k < n,\\ M_n^{-1}, & \text{if } k = n. \end{cases}$$

Using this notation, we have

$$\mathscr{E}_n(\theta_0') = \sum_{0 \le |z| \le n} \nu(z)^{-1} f(|z|, n) \sum_{\substack{|x|=|y|=n \\ x \wedge y \ge z}} \theta_0(x)\theta_0(y)$$
$$= \sum_{0 \le |z| \le n} \theta_0(z)^2 \nu(z)^{-1} f(|z|, n).$$

For unit flows θ on T, define

$$\mathscr{E}_L(\theta) := \begin{cases} \sum_{0 \le |z| \le \operatorname{ht}\Gamma} \theta(z)^2 \nu(z)^{-1} f(|z|, \operatorname{ht}\Gamma), & \text{if } \operatorname{ht}\Gamma < \infty, \\ \lim_{n \to \infty} \sum_{0 \le |z| \le n} \theta(z)^2 \nu(z)^{-1} f(|z|, n), & \text{if } \operatorname{ht}\Gamma = \infty. \end{cases} \tag{3.3}$$

[When $\operatorname{ht}\Gamma = \infty$, this limit exists since the terms form a monotonically increasing sequence as in Section 2.5. This is also a consequence of (3.6).] If it happens that $M_n \to \infty$ as $n \to \infty$, then

$$\mathscr{E}_L(\theta) = \sum_{z \in T} \theta(z)^2 \nu(z)^{-1} \left(M_{|z|}^{-1} - M_{|z|+1}^{-1} \right). \tag{3.4}$$

To show this, denote the sum on the right by S. Certainly $\mathscr{E}_L(\theta) \ge S$. Thus, we must show that if S is finite, then

$$\lim_{n \to \infty} \sum_{|z|=n} \theta(z)^2 \nu(z)^{-1} M_n^{-1} = 0. \tag{3.5}$$

Now

$$\sum_{z \to x} \nu(x) \le \nu(z) \quad \text{and} \quad \sum_{z \to x} \theta(x) = \theta(z).$$

It therefore follows from the Cauchy–Buniakowski–Schwarz inequality that

$$\nu(z) \sum_{z \to x} \theta(x)^2 \nu(x)^{-1} \ge \sum_{z \to x} \nu(x) \sum_{z \to x} \theta(x)^2 \nu(x)^{-1} \ge \Bigl(\sum_{z \to x} \theta(x) \Bigr)^2 = \theta(z)^2,$$

whence

$$\sum_{z \to x} \theta(x)^2 \nu(x)^{-1} \ge \theta(z)^2 \nu(z)^{-1}. \tag{3.6}$$

Thus for $k > n$,

$$\sum_{|x|=k} \theta(x)^2 \nu(x)^{-1} \ge \sum_{|z|=n} \theta(z)^2 \nu(z)^{-1};$$

so, since $M_N \to \infty$,

$$\sum_{|z|=n} \theta(z)^2 \nu(z)^{-1} M_n^{-1} = \lim_{N \to \infty} \sum_{n \le k \le N} \left(M_k^{-1} - M_{k+1}^{-1} \right) \sum_{|z|=n} \theta(z)^2 \nu(z)^{-1}$$
$$\le \sum_{|x| \ge n} \theta(x)^2 \nu(x)^{-1} \left(M_{|x|}^{-1} - M_{|x|+1}^{-1} \right).$$

If $S < \infty$, then this last sum tends to 0 as $n \to \infty$, which demonstrates (3.5).

If $\operatorname{ht}\Gamma < \infty$ and $E \subseteq \{x \in T;\ |x| = \operatorname{ht}\Gamma\}$ or $\operatorname{ht}\Gamma = \infty$ and $E \subseteq \partial' T$, define

$$\operatorname{cap}_L(E) := \sup\left\{\mathscr{E}_L(\mu)^{-1};\ \mu \text{ is a Borel probability measure and } \mu(E) = 1\right\}.$$

Then

$$\operatorname{cap}(\partial\Gamma \times E) = \operatorname{cap}_L(E)$$

since, for each unit flow θ on T, there is a unit flow θ' on $\Gamma \cdot T$ such that $\theta'(\sigma, x)$ depends only on x and $\theta(x) = \sum_{|\sigma|=|x|}\theta'(\sigma, x)$, whence $\mathscr{E}_L(\theta) = \mathscr{E}(\theta')$. This leads to the following corollary of Theorem 3.1, which more directly includes Evans's results [11].

COROLLARY 3.2. *Let L be a spherically symmetric random labelling of a spherically symmetric locally finite tree Γ by a locally finite tree T. For analytic sets E,*

$$\operatorname{cap}_L E \le \mathbb{P}\left[L^{-1}(E) \ne \varnothing\right] \le 4\operatorname{cap}_L E. \tag{3.7}$$

If $M_n \to \infty$ as $n \to \infty$, define conductances

$$C_x := \nu(x)\left(M_{|x|}^{-1} - M_{|x|+1}^{-1}\right)^{-1}, \qquad 0 \ne x \in T,$$

where ν is as in (3.2). *We then have*

$$\begin{aligned}\frac{\mathscr{C}(0 \to \partial' T)}{1 + \left(1 - M_1^{-1}\right)\mathscr{C}(0 \to \partial' T)} &\le \mathbb{P}\left[L^{-1}(\partial' T) \ne \varnothing\right]\\ &\le 4\frac{\mathscr{C}(0 \to \partial' T)}{1 + \left(1 - M_1^{-1}\right)\mathscr{C}(0 \to \partial' T)}.\end{aligned} \tag{3.8}$$

PROOF. The only part left to show is the assertion (3.8) on conductances. Note that the term in (3.4) corresponding to $z = 0$ is $1 - M_1^{-1}$, which differs from the formulas in Section 2.3. Instead of (2.2), we have here

$$\begin{aligned}\operatorname{cap}_L \partial' T &= \left[\left(1 - M_1^{-1}\right) + \inf_\theta \sum_{0 \ne x \in T} \theta(x)^2 C_x^{-1}\right]^{-1}\\ &= \left[\left(1 - M_1^{-1}\right) + \mathscr{C}(0 \to \partial' T)^{-1}\right]^{-1}.\end{aligned}$$

Putting this into (3.7) with $E = \partial' T$ gives (3.8). □

The energy $\mathscr{E}_L$ of measures on ∂T (and hence cap_L) can be expressed in terms analogous to the integrals of Section 2.3. For example, if $M_n \to \infty$, then

$$\mathscr{E}_L(\mu) = \int K_L(v, w)\, d(\mu \times \mu)(v, w), \tag{3.9}$$

where

$$K_L(v, w) := \sum_{0 \le x \le v \wedge w} \nu(x)^{-1}\left(M_{|x|}^{-1} - M_{|x|+1}^{-1}\right);$$

while if ht $\Gamma = \infty$ but $M_n \to M_\infty < \infty$, then (3.9) holds with

$$K_L(v, w) := \sum_{0 \le x \le v \wedge w} \nu(x)^{-1}\left(M_{|x|}^{-1} - M_{|x|+1}^{-1}\right) + \delta_{v,w}\nu(w)^{-1}M_\infty^{-1},$$

where $\nu(w) := \lim_{x \in w} \nu(x)$.

4. Random walks on random trees. If Γ is infinite and locally finite, the *branching number* of Γ, defined as $\operatorname{br} \Gamma := \exp \dim \partial\Gamma$ (where Hausdorff dimension is computed in the metric given in Section 2.3), is an average number of branches per vertex of Γ in several senses ([21]–[23]). In particular, the random walk corresponding to conductances $C_\sigma = \lambda^{-|\sigma|}$ is transient if $\lambda < \operatorname{br} \Gamma$ and recurrent if $\lambda > \operatorname{br} \Gamma$, and the critical probability for Bernoulli percolation (with $p_\sigma \equiv p$) is $p_c(\Gamma) = 1/\operatorname{br} \Gamma$ [22]. What happens at *criticality* (i.e., $\lambda = \operatorname{br} \Gamma$ or $p = 1/\operatorname{br} \Gamma$) was left open in [22]. It is easy to construct examples of either kind of behavior at criticality for random walk ([22], page 944), whence for percolation as well [since an electrical network with a constant multiple of the conductances above, namely, $(1 - \lambda^{-1})^{-1}C_\sigma$, is associated to the Bernoulli percolation with constant survival probability $p = \lambda^{-1}$ for $\lambda > 1$]. What we shall do here is to show that the type of behavior at criticality can often be determined for random trees. We call λ above the *drift parameter* and p the *survival parameter*.

4.1. *Trees generated by percolation and Galton–Watson branching processes.* A simple but quite general situation arises as follows. A denumerable graph without loops or cycles is called a *forest*.

PROPOSITION 4.1. *Let Γ be a locally finite tree and $\Gamma(\omega_p)$ be the random forest generated by Bernoulli percolation with survival parameter p. If critical random walk on Γ is transient, then, for every $p \in [p_c(\Gamma), 1]$, random walk with drift λ is a.s. transient on some component of $\Gamma(\omega_p)$ iff $\lambda \in]0, p/p_c(\Gamma)]$. On the other hand, if critical random walk on Γ is recurrent, then, for every $p \in]p_c(\Gamma), 1]$, random walk with drift λ is a.s. transient on some component of $\Gamma(\omega_p)$ iff $\lambda \in]0, p/p_c(\Gamma)[$.*

PROOF. Suppose that critical random walk on Γ is transient (respectively, recurrent). Then percolation occurs (respectively, does not occur) at criticality. Therefore percolation occurs (respectively, does not occur) a.s. on [some (respectively, any) component of] $\Gamma(\omega_p)$ when done with survival parameter $p_c(\Gamma)/p$ (cf. the proof of [22], Proposition 6.1), that is, at criticality ([22]). Hence random walk at the critical drift $\lambda = p/p_c(\Gamma)$ is a.s. transient (respectively, recurrent) on some (respectively, every) component of $\Gamma(\omega_p)$. It is evident from this what happens at other values of λ. □

This proposition takes a neater form when we need consider only the component of 0. Such is the case with Galton–Watson branching processes.

