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OCCUPATION TIME LARGE DEVIATIONS OF TWO-DIMENSIONAL SYMMETRIC SIMPLE EXCLUSION PROCESS

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We prove a large deviations principle for the occupation time of a site in the two-dimensional symmetric simple exclusion process. The decay probability rate is of order $t/\log t$ and the rate function is given by $\Upsilon_\alpha(\beta) = (\pi/2)\{\sin^{-1}(2\beta - 1) - \sin^{-1}(2\alpha - 1)\}^2$. The proof relies on a large deviations principle for the polar empirical measure which contains an interesting log scale spatial average. A contraction principle permits us to deduce the occupation time large deviations from the large deviations for the polar empirical measure.

1. Introduction.

Description of the result. We consider the nearest-neighbor symmetric simple exclusion process on the lattice $\mathbb{Z}^2$. This process can be informally described as follows. Distribute particles on $\mathbb{Z}^2$ in such a way that each site is occupied by at most one particle. Each particle, independently from the others, waits a mean-1 exponential time at the end of which it chooses one of its neighbor sites with uniform probability. If the site selected is unoccupied, the particle jumps to this site, otherwise, it stays where it is. In both cases, the particle waits a new mean-1 exponential time and repeats the same procedure.

This stochastic dynamics has a one-parameter family of invariant states. Fix $0 \le \alpha \le 1$. The Bernoulli product measure with density α, denoted by ν_α, is an ergodic, invariant, reversible state for the exclusion process. This measure is obtained by placing at each site, independently from the others, a particle with probability α.

Assume until the end of this Introduction that the initial state is chosen according to ν_α. Denote by $\eta_s(0)$ the state of the origin at time $s \ge 0$ so that $\eta_s(0) = 1$ if the origin is occupied by a particle at time s and $\eta_s(0) = 0$ otherwise. Let $V_t = t^{-1} \int_0^t \eta_s(0)\, ds$ be the proportion of time that the origin is occupied in the time interval $[0, t]$. Since ν_α is ergodic, ν_α a.s., V_t converges to α. Kipnis [8] proved that $\sqrt{t/\log t}[V_t - \alpha]$ converges in distribution to a mean-0 Gaussian variable with variance $(2/\pi)\alpha(1-\alpha)$ and Landim [11] proved that the decay rate of the large deviations is $t/\log t$. We prove in this article a large deviations result

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for V_t in which the decay probability rate is $t/\log t$, while the rate function is given by

$$\Upsilon_\alpha(\beta) = \frac{\pi}{2}\{\sin^{-1}(2\beta - 1) - \sin^{-1}(2\alpha - 1)\}^2.$$

The $t/\log t$ order can be anticipated because it is the order of the expected number of distinct particles which visit the origin in the time interval $[0, t]$, while the rate function is related to the variational problem

$$\inf_m \int_0^1 dr \frac{m'(r)^2}{m(r)[1 - m(r)]}, \tag{1.1}$$

where the infimum is carried over all smooth functions $m : [0, 1] \to \mathbb{R}$ such that $m(0) = \beta$, $m(1) = \alpha$. The expression $m'(r)^2/m(r)[1 - m(r)]$ appears recurrently in the context of the exclusion process. For instance, in the investigation of the large deviations of the empirical measure from the hydrodynamic limit, considered by Kipnis, Olla and Varadhan [10], this is the expression obtained if we consider paths $\pi(t, du) = \pi(du)$ which do not depend on time.

Interest and motivation. The interest of this article relies not only on the result itself, but also on the method of the proof.

On the one hand, this is the first example of an interacting particle system where the large deviations rate function associated to the occupation time can be explicitly computed. This is even more surprising in view of the decay rate which is of order $t/\log t$.

On the other hand, it was proved in [11] that in dimension one, the large deviations of the occupation time for the symmetric simple exclusion process, whose decay rate is of order $\sqrt{t}$, is closely related to the large deviations of the empirical measure from the hydrodynamic limit. We show in this article that in dimension two the large deviations of the occupation time are related to the large deviations of the polar empirical measure. To explain this connection, fix $T > 0$ and denote by $\mu^{1,T} = \mu^{1,T}(\eta)$ the measure on $\mathbb{R}_+$ obtained from a configuration of particles $\eta = \{\eta(x), x \in \mathbb{Z}^2\}$ by the formula

$$\mu^{1,T}(\eta) = \frac{1}{2\pi \log T} \sum_{x \in \mathbb{Z}^2_*} \eta(x) \frac{1}{|x|^2} \delta_{\sigma_T(x)},$$

where $\sigma_T(x) = \log|x|/\log T$, δ_r is the Dirac measure concentrated on r and $\mathbb{Z}^2_* = \mathbb{Z}^2 - \{0\}$. We prove in Lemma 3.1 and Corollary 3.2 that the occupation time of the origin and the time average of the $\mu^{1,T}$ measure of the interval $[0, \varepsilon]$ are superexponentially close in the scale $T/\log T$,

$$\limsup_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\log T}{T} \log P\left[\left|\frac{1}{T}\int_0^T ds \left\{\eta_s(0) - \frac{\mu^{1,T}(\eta_s)([0, \varepsilon])}{\varepsilon}\right\}\right| > \delta\right] = -\infty$$

for every $\delta > 0$. This result shows that in dimension two the large deviations of the occupation time is closely related to the large deviations of the time average of the polar empirical measure.

Consider a continuous function with compact support $H : \mathbb{R}_+ \to \mathbb{R}$. The integral of H with respect to $\mu^{1,T}(\eta)$ is given by

$$\frac{1}{2\pi \log T} \sum_{x \in \mathbb{Z}_*^2} \eta(x) \frac{1}{|x|^2} H(\sigma_T(x)).$$

Observe that when the support of H falls in interval $[r - \delta, r + \delta]$ in $\mathbb{R}_+$, a space average over $O(T^{2(r+\delta)})$ random variables $\eta(x)$ appears in the above expression. The number of variables increases with r and is so large at the macroscopic hydrodynamic scale $r = 1/2$ that, even at the level of large deviations, the polar empirical measure $\mu^{1,T}$ is fixed and equal to $\alpha \, d\lambda$ for $r \geq 1/2$. Here λ stands for the Lebesgue measure. All relevant fluctuations occur therefore in the space interval $[0, 1/2)$, which is seen to be of smaller order than the hydrodynamic scale.

A subhydrodynamic scaling appears in several problems related to two-dimensional exclusion processes, for instance, in the integration by parts formula for mean-0 local functions, with important consequence in the equilibrium fluctuations of the empirical measure for asymmetric processes or in the asymptotic behavior of a second class particle in the symmetric case. We hope that the polar empirical measure approach and the estimates presented in this article will shed new light on these problems.

Once the connection between the large deviations of the occupation time and the large deviations of the time average of the polar measure $\mu^{1,T}$ has been established, a natural strategy delineates. We first prove in Sections 5 and 6 a large deviations principle for the time average of the polar empirical measure $\mu^{1,T}(\eta)$, and we then deduce from this result and through a contraction argument in Section 7 a large deviations principle for the occupation time.

Prospectives. We already mentioned problems related to subhydrodynamic scalings. The results and the methods presented in this article also raise the following issues. It is conceivable that a similar approach would permit us to derive explicit formulas for the rate functions of the large deviations principle in dimension $d \geq 3$. This method probably applies also to the case of independent random walks in dimension two. In this case the variational problem (1.1) would be replaced by

$$\inf_m \int_0^1 dr \, \frac{m'(r)^2}{m(r)},$$

where the infimum is carried over all smooth functions $m : [0, 1] \to \mathbb{R}$ such that $m(0) = \beta$, $m(1) = \alpha$. A simple computation shows that the solution of the variational problem is $4(\sqrt{\beta} - \sqrt{\alpha})^2$, which, up to a constant, is the rate function obtained by Cox and Griffeath [6]. This would give a new interpretation to the rate function.

Historical background. Cox and Griffeath [6] were the first authors to consider the question of large deviations of the occupation time in interacting particle systems. They considered the case of independent random walks and proved a large deviations principle in which the decay rates are $\sqrt{t}$ in dimension one, $t/\log t$ in dimension two and t in dimension $d \geq 3$. Bramson, Cox and Griffeath [4] examined the same problem for the voter model and proved upper and lower bounds for the large deviations in dimension $d \geq 3$, showing that the decay rate is $\sqrt{t}$ in $d = 3$, $t/\log t$ in $d = 4$ and t in $d \geq 5$.

As is evidenced in the literature, the voter and the superrandom walk models have dimensional-dependent phenomena parallel to the simple exclusion and the independent random walk models, with the critical dimension shifted by 2. What would be the results for the four $(2 + 2)$-dimensional voter/superrandom walk models, corresponding to that obtained here for the critical dimension two? Substantial modifications of our approach would be needed when proving super-exponential estimates because our method of proof relies on the reversibility which the voter/superrandom walk models lack. Also, the occupation-time difference between two neighboring sites has been worked out (see [5, 7, 12] and references therein) for independent random walks and some superrandom walk type of models. What would be the corresponding results for the simple exclusion model and the voter model? It would be interesting to have these results worked out because they demand deep insight and technique to accomplish, although the simple parallelism seems to go on strong for many qualitative predictions

Landim [11] proved a large deviations principle for the occupation time in the symmetric simple exclusion process in dimension $d \neq 2$. In dimension two he proved that $t/\log t$ is the right decay rate by obtaining lower and upper bounds. In dimension one, the proof relies on the fact that the large deviations of the occupation time are connected to the large deviations of the empirical measure from the hydrodynamic limit. This method was used by Benois [2] to give a new proof of the large deviations principle in the case of independent random walks in dimension one.

Finally, it should be observed (see the end of Section 5) that it follows from our arguments that a superexponential two-blocks estimate is *not* needed in the proof of the large deviations of the empirical measure from the hydrodynamic limit in symmetric simple exclusion processes, which is the problem considered in [10].

2. Notation and results. The nearest-neighbor symmetric simple exclusion process is a continuous time Markov process on $\mathcal{X} = \{0, 1\}^{\mathbb{Z}^d}$ that represents the evolution of random walks on $\mathbb{Z}^d$ with a hard core interaction which prevents more than one particle per site. The configurations of $\mathcal{X}$ are denoted by the Greek letter eta (η) so that $\eta(x)$ is equal to 1 or 0 if site x is occupied or not for η.

Fix $T > 0$. The generator L_T of the *speeded-up* symmetric simple exclusion process is given by

$$(L_T f)(\eta) = (T/2) \sum_{x,y \in \mathbb{Z}^d, |x-y|=1} \eta(x)[1-\eta(y)][f(\sigma^{x,y}\eta) - f(\eta)].$$

In this formula, f is a local function and $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y, \\ \eta(x), & \text{if } z = y, \\ \eta(y), & \text{if } z = x. \end{cases}$$

For each $0 \le \alpha \le 1$, denote by ν_α the Bernoulli product measure on $\mathcal{X}$ with marginals given by

$$\nu_\alpha\{\eta, \eta(x) = 1\} = \alpha$$

for $x \in \mathbb{Z}^d$. A simple computation shows that $\{\nu_\alpha, 0 \le \alpha \le 1\}$ is a one-parameter family of reversible invariant measures. For $0 \le \alpha \le 1$, denote by $\mathbb{P}_\alpha = \mathbb{P}_{T,\alpha}$ the probability on the path space $D(\mathbb{R}_+, \mathcal{X})$ corresponding to the nearest-neighbor speeded-up symmetric simple exclusion process with generator L_T starting from ν_α.

For $t > 0$, consider the occupation time of the origin:

$$V_t = \frac{1}{t}\int_0^t \eta_s(0)\, ds.$$

The main result of this article states a large deviations principle for the occupation time in dimension $d = 2$.

