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OCCUPATION DENSITIES FOR SPDEs WITH REFLECTION[1]

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We consider the solution (u, η) of the white-noise driven stochastic partial differential equation with reflection on the space interval $[0, 1]$ introduced by Nualart and Pardoux, where η is a reflecting measure on $[0, \infty) \times (0, 1)$ which forces the continuous function u, defined on $[0, \infty) \times [0, 1]$, to remain nonnegative and η has support in the set of zeros of u. First, we prove that at any fixed time $t > 0$, the measure $\eta([0, t] \times d\theta)$ is absolutely continuous w.r.t. the Lebesgue measure $d\theta$ on $(0, 1)$. We characterize the density as a family of additive functionals of u, and we interpret it as a renormalized local time at 0 of $(u(t, \theta))_{t\geq 0}$. Finally, we study the behavior of η at the boundary of $[0, 1]$. The main technical novelty is a projection principle from the Dirichlet space of a Gaussian process, vector-valued solution of a linear SPDE, to the Dirichlet space of the process u.

1. Introduction. We are concerned with the solution (u, η) of the stochastic partial differential equation with reflection of the Nualart–Pardoux type, see [9],

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\partial^2 W}{\partial t\, \partial \theta} + \eta(t, \theta), \\
u(0, \theta) = x(\theta), \qquad u(t, 0) = u(t, 1) = 0, \\
u \geq 0, \qquad d\eta \geq 0, \qquad \displaystyle\int u\, d\eta = 0,
\end{cases}
\tag{1}
$$

where u is a continuous function of $(t, \theta) \in \overline{\mathcal{O}} := [0, +\infty) \times [0, 1]$, forced to remain nonnegative by the positive measure η on $\mathcal{O} := [0, +\infty) \times (0, 1)$, $x : [0, 1] \mapsto [0, \infty)$ and $\{W(t, \theta) : (t, \theta) \in \overline{\mathcal{O}}\}$ is a Brownian sheet. We denote by ν the law of a Bessel bridge $(e_\theta)_{\theta\in[0,1]}$ of dimension 3 between 0 and 0; see [10].

The main aim of this paper is to prove the following properties of the reflecting measure η:

1. For all $t \geq 0$, the measure $\eta([0, t], d\theta)$ is absolutely continuous with respect to the Lebesgue measure $d\theta$ on $(0, 1)$,

$$
\eta([0, t], d\theta) = \eta([0, t], \theta)\, d\theta. \tag{2}
$$

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The process $(\eta([0,t],\theta))_{t\geq 0}$, $\theta \in (0,1)$, is an additive functional of u, increasing only on $\{t : u(t,\theta) = 0\}$, with Revuz measure

$$\frac{1}{2\sqrt{2\pi\theta^3(1-\theta)^3}}\nu\big(dx|x(\theta)=0\big). \tag{3}$$

2. For all $t \geq 0$,

$$\eta([0,t],\theta) = \frac{3}{4}\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^3}\int_0^t \mathbb{1}_{[0,\varepsilon]}(u(s,\theta))\,ds, \tag{4}$$

in probability.

3. There exists a family of additive functionals of u, $(l^a(\cdot,\theta))_{a\in[0,\infty),\theta\in(0,1)}$, such that $l^a(\cdot,\theta)$ increases only on $\{t : u(t,\theta) = a\}$ and such that the following occupation times formula holds for all $F \in B_b(\mathbb{R})$:

$$\int_0^t F(u(s,\theta))\,ds = \int_0^\infty F(a)l^a(t,\theta)\,da, \qquad t\geq 0. \tag{5}$$

4. For all $t \geq 0$,

$$\eta([0,t],\theta) = \frac{1}{4}\lim_{a\downarrow 0}\frac{1}{a^2}l^a(t,\theta) \tag{6}$$

in probability.

5. For all $t \geq 0$ and $a \in (0,1)$,

$$\lim_{\varepsilon\downarrow 0}\sqrt{\varepsilon}\int_0^a\left(1\wedge\frac{\theta}{\varepsilon}\right)\eta([0,t],d\theta) = \sqrt{\frac{2}{\pi}}t \tag{7}$$

and symmetrically,

$$\lim_{\varepsilon\downarrow 0}\sqrt{\varepsilon}\int_a^1\left(1\wedge\frac{1-\theta}{\varepsilon}\right)\eta([0,t],d\theta) = \sqrt{\frac{2}{\pi}}t, \tag{8}$$

in probability.

Recall that if B is a linear Brownian motion and (X, L) is the unique continuous solution of the Skorohod problem

$$dX = dB + dL, \qquad X(0) = x \geq 0, \qquad L(0) = 0,$$

$$X \geq 0, \qquad t \mapsto L(t) \text{ nondecreasing}, \qquad \int_0^\infty X(t)\,dL(t) = 0,$$

then it turns out that $2L$ is the local time of X at 0 and

$$L(t) = \frac{1}{2}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \mathbb{1}_{[0,\varepsilon]}(X(s))\,ds. \tag{9}$$

In the infinite-dimensional equation (1), the reflecting term η is a random measure on space-time. In [12], the following decomposition formula was proved:

$$\eta(ds,d\theta) = \delta_{r(s)}(d\theta)\eta\big(ds,(0,1)\big), \tag{10}$$

where δ_a is the Dirac mass at $a \in (0,1)$ and $r(s) \in (0,1)$, for $\eta(ds,(0,1))$-a.e. s, is the unique $r \in (0,1)$ such that $u(s,r) = 0$. This formula was used in [12] to write equation (1) as the following Skorohod problem in the infinite-dimensional convex set K_0 of continuous nonnegative $x : [0,1] \mapsto [0,\infty)$:

$$du = \frac{1}{2}\frac{\partial^2 u}{\partial \theta^2}\, dt + dW + \frac{1}{2} n(u) \cdot dL,$$

interpreting the set of $x \in K_0$ having a unique zero in $(0,1)$ as the boundary of K_0, the increasing process $t \mapsto L_t := 2\eta([0,t],(0,1))$ as the local time of u at this boundary and the measure $n(u) = \delta_{r(s)}$ as the normal vector field to this boundary at $u(s,\cdot)$.

On the other hand, the absolute-continuity result (2) suggests an interpretation of η as sum of reflecting processes $t \mapsto \eta([0,t],\theta)$, each depending only on $(u(t,\theta))_{t\geq 0}$ and increasing only on $\{t : u(t,\theta) = 0\}$. Therefore, by (2) equation (1) can also be interpreted as the following infinite system of one-dimensional Skorohod problems, parametrized by $\theta \in (0,1)$ and coupled through the interaction given by the second derivative w.r.t. θ:

$$\begin{cases} u(t,\theta) = x(\theta) + \dfrac{1}{2}\displaystyle\int_0^t \frac{\partial^2 u}{\partial \theta^2}(s,\theta)\, ds + \frac{\partial W}{\partial \theta}(t,\theta) + \eta([0,t],\theta), \\ u(t,0) = u(t,1) = 0, \\ u \geq 0,\ \eta(dt,\theta) \geq 0, \qquad \displaystyle\int_0^\infty u(t,\theta)\eta(dt,\theta) = 0 \qquad \forall \theta \in (0,1), \end{cases} \tag{11}$$

see (49). This interpretation is reminiscent of the result of Funaki and Olla in [6], where the equilibrium fluctuations around the hydrodynamic limit of a particle system with reflection on a wall are proved to be governed by the SPDE (1).

By (5), $(u(t,\theta))_{t\geq 0}$ admits for all $a \geq 0$ a local time at a, $(l^a(t,\theta))_{t\geq 0}$. However, by (6), the reflecting term $\eta([0,\cdot],\theta)$ which appears in (11) is not proportional to $l^0(\cdot,\theta)$, which in fact turns out to be identically 0, and is rather a renormalized local time. The necessity of such renormalization is linked with the unusual rescaling of (4). These two properties of η seem to be significant differences w.r.t. the finite-dimensional Skorohod problems.

The formulae (7) and (8) give information about the behavior of η near the boundary of [0, 1]. In particular, (7) and (8) prove that for any $t > 0$ and any initial condition x, the mass of η on $[0,t] \times (0,1)$ is infinite. This solves a problem posed by Nualart and Pardoux in [9]. Notice also that the right-hand sides of (7) and (8) are independent of the initial condition x.

In [12] it was proved that for all $I \subset\subset (0,1)$, the process $t \mapsto \eta([0,t] \times I)$, where η is the reflecting term of (1), is an additive functional of u, with Revuz measure

$$\frac{1}{2}\int_I \frac{1}{\sqrt{2\pi\theta^3(1-\theta)^3}}\, \nu\big(dx \,|\, x(\theta) = 0\big)\, d\theta. \tag{12}$$

At a heuristic level, the information given by the formulae (2), (4) and (6)–(8) are already contained in (12) and in the properties of the invariant measure ν of u: for instance, if the limit in the right-hand side of (4) exists for all $\theta \in (0,1)$, then, by the properties of ν, the Revuz-measure of the limit is (3) and, therefore, (2) holds by (12) and by the injectivity of the Revuz-correspondence.

However, the existence of such limit is not implied by the structure of (12) alone. According to the theory of Dirichlet forms, a sufficient condition for the convergence of a family of additive functionals of a Markov process, as for instance in (4), is the convergence in the Dirichlet space of the corresponding one-potentials; see Chapter 5 of [4]. In our case, this amounts to introduce the potentials

$$U_\varepsilon(x) := \frac{3}{4}\int_0^\infty e^{-t}\frac{1}{\varepsilon^3}\mathbb{E}\big[\mathbb{1}_{[0,\varepsilon]}(u(s,\theta))\big]\,ds,$$

where $x:[0,1]\mapsto[0,\infty)$ is continuous and u is the corresponding solution of (1), and to prove that U_ε has a limit as $\varepsilon\to 0$ with respect to the Dirichlet form in $L^2(\nu)$,

$$\mathcal{E}(\varphi,\psi) := \tfrac{1}{2}\int\langle\nabla\varphi,\nabla\psi\rangle\,d\nu, \qquad \varphi,\psi\in W^{1,2}(\nu),$$

where ∇ and $\langle\cdot,\cdot\rangle$ denote, respectively, the gradient and the canonical scalar product in $H := L^2(0,1)$. Indeed, as proved in [12], u is the diffusion properly associated with $\mathcal{E}$ in $L^2(\nu)$.