THEOREM 4.2. *Let $m > 1$ be the mean number of offspring per particle in a Galton–Watson process. Given nonextinction, critical random walk on the associated genealogical tree is a.s. recurrent.*

PROOF. From [22], Proposition 6.4, the critical drift is a.s. m (given nonextinction). Now percolation with survival probability $1/m$ on a Galton–Watson tree yields a component of 0 whose law is the same as that generated by a nontrivial Galton–Watson process of mean 1. This is finite a.s. ([3], page 7), whence percolation a.s. does not occur, so that the random walk is a.s. recurrent. □

The same method, with the aid of [22], Proposition 6.5 produces a generalization to multi-type branching processes. Here, the singular (deterministic) case is already interesting.

THEOREM 4.3. *Given nonextinction, critical random walk is a.s. recurrent on the genealogical tree of any supercritical positive regular multi-type branching process.*

It is interesting to consider whether critical random walk is positive or null recurrent in the setting of Theorem 4.2. It turns out that both are possible and that the "magnitude" of the tail of the distribution of the first generation size does not determine the recurrence type in all cases.

PROPOSITION 4.4. *Let $m > 1$ be the mean number of offspring per particle in a Galton–Watson process. Given nonextinction, critical random walk on the associated genealogical tree is a.s. null recurrent if $\mathbb{E}[Z_1 \log Z_1;\ Z_1 \geq 2] < \infty$, where Z_1 is the number of particles at time 1. However, for any $m > 1$ and $p \in]\frac{1}{2}, 1[$, there is a Galton–Watson process with mean m and $\mathbb{E}[Z_1(\log Z_1)^p;\ Z_1 \geq 2] < \infty$ such that critical random walk is a.s. positive recurrent and another process with mean m and $\mathbb{E}[Z_1(\log Z_1)^p;\ Z_1 \geq 2] = \infty$ such that the walk is a.s. null recurrent.*

PROOF. As in the proof of Proposition 2.5, the walk is null recurrent iff

$$\sum_{n \geq 1} Z_n m^{-n} = +\infty,$$

where Z_n is the number of particles at distance n from 0. By the theorem of Kesten and Stigum ([2], page 23, Theorem 2.2.1), $Z_n m^{-n}$ converges to a nonzero random variable a.s. given nonextinction when the condition $\mathbb{E}[Z_1 \log Z_1] < \infty$ is satisfied. This plainly implies that the above sum diverges.

On the other hand, given $m > 1$ and $p \in]\frac{1}{2}, 1[$, choose $a_n \geq 0$ for $n \geq 0$ so that $\Sigma a_n = 1$, $\Sigma \lfloor m^n \rfloor a_n = m$ and

$$a_n = \lfloor m^n \rfloor^{-1} n^{-(p+3)/2}$$

for all sufficiently large n. Consider the Galton–Watson process such that

$$\mathbb{P}[Z_1 = \lfloor m^n \rfloor] = \sum_{\lfloor m^k \rfloor = \lfloor m^n \rfloor} a_k, \qquad n \geq 0. \tag{4.1}$$

Note that $\mathbb{P}[Z_1 = 0] = 0$. It is evident that $\mathbb{E}[Z_1] = m$ and that for some finite c,

$$\begin{aligned}\mathbb{E}\left[Z_1(\log Z_1)^p\right] &= c + \sum n^{-(p+3)/2}(n \log m)^p \\ &= c + (\log m)^p \sum n^{-(3-p)/2} \\ &< \infty.\end{aligned}$$

We claim that $\Sigma Z_n m^{-n} < \infty$ a.s., so that the walk is positive recurrent a.s. In order to prove this, set

$$t_N^{(1)} := \mathbb{E}\left[Z_1 \log_m Z_1; Z_1 \leq m^N\right], \qquad t_N^{(2)} := N \cdot \mathbb{E}\left[Z_1; Z_1 > m^N\right],$$

$$t_N := t_N^{(1)} + t_N^{(2)}, \qquad c_N := m^N e^{-t_N/m}.$$

Since $\mathbb{E}[Z_1(\log Z_1)^{1/2}] < \infty$ (because $p > \frac{1}{2}$), Corollary 2.5.7 of [2], page 45, states that Z_n/c_n has a finite nonzero limit a.s. In our case, we have, for some constants c, c' and c'' and for all large N,

$$\begin{aligned}t_N &= c + \sum_{n \leq N} n \lfloor m^n \rfloor a_n + N \sum_{n > N} \lfloor m^n \rfloor a_n \\ &= c' + \sum_{n \leq N} n^{-(p+1)/2} + N \sum_{n > N} n^{-(p+3)/2} \\ &\sim c'' N^{(1-p)/2},\end{aligned}$$

whence

$$\sum c_n m^{-n} = \sum e^{-t_n/m} < \infty.$$

This ensures the a.s. convergence of $\Sigma Z_n m^{-n}$ as well.

Finally, to prove the last part of the proposition, given $m > 1$ and $p \in]\frac{1}{2}, 1[$, define b_k inductively by $b_1 = 1$ and $b_{k+1} = e^{b_k}$. Set

$$n_k = \lfloor b_k / k^2 \rfloor^2$$

and choose $a_n \geq 0$ so that $\Sigma a_n = 1$, $\Sigma \lfloor m^n \rfloor a_n = m$ and

$$a_n = \begin{cases} \left(b_{2k} \lfloor m^n \rfloor\right)^{-1}, & \text{if } n = n_{2k}, \\ 0, & \text{if } \forall\, k, n \neq n_{2k}, \end{cases}$$

for all sufficiently large n. Again, the corresponding Galton–Watson process defined by (4.1) satisfies $\mathbb{E}[Z_1] = m$ and, for some finite c, c' and c'',

$$\begin{aligned}\mathbb{E}\left[Z_1(\log Z_1)^{1/2}\right] &= c + c' \sum a_{n_{2k}} \lfloor m^{n_{2k}} \rfloor n_{2k}^{1/2} \\ &= c'' + c' \sum b_{2k}^{-1} b_{2k}/(2k)^2 \\ &< \infty \\ &= \mathbb{E}\left[Z_1(\log Z_1)^p\right].\end{aligned}$$

Define $t_n^{(1)}$, $t_n^{(2)}$ and c_n as before. In view of the theorem cited previously ([2], page 45), it now suffices to show that $\Sigma c_n m^{-n} = \infty$ in order to establish a.s. null recurrence. To this end, we calculate, for all large K,

$$t_{n_{2K}}^{(1)} \leq \frac{b_{2K}}{K^2}, \qquad t_{n_{2K+1}}^{(2)} = n_{2K+1} \sum_{k>K} b_{2k}^{-1} \leq 1,$$

whence

$$t_n \leq \frac{b_{2K}}{K} \quad \text{for } n_{2K} \leq n \leq n_{2K+1}.$$

Since

$$n_{2K+1} - n_{2K} \geq e^{b_{2K}/K},$$

we obtain that for large K_0,

$$\sum c_n m^{-n} = \sum e^{-t_n/m} \geq \sum_{K \geq K_0} \sum_{n_{2K} \leq n < n_{2K+1}} e^{-b_{2K}/K} \geq \sum_{K \geq K_0} 1 = \infty. \qquad \square$$

4.2. *Random walks on randomly perturbed trees*: *Statements of results*. We consider next the Nash–Williams criterion and Griffeath's conjecture. A simple version of this criterion for trees states that if conductances satisfy

$$\sum_{n \geq 1} \left(\sum_{|\sigma|=n} C_\sigma \right)^{-1} = \infty,$$

then the associated random walk is recurrent. The proof (by contraposition) is short: If the walk is transient and θ_0 is the unit current flow on Γ', then by the weighted form of the arithmetic-harmonic mean inequality, we find that

$$\begin{aligned}
\sum_{n \geq 1} \left(\sum_{|\sigma|=n} C_\sigma \right)^{-1} &\leq \sum_{n \geq 1} \left(\sum_{\substack{|\sigma|=n \\ \sigma \in \Gamma'}} \theta_0(\sigma) \big(\theta_0(\sigma) C_\sigma^{-1}\big)^{-1} \right)^{-1} \\
&\leq \sum_{n \geq 1} \sum_{\substack{|\sigma|=n \\ \sigma \in \Gamma'}} \theta_0(\sigma) \big(\theta_0(\sigma) C_\sigma^{-1}\big) \\
&= \sum_{0 \neq \sigma \in \Gamma'} \theta_0(\sigma)^2 C_\sigma^{-1} < \infty,
\end{aligned}$$

as asserted. This proof also shows, by (2.2), that

$$\mathscr{C}(0 \to \partial' \Gamma) \leq \left(\sum_{n \geq 1} \left(\sum_{|\sigma|=n} C_\sigma \right)^{-1} \right)^{-1}. \tag{4.2}$$

In case C_σ depends only on $|\sigma|$, the Nash–Williams criterion takes the following form: Write $C_{|\sigma|}$ for C_σ and let M_n denote the number of vertices at distance n from the root. If $\Sigma M_n^{-1} C_n^{-1} = \infty$, then the walk is recurrent.

The Nash–Williams criterion is useful because it is so easy to apply. Furthermore, in the special case where Γ is spherically symmetric, the criterion is necessary for recurrence as well as sufficient (provided Γ is infinite, of course). Indeed, in this case $\theta_0(\sigma) = M_{|\sigma|}^{-1}$, so $\mathscr{C}(0 \to \partial\Gamma) = (\Sigma M_n^{-1} C_n^{-1})^{-1}$. Thus, it is natural to ask, as Griffeath did, how much this special case may be perturbed before losing the necessity of the Nash–Williams criterion. In particular, what if M_n is the *mean* number of particles of a branching process in a (time-) varying environment (BPVE)?