THEOREM 2.1. *Consider the two-dimensional symmetric simple exclusion process and fix a density* α *in* $(0, 1)$. *For every closed subset* F *of* $[0, 1]$ *and every open subset* G *of* $[0, 1]$,

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\big[V_1 \in F\big] \le -\inf_{\beta \in F} \Upsilon_\alpha(\beta),$$

$$\liminf_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\big[V_1 \in G\big] \ge -\inf_{\beta \in G} \Upsilon_\alpha(\beta),$$

where $\Upsilon_\alpha : [0, 1] \to \mathbb{R}_+$ *is the rate function given by*

$$\Upsilon_\alpha(\beta) = \frac{\pi}{2}\{\sin^{-1}(2\beta - 1) - \sin^{-1}(2\alpha - 1)\}^2.$$

We believe that

$$\lim_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[V_1 = 1] = -\Upsilon_\alpha(1),$$

but do not have a proof. Theorem 2.1 provides an upper bound for this probability of a complete traffic jam at one site, but no lower bound as $\{1\}$ is not an open set. Arratia [1] showed that the order $T/\log T$ is correct. The corresponding problem for other dimensions is also interesting.

The article is organized as follows. In the investigation of large deviations of Markov processes, a major ingredient is to find the relevant perturbations that create the fluctuations. In the present context, the fluctuations are closely connected to the polar empirical measure associated to a configuration, which is described in Section 3. Superexponential estimates are established in Section 4. A large deviation principle for the polar empirical measure is established in Sections 5 and 6. In Section 7, we retrieve a large deviation principle for the occupation time V_1 from the large deviations for the empirical measure through a contraction principle.

3. Polar empirical measure. We show in this section that the correct perturbations involved in the investigation of the large deviations of the occupation time are related to a macroscopic logarithmic scale.

We prove in Lemma 3.1 and Corollary 3.2, through a superexponential estimate, that in dimension two the occupation-time large deviations are given by the large deviations of the density of particles in a ball centered at the origin with radius T^ε, $\varepsilon \downarrow 0$. This result leads to the following strategy of proof. We introduce in (3.7) and (3.9) a polar empirical measure and state in Theorem 3.3 a large deviations principle for it, which is proved in Sections 5 and 6. In Section 7 we recover the large deviations principle for the occupation time from the large deviations for the polar empirical measure through a contraction principle.

Denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^2$ for $x = (x_1, x_2)$, $|x|^2 = x_1^2 + x_2^2$. Fix a positive function $q : (0, 1] \to (0, 1]$, decreasing to 0 slower than the identity: $\lim_{\varepsilon \to 0} q(\varepsilon)/\varepsilon = \infty$. Denote by $\mathbb{T}_\pi$ the one-dimensional torus $[-\pi, \pi)$. For $\theta \in \mathbb{T}_\pi$, denote by $\mathfrak{A}_T^{\theta,\varepsilon}$ the annular region

$$\mathfrak{A}_T^{\theta,\varepsilon} = \{v \in \mathbb{R}^2 : T^\varepsilon \le |v| \le 2T^\varepsilon, |\Theta(v) - \theta| \le q(\varepsilon)\},$$

where $\Theta(v)$ stands for the angle of v. For any region $\mathfrak{A} \subset \mathbb{R}^2$, let $|\mathfrak{A}|$ be the number of sites ($\mathbb{Z}^2$ lattice points) in $\mathfrak{A}$.

LEMMA 3.1. *For any $\delta > 0$, $\theta \in \mathbb{T}_\pi$ and $t > 0$,*

$$\limsup_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log T}{T} \log \mathbb{P}_\alpha \left[\left| \int_0^t ds \left\{ \eta_s(0) - \frac{1}{|\mathfrak{A}_T^{\theta,\varepsilon}|} \sum_{y \in \mathfrak{A}_T^{\theta,\varepsilon}} \eta_s(y) \right\} \right| > \delta \right] \tag{3.1}$$
$$= -\infty.$$

PROOF. From the Feynman–Kac formula and the variational formula for the largest eigenvalue of a reversible operator (see [9], Chapter 11), to prove (3.1), it is enough to show that

$$\limsup_{\varepsilon\to 0}\limsup_{T\to\infty}\sup_{f}\Bigg\{\int\bigg(\eta(0)-\frac{1}{|\mathfrak{A}_T^{\theta,\varepsilon}|}\sum_{x\in\mathfrak{A}_T^{\theta,\varepsilon}}\eta(x)\bigg)f\,d\nu_\alpha - a\log T\,D(\nu_\alpha,f)\Bigg\}=0 \tag{3.2}$$

for every $a>0$. In this formula, the supremum is carried over all densities f with respect to ν_α and $D(\nu_\alpha,f)$ is the modified Dirichlet form of f with respect to ν_α given by

$$D(\nu_\alpha,f)=\tfrac{1}{4}\sum_{|x-y|=1}\int\{\sqrt{f(\sigma^{x,y}\eta)}-\sqrt{f(\eta)}\}^2\,d\nu_\alpha.$$

Fix a density f and two sites x and y. A change of variables $\xi=\sigma^{x,y}\eta$ and Schwarz inequality show that

$$\begin{aligned}\int[\eta(x)-\eta(y)]f(\eta)\,d\nu_\alpha&=\tfrac{1}{2}\int[\eta(x)-\eta(y)]\{f(\eta)-f(\sigma^{x,y}\eta)\}\,d\nu_\alpha\\&\le\bigg(\int\{\sqrt{f(\sigma^{x,y}\eta)}-\sqrt{f(\eta)}\}^2\,d\nu_\alpha\bigg)^{1/2}\end{aligned}$$

because f is a density with respect to ν_α and $\eta(z)$ is bounded by 1. Applying this estimate to the first term inside braces in (3.2), we bound it by

$$\begin{aligned}&\frac{1}{|\mathfrak{A}_T^{\theta,\varepsilon}|}\sum_{x\in\mathfrak{A}_T^{\theta,\varepsilon}}\bigg(\int\{\sqrt{f(\sigma^{0,x}\eta)}-\sqrt{f(\eta)}\}^2\,d\nu_\alpha\bigg)^{1/2}\\&\quad\le\bigg(\frac{1}{|\mathfrak{A}_T^{\theta,\varepsilon}|}\sum_{x\in\mathfrak{A}_T^{\theta,\varepsilon}}\int\{\sqrt{f(\sigma^{0,x}\eta)}-\sqrt{f(\eta)}\}^2\,d\nu_\alpha\bigg)^{1/2}.\end{aligned}\tag{3.3}$$

Just before (4.2) below, for each x in $\mathbb{Z}_*^2$, we define a path Ξ_x connecting the origin to x. By a path, we understand a sequence $0=z(0),z(1),\dots,z(n)=x$ such that $z(k)\in\mathbb{Z}^2$, $n=|x|_1\equiv|x_1|+|x_2|$ and $|z(i+1)-z(i)|=1$ for $0\le i<n$. For each bond b in $\mathbb{Z}^2$, we denote by $m(b;\mathfrak{A}_T^{\theta,\varepsilon})$ the number of paths Ξ_x that use the bond b to connect the origin to $x\in\mathfrak{A}_T^{\theta,\varepsilon}$. By Lemma 4.4, the paths Ξ_x can be defined in such a way that $m(b;\mathfrak{A}_T^{\theta,\varepsilon})\le CT^{2\varepsilon}|b|^{-1}$, where for a bond $b=\{x,y\}$, $|\{x,y\}|=|x|\vee|y|$.

Fix a site x in $\mathfrak{A}_T^{\theta,\varepsilon}$ and recall the definition of the path Ξ_x. The configuration $\sigma^{0,x}\eta$ can be written as

$$\sigma^{z(0),z(1)}\cdots\sigma^{z(n-2),z(n-1)}\sigma^{z(n-1),z(n)}\sigma^{z(n-2),z(n-1)}\cdots\sigma^{z(1),z(2)}\sigma^{z(0),z(1)}\eta.$$

In particular, introducing intermediate terms in (3.3), by a change of variables and by Schwarz inequality, for each fixed x in $\mathfrak{A}_T^{\theta,\varepsilon}$, we may estimate the integral appearing in the right-hand side of (3.3) by

$$
\begin{aligned}
&2\left\{\sum_{i=0}^{n-1}\frac{1}{|\{z(i),z(i+1)\}|}\right\}\\
(3.4)\qquad &\quad\times\left\{\sum_{i=0}^{n-1}|\{z(i),z(i+1)\}|\int\{\sqrt{f(\sigma^{z(i),z(i+1)}\eta)}-\sqrt{f(\eta)}\}^2\,d\nu_\alpha\right\}.
\end{aligned}
$$

By (4.3), the sum in the first line is bounded by

$$
\sup_{x\in\Lambda_{2T^\varepsilon}}\sum_{b\in\Xi_x}\frac{1}{|b|}\le C_0\varepsilon\log T \tag{3.5}
$$

for some finite constant C_0, where $\Lambda_l=\{-l,\dots,l\}^2$. Therefore, after changing the order of summation, we obtain that the square of the right-hand side of (3.3) is bounded above by

$$
\frac{C_0\varepsilon\log T}{q(\varepsilon)T^{2\varepsilon}}\sum_{b\in B(2T^\varepsilon)}|b|\int\{\sqrt{f(\sigma^b\eta)}-\sqrt{f(\eta)}\}^2\,d\nu_\alpha\sum_{x\in\mathfrak{A}_T^{\theta,\varepsilon},b\in\Xi_x} \tag{3.6}
$$

for some finite constant C_0. In this formula, the sum is performed over all bonds b located in $B(2T^\varepsilon)$, the ball of radius $2T^\varepsilon$ centered at the origin. By Lemma 4.4, the second sum is less than or equal to $CT^{2\varepsilon}|b|^{-1}$. This expression is thus bounded above by

$$
\frac{C_0\varepsilon\log T}{q(\varepsilon)}\sum_{b\in B(2T^\varepsilon)}\int\{\sqrt{f(\sigma^b\eta)}-\sqrt{f(\eta)}\}^2\,d\nu_\alpha\le\frac{C_0\varepsilon\log T\,D(\nu_\alpha,f)}{q(\varepsilon)}.
$$

It follows from this estimate that the left-hand side of (3.2) is bounded above by

$$
\limsup_{\varepsilon\to0}\limsup_{T\to\infty}\sup_{K>0}\left\{C_0\sqrt{\frac{\varepsilon K\log T}{q(\varepsilon)}}-aK\log T\right\}=\limsup_{\varepsilon\to0}\frac{C_0\varepsilon}{aq(\varepsilon)}=0,
$$

since $\lim_{\varepsilon\to0}\varepsilon/q(\varepsilon)=0$. This concludes the proof of the lemma. □

In Lemma 3.1, the set $\mathfrak{A}_T^{\theta,\varepsilon}$ can be replaced by several distinct sets. The same proof applies for balls of radius smaller than $a_1T^{a_2\varepsilon}$ for fixed a_1, $a_2>0$ or to annular regions of type $a_1T^{a_2\varepsilon}\le|v|\le b_1T^{b_2\varepsilon}$. The only important assumption is that the angle, if any, must be of order strictly greater than ε.

The arguments presented in the proof of Lemma 3.1 also give the following estimate which shows that a large deviations principle for the occupation time $\int_0^t\eta_s(0)\,ds$ follows by a contraction argument from a large deviations principle for the polar empirical measure μ^T, which is introduced just after the statement.

COROLLARY 3.2. *For every $t > 0$ and $\delta > 0$,*

$$\limsup_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{\log T}{T}\log\mathbb{P}_\alpha\left[\left|\int_0^t ds\left\{\eta_s(0)-\frac{1}{Z_{T,\varepsilon}}\sum_{y\in B_*(T^\varepsilon)}\frac{\eta_s(y)}{|y|^2}\right\}\right|>\delta\right]$$
$$=-\infty,$$

where $Z_{T,\varepsilon}$ is a normalizing constant

$$Z_{T,\varepsilon}=\sum_{y\in B_*(T^\varepsilon)}\frac{1}{|y|^2}$$

and $B_*(T^\varepsilon)$ is the ball $B(T^\varepsilon)=\{x\in\mathbb{Z}^2,\ |x|\le T^\varepsilon\}$ without the origin.