However, due to the strong irregularity of the reflecting measure η in (1), a direct computation of the norm of the gradient of U_ε seems to be out of reach. In order to overcome this difficulty, we take advantage of a connection between equation (1) and the following $\mathbb{R}^3$-valued linear SPDE with additive white-noise:

$$\begin{cases} \dfrac{\partial z_3}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 z_3}{\partial\theta^2} + \dfrac{\partial^2\overline{W}_3}{\partial t\,\partial\theta}, \\ z_3(t,0) = z_3(t,1) = 0, \\ z_3(0,\theta) = \overline{x}(\theta), \end{cases} \tag{13}$$

where $\overline{x}\in H^3$ and $\overline{W}_3$ is the $\mathbb{R}^3$-valued Gaussian process whose components are three independent copies of W. The process z_3 is also called the $\mathbb{R}^3$-valued random string (see [5] and [8]) and is the diffusion properly associated with the Dirichlet form in $L^2(\mu_3)$,

$$\Lambda^3(F,G) := \tfrac{1}{2}\int_{H^3}\langle\overline{\nabla}F,\overline{\nabla}G\rangle_{H^3}\,d\mu_3, \qquad F,G\in W^{1,2}(\mu_3),$$

where μ_3 is the law in H^3 of a standard $\mathbb{R}^3$-valued Brownian bridge, and $\overline{\nabla}F : H^3\mapsto H^3$ is the gradient of F in H^3; see [1] and [11]. Then, in [12] it was noticed that the Dirichlet form $\mathcal{E}$ is the image of Λ^3 under the map $\Phi_3 : H^3 \mapsto H$, $\Phi_3(y)(\theta) := |y(\theta)|_{\mathbb{R}^3}$, that is, ν is the image μ_3 under Φ_3 and

$$W^{1,2}(\nu) = \{\varphi \in L^2(\nu) : \varphi \circ \Phi_3 \in W^{1,2}(\mu_3)\},$$
$$\mathcal{E}(\varphi, \psi) = \Lambda^3(\varphi \circ \Phi_3, \psi \circ \Phi_3) \qquad \forall \varphi, \psi \in W^{1,2}(\nu).$$

This connection involves directly the Dirichlet forms $\mathcal{E}$ and Λ^3, but not the corresponding processes. In particular, it does not imply that u is equal in law to $|z_3|$. Nevertheless, in this paper we prove that this connection gives a useful projection principle from $W^{1,2}(\mu_3)$ onto $W^{1,2}(\nu)$ and that, in particular, the convergence in $W^{1,2}(\mu_3)$ of the one-potentials of z_3

$$\overline{U}_\varepsilon(\overline{x}) := \frac{3}{4} \int_0^\infty e^{-t} \frac{1}{\varepsilon^3} \mathbb{E}\big[\mathbb{1}_{[0,\varepsilon]}\big(|z_3(s,\theta)|\big)\big]\, ds,$$

as $\varepsilon \to 0$, implies the convergence of the one-potentials U_ε of u in $W^{1,2}(\nu)$, and therefore, that (4) holds. Also, the formulae (6)–(8) are proved similarly. Therefore, precise and nontrivial information about u can be obtained from the study of the Gaussian process z_3.

We recall that an analogous connection has been proved in [13] to hold between the $\mathbb{R}^d$-valued solution of a linear white-noise driven SPDE, $d \geq 4$, and the solution of a real-valued nonlinear white-noise driven SPDE with a singular drift.

The paper is organized as follows. Section 2 contains the main definitions and the preliminary results on potentials of the random string in dimension 3. In Section 3 the occupation densities and the occupation times formula (5) are obtained for the SPDE with reflection (1). The main results, together with some corollaries, are then proved in Section 4.

2. The three-dimensional random string. We denote by $(g_t(\theta, \theta') : t > 0, \theta, \theta' \in (0,1))$ the fundamental solution of the heat equation with homogeneous Dirichlet boundary condition, that is,

$$\begin{cases} \dfrac{\partial g}{\partial t} = \dfrac{1}{2} \dfrac{\partial^2 g}{\partial \theta^2}, \\ g_t(0, \theta') = g_t(1, \theta') = 0, \\ g_0(\theta, \cdot) = \delta_\theta, \end{cases}$$

where δ_a is the Dirac mass at $a \in (0,1)$. Moreover, we set $H := L^2(0,1)$ with the canonical scalar product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, $K_0 := \{x \in H : x \geq 0\}$,

$$C_0 := C_0(0,1) := \{c : [0,1] \mapsto \mathbb{R} \text{ continuous}, \ c(0) = c(1) = 0\},$$

$$A : D(A) \subset H \mapsto H, \qquad D(A) := W^{2,2} \cap W_0^{1,2}(0,1), \qquad A := \frac{1}{2} \frac{d^2}{d\theta^2},$$

and for all Fréchet differentiable $F : H \mapsto \mathbb{R}$ we denote by $\nabla F : H \mapsto H$ the gradient in H. We set $\mathcal{O} := [0, +\infty) \times (0, 1)$ and $\overline{\mathcal{O}} := [0, +\infty) \times [0, 1]$. We denote by $(e^{tA})_{t \geq 0}$ the semigroup generated by A in H, that is,

$$e^{tA} h(\theta) := \int_0^1 g_t(\theta, \theta') h(\theta') \, d\theta', \qquad h \in H.$$

Let W be a two-parameter Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is, a Gaussian process with zero mean and covariance function

$$\mathbb{E}[W(t, \theta) W(t', \theta')] = (t \wedge t')(\theta \wedge \theta'), \qquad (t, \theta), (t', \theta') \in \overline{\mathcal{O}}.$$

Let $\overline{W}_3 := (\overline{W}_3^i)_{i=1,2,3}$ be a $\mathbb{R}^3$-valued process, whose components are three independent copies of W, defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{F}_t$ the σ-field generated by the random variables $(W(s, \theta) : (s, \theta) \in [0, t] \times [0, 1])$.

We set for $\overline{x} \in H^3 = L^2((0, 1); \mathbb{R}^3)$,

$$\begin{aligned} w_3(t, \theta) &:= \int_0^t \int_0^1 g_{t-s}(\theta, \theta') \overline{W}_3(ds, d\theta'), \\ z_3(t, \theta) &:= e^{tA} \overline{x}(\theta) + w_3(t, \theta), \qquad Z_3(t, \overline{x}) := z_3(t, \cdot). \end{aligned}$$

Then z_3 is the unique solution of the $\mathbb{R}^3$-valued linear SPDE with additive white-noise (13) above, with $\overline{x} \in H^3$. The process z_3 is also called the $\mathbb{R}^3$-valued random string; see [5] and [8]. Recall that the law of $Z_3(t, \overline{x})$ is the Gaussian measure $\mathcal{N}(e^{tA}\overline{x}, Q_t)$ on H^3, with mean $e^{tA}\overline{x}$ and covariance operator $Q_t : H^3 \mapsto H^3$:

$$Q_t \overline{h}(\theta) = \int_0^1 q_t(\theta, \theta') \, \overline{h}(\theta') \, d\theta', \tag{14}$$

for all $t \in [0, \infty]$, $\theta \in (0, 1)$, $\overline{h} \in H^3$, where

$$q_t(\theta, \theta') := \int_0^t g_{2s}(\theta, \theta') \, ds, \qquad t \in [0, \infty], \ \theta, \theta' \in (0, 1). \tag{15}$$

Recall that A has complete orthonormal system $\{\varepsilon_k\}_k$ of eigenvectors in H,

$$\varepsilon_k(\theta) := \sqrt{2} \sin(\pi k \theta), \qquad \theta \in [0, 1], \qquad A e^k = -\frac{(\pi k)^2}{2} \varepsilon_k, \qquad k \in \mathbb{N}. \tag{16}$$

We denote by $(\overline{\beta}(\theta))_{\theta \in [0,1]}$ a three-dimensional standard Brownian bridge and by μ_3, the law of $\overline{\beta}$. Recall that μ_3 is equal to the Gaussian measure $\mathcal{N}(0, Q_\infty)$ on H^3, $Q_\infty = (-2A)^{-1}$, and

$$q_\infty(\theta, \theta') = \theta \wedge \theta' - \theta\theta'. \tag{17}$$

We set also for all $t \in [0, \infty)$, $\theta, \theta' \in (0, 1)$,

$$q^t(\theta, \theta') := \int_t^\infty g_{2s}(\theta, \theta') \, ds = q_\infty(\theta, \theta') - q_t(\theta, \theta'). \tag{18}$$

Recall that Z_3 is the diffusion associated with the Dirichlet form in $L^2(\mu_3)$,

$$\Lambda^3(F,G) := \frac{1}{2}\int_{H^3} \langle \overline{\nabla} F, \overline{\nabla} G\rangle_{H^3}\, d\mu_3, \qquad F, G \in W^{1,2}(\mu_3),$$

where $\overline{\nabla} F : H^3 \mapsto H^3$ is the gradient of F in H^3; see [1] and [11]. For all $f : H^3 \mapsto \mathbb{R}$ bounded and Borel and for all $\overline{x} \in H^3$, we set

$$P_3(t) f(\overline{x}) := \mathbb{E}\big[f\big(Z_3(t,\overline{x})\big)\big], \qquad t \geq 0,$$
$$R_3(1) f(\overline{x}) := \int_0^\infty e^{-t} P_3(t) f(\overline{x})\, dt.$$

Since $Z_3(t,\overline{x}) \in (C_0)^3$ a.s. for all $t > 0$ and all $\overline{x} \in H^3$, then $R_3(1) f$ is unambiguously defined also for $f : (C_0)^3 \mapsto \mathbb{R}$ bounded and Borel.

The main result of this section is the following:

PROPOSITION 1.

1. *For all* $\theta \in (0,1)$, $a \in \mathbb{R}^3$, *the function* $U_3^{\theta,a} : H^3 \mapsto \mathbb{R}$,

$$U_3^{\theta,a}(\overline{x}) := \int_0^\infty e^{-t} \frac{1}{(2\pi q_t(\theta,\theta))^{3/2}} \exp\left(-\frac{|e^{tA}\overline{x}(\theta) - a|^2}{2q_t(\theta,\theta)}\right) dt, \tag{19}$$

is well defined and belongs to $C_b(H^3) \cap W^{1,2}(\mu_3)$. *If* $(a_n, \theta_n) \to (a, \theta) \in \mathbb{R}^3 \times (0,1)$, *then*

$$\lim_{n\to\infty} \int_{H^3} \big[|U_3^{\theta_n,a_n} - U_3^{\theta,a}|^2 + \|\overline{\nabla} U_3^{\theta_n,a_n} - \overline{\nabla} U_3^{\theta,a}\|^2\big]\, d\mu_3 = 0. \tag{20}$$

Moreover, $(\theta^{3/2}(1-\theta)^{3/2} U_3^{\theta,a})_{\theta\in(0,1),\, a\in\mathbb{R}^3}$ *is uniformly bounded, that is,*

$$\sup_{\theta\in(0,1),\, a\in\mathbb{R}^3} \theta^{3/2}(1-\theta)^{3/2} \sup_{\overline{x}\in H^3} U_3^{\theta,a}(\overline{x}) < \infty. \tag{21}$$

2. *Set* $\overline{\gamma}^\theta(\overline{x}) := |\overline{x}(\theta)|/\sqrt{\theta}$, $\overline{x} \in (C_0)^3$. *Then* $\Gamma_3^\theta := R_3(1)\overline{\gamma}^\theta$ *converges to* $\sqrt{8/\pi}$ *in* $W^{1,2}(\mu_3)$ *as* $\theta \to 0$ *or* $\theta \to 1$.