To facilitate discussion, for the remainder of this paper, let Z_n be the number of particles at time n generated by a branching process beginning with one particle ($Z_0 = 1$) such that at time k, each of the previously existing Z_{k-1} particles gives birth independently to a random number of children distributed according to the law of a random variable, L_k. (The particles existing at time $k - 1$ then cease to exist.) Thus, $M_n = \mathbb{E}[Z_n]$. The case originally considered by Griffeath is that of simple random walk (SRW), that is, $C_k \equiv 1$, on the random genealogical tree generated by

$$L_k = \begin{cases} 1, & \text{with probability } \left(1 - \dfrac{\alpha}{k}\right) \vee 0, \\ 2, & \text{with probability } \dfrac{\alpha}{k} \wedge 1, \end{cases}$$

where α is a positive parameter. Here, for some positive constant c,

$$M_n = \prod_{1 \le k \le n} \left[\left(1 + \frac{\alpha}{k}\right) \wedge 2\right] \sim cn^{\alpha}.$$

Since $Z_n = O(M_n)$ a.s. (see Section 4.3), SRW is critical a.s. for every value of the parameter α. Now $\Sigma M_n^{-1} = \infty$ iff $\alpha \le 1$. Griffeath's conjecture is that this "mean Nash–Williams" condition is both necessary and sufficient for a.s. recurrence, with a.s. transience in the contrary case. As we mentioned in the introduction, this was established for $\alpha \le 1$ and $\alpha > 2$ in both [5] and [19]. Indeed, the conjecture is entirely correct and is a consequence of the following more general result. Write

$$A := \sup_n \|L_n\|_\infty,$$

which may be infinite. We say that a BPVE is *without extinction* if the probability of extinction is zero, that is, $\forall\, n$, $L_n \ge 1$ a.s.

THEOREM 4.5. *Given a BPVE, a random walk on the associated genealogical tree endowed with conductances $\{C_{|\sigma|}\}$ is a.s. recurrent if*

$$\sum_{n=1}^{\infty} M_n^{-1} C_n^{-1} = \infty. \tag{4.3}$$

If (4.3) *fails and either $A < \infty$ and the BPVE is without extinction or $M_n \nrightarrow \infty$, then the random walk is a.s. transient given nonextinction of the BPVE.*

REMARK. The sufficiency of (4.3) can be used to rederive recurrence in Theorems 4.2 and (by a similar method) 4.3.

Since the question of transience of a random walk on a tree depends only on the reduced tree, it is natural to stipulate the lack of extinction in Theorem 4.5. Indeed, the theorem may fail badly without such a stipulation.

PROPOSITION 4.6. *There is a BPVE with $A < \infty$ and with positive probability of nonextinction such that SRW is a.s. recurrent despite failure of* (4.3) (*for* $C_n \equiv 1$).

Now, in principle, the formulas of Section 4.3 allow one to examine the reduced genealogical tree of any BPVE by studying a BPVE without extinction, and thus one could modify (4.3) to apply to the general case where $A < \infty$. In practice, however, one could not make the calculations necessary to verify the resultant criterion. Thus, it is somewhat remarkable that there does exist an easily verifiable criterion which determines the type of the random walk in every case where $A < \infty$. To state this criterion, denote

$$l_n := \frac{\mathbb{E}[L_n^2]}{\mathbb{E}[L_n]^2} - 1 = \frac{\mathrm{Var}(L_n)}{\mathbb{E}[L_n]^2},$$

$$\gamma_n := \left(1 + \sum_{k=1}^{n} C_k^{-1}\right) C_n$$

and

$$C_0 := M_0 = 1.$$

THEOREM 4.7. *Given a BPVE with $A < \infty$, a random walk on the associated genealogical tree endowed with conductances $\{C_{|\sigma|}\}$ is a.s. recurrent if*

$$\sum_{n=1}^{\infty} M_n^{-1} C_n^{-1} (1 + \gamma_n l_{n+1}) = \infty \tag{4.4}$$

and is a.s. transient given nonextinction if (4.4) *fails.*

REMARK. If the BPVE is without extinction and $A < \infty$, then (4.4) is equivalent to (4.3). This will be shown directly in Theorem 4.9.

The situation changes considerably when $A = \infty$, even if the BPVE is without extinction. The following proposition shows, in fact, that the restriction $A < \infty$ cannot be weakened much in Theorem 4.5 and that there is no general "0–1 law" for SRW. It also shows that the mean Nash–Williams criterion (4.3) is no longer necessary for a.s. recurrence under large perturbations.

PROPOSITION 4.8. *There is a BPVE without extinction such that*

$$\sup_n \mathbb{E}\left[\exp\left(\frac{L_n}{(\log L_n)^2}\right); L_n \geq 2\right] < \infty$$

and SRW is recurrent with probability in $]0,1[$. *There is also a BPVE without extinction such that*

$$\sup_n \mathbb{E}\left[\frac{L_n^2}{(\log L_n)^3}; L_n \geq 2\right] < \infty$$

and SRW is a.s. recurrent despite failure of (4.3).

Nevertheless, the requirement $A < \infty$ can be alleviated provided we accept a weaker conclusion.

THEOREM 4.9. *Given a BPVE violating (4.4), that is, such that*

$$\sum_{n=1}^{\infty} M_n^{-1} C_n^{-1}(1 + \gamma_n l_{n+1}) < \infty,$$

random walk on the associated genealogical tree endowed with conductances $\{C_{|\sigma|}\}$ *is a.s. transient given* $\lim Z_n/M_n > 0$, *which is an event of positive probability. Moreover,*

$$\frac{1}{2}\left[\sum_{n=0}^{\infty} M_n^{-1} C_n^{-1}(1 + \gamma_n l_{n+1})\right]^{-1} \leq \mathbb{E}[\mathscr{C}(0 \to \partial'\Gamma); \partial'\Gamma \neq \varnothing]$$

$$\leq \left(\sum_{n=1}^{\infty} M_n^{-1} C_n^{-1}\right)^{-1}.$$

If the BPVE is without extinction, then

$$\frac{1}{2A}\left(\sum_{n=0}^{\infty} M_n^{-1} C_n^{-1}\right)^{-1} \leq \frac{1}{2}\left[\sum_{n=0}^{\infty} M_n^{-1} C_n^{-1}(1 + \gamma_n l_{n+1})\right]^{-1}.$$

In order to prove the preceding results, we need to examine BPVE's in some detail. A few of the results about BPVE's which follow will not be used in our study of random walks; rather, they are included for the sake of completeness and their own interest.

4.3. *Branching processes in varying environments.* We introduce the following notation for the remainder of this paper. Write

$$f_n(s) := \mathbb{E}[s^{L_n}], \qquad 0 \leq s \leq 1, n \geq 1,$$

for the probability generating function of the progeny distribution of a BPVE at time n. Let $\{L_{n,i}\}_{n \geq 1,\ i \geq 1}$ be independent random variables with $L_{n,i}$

having the same law as L_n and such that

$$Z_{n+1} = \sum_{i=1}^{Z_n} L_{n+1,i}.$$

Write

$$F_n(s) := \mathbb{E}[s^{Z_n}]$$

for the probability generating function of Z_n. Since

$$\mathbb{E}[s^{Z_{n+1}}] = \mathbb{E}\Big[\mathbb{E}\big[s^{Z_{n+1}}|Z_n\big]\Big] = \mathbb{E}\Big[\mathbb{E}\big[s^{\sum_{i=1}^{Z_n} L_{n+1,i}}|Z_n\big]\Big]$$
$$= \mathbb{E}\Big[\mathbb{E}\big[s^{L_{n+1,1}}\big]^{Z_n}\Big] = \mathbb{E}\Big[f_{n+1}(s)^{Z_n}\Big] = F_n(f_{n+1}(s)),$$

we have

$$F_n = f_1 \circ f_2 \circ \cdots \circ f_n.$$

As is well known, $\{Z_n M_n^{-1}\}$ is a nonnegative martingale and, as such, converges a.s. to a finite random variable W:

$$W := \lim_{n\to\infty} Z_n/M_n.$$

For $n \geq 0$, let

$$q_n := \mathbb{P}\Big[\lim_{k\to\infty} Z_k = 0 | Z_n = 1\Big]$$

be the probability of (descendant) extinction for a single particle born at time n and

$$\bar{q}_n := 1 - q_n.$$

In principle, one can calculate q_n as

$$q_n = \lim_{N\to\infty} f_{n+1} \circ f_{n+2} \circ \cdots \circ f_{n+N}(0).$$

Let

$$m_n := f_n'(1) = \mathbb{E}[L_n]$$

be the mean of L_n.

At generation n, the i-th particle ($i \leq Z_{n-1}$) gives birth to $L_{n,i}$ new particles, which we label (n, i, j) for $1 \leq j \leq L_{n,i}$. Let

$$L_{n,i}^* := \sum_{j=1}^{L_{n,i}} Y_{n,i,j},$$

where $Y_{n,i,j}$ is the indicator function of particle (n, i, j) having an infinite line of descent, that is, belonging to the reduced genealogical tree. Let

$$Z_0^* := \mathbf{1}_{\{Z_n \nrightarrow 0\}}$$

be the indicator of nonextinction,

$$Z_{n+1}^* := \sum_{i=1}^{Z_n} L_{n+1,i}^*$$

and

$$Z_\infty^* := \lim_{n\to\infty} Z_n^* \in [0,\infty].$$

When the index i does not matter, we shall omit it and write L_n, L_n^* and $Y_{n,j}$.

It will be essential for us to reduce the study of a BPVE on the event of nonextinction to that of a BPVE without extinction. Such a reduction depends on part (i) of the following proposition. Parts (i) and (ii) generalize well-known properties of Galton–Watson processes ([3], pages 49 and 52).