Lemma 3.1 shows that a large deviations principle for the occupation time $\int_0^t \eta_s(0)\,ds$ follows by a contraction argument from a large deviations principle for the macroscopic average $Z_{T,\varepsilon}^{-1}\int_0^t\sum_{y\in B_*(T^\varepsilon)}\eta_s(y)/|y|^2\,ds$. This observation leads us to introduce the polar empirical measure associated to a configuration η of $\{0,1\}^{\mathbb{Z}^2}$.

Let $\mathbb{Z}_*^2=\mathbb{Z}^2-\{0\}$ and for each $T>1$ define the projection $\sigma_T:\mathbb{Z}^2\to[0,\infty)$ by

$$\sigma_T(x)=\frac{\log|x|}{\log T},\qquad \sigma_T(0)=0.$$

For $c>0$, denote by $\mathcal{M}_c$ the set of positive Radon measures μ on $\mathbb{R}_+$ such that $\mu([a,b])\le(b-a)+c$ for every $0\le a\le b<\infty$:

$$\mathcal{M}_c=\{\mu(dr),\ \mu([a,b])\le(b-a)+c \text{ for } 0\le a\le b<\infty\}.$$

The condition on the measure of intervals makes the set $\mathcal{M}_c$, which is endowed with the vague topology, a compact separable metric space. Let $\mathcal{M}_0$ be the subspace of $\mathcal{M}$ of all measures which are absolutely continuous with respect to the Lebesgue measure and whose density is bounded by 1. The subspace $\mathcal{M}_0$ is closed (thus compact) and a sequence μ_n in $\mathcal{M}_0$ converges to μ if and only if

$$\lim_{n\to\infty}\int H(r)\mu_n(dr)=\int H(r)\mu(dr)$$

holds for all continuous functions H with compact support in $(0,\infty)$.

Denote by $\mathfrak{P}:\mathbb{R}_*^2\to(0,\infty)\times\mathbb{T}_\pi$ the polar coordinates map $\mathfrak{P}(u)=(|u|,\Theta(u))$. For $T>0$ and a configuration η, let $\mu^T=\mu^T(\eta)$ be the empirical measure on $\mathbb{R}_+\times\mathbb{T}_\pi$ associated to $\eta\circ\mathfrak{P}^{-1}$,

$$\mu^T(\eta)=\frac{1}{\log T}\sum_{x\in\mathbb{Z}_*^2}\eta(x)\frac{1}{|x|^2}\delta_{\sigma_T(x),\Theta(x)},\tag{3.7}$$

where δ_v is the Dirac measure concentrated on v. Let $\mu^{1,T}$ be the projection of μ^T on the first coordinate. Thus, $\mu^{1,T}$ is the measure on $\mathbb{R}_+$ given by

$$\mu^{1,T}(\eta) = \frac{1}{2\pi \log T} \sum_{x \in \mathbb{Z}^2_*} \eta(x) \frac{1}{|x|^2} \delta_{\sigma_T(x)}.$$

Notice the factor 2π on the denominator to normalize the sum. For a Radon measure μ on $\mathbb{R}_+ \times \mathbb{T}_\pi$ (resp. $\mathbb{R}_+$) and a continuous bounded function $H : \mathbb{R}_+ \times \mathbb{T}_\pi \to \mathbb{R}$ (resp. $H : \mathbb{R}_+ \to \mathbb{R}$), denote by $\langle\langle H, \mu \rangle\rangle$ the integral of H with respect to μ:

$$\langle\langle H, \mu \rangle\rangle = \int H(r, \theta)\mu(dr, d\theta), \qquad \left(\text{resp. } \langle\langle H, \mu \rangle\rangle = \int H(r)\mu(dr)\right).$$

In particular, for $H : \mathbb{R}_+ \to \mathbb{R}$ that depends only on the radius r,

$$\begin{aligned} \langle\langle H, \mu^T(\eta) \rangle\rangle &= \frac{1}{\log T} \sum_{x \in \mathbb{Z}^2_*} H(\sigma_T(x)) \frac{1}{|x|^2} \eta(x) \\ &= 2\pi \langle\langle H, \mu^{1,T}(\eta) \rangle\rangle. \end{aligned} \tag{3.8}$$

The strategy of the proof of Theorem 2.1 can now be explained. To prove the large deviations principle for the occupation time, we first prove a large deviations principle for the time average of the empirical measure $\mu^{1,T}$ and then apply a contraction principle to recover, from this result, the large deviations principle for the occupation time.

Let $\bar{\mu}^T$ be the Radon measure on $\mathbb{R}_+$ defined by

$$\bar{\mu}^T = \int_0^1 \mu^{1,T}(\eta_s)\, ds. \tag{3.9}$$

Overestimating $\eta(x)$ by 1, it is not difficult to show that the random measures $\bar{\mu}^T$ belong to $\mathcal{M}_c$ for T large enough. More precisely, there exists a finite universal constant C_0 such that

$$\bar{\mu}^T([a, b]) \le (b - a) + \frac{C_0}{\log T}$$

for all $0 \le a \le b < \infty$, $T > 1$. In particular, for each $c > 0$, there exists a finite $T(c)$ such that $\bar{\mu}^T$ belongs to $\mathcal{M}_c$ for all $T > T(c)$. The same statement remains in force for $\mu^{1,T}$ in place of $\bar{\mu}^T$. This property of the random measures $\bar{\mu}^T$ explains the introduction of the spaces $\mathcal{M}_c$.

From now on we fix some $c > 0$ and keep it until the end of the article.

We prove in Sections 5 and 6 a large deviations principle for $\bar{\mu}^T$. To state the result requires some notation. For any α in $(0, 1)$, let $C^2(\mathbb{R}_+, \alpha)$ be the space of twice continuously differentiable functions $\gamma : [0, \infty) \to (0, 1)$ such that γ' has a compact support in $(0, \frac{1}{2})$ and such that $\gamma(r) = \alpha$ for $r \ge 1/2$. There exists, therefore, $0 < \beta < 1$ and $0 < \varepsilon < 1/4$ such that $\gamma(r) = \beta$ for $r \le \varepsilon$ and $\gamma(r) = \alpha$ for $r \ge \frac{1}{2} - \varepsilon$.

For each γ in $C^2(\mathbb{R}_+, \alpha)$, $\Gamma = \Gamma_{\gamma,\alpha}$,

$$\Gamma(u) = \frac{1}{2} \log \frac{\gamma(u)[1-\alpha]}{[1-\gamma(u)]\alpha}. \tag{3.10}$$

Notice that Γ is a twice continuously differentiable function with compact support in $[0, 1/2)$ and whose derivative has compact support on $(0, 1/2)$. Denote by $\Sigma(\mathbb{R}_+)$ the space of functions $\{\Gamma'_{\gamma,\alpha}, \gamma \in C^2(\mathbb{R}_+, \alpha)\}$. Notice that $\Sigma(\mathbb{R}_+)$ is a vector space [which is not the case of $C^2(\mathbb{R}_+, \alpha)$ or $\{\Gamma_{\gamma,\alpha}, \gamma \in C^2(\mathbb{R}_+, \alpha)\}$] and that every function h in $\Sigma(\mathbb{R}_+)$ is continuously differentiable and has compact support in $(0, 1/2)$. Moreover, given h in $\Sigma(\mathbb{R}_+)$ there exists one and only one γ in $C^2(\mathbb{R}_+, \alpha)$ such that $h = \Gamma'_{\gamma,\alpha}$. This means that the map from $C^2(\mathbb{R}_+, \alpha)$ to $\Sigma(\mathbb{R}_+)$ is one to one.

For $0 < \alpha < 1$, let $F(a) = a(1-a)$ and $I_\alpha : \mathcal{M}_c \to \mathbb{R}_+$ be given by

$$I_\alpha(\mu) = \pi \sup_{h \in \Sigma(\mathbb{R}_+)} \{-\langle\langle h', m \rangle\rangle - \langle\langle h^2, F(m) \rangle\rangle\}$$

if $\mu(dr) = m(r)\,dr$ is absolutely continuous with respect to the Lebesgue measure and has a density such that $m(r) = \alpha$ for $r \ge 1/2$. In all other cases, $I_\alpha(\mu) = \infty$. We remark that Lemma 5.1, which states that in the large deviations regime the measure $\bar{\mu}^T(dr)$ is fixed on $[1/2, \infty)$ and equal to $\alpha\,dr$, justifies the concentration of I_α on such measures. We prove in Section 6 that the rate function I_α is lower semicontinuous, convex and such that

$$I_\alpha(\mu) = \frac{\pi}{4} \int_{\mathbb{R}_+} \frac{m'(r)^2}{m(r)[1-m(r)]}\,dr.$$

THEOREM 3.3. *For every closed subset F of $\mathcal{M}_c$ and every open subset G of $\mathcal{M}_c$,*

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T \in F] \le -\inf_{\mu \in F} I_\alpha(\mu),$$
$$\liminf_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T \in G] \ge -\inf_{\mu \in G} I_\alpha(\mu).$$

4. Superexponential estimates. The proof of the large deviations principle for the polar empirical measure stated in Theorem 3.3 relies on a superexponential estimate, similar to Lemma 3.1, which permits us to replace the average of local functions by functions of the polar empirical density. To state this result, we need some notation.

For $r_0 > 0$, θ_0 in $\mathbb{T}_\pi$, $0 < \varepsilon < r_0$ and a configuration η, let

$$\iota_+ = \iota_+(\varepsilon, r_0, T) = \frac{1}{\log T} \log \frac{T^{r_0} + T^{\varepsilon}}{T^{r_0}},$$
$$\iota_- = \iota_-(\varepsilon, r_0, T) = \frac{1}{\log T} \log \frac{T^{r_0}}{T^{r_0} - T^{\varepsilon}}$$

and denote by $\Psi_{\varepsilon,T}^{r_0,\theta_0} : \mathbb{R}_+ \times \mathbb{T}_\pi \to \mathbb{R}_+$ the function defined by

$$\Psi_{\varepsilon,T}^{r_0,\theta_0}(r,\theta) = \frac{1}{2(\iota_+ + \iota_-)q(\varepsilon)} \mathbf{1}\{r_0 - \iota_- \le r \le r_0 + \iota_+\}\mathbf{1}\{|\theta - \theta_0| \le q(\varepsilon)\}.$$

Denote, furthermore, by $M_{T,\varepsilon}^{r,\theta}(\eta)$ the average number of particles in the polar cube $[r - \iota_-, r + \iota_+] \times [\theta - q(\varepsilon), \theta + q(\varepsilon)]$:

$$\begin{aligned} M_{T,\varepsilon}^{r,\theta}(\eta) &= \langle\langle \Psi_{\varepsilon,T}^{r,\theta}, \mu^T \rangle\rangle \\ &= \frac{1}{2(\iota_+ + \iota_-)q(\varepsilon)\log T} \sum_{\substack{T^r - T^\varepsilon \le |z| \le T^r + T^\varepsilon \\ \theta - q(\varepsilon) \le \Theta(z) \le \theta + q(\varepsilon)}} \frac{\eta(z)}{|z|^2}. \end{aligned}$$

In the previous formula, the sum is carried over all sites z in $\mathbb{Z}_*^2$ satisfying $T^r - T^\varepsilon \le |z| \le T^r + T^\varepsilon$ and $\theta - q(\varepsilon) \le \Theta(z) \le \theta + q(\varepsilon)$. Notice that $(\iota_+ + \iota_-) \log T = \log\{T^r + T^\varepsilon / T^r - T^\varepsilon\}$, so that the previous sum is an average up to smaller order terms.