PROOF. Let $\omega_3 := 4\pi/3$, and recall that ω_3 is the Lebesgue measure of $\{\alpha \in \mathbb{R}^3 : |\alpha| \leq 1\}$. If $\lambda \in \mathbb{R}$, we denote by $\lambda \cdot I$ the linear application $\mathbb{R}^3 \ni \alpha \mapsto \lambda \cdot \alpha \in \mathbb{R}^3$.

Step 1. Let $\overline{x} \in H^3$ be fixed. Notice that $z_3(t,\theta)$ has law $\mathcal{N}(e^{tA}\overline{x}(\theta), q_t(\theta, \theta) \cdot I)$, where $q_t(\theta,\theta)$ is defined as in (15). We denote by $(G_t(a,b) : t, a, b > 0)$ the fundamental solution of the heat equation on $(0,+\infty)$ with homogeneous Dirichlet boundary condition at $\{0\}$. By the reflection principle we have the explicit representation

$$G_t(a,b) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(a-b)^2}{2t}\right)\left(1 - \exp\left(-\frac{2ab}{t}\right)\right).$$

We set $\tau_\theta := \inf\{t > 0 : \theta + B_t \in \{0, 1\}\}$, $\theta \in (0, 1)$. Then we have

$$g_t(\theta, \theta') = G_t(\theta, \theta') - \mathbb{E}\big[\mathbb{1}_{(t>\tau_{\theta'}, \theta'+B_{\tau_{\theta'}}=1)} G_{t-\tau_{\theta'}}(\theta, 1)\big].$$

Let $c_0 := 1 - \exp(-1) \in (0, 1)$. Then, for all $t > 0$ and $a \geq 0$,

$$\frac{c_0}{\sqrt{2\pi t}}\Big(1 \wedge \frac{2a^2}{t}\Big) \leq G_t(a, a) \leq \frac{1}{\sqrt{2\pi t}}\Big(1 \wedge \frac{2a^2}{t}\Big).$$

For all $\theta \in [0, 1]$ we have $\mathbb{P}(\theta + B_{\tau_\theta} = 1) = \theta$, since

$$\begin{aligned}\theta &= \mathbb{E}\big[\theta + B_{\tau_\theta}\big] \\ &= 0 \cdot \mathbb{P}\big(\theta + B_{\tau_\theta} = 0\big) + 1 \cdot \mathbb{P}\big(\theta + B_{\tau_\theta} = 1\big) \\ &= \mathbb{P}\big(\theta + B_{\tau_\theta} = 1\big).\end{aligned}$$

Let now $\theta \in [0, 1/2]$. Then

$$\begin{aligned}&\int_0^t \mathbb{E}\big[\mathbb{1}_{(2s>\tau_\theta, \theta+B_{\tau_\theta}=1)} G_{2s-\tau_\theta}(\theta, 1)\big]\, ds \\ &\quad = \mathbb{E}\bigg[\int_0^{(2t-\tau_\theta)^+} \frac{\mathbb{1}_{(\theta+B_{\tau_\theta}=1)}}{2\sqrt{2\pi r}} \exp\Big(-\frac{(\theta-1)^2}{2r}\Big)\Big(1 - \exp\Big(-\frac{2\theta}{r}\Big)\Big)\, dr\bigg] \\ &\quad \leq \mathbb{E}\big[\mathbb{1}_{(\theta+B_{\tau_\theta}=1)}\big] \int_0^{2t} \frac{1}{2\sqrt{2\pi r}} \exp\Big(-\frac{(\theta-1)^2}{2r}\Big)\Big(1 - \exp\Big(-\frac{2\theta}{r}\Big)\Big)\, dr \\ &\quad \leq \frac{c_1}{\sqrt{\pi}} t\theta^2, \qquad c_1 := \sup_{r>0} \sqrt{\frac{2}{r^3}} \exp\Big(-\frac{1}{8r}\Big) < \infty.\end{aligned}$$

For all $t > 0$ and $\theta \in [0, 1/2]$ we obtain

$$\begin{aligned}q_t(\theta, \theta) &\geq \int_0^t G_{2s}(\theta, \theta)\, ds - \frac{c_1}{\sqrt{\pi}} t\theta^2 \\ &\geq \int_0^t \frac{c_0}{2\sqrt{\pi s}}\Big(1 \wedge \frac{\theta^2}{s}\Big)\, ds - \frac{c_1}{\sqrt{\pi}} t\theta^2 \\ &= \frac{c_0}{\sqrt{\pi}}\Big(\mathbb{1}_{(t\leq\theta^2)}\sqrt{t} + \mathbb{1}_{(\theta^2\leq t)}\Big(2\theta - \frac{\theta^2}{\sqrt{t}}\Big)\Big) - t\frac{c_1}{\sqrt{\pi}}\theta^2 \\ &\geq \frac{1}{\sqrt{\pi}}\big(c_0 \mathbb{1}_{(t\leq\theta^2)}\sqrt{t} + c_0 \mathbb{1}_{(\theta^2\leq t)}\theta - c_1 t\theta^2\big).\end{aligned}$$

Let $t_0 := (c_0/2c_1) \wedge (c_0/2c_1)^2$. If $t \geq t_0$, then $q_t(\theta, \theta) \geq q_{t_0}(\theta, \theta)$. If $t \leq t_0$, then

$$\begin{aligned}\frac{\theta}{q_t(\theta, \theta)} &\leq \sqrt{\pi}\Big(\frac{\theta}{\sqrt{t}(c_0 - c_1\sqrt{t}\theta^2)} \mathbb{1}_{(t\leq\theta^2)} + \frac{\theta}{\theta(c_0 - c_1 t\theta)} \mathbb{1}_{(\theta^2\leq t)}\Big) \\ &\leq \frac{2\sqrt{\pi}}{c_0}\Big(\frac{1}{\sqrt{t}} + 1\Big).\end{aligned}$$

By symmetry, we obtain that there exists $C_0 > 0$ such that for all $\theta \in (0, 1)$,

$$\left(\frac{\theta(1-\theta)}{q_t(\theta,\theta)}\right)^{3/2} \le C_0\left(\frac{1}{t^{3/4}} \wedge 1\right), \qquad t > 0. \tag{22}$$

Step 2. Fix $\theta \in (0, 1)$. By (22), $U_3^{\theta,a}$ is well defined and in $C_b(H^3)$. Moreover, for all $\overline{x} \in H^3$,

$$\begin{aligned}
&\theta^{3/2}(1-\theta)^{3/2}U_3^{\theta,a}(\overline{x})\\
&\quad = \int_0^\infty e^{-t}\left(\frac{\theta(1-\theta)}{2\pi q_t(\theta,\theta)}\right)^{3/2} \exp\left(-\frac{|e^{tA}\overline{x}(\theta)-a|^2}{2q_t(\theta,\theta)}\right)dt\\
&\quad \le \frac{C_0}{(2\pi)^{3/2}}\int_0^\infty e^{-t}\left(\frac{1}{t^{3/4}} \wedge 1\right)dt < \infty,
\end{aligned}$$

so that (21) is proved. For all $\varepsilon > 0$, $\theta \in (0, 1)$ and $a \in \mathbb{R}^3$, we set

$$f_{\theta,a}^{\varepsilon}(\overline{y}) := \frac{1}{\omega_3\varepsilon^3}\mathbb{1}_{(|\overline{y}(\theta)-a|\le\varepsilon)}, \qquad \overline{y} \in (C_0)^3.$$

Let $\overline{x} \in H^3$. Then

$$\begin{aligned}
R_3(1)f_{\theta,a}^{\varepsilon}(\overline{x}) &= \int_0^\infty e^{-t}\frac{1}{\omega_3\varepsilon^3}\mathbb{P}\big(|w_3(t,\theta)+e^{tA}\overline{x}(\theta)-a| \le \varepsilon\big)\,dt\\
&= \int_0^\infty e^{-t}\frac{1}{\omega_3\varepsilon^3}\int_{\mathbb{R}^3}\mathbb{1}_{(|\alpha|\le\varepsilon)}\mathcal{N}\big(e^{tA}\overline{x}(\theta)-a, q_t(\theta,\theta)\cdot I\big)(d\alpha)\,dt\\
&= \int_0^\infty dt\, e^{-t}\frac{1}{\omega_3\varepsilon^3}\\
&\qquad \times \int_{(|\alpha|\le\varepsilon)}\frac{1}{(2\pi q_t(\theta,\theta))^{3/2}}\exp\left(-\frac{|\alpha-e^{tA}\overline{x}(\theta)+a|^2}{2q_t(\theta,\theta)}\right)d\alpha \qquad (23)\\
&= \frac{1}{\omega_3\varepsilon^3}\int_{(|\alpha|\le\varepsilon)}\left[\int_0^\infty e^{-t}\frac{1}{(2\pi q_t(\theta,\theta))^{3/2}}\right.\\
&\qquad\qquad \left.\times\exp\left(-\frac{|\alpha-e^{tA}\overline{x}(\theta)+a|^2}{2q_t(\theta,\theta)}\right)dt\right]d\alpha\\
&= \frac{1}{\omega_3\varepsilon^3}\int_{(|\alpha|\le\varepsilon)}U_3^{\theta,a+\alpha}(\overline{x})\,d\alpha.
\end{aligned}$$

By (22) and the dominated convergence theorem, we have that, for all $(\theta, a) \in (0, 1) \times \mathbb{R}^3$,

$$\lim_{\varepsilon\to 0} R_3(1)f_{\theta,a}^{\varepsilon}(\overline{x}) = U_3^{\theta,a}(\overline{x}) \qquad \forall\,\overline{x} \in H^3, \tag{24}$$

uniformly for $\overline{x}$ in bounded sets of H^3, and by (22),

$$|R_3(1)f_{\theta,a}^{\varepsilon}(\overline{x})| \le \int_0^\infty e^{-t}\frac{1}{(2\pi q_t(\theta,\theta))^{3/2}}\,dt < \infty. \tag{25}$$