PROPOSITION 4.10. *Let Γ be the genealogical tree of a BPVE with $q_0 < 1$ and let Γ' be the reduced subtree of Γ. The following properties then hold*:

(i) *The law of Γ' given nonextinction is the same as that of the genealogical tree of a BPVE with probability generating functions $\{f_n^*\}_{n\ge 1}$ given by*

$$f_n^*(s) := \bar{q}_{n-1}^{-1}[f_n(q_n + \bar{q}_n s) - q_{n-1}].$$

(ii) *The law of Γ given extinction is the same as that of the genealogical tree of a BPVE with probability generating functions $\{\tilde{f}_n\}_{n\ge 1}$ given by*

$$\tilde{f}_n(s) := q_{n-1}^{-1} f_n(q_n s).$$

(iii) *The joint probability generating function of $L_n - L_n^*$ and L_n^* is*

$$\mathbb{E}[s^{L_n - L_n^*} t^{L_n^*}] = f_n(q_n s + \bar{q}_n t).$$

The joint probability generating function of $Z_n - Z_n^*$ and Z_n^* is

$$\mathbb{E}[s^{Z_n - Z_n^*} t^{Z_n^*}] F_n(q_n s + \bar{q}_n t).$$

(iv) *The law of Γ given nonextinction is the same as that of a tree $\hat{\Gamma}$ generated as follows*: *Let Γ^* be the genealogical tree of a BPVE with probability generating functions $\{f_n^*\}_{n\ge 1}$ as in* (i). *To each vertex σ of Γ^* having degree $d_\sigma + 1$, attach U_σ independent copies of the genealogical tree of a BPVE with probability generating functions $\{\tilde{f}_n\}_{n\ge 1}$ as in* (ii), *where U_σ has the probability generating function*

$$\mathbb{E}[s^{U_\sigma}] = \frac{f_{|\sigma|}^{(d_\sigma)}(q_{|\sigma|} s)}{f_{|\sigma|}^{(d_\sigma)}(q_{|\sigma|})},$$

where d_σ derivatives of $f_{|\sigma|}$ are indicated; *all U_σ and all trees added are mutually independent given Γ^*. The resultant tree is $\hat{\Gamma}$.*

PROOF. We begin by establishing (iii). We have

$$\begin{aligned}
\mathbb{E}[s^{L_n - L_n^*} t^{L_n^*}] &= \mathbb{E}\big[\mathbb{E}[s^{L_n - L_n^*} t^{L_n^*} | L_n]\big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[s^{\sum_{j=1}^{L_n}(1 - Y_{n,j})} t^{\sum_{j=1}^{L_n} Y_{n,j}} | L_n\Big]\Big] \\
&= \mathbb{E}\Big[\mathbb{E}[s^{1 - Y_{n,1}} t^{Y_{n,1}}]^{L_n}\Big] \\
&= \mathbb{E}\Big[(q_n s + \bar{q}_n t)^{L_n}\Big] \\
&= f_n(q_n s + \bar{q}_n t).
\end{aligned}$$

A precisely parallel calculation yields the other part of (iii).

In order to show (i), it is required to show that, given $L_{n,i}^* \neq 0$ and given the σ-field $\mathscr{F}_{n,i}$ generated by $L_{n,k}^*$, $k \neq i$, and $L_{m,k}^*$, $m < n$, $k \geq 1$, the probability generating function of $L_{n,i}^*$ is f_n^*. This is accomplished through the following calculation:

$$\begin{aligned}
\mathbb{E}\big[s^{L_{n,i}^*} | L_{n,i}^* \neq 0, \mathscr{F}_{n,i}\big] &= \mathbb{E}\big[s^{L_{n,i}^*} | L_{n,i}^* \neq 0\big] \\
&= \bar{q}_{n-1}^{-1} \mathbb{E}\big[s^{L_n^*} \mathbf{1}_{L_n^* \neq 0}\big] \\
&= \bar{q}_{n-1}^{-1} \mathbb{E}\big[s^{L_n^*} - \mathbf{1}_{L_n^* = 0}\big] \\
&= \bar{q}_{n-1}^{-1} \big\{\mathbb{E}[1^{L_n - L_n^*} s^{L_n^*}] - \mathbb{P}[L_n^* = 0]\big\} \\
&= \bar{q}_{n-1}^{-1} [f_n(q_n + \bar{q}_n s) - q_{n-1}] \\
&= f_n^*(s)
\end{aligned}$$

in light of (iii). In a similar fashion, (ii) follows from the following calculation:

$$\begin{aligned}
\mathbb{E}\big[s^{L_{n,i}} | Z_k \to 0, \mathscr{F}_{n,i}\big] &= \mathbb{E}\big[s^{L_{n,i}} | L_{n,i}^* = 0\big] \\
&= q_{n-1}^{-1} \mathbb{E}\big[s^{L_n} \mathbf{1}_{L_n^* = 0}\big] \\
&= q_{n-1}^{-1} \mathbb{E}[s^{L_n - L_n^*} 0^{L_n^*}] \\
&= q_{n-1}^{-1} f_n(q_n s) \\
&= \tilde{f}_n(s).
\end{aligned}$$

Now (iv) will follow once we demonstrate that the probability generating function given for U_σ is the same as that of $L_{|\sigma|} - L_{|\sigma|}^*$ given $L_{|\sigma|}^* = d_\sigma$. Again, by using (iii), we have for some constants c and c',

$$\mathbb{E}\big[s^{L_n - L_n^*} | L_n^* = d\big] = c \left(\frac{\partial}{\partial t}\right)^d f_n(q_n s + \bar{q}_n t)\Big|_{t=0} = c' f_n^{(d)}(q_n s).$$

Normalization requires that $c' = 1/f_n^{(d)}(q_n)$, which completes the proof. □

Because of this proposition, it will be useful to denote

$$\begin{aligned}
M_n^* &:= \mathbb{E}[Z_n^* | Z_0^* = 1] = \bar{q}_0^{-1} \mathbb{E}[Z_n^*], \\
m_n^* &:= (f_n^*)'(1) = \mathbb{E}[L_{n,i}^*]
\end{aligned}$$

and

$$W^* := \lim_{n\to\infty} Z_n^* M_n^{*-1}.$$

(Note that this limit exists a.s. because of Proposition 4.10.)

We record the following calculation, the first part of which is well known.

PROPOSITION 4.11. *For any BPVE, we have*

$$M_n = \prod_{k=1}^{n} m_k,$$

$$\mathbb{E}\left[\left(\frac{Z_n}{M_n}\right)^2\right] = 1 + \sum_{k=0}^{n-1} M_k^{-1} l_{k+1}.$$

If $q_0 < 1$, *then*

$$m_n^* = \bar{q}_n \bar{q}_{n-1}^{-1} m_n,$$

$$M_n^* = \prod_{k=1}^{n} m_k^* = \bar{q}_n \bar{q}_0^{-1} M_n,$$

$$\mathbb{E}\left[\left(\frac{Z_n^*}{M_n^*}\right)^2\right] = \bar{q}_0^2 \mathbb{E}\left[\left(\frac{Z_n}{M_n}\right)^2\right] + \frac{\bar{q}_0 q_n}{M_n^*}.$$

PROOF. The first equation follows by induction from

$$M_{n+1} = \mathbb{E}[Z_{n+1}] = \mathbb{E}[\mathbb{E}[Z_{n+1}|Z_n]] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{Z_n} L_{n+1,i}|Z_n\right]\right]$$
$$= \mathbb{E}[Z_n \mathbb{E}[L_{n+1}]] = M_n m_{n+1}.$$

Likewise,

$$\mathbb{E}[Z_{n+1}^2|Z_n] = \mathbb{E}[Z_{n+1}|Z_n]^2 + \operatorname{Var}(Z_{n+1}|Z_n)$$
$$= Z_n^2 \mathbb{E}[L_{n+1}]^2 + \sum_{i=1}^{Z_n} \operatorname{Var}(L_{n+1,i})$$
$$= Z_n^2 \mathbb{E}[L_{n+1}]^2 + Z_n\left(\mathbb{E}[L_{n+1}^2] - \mathbb{E}[L_{n+1}]^2\right),$$

so

$$\mathbb{E}[Z_{n+1}^2 M_{n+1}^{-2}] = \mathbb{E}[Z_n^2 M_n^{-2}] + M_n l_{n+1} \mathbb{E}[L_{n+1}]^2 M_{n+1}^{-2}$$
$$= \mathbb{E}[Z_n^2 M_n^{-2}] + M_n^{-1} l_{n+1},$$

which yields the second equation. The third equation follows from Proposition 4.10 and, in turn, entails the fourth equation.

Finally, let $F_n^*(s) := \mathbb{E}[s^{Z_n^*}|Z_0^* = 1]$ be the probability generating function of Z_n^* given nonextinction. By Proposition 4.10, we have

$$F_n^*(s) = \bar{q}_0^{-1} \mathbb{E}\left[1^{Z_n - Z_n^*} s^{Z_n^*} - \mathbf{1}_{Z_0^*=0}\right] = \bar{q}_0^{-1}[F_n(q_n + \bar{q}_n s) - q_0],$$

whence

$$\begin{aligned}
\mathbb{E}\left[(Z_n^*/M_n^*)^2\right] &= \bar{q}_0 M_n^{*-2}\mathbb{E}\left[Z_n^{*2}|Z_0^* = 1\right] \\
&= \bar{q}_0 M_n^{*-2}(F_n^{*\prime\prime}(1) + M_n^*) \\
&= \bar{q}_0 M_n^{*-2}\left(\bar{q}_0^{-1}\bar{q}_n^2 F_n''(1)\right) + \frac{q_0}{M_n^*} \\
&= \bar{q}_0^2 M_n^{-2}\left(\mathbb{E}[Z_n^2] - M_n\right) + \frac{\bar{q}_0}{M_n^*} \\
&= \bar{q}_0^2 \mathbb{E}\left[\left(\frac{Z_n}{M_n}\right)^2\right] + \frac{\bar{q}_0 q_n}{M_n^*},
\end{aligned}$$

which is the fifth equation. □

We next show that all BPVE's behave regularly in a number of ways. When $A < \infty$, we demonstrate additional nice behavior in Proposition 4.13 and Theorem 4.14. However, lest the reader be lulled into feeling that all BPVE's are just like Galton–Watson processes, we forewarn him of the pathologies to come in Proposition 4.16 (see also [25]). Three parts of the following theorem are already known: Almost sure convergence in (4.5) is due to Lindvall [20], although he did not identify the limit. Similarly, convergence in (4.13) is due to Jirina [16] without identification of the limit. The criterion (4.11) is essentially due to Church [6]. Our proofs, however, are based on an almost entirely new and efficient method which focuses on (4.7).