Denote by $\{e_1, e_2\}$ the canonical basis of $\mathbb{R}^2$. Fix a continuous function $H : \mathbb{R}_+ \to \mathbb{R}$ with compact support in $(0, 1/2)$, $1 \le j \le 2$, $\varepsilon > 0$, and let

$$\begin{aligned} W_{T,\pm e_j}^{H,\varepsilon}(\eta) = \frac{1}{\log T} \sum_{x \in \mathbb{Z}_*^2} H(\sigma_T(x)) \frac{1}{|x|^2} \\ \times \left\{ \eta(x)\eta(x \pm e_j) \frac{(x \cdot e_j)^2}{|x|^2} - \left(M_{T,\varepsilon}^{\sigma_T(x),\Theta(x)}(\eta)\right)^2 \right\}. \end{aligned}$$

LEMMA 4.1. *For any* $\delta > 0$, $j = 1, 2$ *and* $t > 0$,

$$\limsup_{\varepsilon \to 0} \limsup_{T \to \infty} \frac{\log T}{T} \log \mathbb{P}_\alpha \left[\left| \int_0^t ds\, W_{T,\pm e_j}^{H,\varepsilon}(\eta_s) \right| > \delta \right] = -\infty.$$

PROOF. The same proof of Lemma 3.1 gives that for any $0 < a < 1/4$,

$$\begin{aligned} \limsup_{\varepsilon \to 0} \limsup_{T \to \infty} \sup_{T^a \le |x| \le T^{1/2 - a}} \frac{\log T}{T} \\ \times \log \mathbb{P}_\alpha \left[\left| \int_0^t ds \left\{ \eta_s(x) - M_{T,\varepsilon}^{\sigma_T(x),\Theta(x)}(\eta_s) \right\} \right| > \delta \right] \\ = -\infty. \end{aligned}$$

Introducing intermediary terms, as in Lemma 3.2 in [11], we may deduce the statement of the lemma from this result. □

REMARK 4.2. The term $M_{T,\varepsilon}^{\sigma_T(x),\Theta(x)}(\eta)$ is not an average over a macroscopic polar cube. This lemma is thus not replacing local functions by empirical density over macroscopic regions and must be interpreted as a superexponential one-block estimate. However, we will see that almost no technical tools are needed in the proof of the lower bound and that a convexity argument for simple exclusion processes permits us to go from microscopic boxes to macroscopic boxes in the proof of the upper bound.

The second ingredient in the proof of a large deviations principle is the description of the relevant perturbations of the original dynamics which create the fluctuations. To introduce these large deviations dynamics, recall the definition of the space $C^2(\mathbb{R}_+,\alpha)$ and of the logarithmic rescaling $\sigma_T(\cdot)$ introduced in the previous section.

For $T > 0$ and γ in $C^2(\mathbb{R}_+,\alpha)$, denote by $L_{T,\gamma}$ the generator of the inhomogeneous exclusion process in which a particle jumps from x to y at rate $\exp\{\Gamma(\sigma_T(y)) - \Gamma(\sigma_T(x))\}$, $\Gamma = \Gamma_{\gamma,\alpha}$:

$$(L_{T,\gamma} f)(\eta) = \frac{T}{2} \sum_{|x-y|=1} \eta(x)\{1-\eta(y)\} e^{\Gamma(\sigma_T(y))-\Gamma(\sigma_T(x))}[f(\sigma^{x,y}\eta) - f(\eta)].$$

Denote by $\nu_{T,\gamma}$ the product measure on $\mathcal{X}$, with marginals given by

$$\nu_{T,\gamma}\{\eta, \eta(x) = 1\} = \gamma(\sigma_T(x)).$$

Notice that $\nu_{T,\gamma}$ coincides with ν_α outside a ball of radius $T^{1/2-\varepsilon}$ centered at the origin, for some $\varepsilon > 0$. A simple computation shows that $\nu_{T,\gamma}$ is an invariant reversible measure for the Markov process with generator $L_{T,\gamma}$. Denote by $\mathbb{P}_{T,\gamma}$ the probability on the path space $D(\mathbb{R}_+, \mathcal{X})$ that corresponds to the stationary Markov process with generator $L_{T,\gamma}$ starting from $\nu_{T,\gamma}$.

The superexponential estimate stated in Lemma 4.1 holds if we replace the exclusion dynamics by the dynamics induced by the generator $L_{T,\gamma}$ and replace the initial state by $\nu_{T,\gamma}$ for some γ in $C^2(\mathbb{R}_+,\alpha)$.

COROLLARY 4.3. *Fix γ in $C^2(\mathbb{R}_+,\alpha)$. The superexponential estimate holds for $\mathbb{P}_{T,\gamma}$.*

PROOF. In view of Lemma 4.1, to prove this corollary, it is enough to show that the Radon–Nikodym derivative $d\mathbb{P}_{T,\gamma}/d\mathbb{P}_\alpha$ restricted to $\mathcal{F}_t$, the σ-algebra generated by $\{\eta_s, 0 \le s \le t\}$, is bounded in L^∞ by $\exp\{C_0(T/\log T)\}$ for some finite constant C_0. The derivative $d\mathbb{P}_{T,\gamma}/d\mathbb{P}_\alpha|_{\mathcal{F}_t}$ can be decomposed into a stationary and a dynamical part. An elementary computation shows that the stationary part is equal to

$$\frac{d\nu_{T,\gamma}}{d\nu_\alpha} = \exp \sum_{x\in\mathbb{Z}_*^2} \left\{ \eta_0(x) \log\left(\frac{\gamma(\sigma_T(x))}{\alpha}\right) + [1-\eta_0(x)] \log\left(\frac{1-\gamma(\sigma_T(x))}{1-\alpha}\right) \right\},$$

while the dynamical piece is equal to

$$\begin{aligned} &\exp\left\{ \sum_{x\in\mathbb{Z}_*^2} \Gamma_T(x)\{\eta_t(x) - \eta_0(x)\} \right\} \\ &\qquad \times \exp\left\{ -(T/2)\int_0^t ds \sum_{|x-y|=1} \eta_s(x)[1-\eta_s(y)]\{e^{\Gamma_T(y)-\Gamma_T(x)} - 1\} \right\}, \end{aligned}$$

where $\Gamma_T(x) = \Gamma(\sigma_T(x))$.

By definition of γ and σ_T, $\gamma(\sigma_T(x)) = \alpha$ provided $|x| > T^{(1/2)-\lambda}$ for some $\lambda > 0$. In particular, since ν_α is a product measure,

$$\left| \frac{d\nu_{T,\gamma}}{d\nu_\alpha} \right| \le C(\gamma)\, T^{1-2\lambda}$$

for some finite constant which depends only on the profile γ. On the other hand, since $\Gamma_T(x)$ vanishes for $|x| > T^{1/2-\lambda}$,

$$\left| \sum_{x\in\mathbb{Z}_*^2} \Gamma_T(x)\{\eta_T(x) - \eta_0(x)\} \right| \le C(\gamma) T^{1-2\lambda}$$

for some finite constant $C(\gamma)$ depending only on γ. It remains to estimate the time integral expression in the Radon–Nikodym derivative. The argument relies on the fact that $|\sigma_T(x) - \sigma_T(y)| \le C\{[1+|x|]\log T\}^{-1}$ for some finite constant C if $|y-x| = 1$ so that $|\Gamma_T(x) - \Gamma_T(y)| \le C(\gamma)\{[1+|x|]\log T\}^{-1}$ for some finite constant depending only on γ. Expanding the exponential up to the second order, we get that

$$\begin{aligned} &(T/2)\int_0^t ds \sum_{|x-y|=1} \eta_s(x)[1-\eta_s(y)]\{e^{\Gamma_T(y)-\Gamma_T(x)} - 1\} \\ &\qquad = (T/2)\int_0^t ds \sum_{|x-y|=1} \eta_s(x)[1-\eta_s(y)]\{\Gamma_T(y) - \Gamma_T(x)\} + O(T/\log T) \end{aligned}$$

because $\sum_{T^\lambda \le |x| \le T^{1/2-\lambda}} [1+|x|]^{-2} \le C(\gamma) \log T$. The first term on the right-hand side can be rewritten as

$$\begin{aligned} &\frac{T}{2}\int_0^t ds \sum_x \eta_s(x)(\Delta\Gamma_T)(x) \\ &\qquad = \frac{T}{2\log T}\int_0^t ds \sum_{x\in\mathbb{Z}_*^2} \eta_s(x)\Gamma'(\sigma_T(x))\Delta \log|x| + O\left(\frac{T}{\log T}\right), \end{aligned}$$

because $\eta(x)\eta(y)\{\Gamma_T(y) - \Gamma_T(x)\}$ is antisymmetric. In this formula, ΔF stands for the discrete Laplacian defined by

$$(\Delta F)(x) = \sum_{j=1}^{2} \{F(x+e_j) + F(x-e_j) - 2F(x)\}. \tag{4.1}$$

Expanding $\log|x|$ up to the third order and taking advantage of the fact that $\log|x|$ is a harmonic function on $\mathbb{R}^2$, it is easy to show that the previous expression is of order $T/\log T$, which concludes the proof of the corollary. $\square$

We conclude this section with some results on paths that link the origin to sites x of $\mathbb{Z}^2$, leading to Lemma 4.4, which was used in Lemma 3.1. Fix a site $x = (x_1, x_2)$ in $\mathbb{Z}^2$ and recall that a path from the origin to a site x is a sequence of sites $0 = z(0), \dots, z(|x|_1) = x$ such that $|z(i+1) - z(i)| = 1$ for $0 \le i < |x|_1 = |x_1| + |x_2|$. Assume without loss of generality that $0 \le x_2 \le x_1$. We define a path Ξ_x from the origin to x, imposing that each site $z(i) = (z(i)_1, z(i)_2)$ is such that $z(i)_2/z(i)_1 \le x_2/x_1$ and requiring the path to increase its second coordinate whenever possible. These two conditions define a unique path from the origin to x.

This path Ξ_x can be defined as follows. Let $m_0 = 0$ and for each $1 \le j \le x_2$, let m_j be the increasing sequence of positive integers defined by

$$\frac{j}{m_j} \le \frac{x_2}{x_1} < \frac{j}{m_j - 1}. \tag{4.2}$$

Then, for $0 \le i \le x_2$ and $m_i + i \le n \le m_{i+1} + i$, $z(n) = (n-i, i)$.

Notice that for x in

$$\Omega_1 = \{(j,k) \in \mathbb{Z}^2 : 0 \le k \le j\},$$

the path Ξ_x remains in Ω_1. We may, of course, extend the definition of the path to the other seven possible cases.

Recall that for a bond $b = (x, y)$ in $\mathbb{Z}^2_*$, $|b| = |x| \vee |y|$. We claim that

$$\sum_{b \in \Xi_x} \frac{1}{|b|} \le \sqrt{2}\{2 + \log|x|\}. \tag{4.3}$$

Indeed, assume that $0 \le x_2 \le x_1$ and let $\Xi_x = \{z(i),\ 0 \le i \le |x|_1\}$. Since $|z(i)| \le |z(i+1)|$,

$$\sum_{b \in \Xi_x} \frac{1}{|b|} = \sum_{i=1}^{|x|_1} \frac{1}{|z(i)|}.$$

By construction $z(i) = (k, i-k)$ for some $0 \le k \le i$. In particular, $|z(i)| \ge$

$|(i/2, i/2)| = i/\sqrt{2}$. Therefore,

$$\sum_{b \in \Xi_x} \frac{1}{|b|} \le \sqrt{2} \sum_{i=1}^{|x|_1} \frac{1}{i}$$
$$\le \sqrt{2}\{1 + \log |x|_1\} \le \sqrt{2}\bigl\{1 + \log \sqrt{2} + \log |x|\bigr\},$$

which proves (4.3)

LEMMA 4.4. *For $l \ge 1$, let $\Lambda_l = \{-l, \dots, l\}^2$. There exists a finite constant C_0 such that for any bond b,*

$$|\{x \in \Lambda_l : b \in \Xi_x\}| \le C_0 \frac{l^2}{|b|}.$$

PROOF. We first deduce some properties of the sites that belong to a path Ξ_x. Fix x in Ω_1 and assume that z belongs to Ξ_x. By (4.2), $z = (n, i)$ for some $0 \le i \le x_2$ and some $m_i \le n \le m_{i+1}$. In particular,

$$\frac{z_2}{z_1} \le \frac{i}{m_i} \le \frac{x_2}{x_1} \quad \text{and} \quad \frac{z_2 + 1}{z_1 - 1} \ge \frac{i + 1}{m_{i+1} - 1} > \frac{x_2}{x_1}.$$

Therefore, all sites z in the path Ξ_x are such that $z_2/z_1 \le x_2/x_1 < (z_2 + 1)/(z_1 - 1)$.