Step 3. We want to prove now that $U_3^{\theta,a}$ is in $W^{1,2}(\mu_3)$: to this aim we shall prove that $R_3(1)f^{\varepsilon}_{\theta,a}$ converges to $U_3^{\theta,a}$ in $W^{1,2}(\mu_3)$. Recall that $e^{tA}(C_0)\subseteq C_0$ for all $t\geq 0$. We define for all $\overline{x}\in(C_0)^3$, $a\in\mathbb{R}^3\setminus\{\overline{x}(\theta)\}$,

$$\mathcal{U}^{\theta,a}_{\overline{h}}(\overline{x}):=-\int_0^\infty e^{-t}\frac{e^{tA}\overline{h}(\theta)}{(2\pi)^{3/2}(q_t(\theta,\theta)^2}\psi\big((e^{tA}\overline{x}(\theta)-a)/\sqrt{q_t(\theta,\theta)}\big)\,dt \tag{26}$$

$$\text{where } \psi:\mathbb{R}^3\mapsto\mathbb{R}^3,\ \psi(a):=a\exp\biggl(-\frac{|a|^2}{2}\biggr).$$

By standard estimates on the Green function g, we obtain for all $\overline{h}\in H^3$, $t>0$ and $\theta\in(0,1)$,

$$\biggl[\int_0^1|g_t(\theta,\theta')|^2\,d\theta'\biggr]^{1/2}\leq\frac{1\wedge(2\theta/t)}{t^{1/4}},\qquad |e^{tA}\overline{h}(\theta)|\leq\frac{1\wedge(2\theta/t)}{t^{1/4}}\|\overline{h}\|, \tag{27}$$

so that

$$\biggl|\sup_{\|\overline{h}\|=1}\mathcal{U}^{\theta,a}_{\overline{h}}(\overline{x})\biggr|\leq\int_0^\infty\frac{e^{-t}}{(q_t(\theta,\theta))^2t^{1/4}}\big|\psi\big((e^{tA}\overline{x}(\theta)-a)/\sqrt{q_t(\theta,\theta)}\big)\big|\,dt.$$

Since $\overline{\beta}$ has law $\mu_3=\mathcal{N}(0,Q_\infty)$, then $e^{tA}\overline{\beta}$ has law $\mathcal{N}(0,e^{tA}Q_\infty e^{tA})=\mathcal{N}(0,Q_\infty-Q_t)$. Since $|\psi|\leq 1$,

$$\begin{aligned}
&\biggl(\mathbb{E}\biggl[\biggl|\sup_{\|\overline{h}\|=1}\mathcal{U}^{\theta,a}_{\overline{h}}(\overline{\beta})\biggr|^2\biggr]\biggr)^{1/2}\\
&\quad\leq\int_0^\infty\frac{e^{-t}}{(q_t(\theta,\theta))^2t^{1/4}}\biggl(\int_{\mathbb{R}^3}\big|\psi\big(\alpha'/\sqrt{q_t(\theta,\theta)}\big)\big|^2\mathcal{N}\big(a,q^t(\theta,\theta)\cdot I\big)(d\alpha')\biggr)^{1/2}dt\\
&\quad\leq\int_0^\infty\frac{e^{-t}}{(q_t(\theta,\theta))^2t^{1/4}}\biggl\{1\wedge\biggl[\biggl(\frac{q_t(\theta,\theta)}{2\pi q^t(\theta,\theta)}\biggr)^{3/4}\|\psi\|_{L^2(\mathbb{R}^3)}\biggr]\biggr\}\,dt\\
&\quad\leq\frac{1}{(q^1(\theta,\theta))^{3/4}}\int_0^1\frac{1}{(q_t(\theta,\theta))^{5/4}t^{1/4}}\,dt+\frac{1}{(q_1(\theta,\theta))^2}\int_1^\infty e^{-t}\,dt,
\end{aligned}$$

so that by (22), we obtain, for all $\delta\in(0,1/2]$,

$$\sup_{a\in\mathbb{R}^3}\sup_{\theta\in[\delta,1-\delta]}\biggl(\mathbb{E}\biggl[\biggl|\sup_{\|\overline{h}\|=1}\mathcal{U}^{\theta,a}_{\overline{h}}(\overline{\beta})\biggr|^2\biggr]\biggr)^{1/2}<\infty.$$

Therefore, setting for μ_3-a.e. $\overline{x}$,

$$\mathcal{U}^{\theta,a}:=-\int_0^\infty e^{-t}\frac{g_t(\theta,\cdot)}{(2\pi)^{3/2}(q_t(\theta,\theta))^2}\psi\big((e^{tA}\overline{x}(\theta)-a)/\sqrt{q_t(\theta,\theta)}\big)\,dt,$$

we have that $\mathcal{U}^{\theta,a} \in L^2(H^3, \mu_3; H^3)$, and $\langle \mathcal{U}^{\theta,a}, \overline{h}\rangle = \mathcal{U}^{\theta,a}_{\overline{h}}$ in $L^2(\mu_3)$, for all $\overline{h} \in H^3$. Arguing analogously, we have

$$\begin{aligned}
&\big(\mathbb{E}[\|\mathcal{U}^{\theta,a+\alpha}(\overline{\beta}) - \mathcal{U}^{\theta,a}(\overline{\beta})\|^2]\big)^{1/2} \\
&\quad\le \int_0^\infty \frac{e^{-t}}{(q_t(\theta,\theta))^2 t^{1/4}} \\
&\qquad \times \left\{1 \wedge \left[\left(\frac{q_t(\theta,\theta)}{2\pi q^t(\theta,\theta)}\right)^{3/4} \|\psi(\cdot + \alpha/\sqrt{q_t(\theta,\theta)}) - \psi\|_{L^2(\mathbb{R}^3)}\right]\right\} dt
\end{aligned} \tag{28}$$

and by (22), we obtain, for all $\delta \in (0, 1/2]$,

$$\lim_{\alpha\to 0} \sup_{a\in\mathbb{R}^3} \sup_{\theta\in[\delta,1-\delta]} \big(\mathbb{E}[\|\mathcal{U}^{\theta,a+\alpha}(\overline{\beta}) - \mathcal{U}^{\theta,a}(\overline{\beta})\|^2]\big)^{1/2} = 0. \tag{29}$$

Therefore, for all $\theta \in (0, 1)$, we can differentiate under the integral sign in (23) and obtain

$$\overline{\nabla} R_3(1) f^\varepsilon_{\theta,a} = \frac{1}{\omega_3 \varepsilon^3} \int_{(|\alpha|\le\varepsilon)} \mathcal{U}^{\theta,a+\alpha}\, d\alpha \qquad \text{in } L^2(H^3, \mu_3; H^3).$$

Therefore, by (28),

$$\begin{aligned}
&\int_{H^3} \|\overline{\nabla} R_3(1) f^\varepsilon_{\theta,a} - \mathcal{U}^{\theta,a}\|^2 \, d\mu_3 \\
&\qquad \le \frac{1}{\omega_3\varepsilon^3} \int_{(|\alpha|\le\varepsilon)} \int_{H^3} \|\mathcal{U}^{\theta,a+\alpha} - \mathcal{U}^{\theta,a}\|^2 \, d\mu_3 \, d\alpha \to 0
\end{aligned}$$

as $\varepsilon \to 0$, $R_3(1) f^\varepsilon_{\theta,a}$ converges to $U_3^{\theta,a}$ in $L^2(\mu_3)$ and $\overline{\nabla} R_3(1) f^\varepsilon_{\theta,a}$ converges to $\mathcal{U}^{\theta,a}$ in $L^2(H^3, \mu_3; H^3)$ as $\varepsilon \to 0$. Since $W^{1,2}(\mu_3)$ is complete, then $U_3^{\theta,a} \in W^{1,2}(\mu_3)$, $\overline{\nabla} U_3^{\theta,a} = \mathcal{U}^{\theta,a}$ in $L^2(H^3, \mu_3; H^3)$ and $R_3(1) f^\varepsilon_{\theta,a}$ converges to $U_3^{\theta,a}$ in $W^{1,2}(\mu_3)$ as $\varepsilon \to 0$. Moreover, by (29), for all $\delta \in (0, 1/2]$,

$$\lim_{\varepsilon\downarrow 0} \sup_{a\in\mathbb{R}^3} \sup_{\theta\in[\delta,1-\delta]} \int_{H^3} \|\overline{\nabla} R_3(1) f^\varepsilon_{\theta,a} - \overline{\nabla} U_3^{\theta,a}\|^2 \, d\mu_3 = 0. \tag{30}$$

Step 4. We want to prove (20). By the dominated convergence theorem the map $\mathbb{R}^3 \times (0, 1) \ni (a, \theta) \mapsto U_3^{\theta,a} \in L^2(\mu_3)$ is continuous. Notice now that

$$P_3(t) f^\varepsilon_{\theta,a}(\overline{x}) = \int_{H^3} f^\varepsilon_{\theta,a}(\overline{y}) \mathcal{N}(e^{tA}\overline{x}, Q_t)(d\overline{y}),$$

and by standard Gaussian computations we obtain, for all $t > 0$,

$$\overline{\nabla} P_3(t) f^\varepsilon_{\theta,a}(\overline{x}) = \int_{H^3} \frac{1}{\omega_3\varepsilon^3} \mathbb{1}_{(|\overline{y}(\theta)-a|\le\varepsilon)} [Q_t^{-1} e^{tA}(\overline{y} - e^{tA}\overline{x})] \mathcal{N}(e^{tA}\overline{x}, Q_t)(d\overline{y}).$$

Denoting by Tr the trace in H^3, by (16) we have, for all $t > 0$,

$$\begin{aligned}
&\left[\int_{H^3} \|Q_t^{-1} e^{tA}(\overline{y} - e^{tA}\overline{x})\|^2 \mathcal{N}(e^{tA}\overline{x}, Q_t)(d\overline{y})\right]^{1/2} \\
&\qquad = \left[\mathrm{Tr}[e^{tA} Q_t^{-1} e^{tA}]\right]^{1/2} \\
&\qquad = \left[3 \sum_{k=1}^{\infty} \frac{(\pi k)^2 e^{-(\pi k)^2 t}}{1 - e^{-(\pi k)^2 t}}\right]^{1/2} \\
&\qquad \le \sqrt{3}\left[\frac{\pi^2 e^{-\pi^2 t}}{1 - e^{-\pi^2 t}} + \int_1^{\infty} \frac{(\pi k)^2 e^{-(\pi k)^2 t}}{1 - e^{-(\pi k)^2 t}}\, dk\right]^{1/2} \\
&\qquad \le \sqrt{3}\left[\frac{1}{t} + \frac{1}{t^{3/2}} \int_1^{\infty} \frac{(\pi k)^2 e^{-(\pi k)^2}}{1 - e^{-(\pi k)^2}}\, dk\right]^{1/2} \\
&\qquad \le C(t^{-3/4} \wedge 1),
\end{aligned}$$

for some constant $C > 0$. Then we can write

$$\begin{aligned}
&\overline{\nabla} R_3(1) f^{\varepsilon}_{\theta,a}(\overline{x}) \\
&\quad = \int_{(C_0)^3} \frac{1}{\omega_3 \varepsilon^3} \mathbb{1}_{(|\overline{y}(\theta) - a| \le \varepsilon)} \\
&\qquad\qquad \times \left[\int_0^{\infty} dt\, e^{-t} [Q_t^{-1} e^{tA}(\overline{y} - e^{tA}\overline{x})] \mathcal{N}(e^{tA}\overline{x}, Q_t)(d\overline{y})\right]
\end{aligned}$$

and obtain, by the dominated convergence theorem, that the map

$$(0, 1) \times \mathbb{R}^3 \ni (\theta, a) \mapsto \overline{\nabla} R_3(1) f^{\varepsilon}_{\theta,a} \in L^2(H^3, \mu_3; H^3)$$

is continuous. Since the locally uniform limit of continuous functions is continuous, (30) yields (20).