THEOREM 4.12. *If $q_0 < 1$, then the following hold*:

$$Z_n \to Z_\infty^* \quad a.s.; \tag{4.5}$$

$$\bar{q}_n Z_n \to_P Z_\infty^*; \tag{4.6}$$

$$\frac{Z_n^*}{\bar{q}_n Z_n} \to_P \mathbf{1} \quad \textit{given } Z_0^* = 1; \tag{4.7}$$

(4.8) *convergence in* (4.6) *and* (4.7) *is a.s. given* $W > 0$;

$$W^* = \bar{q}_0 W \quad a.s.; \tag{4.9}$$

$$\mathbb{P}[0 < Z_\infty^* < \infty] > 0 \quad \Rightarrow \quad \bar{q}_n \to 1; \tag{4.10}$$

$$\mathbb{P}[0 < Z_\infty^* < \infty] > 0 \quad \Leftrightarrow \quad \sum \mathbb{P}[L_n \neq 1] < \infty; \tag{4.11}$$

$$M_n \sum_{k>n} \mathbb{P}[L_k \neq 1] \to 0 \quad \Rightarrow \quad Z_\infty^* < \infty \quad a.s.; \tag{4.12}$$

$$M_n \to \mathbb{E}[Z_\infty^*]; \tag{4.13}$$

$$\bar{q}_n M_n \uparrow \mathbb{E}[Z_\infty^*]; \tag{4.14}$$

$$\lim M_n = \infty \quad \Rightarrow \quad \lim M_n^* = \infty \textit{ and } \bar{q}_n^{-1} = o(M_n); \tag{4.15}$$

$$\lim M_n < \infty \quad \Rightarrow \quad \bar{q}_n \to 1. \tag{4.16}$$

PROOF. We make the convention that

$$\frac{Z_n^*}{\bar{q}_n Z_n} := 0 \quad \text{on } Z_0^* = 0.$$

By compactness, there exist $[0, \infty]$-valued random variables Z_∞, $\tilde{Z}_\infty$ and Y, a number γ and a sequence $\{n_k\}$ tending to ∞ such that $Z_{n_k} \Rightarrow Z_\infty$, $\bar{q}_{n_k} Z_{n_k} \Rightarrow \tilde{Z}_\infty$, $Z_{n_k}^*/(\bar{q}_{n_k} Z_{n_k}) \Rightarrow Y$, and $\bar{q}_{n_k} \to \gamma$. Now

$$q_0 = \mathbb{P}[Z_n^* = 0] = \mathbb{E}[\mathbb{P}[Z_n^* = 0 | Z_n]] = \mathbb{E}\left[q_n^{Z_n}\right]. \tag{4.17}$$

Therefore, if $\gamma < 1$, we obtain

$$q_0 \geq \mathbb{P}[Z_\infty = 0] + \mathbb{E}\left[(1 - \gamma)^{Z_\infty}; 0 < Z_\infty < \infty\right].$$

Since $\mathbb{P}[Z_\infty = 0] = q_0$, it follows that $\mathbb{P}[0 < Z_\infty < \infty] = 0$:

$$\gamma < 1 \quad \Rightarrow \quad \mathbb{P}[Z_\infty = \infty] = \bar{q}_0. \tag{4.18}$$

We next demonstrate that

$$\mathbb{P}\left[\tilde{Z}_\infty = \infty\right] = \mathbb{P}[Z_\infty = \infty]. \tag{4.19}$$

If this is not the case, then necessarily $\gamma = 0$ and, by (4.18),

$$\mathbb{P}\left[\tilde{Z}_\infty < \infty\right] > q_0 \quad \text{and} \quad \mathbb{P}\left[\tilde{Z}_\infty = 0\right] \geq q_0. \tag{4.20}$$

Now

$$1 - x \geq \exp\left(\frac{-ex}{e-1}\right) \quad \text{for} \quad 0 \leq x \leq \frac{e-1}{e}, \tag{4.21}$$

by convexity of the exponential, whence, by (4.17),

$$q_0 = \mathbb{E}\left[\left(1 - \bar{q}_{n_k}\right)^{Z_{n_k}}\right] \geq \mathbb{E}\left[\exp\left(\frac{-e\bar{q}_{n_k} Z_{n_k}}{e-1}\right)\right]$$

for all sufficiently large k. Taking the limit as $k \to \infty$ gives

$$q_0 \geq \mathbb{E}\left[\exp\left(\frac{-e\tilde{Z}_\infty}{e-1}\right)\right] > q_0$$

by (4.20), which is a contradiction.

We now claim that the Laplace transform of Y is

$$\mathbb{E}[e^{-xY}] = e^{-x}\bar{q}_0 + q_0, \qquad x \geq 0.$$

Indeed, for $x \geq 0$, we have

$$\begin{aligned}
\mathbb{E}[e^{-xY}] &= \lim_{k\to\infty} \mathbb{E}\left[\exp\left(\frac{-xZ^*_{n_k}}{\bar{q}_{n_k}Z_{n_k}}\right)\right] \\
&= \lim \mathbb{E}\left[\mathbb{E}\left[\exp\left(\frac{-xZ^*_{n_k}}{\bar{q}_{n_k}Z_{n_k}}\right)|Z_{n_k}\right]\right] \\
&= \lim \mathbb{E}\left[\left(q_{n_k} + \bar{q}_{n_k}\exp\left(\frac{-x}{\bar{q}_{n_k}Z_{n_k}}\right)\right)^{Z_{n_k}}\right] \\
&= \lim \mathbb{E}\left[\left(1 + \bar{q}_{n_k}\left(\exp\left(\frac{-x}{\bar{q}_{n_k}Z_{n_k}}\right) - 1\right)\right)^{Z_{n_k}}; Z^*_0 = 1\right] + q_0.
\end{aligned}$$

Our method of calculation of this limit depends on whether $\gamma < 1$ or $\gamma = 1$. Suppose first that $\gamma < 1$. Then by (4.18) and (4.19), we have $\mathbb{P}[\tilde{Z}_\infty = \infty] = \mathbb{P}[Z_\infty = \infty] = \bar{q}_0$. Therefore ([4], page 341, Theorem 25.3), we have

$$Z_{n_k} \to_P \infty \quad \text{and} \quad \bar{q}_{n_k}Z_{n_k} \to_P \infty \quad \text{on } Z^*_0 = 1,$$

whence

$$\begin{aligned}
\mathbb{E}[e^{-xY}] &= \lim \mathbb{E}\left[\left(1 - \frac{x}{Z_{n_k}}(1 + o(1))\right)^{Z_{n_k}}; Z^*_0 = 1\right] + q_0 \\
&= e^{-x}\bar{q}_0 + q_0.
\end{aligned}$$

Now suppose instead that $\gamma = 1$. By taking a subsequence, we may assume that $\bar{q}_{n_k} \uparrow 1$. For fixed $x \geq 0$, consider the function

$$G(r,t) := \left(1 + r\left(\exp\left(\frac{-x}{rt}\right) - 1\right)\right)^t$$

for $0 < r \leq 1$ and $1 \leq t \leq \infty$ [where $G(r,\infty) := e^{-x}$]. The derivative

$$\frac{\partial}{\partial r}G(r,t)^{1/t} = \exp\left(\frac{-x}{rt}\right)\left(1 + \frac{x}{rt} - \exp\frac{x}{rt}\right)$$

is nonpositive, so that

$$G(\bar{q}_{n_k}, t) \downarrow e^{-x}$$

as $k \to \infty$. By Dini's theorem ([29], Proposition 9.11), this convergence is uniform on $[1,\infty]$, whence

$$\begin{aligned}
\mathbb{E}[e^{-xY}] &= \lim_{k\to\infty} \mathbb{E}\left[G(\bar{q}_{n_k}, Z_{n_k}); Z^*_0 = 1\right] + q_0 \\
&= \mathbb{E}[e^{-x}; Z^*_0 = 1] + q_0 \\
&= e^{-x}\bar{q}_0 + q_0.
\end{aligned}$$

This establishes our claim.

By uniqueness of the Laplace transform, it follows that we may take $Y = Z_0^*$. In particular, the law of Y is independent of the sequence $\{n_k\}$, so that $Z_n^*/(\bar{q}_n Z_n) \Rightarrow Z_0^*$. Since Z_0^* is constant on nonextinction, we have convergence in probability, not merely in distribution ([4], page 341, Theorem 25.3), from which (4.7) and (4.6) follow. In particular, Z_∞^* and $\tilde{Z}_\infty$ have the same distribution and so (4.10) follows from (4.18) and (4.19). If $\bar{q}_n \to 1$, then (4.6) entails

$$Z_n \to_P Z_\infty^*, \tag{4.22}$$

while if $\bar{q}_n \nrightarrow 1$, then (4.22) is a consequence of (4.6), (4.10) and the fact that $Z_n \geq \bar{q}_n Z_n$.

We next demonstrate (4.5) from (4.22). If $\mathbb{P}[0 < Z_\infty^* < \infty] = 0$, then (4.5) follows from the fact that $Z_n \geq Z_n^*$. On the other hand, if $\mathbb{P}[0 < Z_\infty^* < \infty] > 0$, then $F_n(s) \to F_\infty(s)$, where $F_\infty(s) := \mathbb{E}[s^{Z_\infty^*}]$ is, like every F_n, strictly increasing on $[0,1]$. Define $t_0 := F_\infty(\frac{1}{2})$ and $s_n := F_n^{-1}(t_0)$. Then $s_n \to F_\infty^{-1}(t_0) = \frac{1}{2}$ and $f_{n+1}(s_{n+1}) = s_n$. Therefore $\{s_n^{Z_n}\}$ is a martingale converging in probability and hence a.s. to $(\frac{1}{2})^{Z_\infty^*}$. This yields (4.5).

We deduce from (4.7) and Proposition 4.11 that

$$\mathbf{1}_{Z_0^*=1} \leftarrow_P \frac{Z_n^*/M_n^*}{\bar{q}_n Z_n/M_n^*} = \frac{Z_n^*/M_n^*}{\bar{q}_0 Z_n/M_n} \to \frac{W^*}{\bar{q}_0 W} \quad \text{a.s., on } W > 0 \vee W^* > 0,$$

from which (4.9) and (4.8) follow. By the monotone convergence theorem,

$$\mathbb{E}[Z_\infty^*] = \bar{q}_0 \lim M_n^* = \lim \bar{q}_n M_n,$$

which is (4.14). Furthermore, if $\mathbb{E}[Z_\infty^*] = \infty$, then $\liminf M_n \geq \lim \bar{q}_n M_n = \infty$, while if $\mathbb{E}[Z_\infty^*] < \infty$, then $\mathbb{P}[0 < Z_\infty^* < \infty] > 0$, whence $\bar{q}_n \to 1$ by (4.10), so $\lim M_n = \mathbb{E}[Z_\infty^*]$. We thus obtain (4.13), (4.15) and (4.16).