Fix now a bond $b = (z, y)$ and assume, without loss of generality, that $0 \le z_2 \le z_1$ and that $y = z + e_1$, where $\{e_1, e_2\}$ stands for the canonical basis of $\mathbb{R}^2$. Fix a site x in Λ_l whose path uses the bond b. By the previous remark, $z_2/z_1 \le x_2/x_1 < (z_2 + 1)/(z_1 - 1)$. It remains to count the number of sites x in Λ_l which satisfy these restrictions and such that $x_1 \ge z_1$ to conclude the proof of the lemma. $\square$

5. Large deviations upper bound. We prove in this section the upper bound of the large deviations principle for $\bar{\mu}^T$. We start with a useful elementary observation. Let $H : \mathbb{R}_+ \times \mathbb{T}_\pi \to \mathbb{R}$ be a continuous function with compact support in $(0, \infty) \times \mathbb{T}_\pi$. Then

$$\frac{1}{\log T} \sum_{x \in \mathbb{Z}_*^2} H\bigl(\sigma_T(x), \Theta(x)\bigr) \frac{1}{|x|^2} = \int_{\mathbb{R}_+} dr \int_{\mathbb{T}_\pi} d\theta \, H(r, \theta) + o_T(1), \tag{5.1}$$

where $o_T(1)$ is a constant which depends on H and T and which vanishes as $T \uparrow \infty$. In the case where the function H is of class $C^1(\mathbb{R}_+ \times \mathbb{T}_\pi)$, it can be shown that the remainder $o_T(1)$ is absolutely bounded by $C(H)/\log T$.

The proof of the upper bound relies on the following result which states that in the large deviations regime the measure $\bar{\mu}^T(dr)$ is fixed on $[1/2, \infty)$ and equal to $\alpha \, dr$.

LEMMA 5.1. *For r in $\mathbb{R}_+$ and $\varepsilon > 0$, let $\Phi_{r,\varepsilon}(r') = \varepsilon^{-1}\mathbf{1}\{[r, r+\varepsilon]\}(r')$. For every $r \ge 1/2$, $\varepsilon > 0$ and $\delta > 0$,*

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\big[|\langle\langle \Phi_{r,\varepsilon}, \bar{\mu}^T\rangle\rangle - \alpha| > \delta\big] = -\infty.$$

PROOF. Fix $r \ge 1/2$, $\varepsilon > 0$ and $\delta > 0$. In view of (5.1), it is enough to show that

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\left[\left|\frac{1}{2\pi}\int_0^1 ds\, \frac{1}{\log T} \sum_{x\in\mathbb{Z}_*^2} \Phi_{r,\varepsilon}(\sigma_T(x)) \frac{\eta_s(x) - \alpha}{|x|^2}\right| > \frac{\delta}{2}\right]$$
$$= -\infty.$$

It is enough to prove this statement without the absolute value. Multiplying both sides of the inequality by a, by an exponential Chebyshev inequality, to prove the theorem it is enough to show that

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{E}_\alpha\left[\exp\left\{\frac{a}{2\pi}\int_0^1 ds\, \frac{T}{(\log T)^2} \sum_{x\in\mathbb{Z}_*^2} \Phi_{r,\varepsilon}(\sigma_T(x)) \frac{\eta_s(x) - \alpha}{|x|^2}\right\}\right]$$
$$\le 0$$

for every $a > 0$. Fix $T > 0$ and consider the previous expression. By Jensen's inequality and since ν_α is a product stationary state, it is bounded above by

$$\frac{\log T}{T} \sum_{x\in\mathbb{Z}_*^2} \log E_\alpha\left[\exp\left\{\frac{a}{2\pi} \frac{T}{(\log T)^2} \Phi_{r,\varepsilon}(\sigma_T(x)) \frac{\eta(x) - \alpha}{|x|^2}\right\}\right].$$

Since $\Phi_{r,\varepsilon}(r') = 0$ if $r' \le 1/2$, the previous sum is carried over sites x such that $\sigma_T(x) \ge 1/2$, that is, over sites x such that $|x|^2 \ge T$. In particular, the expression multiplying $\eta(x) - \alpha$ is less than or equal to $Ca/(\log T)^2$, which vanishes as $T \uparrow \infty$. Since $\exp\{b\} \le 1 + b + b^2 \exp\{|b|\}$, $E_\alpha[\eta(x)] = \alpha$ and $\log(1+b) \le b$, expanding the exponential up to the second order, we obtain that the previous expression is bounded by

$$\frac{T}{(\log T)^3} \frac{a^2 F(\alpha)}{4\pi^2} \sum_{x\in\mathbb{Z}_*^2} \Phi_{r,\varepsilon}(\sigma_T(x))^2 \frac{1}{|x|^4}.$$

By (5.1) and since $|x|^2 \ge T$, the previous sum is less than or equal to

$$\frac{1}{(\log T)^2} \frac{a^2 F(\alpha)}{2\pi} \left(\frac{1}{\varepsilon} + o_T(1)\right).$$

This proves the lemma. $\square$

The same argument shows that

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\left[\left|\langle\langle H, \bar\mu^T\rangle\rangle - \alpha\int_{\mathbb{R}_+} H(r)\,dr\right| > \delta\right] = -\infty \tag{5.2}$$

for every $\delta > 0$ and every continuous function $H:\mathbb{R}_+ \to \mathbb{R}$ with compact support in $[1/2, \infty)$.

We are now in a position to prove the upper bound. Fix a closed subset C of $\mathcal{M}_c$ and $\gamma \in C^2(\mathbb{R}_+, \alpha)$. The set C is compact because $\mathcal{M}_c$ is itself compact. Let Γ be the function associated to γ by (3.10) and set

$$W_{T,\gamma}(s) = \frac{1}{2\log T} \sum_{x\in\mathbb{Z}^2_*} \sum_{j=1}^{2} \eta_s(x)\eta_s(x+e_j)\Gamma'(\sigma_T(x))^2 \frac{1}{|x|^2} \frac{(x\cdot e_j)^2}{|x|^2}.$$

In this formula, $x \cdot e_j$ stands for the scalar product between x and e_j. Recall the definition of the function $\Psi^{r,\theta}_{\varepsilon,T}$ introduced just before the superexponential estimate stated in Lemma 4.1. Let

$$\tilde W^\varepsilon_{T,\gamma}(s) = \frac{1}{2\log T} \sum_{x\in\mathbb{Z}^2_*} \Gamma'(\sigma_T(x))^2 \frac{1}{|x|^2} \langle\langle \Psi^{\sigma_T(x),\Theta(x)}_{\varepsilon,T}, \mu^T_s\rangle\rangle^2$$

and let $B^{\delta,\varepsilon}_{T,\gamma}$ be the set defined by

$$B^{\delta,\varepsilon}_{T,\gamma} = \left\{\eta : \left|\int_0^1 ds\,\{W_{T,\gamma}(s) - \tilde W^\varepsilon_{T,\gamma}(s)\}\right| \le \delta\right\}$$

for $\delta > 0$. It follows from the superexponential estimate stated in Lemma 4.1 and from the fact that

$$\limsup_{N\to\infty} \frac{1}{N}\log(a_N + b_N) = \max\left\{\limsup_{N\to\infty}\frac{1}{N}\log a_N, \limsup_{N\to\infty}\frac{1}{N}\log b_N\right\}$$

that for any $\delta > 0$ and any subset A of $\mathcal{M}_c$,

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar\mu^T \in A] \\
&\quad = \max\left\{\limsup_{T\to\infty}\frac{\log T}{T}\log \mathbb{P}_\alpha[\bar\mu^T \in A, B^{\delta,\varepsilon}_{T,\gamma}], C_\Gamma(\delta,\varepsilon)\right\},
\end{aligned}$$

where $C_\Gamma(\delta, \varepsilon)$ decreases to $-\infty$ as $\varepsilon \downarrow 0$ for each fixed Γ in $C^2_K(\mathbb{R}_+)$ and $\delta > 0$.

To estimate the right-hand side, observe that

$$\mathbb{P}_\alpha[\bar\mu^T \in A, B^{\delta,\varepsilon}_{T,\gamma}] = \mathbb{E}_{T,\gamma}\left[\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}} \mathbf{1}\{\bar\mu^T \in A, B^{\delta,\varepsilon}_{T,\gamma}\}\right],$$

where the Radon–Nikodym derivative $d\mathbb{P}_\alpha/d\mathbb{P}_{T,\gamma}$ must be understood as restricted to the σ-algebra $\mathcal{F}_1 = \sigma\{\eta_s, 0 \le s \le 1\}$. For $r_0 > 0$, θ_0 in $\mathbb{T}_\pi$ and $\varepsilon > 0$, denote by $\Psi^{r,\theta}_\varepsilon : \mathbb{R}_+ \times \mathbb{T}_\pi \to \mathbb{R}_+$ the function defined by

$$\Psi^{r_0,\theta_0}_\varepsilon(r,\theta) = \frac{1}{4\varepsilon q(\varepsilon)} \mathbf{1}\{|r - r_0| \le \varepsilon\}\mathbf{1}\{|\theta - \theta_0| \le q(\varepsilon)\}.$$

At the end of the proof we show that on the set $B^{\delta,\varepsilon}_{T,\gamma}$, if $0<\varepsilon<\delta$,

$$\begin{aligned}\log\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}} \le \frac{T}{2\log T}\int_0^1 ds\Big\{&\langle\!\langle \Gamma''+[\Gamma']^2,\mu_s^T\rangle\!\rangle\\ &-\int_0^{1/2} dr\int_{\mathbb{T}_\pi} d\theta\,\Gamma'(r)^2\langle\!\langle\Psi_\varepsilon^{r,\theta},\mu_s^T\rangle\!\rangle^2\Big\}\\ &+\frac{2\delta T}{\log T}+o\Big(\frac{T}{\log T}\Big).\end{aligned}\tag{5.3}$$

Since Γ depends on the first coordinate r only, by definition of $\bar\mu^T$ and by Schwarz inequality, the previous time integral is bounded above by

$$\pi\Big\{\langle\!\langle\Gamma''+[\Gamma']^2,\bar\mu^T\rangle\!\rangle-\int_0^{1/2}dr\,\Gamma'(r)^2\langle\!\langle\Psi_\varepsilon^r,\bar\mu^T\rangle\!\rangle^2\Big\},$$

where $\Psi_\varepsilon^r(r_0)=(2\varepsilon)^{-1}\mathbf{1}\{|r-r_0|\le\varepsilon\}$.