Step 5. We prove now the last assertion. By symmetry, it is enough to consider the case $\theta \to 0$. Recall that $\overline{\gamma}^{\theta}(\overline{x}) = |\overline{x}(\theta)|/\sqrt{\theta}$, $\overline{x} \in (C_0)^3$. Then

$$\begin{aligned}
\Gamma_3^{\theta}(\overline{x}) &:= R_3(1)\overline{\gamma}^{\theta}(\overline{x}) \\
&= \frac{1}{\sqrt{\theta}} \int_0^{\infty} e^{-t} \int_{\mathbb{R}^3} |\alpha| \mathcal{N}\big(e^{tA}\overline{x}(\theta), q_t(\theta, \theta) \cdot I\big)(d\alpha)\, dt \\
&= \int_0^{\infty} e^{-t} \sqrt{\frac{q_t(\theta, \theta)}{\theta}} \int_{\mathbb{R}^3} |\alpha|\, \mathcal{N}\big(e^{tA}\overline{x}(\theta)/\sqrt{q_t(\theta, \theta)}, I\big)(d\alpha)\, dt,
\end{aligned}$$

and

$$\begin{aligned}
\overline{\nabla}\Gamma_3^{\theta}(\overline{x}) = -\int_0^{\infty} e^{-t} \sqrt{\frac{q_t(\theta, \theta)}{\theta}} g_t(\theta, \cdot) \int_{\mathbb{R}^3} &\big(\alpha - e^{tA}\overline{x}(\theta)/\sqrt{q_t(\theta, \theta)}\big)|\alpha| \\
&\times \mathcal{N}\big(e^{tA}\overline{x}(\theta)/\sqrt{q_t(\theta, \theta)}, I\big)(d\alpha)\, dt,
\end{aligned}$$

for all $\overline{x} \in (C_0)^3$. By the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^3} \big(\alpha - e^{tA}\overline{x}(\theta)/\sqrt{q_t(\theta,\theta)}\,\big)|\alpha|\mathcal{N}\big(e^{tA}\overline{x}(\theta)/\sqrt{q_t(\theta,\theta)}, I\big)(d\alpha)\right| \\
&\quad \le \left[\int_{\mathbb{R}^3} |\alpha|^2 \mathcal{N}(0,I)(d\alpha)\right]^{1/2}\left[\int_{\mathbb{R}^3}\left|\alpha + \frac{e^{tA}\overline{x}(\theta)}{\sqrt{q_t(\theta,\theta)}}\right|^2 \mathcal{N}(0,I)(d\alpha)\right]^{1/2} \\
&\quad = \sqrt{3}\sqrt{3 + \frac{|e^{tA}\overline{x}(\theta)|^2}{q_t(\theta,\theta)}}.
\end{aligned}$$

By (27) we obtain

$$\|\overline{\nabla}\Gamma_3^\theta(\overline{x})\| \le \sqrt{3}\int_0^\infty e^{-t}\frac{1}{t^{1/4}}\left(1 \wedge \frac{2\theta}{t}\right)\sqrt{3 + \frac{|e^{tA}\overline{x}(\theta)|^2}{q_t(\theta,\theta)}}\,dt.$$

By the sub-additivity of the square-root, by (22) and since $q^t(\theta,\theta) \le \theta(1-\theta)$,

$$\begin{aligned}
&\big[\mathbb{E}\big(\|\overline{\nabla}\Gamma_3^\theta(\overline{\beta})\|^2\big)\big]^{1/2} \\
&\quad \le \sqrt{3}\int_0^\infty \frac{e^{-t}}{t^{1/4}}\left(1 \wedge \frac{2\theta}{t}\right)\left(\sqrt{3} + \sqrt{\frac{q^t(\theta,\theta)}{q_t(\theta,\theta)}}\,\right)dt \\
&\quad \le \sqrt{3}\int_0^\infty e^{-t}\left(1 \wedge \frac{2\theta}{t}\right)\left(\frac{\sqrt{3}}{t^{1/4}} + \frac{(C_0)^{1/3}}{t^{3/4}}\right)dt \to 0
\end{aligned}$$

as $\theta \to 0$. Since μ_3 is invariant for z_3, we have

$$\mathbb{E}\big(\Gamma_3^\theta(\overline{\beta})\big) = \frac{1}{\sqrt{\theta}}\mathbb{E}\big(|\overline{\beta}(\theta)|\big) =: c_\theta.$$

By the Poincaré inequality for Λ^3, see [2], there exists $C > 0$ such that

$$\mathbb{E}\big(\Gamma_3^\theta(\overline{\beta}) - c_\theta\big)^2 \le \frac{1}{C}\mathbb{E}\big(\|\overline{\nabla}\Gamma_3^\theta(\overline{\beta})\|^2\big) \to 0,$$

as $\theta \to 0$, and since

$$\begin{aligned}
c_\theta &= \frac{4\pi}{\sqrt{\theta}}\int_0^\infty \frac{1}{(2\pi\theta(1-\theta))^{3/2}} r^3 \exp\left(-\frac{r^2}{2\theta(1-\theta)}\right)dr \\
&= \sqrt{1-\theta}\sqrt{\frac{2}{\pi}}\int_0^\infty r^3 \exp\left(-\frac{r^2}{2}\right)dr \\
&= \sqrt{1-\theta}\sqrt{\frac{8}{\pi}} \to \sqrt{\frac{8}{\pi}},
\end{aligned}$$

we obtain that Γ_3^θ converges to $\sqrt{8/\pi}$ in $W^{1,2}(\mu_3)$ as $\theta \to 0$. $\square$

3. Occupation densities. Following [9], we set the following:

DEFINITION 1. A pair (u, η) is said to be a solution of equation (1) with initial value $x \in K_0 \cap C_0$, if:

(i) $\{u(t, \theta) : (t, \theta) \in \overline{\mathcal{O}}\}$ is a continuous and adapted process, that is, $u(t, \theta)$ is $\mathcal{F}_t$-measurable for all $(t, \theta) \in \overline{\mathcal{O}}$, a.s. $u(\cdot, \cdot)$ is continuous on $\overline{\mathcal{O}}$, $u(t, \cdot) \in K_0 \cap C_0$ for all $t \geq 0$, and $u(0, \cdot) = x$.

(ii) $\eta(dt, d\theta)$ is a random positive measure on $\mathcal{O}$ such that $\eta([0, T] \times [\delta, 1 - \delta]) < +\infty$ for all $T, \delta > 0$, and η is adapted, that is, $\eta(B)$ is $\mathcal{F}_t$-measurable for every Borel set $B \subset [0, t] \times (0, 1)$.

(iii) For all $t \geq 0$ and $h \in D(A)$,

$$\begin{aligned} &\langle u(t, \cdot), h\rangle - \langle x, h\rangle - \int_0^t \langle u(s, \cdot), Ah\rangle \, ds \\ &\qquad = -\int_0^1 h'(\theta) W(t, \theta) \, d\theta + \int_0^t \int_0^1 h(\theta)\eta(ds, d\theta) \qquad \text{a.s.} \end{aligned} \tag{31}$$

(iv) $\int_{\mathcal{O}} u \, d\eta = 0$.

In [9], existence and uniqueness solutions of equation (1) were proved.

We denote by $(e(\theta))_{\theta \in [0,1]}$, the 3-Bessel bridge between 0 and 0, see [10], and by ν, the law on K_0 of e. We recall the following result, proved in [12].

THEOREM 1. *Let* $\Phi_3 : H^3 = L^2(0, 1; \mathbb{R}^3) \mapsto K_0$, $\Phi_3(y)(\theta) := |y(\theta)|_{\mathbb{R}^3}$.

1. *The process u is a Strong–Feller Markov process properly associated with the symmetric Dirichlet form $\mathcal{E}$ in $L^2(\nu)$,*

$$\mathcal{E}(\varphi, \psi) := \frac{1}{2} \int_{K_0} \langle \nabla\varphi, \nabla\psi\rangle \, d\nu, \qquad \varphi, \psi \in W^{1,2}(\nu).$$

2. *The Dirichlet form $\mathcal{E}$ is the image of Λ^3 under the map Φ_3, that is, ν is the image of μ_3 under Φ_3 and*

$$\begin{aligned} &W^{1,2}(\nu) = \{\varphi \in L^2(\nu) : \varphi \circ \Phi_3 \in W^{1,2}(\mu_3)\}, \\ &\mathcal{E}(\varphi, \psi) = \Lambda^3(\varphi \circ \Phi_3, \psi \circ \Phi_3) \qquad \forall \varphi, \psi \in W^{1,2}(\nu). \end{aligned} \tag{32}$$

We refer to [4] and [7] for all basic definitions in the theory of Dirichlet forms. Notice that by point 1 in Theorem 1 and by Theorem IV.5.1 in [7], the Dirichlet form $\mathcal{E}$ is quasi-regular. In particular, by the transfer method stated in VI.2 of [7], we can apply several results of [4] in our setting.

We recall the definition of an additive functional of the Markov process u. We denote by $(\mathbb{P}_x : x \in K_0)$ the family the of laws of u on $E := C([0, \infty); K_0)$ and the coordinate process on K_0 by: $X_t : E \mapsto K_0$, $t \geq 0$, $X_t(e) := e(t)$. By a positive continuous additive functional (PCAF) in the strict sense of u, we mean a family of functions $A_t : E \mapsto \mathbb{R}^+$, $t \geq 0$, such that:

(A1) $(A_t)_{t\geq 0}$ is adapted to the minimum admissible filtration $(\mathcal{N}_t)_{t\geq 0}$ of u, see Appendix A.2 in [4].

(A2) There exists a set $\Lambda \in \mathcal{N}_\infty$ such that $\mathbb{P}_x(\Lambda) = 1$ for all $x \in K_0$, $\theta_t(\Lambda) \subseteq \Lambda$ for all $t \geq 0$, and for all $\omega \in \Lambda : t \mapsto A_t(\omega)$ is continuous nondecreasing, $A_0(\omega) = 0$ and for all $t, s \geq 0$,

$$A_{t+s}(\omega) = A_s(\omega) + A_t(\theta_s \omega), \tag{33}$$

where $(\theta_s)_{s\geq 0}$ is the time-translation semigroup on E.