It follows from (4.5) that $0 < Z_\infty^* < \infty$ is a.s. the same event as $\exists\ n, z \geq 1$, $\forall\ k \geq n$, $Z_k = z$. Now

$$\begin{aligned}
&\mathbb{P}[\forall\ k \geq n, Z_k = z] > 0 \\
&\quad \Leftrightarrow \quad 0 < \mathbb{P}[\forall\ k \geq n, Z_k = Z_n | Z_n = z] = \prod_{k>n} \mathbb{P}[L_k = 1]^z \\
&\quad \Leftrightarrow \quad \forall\ k > n, \qquad \mathbb{P}[L_k = 1] > 0 \quad \text{and} \quad \sum_{k>n} (1 - \mathbb{P}[L_k = 1]) < \infty.
\end{aligned}$$

This gives (4.11). In addition, we see by (4.21) that

$$\begin{aligned}
\mathbb{P}[Z_n = Z_\infty^*] &= \mathbb{E}[\mathbb{P}[Z_n = Z_\infty^* | Z_n]] \\
&\geq \mathbb{E}\left[\prod_{k>n} \mathbb{P}[L_k = 1]^{Z_n}\right] \\
&\geq \mathbb{E}\left[\exp\left\{-\frac{e}{e-1} Z_n \sum_{k>n} \mathbb{P}[L_k \neq 1]\right\}\right]
\end{aligned}$$

if $\mathbb{P}[L_k \neq 1] \leq (e-1)/e$ for $k > n$. By Jensen's inequality, a still lower bound is

$$\exp\left\{-\frac{e}{e-1} M_n \sum_{k>n} \mathbb{P}[L_k \neq 1]\right\}.$$

This gives (4.12). □

PROPOSITION 4.13. *If $A < \infty$ and $q_0 < 1$, then*

$$M_n \to \infty \quad \Leftrightarrow \quad Z_n \to \infty \quad a.s. \text{ given nonextinction}.$$

PROOF. Suppose that $M_n \to \infty$. Since

$$m_k = 1 + \mathbb{E}[L_k - 1; L_k \neq 1] \leq 1 + (A-1)\mathbb{P}[L_k \neq 1],$$

it follows that $\Sigma \mathbb{P}[L_k \neq 1] = \infty$, whence $Z_\infty^* = \infty$ a.s. on nonextinction by (4.11). Thus $Z_n \to \infty$ a.s. on nonextinction by (4.5).

On the other hand, $Z_n \to \infty$ a.s. on nonextinction trivially implies that $M_n \to \infty$. □

We now have a result, the first parts of which will be crucial in Section 4.4. The equivalence of (i) with (iv) in Theorem 4.14 is also in [16], although in slightly disguised form.

THEOREM 4.14. *If $A < \infty$, then $W > 0$ a.s. given nonextinction and the following are equivalent*:

(i) $$q_0 < 1;$$

(ii) $$\sum M_n^{-1} l_{n+1} < \infty;$$

(iii) $$\sup_n \mathbb{E}\left[Z_n^2 M_n^{-2}\right] < \infty;$$

(iv) $$\sum m_n^{-1} M_n^{-1} \mathbb{P}[L_n \geq 2] < \infty \quad \text{and} \quad \inf M_n > 0.$$

PROOF. Actually, (ii) ⇔ (iii) by Proposition 4.11. Also, if $M_n \not\to \infty$, then of course $W > 0$ a.s. given nonextinction. Suppose that $M_n \to \infty$ and that $A < \infty$. In light of (4.9), it suffices to prove that $W > 0$ a.s. for processes without extinction in order to show that $W > 0$ a.s. given nonextinction. We now use a suggestion of Bramson and Griffeath. Given a process without extinction and given $r \geq 0$, let $\{Z_{r,n}\}_{n \geq r}$ be a BPVE with $Z_{r,r} \equiv 1$, probability generating functions $f_{r+1}, f_{r+2}, \dots$, means $M_{r,n}$, $n \geq r$, and martingale limit $W_r :=$ $\lim Z_{r,n}/M_{r,n}$. Clearly, $M_{r,n} = M_n/M_r$. By virtue of Proposition 4.11, we have

$$\mathbb{E}\left[Z_{r,n}^2 M_{r,n}^{-2}\right] = 1 + M_r \sum_{k=r}^{n-1} M_k^{-1} l_{k+1}.$$

Because the BPVE is without extinction, $m_k \geq 1$. Setting $c_k := m_k - 1$, we see that

$$\mathbb{E}[L_k^2] = \mathbb{E}[(L_k + 1)(L_k - 1) + 1] \le (A + 1)c_k + 1,$$

whence

$$(4.23)\qquad \begin{aligned} l_k &\le \frac{(A+1)c_k + 1}{(c_k + 1)^2} - 1 = \frac{(A-1)c_k - c_k^2}{m_k^2} \\ &\le \frac{(A-1)c_k}{m_k} = (A-1)(1 - m_k^{-1}), \end{aligned}$$

so

$$\begin{aligned} \mathbb{E}[Z_{r,n}^2 M_{r,n}^{-2}] &\le 1 + M_r \sum_{k=r}^{n-1} M_k^{-1}(A-1)(1 - m_{k+1}^{-1}) \\ &= 1 + (A-1)M_r \sum_{k=r}^{n-1} (M_k^{-1} - M_{k+1}^{-1}) \\ &= 1 + (A-1)(1 - M_r M_n^{-1}) \\ &\le A. \end{aligned}$$

Therefore $Z_{r,n}M_{r,n}^{-1}$ tends to W_r in L^2 ([28], Proposition 4-2-7) and thus $\mathbb{E}[W_r] = 1$ and $\mathbb{E}[W_r^2] \le A$. In conjunction with the Cauchy–Buniakowski–Schwarz inequality, this leads to the bound

$$\mathbb{P}[W_r > 0] \ge A^{-1}.$$

On the other hand, it is clear that

$$W = M_r^{-1} \sum_{i=1}^{Z_r} W_{r,i}$$

for independent copies $W_{r,i}$ of W_r, whence

$$\begin{aligned} \mathbb{P}[W = 0] &= \mathbb{E}[\mathbb{P}[W = 0 | Z_r]] = \mathbb{E}\big[\mathbb{P}[W_r = 0]^{Z_r}\big] \\ &\le \mathbb{E}\big[(1 - A^{-1})^{Z_r}\big]. \end{aligned}$$

As $Z_r \to \infty$ a.s. from Proposition 4.13, we conclude finally that $W > 0$ a.s.

To establish the remainder of the theorem, note that the preceding paragraph shows that

$$q_0 < 1 \quad \Rightarrow \quad \mathbb{E}\left[\left(\frac{Z_n^*}{M_n^*}\right)^2\right] \le A,$$

whence, by Proposition 4.11,

$$q_0 < 1 \quad \Rightarrow \quad \sup_n \mathbb{E}\left[\left(\frac{Z_n}{M_n}\right)^2\right] < \infty.$$

Conversely, if $\sup_n \mathbb{E}[(Z_n/M_n)^2] < \infty$, then $Z_n/M_n \to W$ in L^2, so that $\mathbb{E}[W] = 1$ and $q_0 \le \mathbb{P}[W = 0] < 1$.

Finally, to show the equivalence of (ii) and (iv), note that

$$\mathbb{P}[L_n \ge 2] \le \mathbb{E}\left[\binom{L_n}{2}\right] \le \binom{A}{2}\mathbb{P}[L_n \ge 2]$$

and

$$2\mathbb{E}\left[\binom{L_n}{2}\right] = \mathbb{E}\left[L_n^2\right] - m_n = m_n^2 l_n + M_n^2 - m_n.$$

Thus the series in (iv) converges iff

$$\begin{aligned}\infty &> \sum m_n^{-1} M_n^{-1}\left(m_n^2 l_n + m_n^2 - m_n\right)\\ &= \sum\left(M_{n-1}^{-1} l_n + M_{n-1}^{-1} - M_n^{-1}\right).\end{aligned}$$

If $\inf M_n > 0$, then this plainly implies (ii). Conversely, (ii) implies (i), which means, by Fatou's lemma, that $\inf M_n > 0$ and hence (iv) holds. □

Parts of this theorem are valid in general. For example:

PROPOSITION 4.15. *For any BPVE,*

$$\bar{q}_0 \ge \mathbb{P}[W > 0] \ge \left(1 + \sum_{n=0}^{\infty} M_n^{-1} l_{n+1}\right)^{-1}.$$

PROOF. This follows from Proposition 4.11 and L^2-convergence of the martingale $Z_n M_n^{-1}$. □

The pathologies of BPVE's alluded to earlier are possible when the condition $A < \infty$ is relaxed even the slightest bit.

PROPOSITION 4.16. *For any continuous* φ: $\mathbb{R}^+ \to \mathbb{R}^+$, *there is a BPVE without extinction such that*

$$\sup_n \mathbb{E}[\varphi \circ L_n] < \infty$$

and $\mathbb{P}[Z_n \to \infty]$, $\mathbb{P}[W = 0] \in]0, 1[$.

PROOF. We may assume that φ is strictly increasing, that

$$a_n := \lfloor \varphi^{-1}(n) \rfloor - 1 \uparrow \infty, \qquad 2 \le a_n \le a_{n-1} + 1,$$

and that

$$2 \le b_k := \operatorname{card} I_k \uparrow \infty,$$

where

$$I_k := \left\{n;\, 2^k \le a_n < 2^{k+1}\right\}, \qquad k \ge 1.$$

Set

$$r_n := 3 \cdot 2^{-k} b_k^{-1}, \qquad n \in I_k,$$

and

$$f_n(s) := (1 - r_n)s + r_n s^{1+a_n}.$$

We claim that the BPVE with probability generating functions $\{f_n\}$ satisfies the conditions desired.