Recollecting all previous estimates and minimizing over Γ, $0<\varepsilon<\delta$, we obtain that

$$\begin{aligned}&\limsup_{T\to\infty}\frac{\log T}{T}\log\mathbb{P}_\alpha[\bar\mu^T(\eta)\in A]\\ &\qquad\le\inf_{\gamma\in C^2(\mathbb{R}_+,\alpha),0<\varepsilon<\delta}\max\Big\{\sup_{\mu\in A}\tilde J_{\gamma,\varepsilon,\delta}(\mu),\,C_\Gamma(\delta,\varepsilon)\Big\},\end{aligned}$$

where $\limsup_{\varepsilon\to0}C_\Gamma(\delta,\varepsilon)=-\infty$ for every γ in $C^2(\mathbb{R}_+,\alpha)$ and $\delta>0$. In this formula, $\tilde J_{\gamma,\varepsilon,\delta}(\mu)=\delta-J_{\Gamma'_{\gamma,\alpha},\varepsilon}(\mu)$ and, for any continuously differentiable function $H:\mathbb{R}_+\to\mathbb{R}$ with compact support in $(0,1/2)$, $J_{H,\varepsilon}:\mathcal{M}_c\to\mathbb{R}$ is the functional given by

$$J_{H,\varepsilon}(\mu)=-\pi\Big\{\langle\!\langle H'+H^2,\mu\rangle\!\rangle-\int_0^{1/2}dr\,H(r)^2\langle\!\langle\Psi_\varepsilon^r,\mu\rangle\!\rangle^2\Big\}.\tag{5.4}$$

Remember that Γ is a function of γ and that there is a one-to-one correspondence between $C^2(\mathbb{R}_+,\alpha)$ and $\Sigma(\mathbb{R}_+)$. In particular, in the penultimate formula, the minimization may be performed over $\Sigma(\mathbb{R}_+)$ instead of $C^2(\mathbb{R}_+,\alpha)$. On the other hand, for each fixed γ, ε and δ, $\tilde J_{\gamma,\varepsilon,\delta}$ is a continuous function for the vague topology. Since the set C is compact, by Varadhan's lemma (see [9], Appendix II, Lemma 3.3), we may interchange the supremum with the infimum to obtain

$$\begin{aligned}&\limsup_{T\to\infty}\frac{\log T}{T}\log\mathbb{P}_\alpha[\bar\mu^T(\eta)\in C]\\ &\qquad\le-\inf_{\mu\in C}\sup_{h\in\Sigma(\mathbb{R}_+),0<\varepsilon<\delta}\min\big\{J_{h,\varepsilon}(\mu)-\delta,\,-C_{\Gamma_{\alpha,h}}(\delta,\varepsilon)\big\},\end{aligned}$$

where $\Gamma_{\alpha,h}$ is the space integral of h with the condition that $\Gamma_{\alpha,h}(1/2) = \alpha$. Since $\limsup_{\varepsilon\to 0} C_\Gamma(\delta, \varepsilon) = -\infty$ for every γ in $C^2(\mathbb{R}_+, \alpha)$ and $\delta > 0$, letting first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$, we obtain that

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T(\eta) \in C] \le -\inf_{\mu\in C} I(\mu), \tag{5.5}$$

where

$$I(\mu) = \sup_{h\in\Sigma(\mathbb{R}_+)} \limsup_{\varepsilon\to 0} J_{h,\varepsilon}(\mu).$$

We are almost done. We need to observe, however, that the rate function I does not depend on α and that the proof is thus not yet complete. The dependence on the density α is given by Lemma 5.1, which states that under the large deviations regime the measure $\bar{\mu}^T(dr)$ is equal to $\alpha\, dr$ on the region $[1/2, \infty)$. In other words, it asserts that the rate function of the large deviations principle, denoted by I_α, is identically equal to $+\infty$ on the subset of measures $\mu(dr)$ which are not equal to $\alpha\, dr$ on the set $[1/2, \infty)$.

To turn the previous discussion into a rigorous argument, denote by $C_K([1/2, \infty))$ the space of continuous functions $H : \mathbb{R}_+ \to \mathbb{R}$ with compact support in $[1/2, \infty)$ endowed with the supremum norm. Consider a dense and countable sequence $\{H_k, k \ge 1\}$ in $C_K([1/2, \infty))$. For $\delta > 0$ and $k \ge 1$, denote by $A_{k,\delta}$ the closed subspace of $\mathcal{M}_c$ defined by

$$A_{k,\delta} = \left\{\mu \in \mathcal{M}_c : \left|\langle\langle H_j, \mu\rangle\rangle - \alpha \int H_j(r)\, dr\right| \le \delta \text{ for } 1 \le j \le k \right\}.$$

It follows from (5.2) that

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T(\eta) \in C] = \limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T(\eta) \in C \cap A_{k,\delta}]$$

for all $k \ge 1$, $\delta > 0$. Since $A_{k,\delta}$ is a closed set, by (5.5),

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T(\eta) \in C] \le -\inf_{\mu\in C\cap A_{k,\delta}} I(\mu).$$

The previous bound holds for all $k \ge 1$, $\delta > 0$. As $\delta \downarrow 0$, the sequence of sets $A_{k,\delta}$ decreases to the closed subset A_k of $\mathcal{M}_c$ given by

$$A_k = \left\{\mu \in \mathcal{M}_c, \langle\langle H_j, \mu\rangle\rangle = \alpha \int H_j(r)\, dr \text{ for } 1 \le j \le k \right\}$$

so that

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T(\eta) \in C] \le -\inf_{\mu\in C\cap A_k} I(\mu).$$

As $k \uparrow \infty$, the sequence of sets A_k decreases to the closed subset $\mathcal{M}_{c,\alpha}$ of $\mathcal{M}_c$ given by

$$\mathcal{M}_{c,\alpha} = \left\{\mu \in \mathcal{M}_c, \langle\langle H_j, \mu\rangle\rangle = \alpha \int H_j(r)\, dr, j \geq 1\right\}.$$

In particular,

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T(\eta) \in C] \leq - \inf_{\mu \in C \cap \mathcal{M}_{c,\alpha}} I(\mu).$$

Since $\{H_k, k \geq 1\}$ is a dense subset in $C_K([1/2, \infty))$, $\mathcal{M}_{c,\alpha}$ corresponds to the subset of $\mathcal{M}_c$ of all measures $\mu(dr)$ which are equal to $\alpha\, dr$ on $[1/2, \infty)$. If we now define $I_\alpha : \mathcal{M}_c \to \mathbb{R}_+$ by

$$I_\alpha(\mu) = \begin{cases} I(\mu), & \text{if } \mu \in \mathcal{M}_{c,\alpha}, \\ +\infty, & \text{otherwise,} \end{cases}$$

we have that

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\bar{\mu}^T(\eta) \in C] \leq - \inf_{\mu \in C} I_\alpha(\mu)$$

for every closed subset C of $\mathcal{M}_c$.

To conclude the proof of the upper bound, it remains to check (5.3). To keep the proof short, we derive below an upper bound for $d\mathbb{P}_{T,\gamma}/d\mathbb{P}_\alpha$ taking advantage of the computations presented in the proof of Corollary 4.3. The same arguments give an upper bound for $d\mathbb{P}_\alpha/d\mathbb{P}_{T,\gamma}$. Recall from the proof of the superexponential estimate the expression of the Radon–Nikodym derivative $d\mathbb{P}_{T,\gamma}/d\mathbb{P}_\alpha$. We need to compute precisely the order $(T/\log T)$ term. We have seen there that

$$\begin{aligned} \frac{d\mathbb{P}_{T,\gamma}}{d\mathbb{P}_\alpha} &= \exp\left\{-\frac{1}{2} T \int_0^1 ds \sum_{|x-y|=1} \eta_s(x)[1-\eta_s(y)]\{e^{\Gamma_T(y)-\Gamma_T(x)} - 1\}\right\} \\ &\quad \times \exp\left\{o\left(\frac{T}{\log T}\right)\right\}. \end{aligned}$$

Recall that $\sigma_T(y) - \sigma_T(x) = O(\{|x|\log T\}^{-1})$. Expanding the exponential up to the third order, we obtain that the expression inside the integral is equal to

$$\begin{aligned} &-(1/2)\sum_x \eta_s(x)(\Delta\Gamma_T)(x) - (1/4)\sum_x \eta_s(x) \sum_{|y-x|=1} [\Gamma_T(y) - \Gamma_T(x)]^2 \\ &\quad + (1/4) \sum_{|y-x|=1} \eta_s(x)\eta_s(y)[\Gamma_T(y) - \Gamma_T(x)]^2 + o(1/\log T), \end{aligned} \tag{5.6}$$

where Δ is the discrete Laplacian defined in (4.1). Since $\log|x|$ in a harmonic function in $\mathbb{R}^2$, the first term in this expression is equal to

$$-\frac{1}{2(\log T)^2} \sum_x \eta_s(x)\Gamma''(\sigma_T(x))\frac{1}{|x|^2} + o\left(\frac{1}{\log T}\right).$$

Expanding Γ_T up to the second order, we obtain that the sum of the second and third terms in (5.6) is equal to

$$
\begin{aligned}
&-\frac{1}{2(\log T)^2}\sum_x \eta_s(x)[\Gamma'(\sigma_T(x))]^2\frac{1}{|x|^2}\\
&\quad+\frac{1}{2(\log T)^2}\sum_x\sum_{j=1}^2 \eta_s(x)\eta_s(x+e_j)[\Gamma'(\sigma_T(x))]^2\frac{1}{|x|^2}\frac{(x\cdot e_j)^2}{|x|^2}\\
&\quad+o\biggl(\frac{1}{\log T}\biggr).
\end{aligned}
$$

Recall the definition of the measure μ^T and identity (3.8). With this notation, we may write the Radon–Nikodym derivative $d\mathbb{P}_{T,\gamma}/d\mathbb{P}_\alpha$ as

$$
\begin{aligned}
&\exp\biggl\{-\frac{T}{2\log T}\int_0^1 ds\langle\langle\Gamma''+[\Gamma']^2,\mu_s^T\rangle\rangle+o\biggl(\frac{T}{\log T}\biggr)\\
&\qquad+\frac{T}{2(\log T)^2}\int_0^1 ds\\
&\qquad\times\sum_x\sum_{j=1}^2\eta_s(x)\eta_s(x+e_j)[\Gamma'(\sigma_T(x))]^2\frac{1}{|x|^2}\frac{(x\cdot e_j)^2}{|x|^2}\biggr\}.
\end{aligned}
\tag{5.7}
$$

On the set $B_{T,\gamma}^{\delta,\varepsilon}$ the second term of the previous formula is bounded above by

$$
\frac{T}{2\log T}\int_0^1 ds\int_0^{1/2}dr\int_{\mathbb{T}_\pi}d\theta\,\Gamma'(r)^2\langle\langle\Psi_{\varepsilon,T}^{r,\theta},\mu_s^T\rangle\rangle^2+\frac{\delta T}{\log T}.
$$

Hence, replacing $d\mathbb{P}_{T,\gamma}/d\mathbb{P}_\alpha$ by $d\mathbb{P}_\alpha/d\mathbb{P}_{T,\gamma}$, up to this point we have proved that

$$
\begin{aligned}
\log\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}}\le\frac{T}{2\log T}\int_0^1 ds\biggl\{&\langle\langle\Gamma''+[\Gamma']^2,\mu_s^T\rangle\rangle\\
&-\int_0^{1/2}dr\int_{\mathbb{T}_\pi}d\theta\,\Gamma'(r)^2\langle\langle\Psi_{\varepsilon,T}^{r,\theta},\mu_s^T\rangle\rangle^2\biggr\}\\
+\frac{\delta T}{\log T}+o\biggl(\frac{T}{\log T}\biggr),&
\end{aligned}
$$

which is (5.3) with $\Psi_{\varepsilon,T}^{r,\theta}$ in place of $\Psi_\varepsilon^{r,\theta}$. We do not have $\Psi_\varepsilon^{r,\theta}$ in the formula above because in Lemma 4.1 we were not able to replace local functions by macroscopic functions of the empirical measure. Nevertheless, a convexity argument permits us to substitute $\Psi_{\varepsilon,T}^{r,\theta}$ by $\Psi_\varepsilon^{r,\theta}$. Since Γ is a smooth function, we may replace in the above formula $[\Gamma']^2$ by an average of $[\Gamma']^2$ over an interval of length 2ε, paying the price $\varepsilon T/\log T$. Changing the order of integrals, we may transfer the average to $\langle\langle \Psi^{r,\theta}_{\varepsilon,T}, \mu^T_s \rangle\rangle^2$. Due to the presence of the negative sign, by Jensen inequality, we may introduce the average inside the square. The average of $\langle\langle \Psi^{r,\theta}_{\varepsilon,T}, \mu^T_s \rangle\rangle$ over an r-interval of length 2ε is just $\langle\langle \Psi^{r,\theta}_{\varepsilon}, \mu^T_s \rangle\rangle$ plus a smaller-order term. This concludes the proof of (5.3) and the proof of the upper bound.

REMARK 5.2. The last argument in the proof shows that a superexponential two-blocks estimate is not needed in the proof of the large deviations of the empirical measure from the hydrodynamic limit for symmetric simple exclusion processes considered in [10].

6. Large deviations lower bound. We prove in this section the lower bound of the large deviations principle. It relies on the following law of large numbers.