Two PCAFs in the strict sense A^1 and A^2 are said to be equivalent if

$$\mathbb{P}_x(A_t^1 = A_t^2) = 1 \qquad \forall t > 0, \ \forall x \in K_0.$$

If A is a linear combination of PCAFs in the strict sense of u, then the Revuz-measure of A is a Borel signed measure m on K_0 such that

$$\int_{K_0} \varphi \, dm = \int_{K_0} \mathbb{E}_x \left[\int_0^1 \varphi(X_t) \, dA_t \right] \nu(dx) \qquad \forall \varphi \in C_b(K_0).$$

Moreover, $U \in D(\mathcal{E})$ is the one-potential of a PCAF A in the strict sense with Revuz-measure m if

$$\mathcal{E}_1(U, \varphi) = \int_{K_0} \varphi \, dm \qquad \forall \varphi \in D(\mathcal{E}) \cap C_b(K_0),$$

where $\mathcal{E}_1 := \mathcal{E} + (\cdot, \cdot)_{L^2(\nu)}$. We introduce the following notion of convergence of positive continuous additive functionals in the strict sense of X.

DEFINITION 2. Let $(A_n(t))_{t\geq 0}$, $n \in \mathbb{N} \cup \{\infty\}$, be a sequence of PCAF's in the strict sense of u. We say that A_n converges to A_∞, if:

1. For all $\varepsilon > 0$ and for all $x \in K_0 \cap C_0$,

$$\lim_{n\to\infty} A_n(t + \varepsilon) - A_n(\varepsilon) = A_\infty(t + \varepsilon) - A_\infty(\varepsilon), \tag{34}$$

uniformly for t in compact sets of $[0, \infty)$, $\mathbb{P}_x$-almost surely.

2. For $\mathcal{E}$-q.e. $x \in K_0 \cap C_0$,

$$\lim_{n\to\infty} A_n(t) = A_\infty(t), \tag{35}$$

uniformly for t in compact sets of $[0, \infty)$, $\mathbb{P}_x$-almost surely.

LEMMA 1. *Let $(A_n(t))_{t\geq 0}$, $n \in \mathbb{N} \cup \{\infty\}$, be a sequence of PCAFs in the strict sense of X, and let U_n be the one-potential of A_n, $n \in \mathbb{N} \cup \{\infty\}$. If $U_n \to U_\infty$ in $D(\mathcal{E})$, then A_n converges to A_∞ in the sense of Definition* 2.

PROOF. Since $U_n \to U_\infty$ in $D(\mathcal{E})$, by Corollary 5.2.1 in [4], we have point 2 of Definition 2, that is, there exists an $\mathcal{E}$-exceptional set V such that (35) holds for all $x \in K_0 \setminus V$. By the Strong–Feller property of X, $\mathbb{P}_x$-a.s. $X_t \in E \setminus V$, for all $t > 0$ and for all $x \in K_0$, and by the additivity property, (34) holds for all $x \in K_0$. $\square$

REMARK 1. We recall that if $(A, \mathcal{A})$ is a measurable space, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and X_n is a sequence of $\mathcal{A} \otimes \mathcal{F}$-measurable random variables, such that $X_n(a, \cdot)$ converges in probability for every $a \in A$, then there exists a $\mathcal{A} \otimes \mathcal{F}$-measurable random variable X, such that $X(a, \cdot)$ is the limit in probability of $X_n(a, \cdot)$ for every $a \in A$.

We can now state the main result of this section:

THEOREM 2. *Let $\theta \in (0,1)$, $a \geq 0$.*

1. *For all $(\theta, a) \in (0,1) \times [0, \infty)$, there exists a PCAF in the strict sense of u, $(l^a(t,\theta))_{t \geq 0}$, such that $(l^a(\cdot,\theta))_{\theta \in (0,1), a \in [0,\infty)}$ is continuous in the sense of Definition 2 and jointly measurable, such that for all $a \geq 0$,*

$$l^a(t,\theta) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[a,a+\varepsilon]}(u(s,\theta))\,ds, \qquad t \geq 0,$$

in the sense of Definition 2.

2. *The Revuz-measure of $l^a(\cdot,\theta)$ is*

$$\sqrt{\frac{2}{\pi \theta^3 (1-\theta)^3}}\, a^2 \exp\biggl(-\frac{a^2}{2\theta(1-\theta)}\biggr) \nu\bigl(dx \,|\, x(\theta) = a\bigr), \qquad a \geq 0,$$

and, in particular, $l^0(\cdot,\theta) \equiv 0$. Moreover, $l^a(\cdot,\theta)$ increases only on $\{t : u(t,\theta) = a\}$.

3. *The following occupation times formula holds for all $\theta \in (0,1)$,*

$$\int_0^t F(u(s,\theta))\,ds = \int_0^\infty F(a) l^a(t,\theta)\,da, \qquad F \in B_b(\mathbb{R}),\ t \geq 0. \tag{36}$$

For an overview on existence of occupation densities see [3].

We set $\Lambda_1^3 := \Lambda^3 + (\cdot,\cdot)_{L^2(\mu_3)}$, $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_{L^2(\nu)}$. For all $f : H \mapsto \mathbb{R}$ bounded and Borel and for all $x \in K_0 \cap C_0$, we introduce the one-resolvent of u,

$$R(1)f(x) = \int_0^\infty e^{-t} \mathbb{E}_x[f(X_t)]\,dt,$$

where $\mathbb{E}_x$ denotes the expectation w.r.t. the law of the solution u of (1) with initial value x. The next lemma gives the projection principle from the Dirichlet space $W^{1,2}(\mu_3)$, associated with the Gaussian process z_3, to the Dirichlet space $W^{1,2}(\nu)$ of the solution u of the SPDE with reflection (1).

LEMMA 2. *There exists a unique bounded linear operator $\Pi : W^{1,2}(\mu_3) \mapsto W^{1,2}(\nu)$, such that for all $F, G \in W^{1,2}(\mu_3)$ and $f \in W^{1,2}(\nu)$,*

$$\Lambda_1^3(F, f \circ \Phi_3) = \mathcal{E}_1(\Pi F, f), \tag{37}$$

$$\Lambda_1^3\bigl((\Pi F) \circ \Phi_3, G\bigr) = \Lambda_1^3\bigl(F, (\Pi G) \circ \Phi_3\bigr). \tag{38}$$

In particular, we have that for all $\varphi \in L^2(\nu)$ *and* $F \in W^{1,2}(\mu_3)$,

$$R(1)\varphi = \Pi\big(R_3(1)[\varphi \circ \Phi_3]\big), \tag{39}$$

$$\|\Pi F\|_{\mathcal{E}_1} \le \|F\|_{\Lambda_1^3}. \tag{40}$$

Finally, Π *is Markovian*, *that is*, $\Pi 1 = 1$ *and*

$$F \in W^{1,2}(\mu_3), \qquad 0 \le F \le 1 \quad \Longrightarrow \quad 0 \le \Pi F \le 1. \tag{41}$$

PROOF. Let $\mathcal{D} := \{\varphi \circ \Phi_3 : \varphi \in W^{1,2}(\nu)\} \subset W^{1,2}(\mu_3)$. Let $W^{1,2}(\mu_3)$ be endowed with the scalar product Λ_1^3: then, by (32), $\mathcal{D}$ is a closed subspace of $W^{1,2}(\mu_3)$. Therefore, there exists a unique bounded linear projector $\hat{\Pi} : W^{1,2}(\mu_3) \mapsto \mathcal{D}$, symmetric with respect to the scalar product Λ_1^3. For all $F \in W^{1,2}(\mu_3)$ we set $\Pi F := f$, where f is the unique element of $W^{1,2}(\nu)$ such that $f \circ \Phi_3 = \hat{\Pi} F$. Then (37) and (38) are satisfied by construction. Let now $\varphi, \psi \in W^{1,2}(\nu)$. Then by (32),

$$\begin{aligned}
\mathcal{E}_1\big(R(1)\varphi, \psi\big) &= \int_{K_0} \varphi\psi \, d\nu = \int_{H^3} (\varphi \circ \Phi_3)(\psi \circ \Phi_3)\, d\mu_3 \\
&= \Lambda_1^3\big(R_3(1)[\varphi \circ \Phi_3], \psi \circ \Phi_3\big) = \Lambda_1^3\big(\hat{\Pi} R_3(1)[\varphi \circ \Phi_3], \psi \circ \Phi_3\big) \\
&= \mathcal{E}_1\big(\Pi R_3(1)[\varphi \circ \Phi_3], \psi\big),
\end{aligned}$$

which implies (39). Then, since $\hat{\Pi}$ is a symmetric projector,

$$\|\Pi F\|_{\mathcal{E}_1} = \|\hat{\Pi} F\|_{\Lambda_1^3} \le \|F\|_{\Lambda_1^3},$$

so that (40) is proved. Notice now that $1 \in \mathcal{D}$, so that obviously $\Pi 1 = 1$. Moreover, recall that $\hat{\Pi} F$ is characterized by the property

$$\hat{\Pi} F \in \mathcal{D}, \qquad \Lambda_1^3(F - \hat{\Pi} F, G) = 0 \qquad \forall\, G \in \mathcal{D}.$$

Let $F \in W^{1,2}(\mu_3)$ such that $F \ge 0$. Since $\mathcal{E}$ is a Dirichlet form, then $(\hat{\Pi} F)^- := (-\hat{\Pi} F) \vee 0$ still belongs to $\mathcal{D}$, and since Λ_1^3 is a Dirichlet form,

$$0 = \Lambda_1^3\big(F - \hat{\Pi} F, (\hat{\Pi} F)^-\big) = \Lambda_1^3\big(F, (\hat{\Pi} F)^-\big) + \|(\hat{\Pi} F)^-\|_{\Lambda_1^3}^2 \ge \|(\hat{\Pi} F)^-\|_{\Lambda_1^3}^2,$$

so that $\hat{\Pi} F \ge 0$, and (41) follows. $\square$

PROOF OF THEOREM 2. Let $a \ge 0$. For all $\varepsilon > 0$ we set

$$f^\varepsilon(y) := \frac{1}{\varepsilon} \mathbb{1}_{[a, a+\varepsilon]}(y(\theta)), \qquad y \in K_0 \cap C_0.$$