Since

$$\sum_{n \ge 1} r_n = \sum_{k \ge 1} \sum_{n \in I_k} 3 \cdot 2^{-k} b_k^{-1} = \sum_k 3 \cdot 2^{-k} < \infty,$$

we have

$$\mathbb{P}[\forall\, n, Z_n = 1] = \prod (1 - r_n) > 0.$$

Also,

$$M_n = \prod_{j \le n} (1 + r_j a_j) \ge \prod_{\substack{j \le n \\ j \in I_k}} (1 + 3b_k^{-1}) \ge \prod_{2^{k+1} \le a_n} (1 + 3b_k^{-1})^{b_k}$$

$$\ge \prod_{2^{k+1} \le a_n} 4 \ge \left(\frac{a_n}{4} \right)^2.$$

In particular, $M_n \to \infty$, so that $\mathbb{P}[W = 0] \ge \mathbb{P}[\forall\, n,\, Z_n = 1] > 0$. Furthermore, an easy calculation shows that

$$l_n = \frac{a_n^2 r_n (1 - r_n)}{(1 + r_n a_n)^2}. \tag{4.24}$$

Therefore

$$\sum_{n \ge 1} M_n^{-1} l_{n+1} \le \sum_{n \ge 2} \left(\frac{4}{a_n - 1} \right)^2 a_n^2 r_n < \infty,$$

so that, by Proposition 4.15, we have $\mathbb{P}[W = 0] < 1$. Since $M_n \to \infty$, this implies that $\mathbb{P}[Z_n \to \infty] > 0$. Finally,

$$\begin{aligned}
\mathbb{E}[\varphi \circ L_n] &= (1 - r_n)\varphi(1) + r_n \varphi(1 + a_n) \\
&\le \varphi(1) + r_n \varphi(\lfloor \varphi^{-1}(n) \rfloor) \\
&\le \varphi(1) + n r_n.
\end{aligned}$$

Because r_n is decreasing and $\Sigma r_n < \infty$, we have $n r_n = o(1)$. Therefore $\mathbb{E}[\varphi \circ L_n]$ is bounded. □

Despite the pathological behavior exhibited in Proposition 4.16, there is a remarkable kind of 0–1 law due to Cohn and Hering [7]. Their work extends to complete generality as follows.

THEOREM 4.17. *For any BPVE,* $\mathbb{E}[W] \in \{0, 1\}$.

PROOF. If $q_0 = 1$, then $\mathbb{E}[W] = 0$. If $q_0 < 1$ and $M_n \to \infty$, then the result is in [7]. Finally, if $q_0 < 1$ and $M_n \not\to \infty$, then $\mathbb{E}[W] = 1$ by virtue of (4.5) and (4.13). □

4.4. *Random walks on randomly perturbed trees*: *Proofs*. We are at last ready to prove our results concerning random walks on random trees.

PROOF OF THEOREM 4.5. Since $\{Z_n M_n^{-1}\}$ converges a.s. to a finite number, (4.3) implies that $\Sigma Z_n^{-1} C_n^{-1} = \infty$ a.s., which, in turn, ensures a.s. recurrence by the Nash–Williams criterion.

If (4.3) fails and $M_n \not\to \infty$, then, by Theorem 4.12, M_n has a finite limit. It follows that $\Sigma C_n^{-1} < \infty$, which is to say that every element of $\partial'\Gamma$ has finite resistance. Evidently, the random walk is transient a.s. given nonextinction in this case. The remainder of the theorem will follow once we establish Theorem 4.9. □

In order to demonstrate Theorems 4.7 and 4.9, we use Theorem 2.1. That is, we examine percolation, rather than random walk, on the genealogical tree of the BPVE. Some calculations involving percolation on BPVE's are collected in the following lemma.

LEMMA 4.18. *Given a BPVE and conductances* $\{C_{|\sigma|}\}$, *let* $\tilde{q}$ *be the probability that the root of the genealogical tree has only finitely many descendants which remain under the independently performed Bernoulli percolation associated to* $\{C_{|\sigma|}\}$. *Then*

$$1 - \tilde{q} \geq \left[1 + l_1 + \sum_{n=1}^{\infty} M_n^{-1} C_n^{-1} (1 + \gamma_n l_{n+1})\right]^{-1}.$$

If $A < \infty$, *then*

$$\tilde{q} < 1 \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} M_n^{-1} C_n^{-1} (1 + \gamma_n l_{n+1}) < \infty.$$

PROOF. The component of the root under percolation has the same law as that of the genealogical tree of a BPVE with variables, say, Z'_n, M'_n, L'_n and l'_n. Calculations similar to those in the proof of Proposition 4.11 show that

$$\mathbb{E}[L'_k] = p_k \mathbb{E}[L_k], \qquad \mathbb{E}\left[(L'_k)^2\right] = p_k^2 \mathbb{E}\left[L_k^2\right] + \left(p_k - p_k^2\right)\mathbb{E}[L_k].$$

Therefore

$$M'_n = \prod_{k=1}^{n} \mathbb{E}[L'_k] = \prod_{k=1}^{n} p_k \cdot M_n$$

and

$$(M_n')^{-1}l_{n+1}' = \left(\frac{p_{n+1}^2\mathbb{E}[L_{n+1}^2] + p_{n+1}(1-p_{n+1})\mathbb{E}[L_{n+1}]}{p_{n+1}^2\mathbb{E}[L_{n+1}]^2} - 1\right)M_n^{-1}\prod_{k=1}^{n} p_k^{-1}$$

$$= l_{n+1}M_n^{-1}\prod_{k=1}^{n} p_k^{-1} + (1-p_{n+1})\prod_{k=1}^{n+1} p_k^{-1}\mathbb{E}[L_{n+1}]^{-1}M_n^{-1}$$

$$= l_{n+1}M_n^{-1}\gamma_n C_n^{-1} + C_{n+1}^{-1}M_{n+1}^{-1},$$

whence

$$1 + \sum_{n=0}^{\infty}(M_n')^{-1}l_{n+1}' = 1 + l_1 + \sum_{n=1}^{\infty} M_n^{-1}C_n^{-1}(1+\gamma_n l_{n+1}).$$

The conclusions now follow from Proposition 4.15 and Theorem 4.14. □

PROOF OF THEOREMS 4.7 AND 4.9. Let R_0 be the event of recurrence and $R_{n,i}$ be the event that the walk is recurrent when restricted to the subtree $\Gamma^{\sigma_{n,i}}$, where $\sigma_{n,i}$ is the i-th particle of the n-th generation. Thus, for any n,

$$\mathbf{1}_{R_0} = \prod_{i=1}^{Z_n}\mathbf{1}_{R_{n,i}}. \tag{4.25}$$

Let $\tilde{q}_{n,i}(\Gamma)$ be the probability that no element of $\partial'\Gamma^{\sigma_{n,i}}$ remains under the Bernoulli percolation on $\Gamma^{\sigma_{n,i}}$ which is associated to the conductances $\{C_{|\sigma|};\ \sigma\in\Gamma^{\sigma_{n,i}}\}$. (The survival probabilities of the edges of $\Gamma^{\sigma_{n,i}}$ depend on n for this percolation process, even though the conductances do not, because of the dependence on the root.) Thus,

$$R_{n,i} = \{\tilde{q}_{n,i}(\Gamma) = 1\} \tag{4.26}$$

by Theorem 2.1. Let

$$\tilde{q}_n := \mathbb{E}[\tilde{q}_{n,i}(\Gamma)|Z_n \ge i]$$

be the extinction probability for the initial particle in the compound (or mixture) process of generating a tree by $\{L_k\}_{k>n}$ and following independently by Bernoulli percolation associated to the conductances $\{C_k\}_{k>n}$ on that tree. Lemma 4.18 gives

$$\begin{aligned}1-\tilde{q}_n &\ge \left\{1 + l_{n+1} + \sum_{k>n}\left(\frac{M_k}{M_n}\right)^{-1}\right.\\ &\qquad\left.\times C_k^{-1}\left[1+\left(1+\sum_{j=n+1}^{k}C_j^{-1}\right)C_k l_{k+1}\right]\right\}^{-1}\\ &\ge \left\{1 + l_{n+1} + M_n\sum_{k>n}M_k^{-1}C_k^{-1}(1+\gamma_k l_{k+1})\right\}^{-1}.\end{aligned} \tag{4.27}$$

Suppose now that (4.4) fails. Since

$$M_k^{-1}C_k^{-1}(1+\gamma_k l_{k+1}) \geq M_k^{-1}C_k^{-1}(C_k l_{k+1}) = M_k^{-1} l_{k+1},$$

we have $\Sigma M_k^{-1} l_{k+1} < \infty$. Thus $\mathbb{P}[W > 0] > 0$ by Proposition 4.15. In addition, it follows that $l_{n+1} = o(M_n)$, whence by (4.27), we have

$$\tilde{q}_n \leq 1 - \frac{1}{1+o(M_n)}. \tag{4.28}$$

In order to show that transience is a.s. on $W > 0$, we may and now do assume that $M_n \to \infty$ since the result is otherwise a consequence of the trivial part of Theorem 4.5. Let $\mathscr{F}_n$ be the σ-field generated by $\{L_{k,i};\ k \leq n,\ i \geq 1\}$. The martingale convergence theorem guarantees that

$$\begin{aligned}
\mathbf{1}_{R_0} &= \lim_{n\to\infty} \mathbb{E}\big[\mathbf{1}_{R_0} | \mathscr{F}_n\big] = \lim_{n\to\infty} \mathbb{E}\left[\prod_{i=1}^{Z_n} \mathbf{1}_{R_{n,i}} | \mathscr{F}_n\right] \\
&= \lim_{n\to\infty} \prod_{i=1}^{Z_n} \mathbb{E}\big[\mathbf{1}_{\tilde{q}_{n,i}(\Gamma)=1}\big] \leq \liminf_{n\to\infty} \prod_{i=1}^{Z_n} \mathbb{E}[\tilde{q}_{n,i}(\Gamma)] \\
&= \liminf_{n\to\infty} (\tilde{q}_n)^{Z_n} \leq \liminf_{n\to\infty} \left(1 - \frac{1}{o(M_n)}\right)^{M_n Z_n M_n^{-1}} \leq \mathbf{1}_{W=0};
\end{aligned}$$

we have used (4.25), (4.26) and (4.28) in turn, together with the independence of the events $R_{n,i}$ given $\mathscr{F}_n$. According to this inequality, recurrence can only occur on $W = 0$, as desired.