LEMMA 6.1. *Fix γ in $C^2(\mathbb{R}_+, \alpha)$ and recall the definition of $\Gamma = \Gamma_{\gamma,\alpha}$ given by* (3.10). *As $T \uparrow \infty$, the measure $\bar\mu^T(\eta)$ converges in $\mathbb{P}_{T,\gamma}$ probability to the measure $\gamma(r)\,dr$.*

PROOF. Fix a continuous function $H:[0,1/2] \to \mathbb{R}$ with compact support in $(0,1/2)$. It is enough to show that $\langle\langle H, \bar\mu^T \rangle\rangle$ converges in $\mathbb{P}_{T,\gamma}$ probability to $\int_0^{1/2} H(r)\gamma(r)\,dr$. By definition of $\bar\mu^T$,

$$\begin{aligned}
&\mathbb{E}_{T,\gamma}\left[\left|\langle\langle H, \bar\mu^T \rangle\rangle - \int_0^{1/2} H(r)\gamma(r)\,dr\right|\right] \\
&\qquad = \mathbb{E}_{T,\gamma}\left[\left|\int_0^1 ds\,\frac{1}{2\pi\log T}\sum_{x\in\mathbb{Z}^2_*} H(\sigma_T(x))\frac{\eta_s(x)}{|x|^2} - \int_0^{1/2} H(r)\gamma(r)\,dr\right|\right].
\end{aligned}$$

Since $\nu_{T,\gamma}$ is a stationary state and since by (5.1),

$$\lim_{T\to\infty}\frac{1}{2\pi\log T}\sum_{x\in\mathbb{Z}^2_*} H(\sigma_T(x))\frac{\gamma(\sigma_T(x))}{|x|^2} = \int_0^{1/2} H(r)\gamma(r)\,dr,$$

the previous expression is bounded by

$$E_{\nu_{T,\gamma}}\left[\left|\frac{1}{2\pi\log T}\sum_{x\in\mathbb{Z}^2_*} H(\sigma_T(x))\frac{\eta(x)-\gamma(\sigma_T(x))}{|x|^2}\right|\right] + o_T(1),$$

where $o_T(1)$ is a finite constant depending on H, γ and T which vanishes as $T \uparrow \infty$. By the Schwarz inequality, since $\nu_{T,\gamma}$ is a product measure and $E_{\nu_{T,\gamma}}[\eta(x)] = \gamma(\sigma_T(x))$, the square of the previous expectation is less than or equal to

$$\frac{1}{4\pi^2(\log T)^2}\sum_{x\in\mathbb{Z}^2_*} H(\sigma_T(x))^2 F\big(\gamma(\sigma_T(x))\big)\frac{1}{|x|^4},$$

where $F(a) = a(1-a)$. A computation similar to the one that led to (5.1) shows that the previous expression is equal to

$$\frac{1}{2\pi \log T} \int_0^{1/2} H(r)^2 F(\gamma(r)) \frac{1}{r^2}\, dr + o_T(1),$$

which proves the lemma. $\square$

The lower bound requires an explicit expression for the rate function. Recall from (5.4) and (5.5) the definitions of the functionals $J_{H,\varepsilon}$ and I.

It is not difficult to show that $I_\gamma(\mu) = \infty$ if μ is not absolutely continuous with respect to the Lebesgue measure on the interval $[0, 1/2]$ because $\int_{\mathbb{R}_+} dr\, \langle\langle \Psi_\varepsilon^r, \mu \rangle\rangle^2$ diverges as $\varepsilon \downarrow 0$ if μ is not absolutely continuous. On the other hand, if $\mu(dr) = m(r)\, dr$,

$$I(\mu) = \pi \sup_{h \in \Sigma(\mathbb{R}_+)} \{-\langle\langle h', \mu \rangle\rangle - \langle\langle h^2, F(m) \rangle\rangle\}.$$

By choosing an appropriate sequence of functions h, we may further show that $I(\mu) = \infty$ if the set $\{r \in [0, 1/2], m(r) \notin [0, 1]\}$ has positive Lebesgue measure.

Recall that we denote by $\mathcal{M}_0$ the compact subspace of $\mathcal{M}_c$ of all measures μ which are absolutely continuous with respect to the Lebesgue measure and whose density is positive and bounded above by 1:

$$\mathcal{M}_0 = \{\mu \in \mathcal{M}_c, \mu(dr) = m(r)\, dr, 0 \le m(r) \le 1\}.$$

Recall from the previous section the definitions of the set $\mathcal{M}_{c,\alpha}$ and of the rate function I_α. Let

$$\mathcal{M}_{0,\alpha} = \mathcal{M}_{c,\alpha} \cap \mathcal{M}_0$$

be the set of absolutely continuous measures $\mu(dr) = m(r)\, dr$ whose density m is such that $0 \le m(r) \le 1$ for $0 \le r \le 1/2$, $m(r) = \alpha$ for $r \ge 1/2$. It is easy to check that $\mathcal{M}_0$ is closed for the weak topology. Putting together the previous estimate with the definition of I_α, we obtain that

$$I_\alpha(\mu) = \pi \sup_{h \in \Sigma(\mathbb{R}_+)} \{-\langle\langle h', \mu \rangle\rangle - \langle\langle h^2, F(m) \rangle\rangle\} \tag{6.1}$$

for μ in $\mathcal{M}_{0,\alpha}$ and $I_\alpha(\mu) = \infty$ otherwise.

We claim that the functional $I_\alpha(\cdot)$ is convex and lower semicontinuous. To prove this claim, it is enough to check that for any h in $\Sigma(\mathbb{R}_+)$, the functional J_h defined by

$$J_h(\mu) = \begin{cases} -\langle\langle h'+, \mu \rangle\rangle - \langle\langle h^2, F(m) \rangle\rangle, & \text{if } \mu \in \mathcal{M}_{0,\alpha}, \\ \infty, & \text{otherwise} \end{cases}$$

is convex and lower semicontinuous. It is convex because $\mathcal{M}_{0,\alpha}$ is a convex set and F is a concave function. To show that it is lower semicontinuous, consider a sequence μ_n converging weakly to μ. If μ_n is not in $\mathcal{M}_{0,\alpha}$ for n sufficiently large, there is nothing to prove. Assume, therefore, that there is a subsequence, still denoted by μ_n, in $\mathcal{M}_{0,\alpha}$ converging to μ. Let $\iota_\varepsilon = \varepsilon^{-1}\mathbf{1}\{[0,\varepsilon]\}$ be an approximation of the identity and let $*$ be the convolution in $\mathbb{R}$. By the explicit form of J_h,

$$\begin{aligned} J_h(\mu) &= \lim_{\varepsilon\to 0} J_h(\mu * \iota_\varepsilon) = \lim_{\varepsilon\to 0}\lim_{n\to\infty} J_h(\mu_n * \iota_\varepsilon) \\ &\le \lim_{\varepsilon\to 0}\liminf_{n\to\infty} -\langle\langle h' * \iota_\varepsilon, \mu_n\rangle\rangle - \langle\langle h^2 * \iota_\varepsilon, F(m_n)\rangle\rangle \\ &= \liminf_{n\to\infty} J_h(\mu_n). \end{aligned}$$

In the previous sequence of steps, we used the convexity of $-F$, the Fubini theorem, to transfer the convolution from m to h and the fact that h, h' are continuous functions with compact support so that the convolution converges to the function in $L^1(\mathbb{R}_+)$. We have thus proved that I_α is convex and lower semicontinuous.

We now derive an explicit formula for the rate function I_α. Fix a measure μ in $\mathcal{M}_{0,\alpha}$ such that $I_\alpha(\mu) < \infty$. By the previous observation, $\mu(dr) = m(r)\,dr$ for some density $m(r)$ which is almost surely equal to α on $[1/2,\infty)$. Denote by $\mathcal{H}_1(\mu)$ the Hilbert space induced by $\Sigma(\mathbb{R}_+)$ endowed with the scalar product defined by

$$\langle h, g\rangle_\mu = \int_0^{1/2} dr\, h(r) g(r) F(m(r)).$$

Denote by L_μ the linear functional defined on $\Sigma(\mathbb{R}_+)$ by $L_\mu(h) = -\langle h', m\rangle$. By definition of I_α,

$$\pi \sup_{h\in\Sigma(\mathbb{R}_+)} \{L_\mu(h) - \langle h, h\rangle_\mu\} \le I_\alpha(\mu).$$

Replacing h by ah for a in $\mathbb{R}$ and minimizing over a, we obtain that $\pi^2 L_\mu(h)^2 \le I_\alpha(\mu)\langle h, h\rangle_\mu$. Therefore, by the Riesz representation theorem, there exists g_μ in $\mathcal{H}_1(\mu)$ such that $L_\mu(h) = 2\langle h, g_\mu\rangle_\mu$ for all h in $\mathcal{H}_1(\mu)$. In particular,

$$I_\alpha(\mu) = \pi \sup_{h\in\Sigma(\mathbb{R}_+)} \{2\langle h, g_\mu\rangle_\mu - \langle h, h\rangle_\mu\} = \pi\langle g_\mu, g_\mu\rangle_\mu$$

and for all h in $\Sigma(\mathbb{R}_+)$,

$$\begin{aligned} -\int_{\mathbb{R}_+} h'(r) m(r)\, dr &= L_\mu(h) = 2\langle h, g_\mu\rangle_\mu \\ &= 2\int_{\mathbb{R}_+} h(r) g_\mu(r) F(m(r))\, dr. \end{aligned}$$

It follows from these identities that $m'(r) = 2g_\mu(r)F(m(r))$. From this relation and the penultimate formula we deduce the following explicit formula, for the rate function I_α:

$$I_\alpha(\mu) = \frac{\pi}{4}\int_{\mathbb{R}_+} \frac{m'(r)^2}{F(m(r))}\,dr.$$

We summarize in the next lemma the statements proved above.

LEMMA 6.2. *The rate function I_α defined by* (5.5) *and* (6.1) *is convex and lower semicontinuous. Moreover, if μ in $\mathcal{M}_c$ is such that $I_\alpha(\mu) < \infty$, then $\mu(dr) = m(r)\,dr$ is absolutely continuous with respect to the Lebesgue measure, $0 \le m(r) \le 1$ for $0 \le r \le 1/2$, $m(r) = \alpha$ for $r \ge 1/2$ and*

$$I_\alpha(\mu) = \frac{\pi}{4}\int_{\mathbb{R}_+} \frac{m'(r)^2}{F(m(r))}\,dr.$$

Denote by $\mathcal{M}_*$ the subspace of $\mathcal{M}_{0,\alpha}$ of all measures $\mu(dr) = m(r)\,dr$ whose density m belongs to $C^2(\mathbb{R}_+, \alpha)$ and is bounded away from 0 and 1: $\delta \le m(r) \le 1 - \delta$ for some $\delta > 0$.