By Lemma 2, we have that

$$\begin{aligned} R(1)f^{\varepsilon} &= \Pi\big(R_3(1)[f^{\varepsilon}\circ\Phi_3]\big) \\ &= \frac{1}{\varepsilon}\int_{(a\le|\alpha|\le a+\varepsilon)} \Pi(U_3^{\theta,a+\alpha})\,d\alpha \\ &= \frac{1}{\varepsilon}\int_a^{a+\varepsilon} r^2\,dr\int_{\mathbb{S}^2}\Pi(U_3^{\theta,r\cdot n})\mathcal{H}^2(dn), \end{aligned}$$

where $\mathcal{H}^2$ is the two-dimensional Hausdorff measure, $U_3^{\theta,a\cdot n}$ is the one-potential in $W^{1,2}(\mu_3)$ defined by (19) and Π is the operator defined in Lemma 2. By (20), the map $r\mapsto U_3^{\theta,r\cdot n}\in W^{1,2}(\mu_3)$ is continuous. Let $U^{\theta,a}\in W^{1,2}(\nu)$ be defined by

$$U^{\theta,a} := a^2\int_{\mathbb{S}^2}\Pi(U_3^{\theta,a\cdot n})\mathcal{H}^2(dn), \qquad a\ge 0.$$

By (40) we have that $R(1)f^{\varepsilon}$ converges to $U^{\theta,a}$ in $W^{1,2}(\nu)$ as $\varepsilon\to 0$. For all $\varepsilon>0$ and $\varphi\in W^{1,2}(\nu)\cap C_b(K_0)$, we have

$$\mathcal{E}_1\big(R(1)f^{\varepsilon},\varphi\big) = \int_{K_0} f^{\varepsilon}\varphi\,d\nu = \frac{1}{\varepsilon}\mathbb{E}[\varphi(e)\mathbb{1}_{[a,a+\varepsilon]}(e(\theta))],$$

where the law of e is ν and $\mathcal{E}_1=\mathcal{E}+(\cdot,\cdot)_{L^2(\nu)}$. Letting $\varepsilon\to 0$, we get

$$\begin{aligned} \mathcal{E}_1(U^{\theta,a},\varphi) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}[\varphi(e)\mathbb{1}_{[a,a+\varepsilon]}(e(\theta))] \\ &= \sqrt{\frac{2}{\pi\theta^3(1-\theta)^3}}\,a^2\exp\biggl(-\frac{a^2}{2\theta(1-\theta)}\biggr)\mathbb{E}[\varphi(e)|e(\theta)=a]. \end{aligned} \tag{42}$$

By Lemma 2, Π is a Markovian operator and by (21) in Proposition 1, the family $(U_3^{\theta,a\cdot n}:n\in\mathbb{S}^2)$ is uniformly bounded in the supremum-norm. Therefore, $U^{\theta,a}$ is bounded, and by (42) $U^{\theta,a}$ is the one-potential of a nonnegative finite measure. By Theorem 5.1.6 in [4], there exists a PCAF $(l^a(t,\theta))_{t\ge 0}$ in the strict sense of u, with one-potential equal to $U^{\theta,a}$ and with Revuz-measure given by (42). Notice now that $R(1)f^{\varepsilon}$ is the one-potential of the following PCAF in the strict sense of u,

$$t\mapsto\frac{1}{\varepsilon}\int_0^t\mathbb{1}_{[a,a+\varepsilon]}(u(s,\theta))\,ds, \qquad t\ge 0.$$

Therefore, points 1 and 2 of Theorem 2 are proved by (20), Lemma 1 and Remark 1. To prove the last assertion of point 2, just notice that the following PCAF of u,

$$t\mapsto\int_0^t|u(s,\theta)-a|l^a(ds,\theta),$$

has Revuz-measure

$$\sqrt{\frac{2}{\pi\theta^3(1-\theta)^3}}a^2\exp\left(-\frac{a^2}{2\theta(1-\theta)}\right)\cdot|x(\theta)-a|\nu\big(dx|x(\theta)=a\big)\equiv 0.$$

To prove point 3 it is enough to notice that the PCAF of u in the left-hand side of (36) has one-potential $R(1)F_\theta$, where $F_\theta(y) := F(y(\theta))$, $y \in K_0 \cap C_0$, and the PCAF in the right-hand side has one-potential,

$$\begin{aligned}\int_0^\infty r^2 F(r)\,dr \int_{\mathbb{S}^2} \Pi(U_3^{\theta,r\cdot n})\mathcal{H}^2(dn)\\ = \Pi\left(\int_{\mathbb{R}^3} F(|\alpha|)U_3^{\theta,\alpha}\,d\alpha\right)\\ = \Pi\big(R_3(1)[F_\theta\circ\Phi_3]\big) = R(1)F_\theta.\end{aligned}$$

Since $R(1)F_\theta$ is bounded, then, arguing like in Theorem 5.1.6 of [4], the two processes in (36) coincide as PCAF's in the strict sense. □

4. The reflecting measure $\boldsymbol{\eta}$. Recall that η is the reflecting measure on $\mathcal{O} = [0,\infty)\times(0,1)$ which appears in equation (1). The main result of this section is the following:

THEOREM 3. *Let $\theta \in (0,1)$.*

1. *For all $\theta \in (0,1)$, there exists a PCAF in the strict sense $(l(t,\theta))_{t\ge 0}$ of u, such that $(l(\cdot,\theta))_{\theta\in(0,1)}$ is continuous in the sense of Definition* 2 *and jointly measurable, and such that*
$$l(t,\theta) = \lim_{\varepsilon\downarrow 0}\frac{3}{\varepsilon^3}\int_0^t \mathbb{1}_{[0,\varepsilon]}(u(s,\theta))\,ds,$$
in the sense of Definition 2.
2. *The PCAF $(l(t,\theta))_{t\ge 0}$ has Revuz-measure*
$$\sqrt{\frac{2}{\pi\theta^3(1-\theta)^3}}\nu\big(dx|x(\theta)=0\big),$$
and increases only on $\{t : u(t,\theta) = 0\}$.
3. *We have*
$$l(t,\theta) = \lim_{a\downarrow 0}\frac{1}{a^2}l^a(t,\theta)$$
in the sense of Definition 2.
4. *For all $t \ge 0$ and $x \in K_0$, $\eta([0,t],d\theta)$ is absolutely continuous w.r.t. the Lebesgue measure $d\theta$ and*

$$\eta([0,t],d\theta) = \tfrac{1}{4}l(t,\theta)\,d\theta. \tag{43}$$

5. *For all* $a \in (0,1)$,

$$\lim_{\varepsilon \downarrow 0} \sqrt{\varepsilon} \int_0^a \left(1 \wedge \frac{\theta}{\varepsilon}\right) \eta([0,t], d\theta) = \sqrt{\frac{2}{\pi}}\, t,$$

$$\lim_{\varepsilon \downarrow 0} \sqrt{\varepsilon} \int_a^1 \left(1 \wedge \frac{1-\theta}{\varepsilon}\right) \eta([0,t], d\theta) = \sqrt{\frac{2}{\pi}}\, t$$

in the sense of Definition 2.

PROOF. For all $\varepsilon > 0$ we set

$$g^\varepsilon(y) := \frac{3}{\varepsilon^3} \mathbb{1}_{[0,\varepsilon]}(y(\theta)), \qquad y \in K_0 \cap C_0.$$

By Lemma 2, we have that

$$\begin{aligned} R(1)g^\varepsilon &= \Pi\big(R_3(1)[g^\varepsilon \circ \Phi_3]\big) \\ &= \frac{3}{\varepsilon^3} \int_{(|\alpha| \le \varepsilon)} \Pi(U_3^{\theta,\alpha})\, d\alpha \\ &= \frac{3}{\varepsilon^3} \int_0^\varepsilon r^2\, dr \int_{\mathbb{S}^2} \Pi(U_3^{\theta, r\cdot n}) \mathcal{H}^2(dn). \end{aligned}$$

By Lemma 2, $R(1)g^\varepsilon$ converges in $W^{1,2}(\nu)$ as $\varepsilon \to 0$ to

$$U^\theta := 4\pi\, \Pi(U_3^{\theta,0}), \qquad a \ge 0.$$

For all $\varepsilon > 0$ and $\varphi \in W^{1,2}(\nu) \cap C_b(K_0)$, we have

$$\mathcal{E}_1\big(R(1)g^\varepsilon, \varphi\big) = \int_{K_0} g^\varepsilon \varphi\, d\nu = \frac{3}{\varepsilon^3} \mathbb{E}\big[\varphi(e) \mathbb{1}_{[0,\varepsilon]}(e(\theta))\big],$$

and letting $\varepsilon \to 0$, we get

$$\begin{aligned} \mathcal{E}_1(U^\theta, \varphi) &= \lim_{\varepsilon \to 0} \frac{3}{\varepsilon^3} \mathbb{E}\big[\varphi(e) \mathbb{1}_{[0,\varepsilon]}(e(\theta))\big] \\ &= \sqrt{\frac{2}{\pi \theta^3 (1-\theta)^3}}\, \mathbb{E}[\varphi(e) | e(\theta) = 0]. \end{aligned} \tag{44}$$

Since Π is Markovian, by (21), U^θ is bounded, and by (44), U^θ is the one-potential of a nonnegative finite measure. By Theorem 5.1.6 in [4], there exists a PCAF $(l(t,\theta))_{t \ge 0}$ in the strict sense of u, with one-potential equal to U^θ and with Revuz-measure given by (44). Since $R(1)g^\varepsilon$ is the one-potential of the following PCAF of u,

$$t \mapsto \frac{3}{\varepsilon^3} \int_0^t \mathbb{1}_{[0,\varepsilon]}(u(s,\theta))\, ds, \qquad t \ge 0,$$

then, points 1 and 2 of Theorem 3 are proved by (20), Lemma 1 and Remark 1. To prove the last assertion of point 2, just notice that the following PCAF of u,

$$t \mapsto \int_0^t u(s,\theta) l(ds,\theta),$$

has Revuz-measure

$$\sqrt{\frac{2}{\pi\theta^3(1-\theta)^3}} \cdot x(\theta)\nu\big(dx|x(\theta)=0\big) \equiv 0.$$

From the proof of Theorem 2, we know that the one-potential of $l^a(\cdot,\theta)$ is

$$U^{\theta,a} = a^2 \int_{\mathbb{S}^2} \Pi(U_3^{\theta,a\cdot n}) \mathcal{H}^2(dn), \qquad a \geq 0.$$