The first inequality of Theorem 4.9 is a direct consequence of (4.2):

$$\begin{aligned}
\mathbb{E}[\mathscr{C}(0 \to \partial'\Gamma); \partial'\Gamma \neq \varnothing] &\leq \mathbb{E}\left[\left(\sum_{n\geq 1} Z_n^{-1} C_n^{-1}\right)^{-1}\right] \\
&\leq \left(\sum_{n\geq 1} \mathbb{E}[Z_n]^{-1} C_n^{-1}\right)^{-1} = \left(\sum_{n\geq 1} M_n^{-1} C_n^{-1}\right)^{-1},
\end{aligned}$$

where the second inequality stems from an elementary inequality between iterated means ([15], Theorem 26, page 31), here of orders -1 and $+1$. In the other direction, we have, by virtue of Theorem 2.1 and Lemma 4.18,

$$\begin{aligned}
\mathbb{E}[\mathscr{C}(0 \to \partial'\Gamma); \partial'\Gamma \neq \varnothing] &\geq \mathbb{E}\left[\frac{\mathscr{C}(0 \to \partial'\Gamma)}{1+\mathscr{C}(0 \to \partial'\Gamma)}; \partial'\Gamma \neq \varnothing\right] \\
&\geq \mathbb{E}\left[\frac{1}{2}(1 - \tilde{q}_{0,1}(\Gamma))\right] = \frac{1}{2}(1-\tilde{q}_0) \\
&\geq \frac{1}{2}\left(\sum_{n=0}^{\infty} M_n^{-1} C_n^{-1}(1+\gamma_n l_{n+1})\right)^{-1}.
\end{aligned}$$

In case the probability of extinction is zero, we may use (4.23) to find that

$$\begin{aligned}\sum_{n=0}^{N} M_n^{-1} C_n^{-1} \gamma_n l_{n+1} &= \sum_{n=0}^{N} M_n^{-1} l_{n+1} \sum_{k=0}^{n} C_k^{-1} \\ &\le (A-1) \sum_{n=0}^{N} \left(M_n^{-1} - M_{n+1}^{-1}\right) \sum_{k=0}^{n} C_k^{-1} \\ &= (A-1) \left\{ \sum_{n=0}^{N} M_n^{-1} C_n^{-1} - M_{N+1}^{-1} \sum_{k=0}^{N} C_k^{-1} \right\} \\ &\le (A-1) \sum_{n=0}^{N} M_n^{-1} C_n^{-1},\end{aligned}$$

whence

$$\sum_{n=0}^{\infty} M_n^{-1} C_n^{-1} (1 + \gamma_n l_{n+1}) \le A \sum_{n=0}^{\infty} M_n^{-1} C_n^{-1},$$

which yields the last inequality of Theorem 4.9. This also finishes the proof of Theorem 4.5.

Suppose now that $A < \infty$. If (4.4) holds, then Lemma 4.18 shows that $\tilde{q}_0 = 1$, whence $\tilde{q}_{0,1}(\Gamma) = 1$ a.s. Thus, (4.26) says that recurrence is a.s. On the other hand, if (4.4) fails, then transience is a.s. given $W > 0$ by Theorem 4.9, which is a.s. given nonextinction by Theorem 4.14. This proves Theorem 4.7. □

We turn now to the propositions concerning misbehaving BPVE's.

PROOF OF PROPOSITION 4.6. Consider a BPVE with generating functions $f_1(s) := s$, $f_n(s) := (1/2 - 1/n) + (1/2 + 1/n)s^2$, $n \ge 2$. Then easy calculations show that

$$m_n = 1 + 2/n, \qquad n \ge 2,$$
$$M_n = (n+1)(n+2)/6$$

and

$$l_n = (1/2 + 1/n)^{-1} - 1 \ge 1/5, \qquad n \ge 3,$$

whence

$$\sum M_n^{-1} < \infty \quad \text{and} \quad \sum M_n^{-1}[1 + (n+1) l_{n+1}] = \infty,$$

that is, (4.3) fails and (4.4) holds. Also, $l_n < 1$, so that, by Theorem 4.14, $q_0 < 1$. Finally, SRW is a.s. recurrent by Theorem 4.7. □

PROOF OF PROPOSITION 4.8. Consider a BPVE with generating functions $f_n(s) := s$, $n < 27$, and $f_n(s) := (1 - r_n)s + r_n s^{1+a_n}$, $n \ge 27$, where

$$r_n := \frac{2}{n\left[(\log n)(\log\log n)^2\right]}$$

and

$$a_n := \lfloor (\log n)(\log\log n)^2 \rfloor.$$

For $n \geq 27$, we have

$$\begin{aligned}\mathbb{E}\left[\exp\frac{L_n}{(\log L_n)^2};\, L_n \geq 2\right] &= r_n \exp\frac{1+a_n}{(\log(1+a_n))^2}\\ &\leq r_n \exp(1+\log n)\\ &= o(1).\end{aligned}$$

Since $\Sigma r_n < \infty$, $\mathbb{P}[\forall\ n,\ Z_n = 1] > 0$, whence the probability that SRW is recurrent is greater than 0. To show that recurrence is not a.s., we verify failure of (4.4). Now

$$M_n = \prod_{k=27}^{n}(1 + r_k a_k) = \prod_{k=27}^{n}\left(1+\frac{2}{k}\right) = \frac{(n+1)(n+2)}{27\cdot 28}, \qquad n \geq 27,$$

and

$$l_n \leq a_n^2 r_n = o\left(\frac{1}{\sqrt{n}}\right),$$

by (4.24), whence

$$\sum M_{n-1}^{-1}(1 + nl_n) < \infty.$$

This proves the first part of the proposition.

The proof of the second part uses the generating functions f_n above but with the parameters

$$r_n := \frac{2+\log n}{n\log n\lceil n(\log n)^2\log\log n\rceil}$$

and

$$a_n := \lceil n(\log n)^2 \log\log n\rceil.$$

Now we have, for $n \geq 27$ and some constants c_i,

$$\mathbb{E}\left[\frac{L_n^2}{(\log L_n)^3};\, L_n \geq 2\right] = \frac{r_n(1+a_n)^2}{(\log(1+a_n))^3} = O\left(\frac{(\log n)^2\log\log n}{(\log n)^3}\right) = o(1),$$

$$\begin{aligned}M_n &= \prod_{k=27}^{n}(1+r_k a_k) = \sum_{k=27}^{n}\left(1+\frac{2+\log n}{n\log n}\right) \sim c_1\exp\left(\sum_{k=27}^{n}\frac{2+\log n}{n\log n}\right)\\ &\sim c_2\exp\left(\int_{27}^{n}\frac{2+\log x}{x\log x}\,dx\right) \sim c_3\exp(\log n + 2\log\log n) = c_3 n(\log n)^2,\end{aligned}$$

whence $\Sigma M_n^{-1} < \infty$, that is, (4.3) fails for SRW. Finally, we claim that Z_n has a finite limit a.s., whence SRW is recurrent a.s. We have

$$\begin{aligned}\sum_{k>n} \mathbb{P}[L_k \geq 2] = \sum_{k>n} r_k &< \frac{1}{\log\log n} \sum_{k>n} \frac{2+\log k}{k^2(\log k)^3} \\ &< \frac{1}{\log\log n} \int_n^\infty \frac{2+\log x}{x^2(\log x)^3}\, dx \\ &= \frac{1}{n(\log n)^2 \log\log n},\end{aligned}$$

whence the claim follows from our preceding calculation of M_n, (4.12) and (4.5). □

Finally, we say that a tree, Γ, is *quasispherical* [22] if

$$\operatorname{br}\Gamma = \liminf_{n\to\infty} \operatorname{card}\{\sigma \in \Gamma;\, |\sigma| = n\}^{1/n}.$$

This represents a sort of regularity of Γ [22]. The genealogical trees of Galton–Watson processes are quasispherical a.s. given nonextinction ([22], Proposition 6.4). A simple consequence of Theorem 4.5 is that the trees of BPVE's with $A < \infty$ are also quasispherical a.s. given nonextinction, as is fitting in view of so many of our preceding results.

COROLLARY 4.19. *The genealogical tree of a BPVE satisfying $A < \infty$ has branching number $\liminf M_n^{1/n}$ and is quasispherical a.s. given nonextinction.*

PROOF. As indicated in the remarks introducing Section 4, the branching number is the same as the critical drift. From Theorem 4.7, the critical drift, λ_c, is

$$\lambda_c = \sup\left\{\lambda \geq 1;\ \sum M_n^{-1}\lambda^n\left(1 + l_{n+1}\lambda^{-n}\sum_{k=0}^{n}\lambda^k\right) < \infty\right\}$$

a.s. given nonextinction. Now

$$1 \leq 1 + l_{n+1}\lambda^{-n}\sum_{k=0}^{n}\lambda^k \leq 1 + \frac{A}{m_{n+1}}\frac{\lambda}{\lambda - 1},$$

whence

$$\sum M_n^{-1}\lambda^n \leq \sum M_n^{-1}\lambda^n\left(1 + l_{n+1}\lambda^{-n}\sum_{k=0}^{n}\lambda^k\right) \leq \left(1 + \frac{A}{\lambda - 1}\right)\sum M_n^{-1}\lambda^n.$$

Therefore λ_c is the radius of convergence of $\sum M_n^{-1}\lambda^n$, that is, $\liminf M_n^{1/n}$.

Because $W > 0$ a.s. given nonextinction by Theorem 4.14, this also equals $\liminf Z_n^{1/n}$ a.s. given nonextinction. This is the definition of quasisphericity. □

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