LEMMA 6.3. *Fix μ in $\mathcal{M}_c$ such that $I_\alpha(\mu) < \infty$. There exists a sequence $\mu_n(dr) = m_n(r)\,dr$ of measures in $\mathcal{M}_*$ which converges to μ and such that $I_\alpha(\mu_n)$ converges to $I_\alpha(\mu)$.*

PROOF. Fix a measure μ in $\mathcal{M}_c$ such that $I_\alpha(\mu) < \infty$. Since $I_\alpha(\mu)$ is finite, $\mu(dr) = m(r)\,dr$ for some density m which is equal to α on $[1/2, \infty)$ and such that $0 \le m(r) \le 1$. For each $0 < \theta < 1$, let $m_\theta(r) = \theta m(r) + (1-\theta)\alpha$. Of course, $\mu_\theta(dr) = m_\theta(r)\,dr$ converges vaguely to μ as $\theta \uparrow 1$. On the other hand, since I_α is convex and since $I_\alpha(\alpha\,dr) = 0$, $I_\alpha(\mu_\theta) \le \theta I_\alpha(\mu)$ so that $\limsup_{\theta\to 1} I_\alpha(\mu_\theta) \le I_\alpha(\mu)$. In contrast, by lower semicontinuity of I_α, $I_\alpha(\mu) \le \liminf_{\theta\to 1} I_\alpha(\mu_\theta)$. Up to this point, we obtained a sequence $\mu_n(dr) = m_n(r)\,dr$ such that

$$\lim_{n\to\infty} \mu_n = \mu, \qquad \lim_{n\to\infty} I_\alpha(\mu_n) = I_\alpha(\mu), \qquad 0 < \delta_n \le m_n(r) \le 1 - \delta_n,$$

for some positive sequence δ_n. Fix a measure $\mu(dr) = m(r)\,dr$ in $\mathcal{M}_c$ such that $I_\alpha(\mu) < \infty$, $\delta \le m(r) \le 1 - \delta$ for some $\delta > 0$. Since $I_\alpha(\mu) < \infty$, by Lemma 6.2, $\int_{\mathbb{R}_+} [m'(r)]^2/F(m(r))\,dr < \infty$. Since m is bounded away from 0 and 1, $\int_{\mathbb{R}_+} dr\,[m'(r)]^2 < \infty$. Consider a sequence of smooth functions $m_n(r)$ in $C^2(\mathbb{R}_+, \alpha)$ converging to $m(r)$ in H_1:

$$\lim_{n\to\infty} \int_{\mathbb{R}_+} [m_n'(r) - m'(r)]^2\,dr = 0.$$

Since $\delta \le m(r) \le 1-\delta$, we may take m_n satisfying the same bounds and converging a.s. to m. In particular, $\mu_n(dr) = m_n(r)\,dr$ converges vaguely to μ and

$$\lim_{n\to\infty} I_\alpha(\mu_n) = \lim_{n\to\infty} \frac{\pi}{4}\int_{\mathbb{R}_+} \frac{m_n'(r)^2}{m_n(r)[1-m_n(r)]}\,dr$$
$$= \frac{\pi}{4}\int_{\mathbb{R}_+} \frac{m'(r)^2}{m(r)[1-m(r)]}\,dr = I_\alpha(\mu).$$

This proves the lemma. $\square$

We are now in a position to prove the lower bound. Fix an open subset G of $\mathcal{M}_c$. In view of the previous lemma, it is enough to show that

$$\liminf_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_{\alpha,T}[\bar{\mu}^T \in G] \ge -I_\alpha(\mu)$$

for every μ in $\mathcal{M}_* \cap G$. Fix such μ and denote its density by γ, which belongs to $C^2(\mathbb{R}_+,\alpha)$ by assumption. Let $\mathcal{A}_\gamma = \{\bar{\mu}^T \in G\}$ and denote by $\mathbb{P}^A_{T,\gamma}$ the probability $\mathbb{P}_{T,\gamma}$ conditioned on $\mathcal{A}_\gamma$. With this notation we may write

$$\mathbb{P}_\alpha[\bar{\mu}^T \in G] \ge \mathbb{E}_{T,\gamma}\left[\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}}\mathbf{1}\{\mathcal{A}_\gamma\}\right]$$
$$= \mathbb{E}^A_{T,\gamma}\left[\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}}\right]\mathbb{P}_{T,\gamma}[\mathcal{A}_\gamma].$$

By the law of large numbers stated in Lemma 6.1, $\lim_{T\to\infty}\mathbb{P}_{T,\gamma}[\mathcal{A}_\gamma] = 1$. Hence, by Jensen inequality,

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{\log T}{T}\log \mathbb{P}_\alpha[\bar{\mu}^T \in G]\\
&\quad\ge \liminf_{T\to\infty} \frac{\log T}{T}\log \mathbb{E}^A_{T,\gamma}\left[\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}}\right]\\
&\quad\ge \liminf_{T\to\infty} \frac{\log T}{T}\mathbb{E}^A_{T,\gamma}\left[\log\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}}\right]\\
&\quad= \liminf_{T\to\infty} \frac{\log T}{T}\mathbb{E}_{T,\gamma}\left[\log\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}}\mathbf{1}\{\mathcal{A}_\gamma\}\right].
\end{aligned}$$

We showed in the proof of Corollary 4.3 that the Radon–Nikodym derivative $d\mathbb{P}_\alpha/d\mathbb{P}_{T,\gamma}$ is absolutely bounded by $\exp\{C_0 T/\log T\}$. In particular, by Lemma 6.1, the last term is equal to

$$\liminf_{T\to\infty} \frac{\log T}{T}\mathbb{E}_{T,\gamma}\left[\log\frac{d\mathbb{P}_\alpha}{d\mathbb{P}_{T,\gamma}}\right],$$

which is, up to a sign, the entropy of $\mathbb{P}_{T,\gamma}$ with respect to $\mathbb{P}_\alpha$. In view of formula (5.7) for the Radon–Nikodym derivative $d\mathbb{P}_{T,\gamma}/d\mathbb{P}_\alpha$, the previous limit is equal to

$$\begin{aligned}\liminf_{T\to\infty}\mathbb{E}_{T,\gamma}&\left[\frac{1}{2}\int_0^1 ds\langle\langle\Gamma''+[\Gamma']^2,\mu_s^T\rangle\rangle\right]\\ &-\limsup_{T\to\infty}\mathbb{E}_{T,\gamma}\left[\int_0^1 ds\frac{1}{2\log T}\right.\\ &\qquad\left.\times\sum_x\sum_{j=1}^2\eta_s(x)\eta_s(x+e_j)[\Gamma'(\sigma_T(x))]^2\frac{1}{|x|^2}\frac{(x\cdot e_j)^2}{|x|^2}\right].\end{aligned}$$

Since $\nu_{T,\gamma}$ is a stationary state, these expectations are easily computed. Recall (5.1) to show that the limit is equal to

$$\begin{aligned}\pi\int_0^{1/2}dr\,&\{\Gamma''(r)\gamma(r)+[\Gamma'(r)]^2F(\gamma(r))\}\\ &=\frac{\pi}{4}\int_0^{1/2}dr\frac{[\gamma']^2}{\gamma(1-\gamma)}=I_\alpha(\mu)\end{aligned}$$

because $\Gamma=(1/2)\log\gamma/(1-\gamma)+C$ vanishes at the boundary. This proves the lower bound.

7. Large deviations for occupation times. We prove Theorem 2.1 in this section. The idea of the proof is simple. In view of the definition (3.9) of the polar empirical measure $\bar{\mu}^T$, Corollary 3.2 may be restated as follows. For every $\delta>0$,

$$\limsup_{\varepsilon\to0}\limsup_{T\to\infty}\frac{\log T}{T}\log\mathbb{P}_\alpha\left[\left|\int_0^1 ds\,\eta_s(0)-\langle\langle\Phi_\varepsilon,\bar{\mu}^T\rangle\rangle\right|>\delta\right]=-\infty, \tag{7.1}$$

where $\Phi_\varepsilon(r)=\varepsilon^{-1}\mathbf{1}\{[0,\varepsilon]\}$. In particular, a large deviations principle for the occupation time follows from a large deviations principle for the polar empirical measure $\bar{\mu}^T$, which has been proved in the previous two sections.

The proof relies also on the following simple identity. Recall the definition of the rate function Υ_α given in the statement of Theorem 2.1. An elementary computation shows that for every $0\le a<1/2$,

$$\inf_{\substack{m\in C^2(\mathbb{R}_+,\alpha)\\ m(a)=\beta\\ m(1/2)=\alpha}} I_\alpha(\mu)=\frac{1}{1-2a}\Upsilon_\alpha(\beta). \tag{7.2}$$

Proof of the upper bound. Consider a closed subset F of $[0,1]$. Fix $\delta > 0$ and let $F^\delta = \{\beta, d(\beta, F) \le \delta\}$. In view of (7.1),

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\left[\int_0^1 ds\, \eta_s(0) \in F\right]$$
$$\le \limsup_{\varepsilon\to 0} \limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\langle\langle \Phi_\varepsilon, \bar\mu^T\rangle\rangle \in F^\delta].$$

By the large deviations principle stated in Theorem 3.3, the previous expression is bounded above by

$$-\liminf_{\varepsilon\to 0} \inf_{\mu, \langle\langle \Phi_\varepsilon, \mu\rangle\rangle \in F^\delta} I_\alpha(\mu) = -\liminf_{\varepsilon\to 0} \inf_{\beta\in F^\delta} \inf_{\mu, \langle\langle \Phi_\varepsilon, \mu\rangle\rangle = \beta} I_\alpha(\mu).$$

By Lemma 6.3, we may restrict the last infimum to measures $\mu(dr) = m(r)\,dr$ whose density belongs to $C^2(\mathbb{R}_+, \alpha)$. In such a case, $m(\cdot)$ being continuous and such that $\varepsilon^{-1}\int_0^\varepsilon m(r)\,dr = \beta$, there must exist r' in $[0,\varepsilon]$ such that $m(r') = \beta$. In particular,

$$\inf_{\substack{\mu\in C^2(\mathbb{R}_+,\alpha)\\ \langle\langle \Phi_\varepsilon, \mu\rangle\rangle = \beta}} I_\alpha(\mu) \ge \inf_{a\in[0,\varepsilon]} \inf_{\substack{\mu\in C^2(\mathbb{R}_+,\alpha)\\ m(a)=\beta}} I_\alpha(\mu).$$

By (7.2) and taking a density which is constant on the interval $[0,a]$,

$$\inf_{a\in[0,\varepsilon]} \inf_{\substack{\mu\in C^2(\mathbb{R}_+,\alpha)\\ m(a)=\beta}} I_\alpha(\mu) = \inf_{a\in[0,\varepsilon]} \frac{1}{1-2a}\Upsilon_\alpha(\beta) = \Upsilon_\alpha(\beta).$$

Recollecting all previous estimates, we obtain that

$$\limsup_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\left[\int_0^1 ds\, \eta_s(0) \in F\right] \le -\inf_{\beta\in F^\delta} \Upsilon_\alpha(\beta).$$

To conclude the proof of the upper bound, it remains to let $\delta \downarrow 0$ and recall that $\bigcap_{\delta>0} F^\delta = F$ because F is closed.

Proof of the lower bound. Consider an open subset G of $[0,1]$. Fix $\delta > 0$ and let $G_\delta = \{\beta, d(\beta, G^c) > \delta\}$, where G^c stands for the complement of G. By similar arguments to those used to prove the upper bound,

$$\liminf_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\left[\int_0^1 ds\, \eta_s(0) \in G\right]$$
$$\ge \liminf_{\varepsilon\to 0} \liminf_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha[\langle\langle \Phi_\varepsilon, \bar\mu^T\rangle\rangle \in G_\delta].$$

By Theorem 3.3, the previous expression is bounded below by

$$-\limsup_{\varepsilon\to 0} \inf_{\mu, \langle\langle \Phi_\varepsilon, \mu\rangle\rangle \in G_\delta} I_\alpha(\mu) = -\limsup_{\varepsilon\to 0} \inf_{\beta\in G_\delta} \inf_{\mu, \langle\langle \Phi_\varepsilon, \mu\rangle\rangle = \beta} I_\alpha(\mu).$$

Again, we may restrict the last infimum to the set of measures $\mu(dr) = m(r)\,dr$ whose density is in $C^2(\mathbb{R}_+, \alpha)$. The previous infimum is bounded above by the infimum in which we require $m(r) = \beta$ for $0 \le r \le \varepsilon$. By (7.2), the previous limit is thus bounded below by

$$
\begin{aligned}
-\limsup_{\varepsilon\to 0} \inf_{\beta\in G_\delta} \inf_{\substack{\mu, m(r)=\beta\\ 0\le r\le \varepsilon}} I_\alpha(\mu) &= -\limsup_{\varepsilon\to 0} \frac{1}{1-2\varepsilon} \inf_{\beta\in G_\delta} \Upsilon_\alpha(\beta)\\
&= -\inf_{\beta\in G_\delta} \Upsilon_\alpha(\beta).
\end{aligned}
$$

Recollecting the previous estimates, we get that

$$
\liminf_{T\to\infty} \frac{\log T}{T} \log \mathbb{P}_\alpha\left[\int_0^1 ds\, \eta_s(0) \in G\right] \ge -\inf_{\beta\in G_\delta} \Upsilon_\alpha(\beta).
$$

To conclude the proof of the lower bound, it remains to let $\delta \downarrow 0$ because $\bigcup_{\delta>0} G_\delta = G$.

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