Then $U^{\theta,a}/a^2$ converges as $a \to 0$ to U^θ in $W^{1,2}(\nu)$. Since $U^{\theta,a}/a^2$ is the one-potential of $l^a(\cdot,\theta)/a^2$, by Lemma 1, point 3 of Theorem 3 is proved. Let now $I \subset\subset (0,1)$ be Borel. Notice that the following PCAF in the strict sense of u,

$$t \mapsto \tfrac{1}{4} \int_I l(t,\theta)\, d\theta \tag{45}$$

has Revuz-measure

$$\frac{1}{2} \int_I \frac{1}{\sqrt{2\pi\theta^3(1-\theta)^3}} \nu\big(dx|x(\theta)=0\big)\, d\theta, \tag{46}$$

and one-potential equal to

$$\tfrac{1}{4} U^I := \tfrac{1}{4} \int_I U^\theta \, d\theta.$$

On the other hand, it was proved in Theorem 7 of [12] that the PCAF in the strict sense of u,

$$t \mapsto \eta([0,t] \times I) \tag{47}$$

has Revuz-measure equal to (46). Therefore, by Theorem 5.1.6 in [4], the two PCAFs of u in (45) and (47) coincide, and since U^I is a bounded one-potential then they coincide as PCAFs in the strict sense. Therefore, point 4 is proved. We prove now the last assertion. For all $\varepsilon \in (0, 1/2)$, set $h_\varepsilon : [0,1] \mapsto [0,1]$

$$h_\varepsilon(\theta) := \sqrt{\varepsilon}\bigg(\Big(1 \wedge \frac{\theta}{\varepsilon}\Big) \mathbb{1}_{[0,1/2]}(\theta) + 4\theta(1-\theta)\mathbb{1}_{[1/2,1]}(\theta)\bigg).$$

Then h_ε is concave and continuous on [0, 1], with

$$h_\varepsilon''(d\theta) = -\frac{1}{\sqrt{\varepsilon}} \delta_\varepsilon(d\theta) - \sqrt{\varepsilon} 8 \mathbb{1}_{[1/2,1]}(\theta)\, d\theta,$$

where δ_ε is the Dirac mass at ε. Moreover, $h_\varepsilon(0) = h_\varepsilon(1) = 0$ and $h_\varepsilon \to 0$ uniformly on [0, 1] as $\varepsilon \to 0$. By (31) we have then

$$\lim_{\varepsilon\to 0}\left(\frac{1}{2\sqrt{\varepsilon}}\int_0^t u(s,\varepsilon)\,ds - \sqrt{\varepsilon}\int_0^{1/2}\left(1\wedge\frac{\theta}{\varepsilon}\right)\eta([0,t],\,d\theta)\right) = 0. \tag{48}$$

Recall the definition of $\overline{\gamma}^\theta$ given in point 2 of Proposition 1. We set $\gamma^\varepsilon : K_0 \cap C_0 \mapsto \mathbb{R}$, $\gamma^\varepsilon(x) := x(\varepsilon)/\sqrt{\varepsilon}$. Then, by Lemma 2 we have that $R(1)\gamma^\varepsilon = \Pi(R_3(1)\overline{\gamma}^\varepsilon)$. By point 2 of Proposition 1 and by Lemma 2, we obtain that $R(1)\gamma^\varepsilon$ converges to $\sqrt{8/\pi}$ in $W^{1,2}(\nu)$. Therefore, by Lemma 1,

$$\lim_{\varepsilon\to 0}\frac{1}{2\sqrt{\varepsilon}}\int_0^t u(s,\varepsilon)\,ds = \sqrt{\frac{2}{\pi}}t,$$

in the sense of Definition 2, and by (48) point 5 is proved. □

COROLLARY 1. *For all $x \in K_0 \cap C_0$, a.s. the set*

$$S := \{s > 0 : \exists\theta \in (0,1), u(s,\theta) = 0\}$$

is dense in $\mathbb{R}^+$ and has zero Lebesgue measure.

PROOF. By point 5 in Theorem 3, for all $x \in K_0 \cap C_0$, a.s. for all $t > 0$ we have $\eta([0,t] \times (0,1)) = +\infty$, so that, in particular, $\eta([0,t] \times (0,1)) > 0$. By (iv) in Definition 1 the support of η is contained in the set $\{u = 0\}$, so that for all $t > 0$, there exists $s \in (0,t) \cap S$. By the Markov property, for all $q \in \mathbb{Q}$ and all $t > q$, there exists $s \in (q,t) \cap S$, which implies the density of S in $\mathbb{R}^+$. To prove that S has zero Lebesgue measure, recall that the law of $u(t,\cdot)$ is absolutely continuous w.r.t. ν for all $t > 0$, and $\nu(x : \exists\theta \in (0,1), x(\theta) = 0) = 0$. Then, if $\mathcal{H}^1$ is the Lebesgue measure on $\mathbb{R}$,

$$\mathbb{E}_x[\mathcal{H}^1(S)] = \int_0^\infty \mathbb{E}_x[\mathbb{1}_S(t)]\,dt = \int_0^\infty \mathbb{P}\big(\exists\theta \in (0,1), u(t,\theta) = 0\big)\,dt = 0. \qquad \square$$

Notice now that, by points 2 and 4 of Theorem 3, equation (1) can be formally written in the following form:

$$\begin{cases} u(t,\theta) = x(\theta) + \dfrac{1}{2}\displaystyle\int_0^t \frac{\partial^2 u}{\partial\theta^2}(s,\theta)\,ds + \frac{\partial W}{\partial\theta}(t,\theta) + \frac{1}{4}l(t,\theta), \\ u(t,0) = u(t,1) = 0, \\ u \geq 0,\ l(dt,\theta) \geq 0, \qquad \displaystyle\int_0^\infty u(t,\theta)l(dt,\theta) = 0 \qquad \forall\theta \in (0,1), \end{cases} \tag{49}$$

where, as usual, the first line is rigorously defined after taking the scalar product in H between each term and any $h \in D(A)$. Formula (49) allows to interpret $(u(\cdot,\theta), l(\cdot,\theta))_{\theta\in(0,1)}$ as the solution of a system of one-dimensional Skorohod problems, parametrized by $\theta \in (0,1)$. This fact is reminiscent of the result of Funaki and Olla who proved in [6] that the stationary solution of a certain system of one-dimensional Skorohod problems converges under a suitable rescaling to the stationary solution of (1).

Finally, we show that u satisfies a closed formula and that equation (1) is related to a fully nonlinear equation. Let $(w(t,\theta))_{t\geq 0,\theta\in[0,1]}$ be the stochastic convolution

$$w(t,\theta) := \int_0^t \int_0^1 g_{t-s}(\theta,\theta')W(ds,d\theta'),$$

solution of

$$\begin{cases} w(t,\theta) = \dfrac{1}{2}\displaystyle\int_0^t \frac{\partial^2 w}{\partial\theta^2}(s,\theta)\,ds + \frac{\partial W}{\partial\theta}(t,\theta) \\ w(t,0) = w(t,1) = 0. \end{cases} \tag{50}$$

Subtracting the first line of (49) and the first line of (50), we obtain that

$$(t,\theta) \mapsto \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\int_0^t \big(u(s,\theta) - w(s,\theta)\big)\,ds$$

is in $L^1_{\text{loc}}((0,1);\ C([0,T]))$ for all $T > 0$, that is, it admits a measurable version which is continuous in t for all $\theta \in (0,1)$ and such that the sup-norm in $t \in [0,T]$ is locally integrabile in θ. Then, we can write

$$\begin{cases} u(t,\theta) = x(\theta) + w(t,\theta) + \dfrac{1}{2}\dfrac{\partial^2}{\partial\theta^2}\displaystyle\int_0^t (u-w)(s,\theta)\,ds + \frac{1}{4}l(t,\theta), \\ u(t,0) = u(t,1) = 0, \\ u \geq 0,\ l(dt,\theta) \geq 0, \qquad \displaystyle\int_0^\infty u(t,\theta)l(dt,\theta) = 0 \qquad \forall\theta\in(0,1), \end{cases} \tag{51}$$

where every term is now well defined and continuous in t, and we can apply Skorohod's lemma (see Lemma VI.2.1 in [10]) for fixed $\theta \in (0,1)$, obtaining

$$\frac{1}{4}l(t,\theta) = \sup_{s\leq t}\left[-\left(x(\theta) + w(s,\theta) + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\int_0^s (u-w)(r,\theta)\,dr\right)\right] \vee 0, \tag{52}$$

for all $t \geq 0$, $\theta \in (0,1)$. Therefore, we have the following:

COROLLARY 2. *For all $x \in K_0 \cap C_0$, a.s. u satisfies the closed formula*

$$\begin{aligned} u(t,\theta) &= x(\theta) + w(t,\theta) + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\int_0^t \big(u(s,\theta) - w(s,\theta)\big)\,ds \\ &\quad + \sup_{s\leq t}\left[-\left(x(\theta) + w(s,\theta) + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\int_0^s \big(u(r,\theta) - w(r,\theta)\big)\,dr\right)\right] \vee 0, \end{aligned} \tag{53}$$

for all $t \geq 0, \theta \in (0,1)$. *In particular*, v, *defined by*

$$v(t,\theta) := \int_0^t \big(u(s,\theta) - w(s,\theta)\big)\, ds,$$

is a solution of the following fully nonlinear equation:

$$(54)\qquad \begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 v}{\partial \theta^2} + x(\theta) + \sup\limits_{s \leq t}\Big[-\Big(x(\theta) + w(s,\theta) + \dfrac{1}{2}\dfrac{\partial^2 v}{\partial \theta^2}(s,\theta)\Big)\Big] \vee 0, \\ v(0,\cdot) = 0, \ v(t,0) = v(t,1) = 0, \end{cases}$$

unique in $\mathcal{V} := \{v' : \overline{\mathcal{O}} \mapsto \mathbb{R}$ *continuous*: $\partial v'/\partial t$ *continuous*, $\partial^2 v'/\partial \theta^2 \in L^1_{\mathrm{loc}}((0,1); C([0,T]))$ *for all* $T > 0\}$.

The uniqueness of solutions of equation (54) in $\mathcal{V}$ is a consequence of the pathwise uniqueness of solutions of equation (1), proved in [9]: indeed, if $v' \in \mathcal{V}$ is a solution of (54), then setting

$$u'(t,\theta) := \frac{\partial v'}{\partial t}(t,\theta) + w(t,\theta),$$

$$\eta'(dt, d\theta) := d_t \left\{ \sup_{s \leq t}\Big[-\Big(x(\theta) + w(s,\theta) + \frac{1}{2}\frac{\partial^2 v'}{\partial \theta^2}(s,\theta)\Big)\Big] \vee 0 \right\} d\theta$$

and repeating the above arguments backwards, we obtain that (u', η') is a weak solution of (1), so that $u' = u$ and, therefore, $v' = v$.

Notice that, by point 5 in Theorem 3, by (43)–(54) and by the continuity of $\partial v/\partial t$ on $\overline{\mathcal{O}}$, then, for all $t > 0$, $\partial^2 v/\partial \theta^2(t,\cdot)$ is not in $L^1(0,1)$, so that by the uniqueness a $C^{1,2}(\overline{\mathcal{O}})$ solution of (54) does not exist.

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