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INCIPIENT INFINITE PERCOLATION CLUSTERS IN 2D

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We study several kinds of large critical percolation clusters in two dimensions. We show that from the microscopic (lattice scale) perspective these clusters can be described by Kesten's incipient infinite cluster (IIC), as was conjectured by Aizenman. More specifically, we establish this for incipient spanning clusters, large clusters in a finite box and the inhomogeneous model of Chayes, Chayes and Durrett. Our results prove the equivalence of several natural definitions of the IIC.

We also show that for any $k \geq 1$ the difference in size between the kth and $(k+1)$st largest critical clusters in a finite box goes to infinity in probability as the size of the box goes to infinity. In addition, the distribution of the Chayes–Chayes–Durrett cluster is shown to be singular with respect to the IIC.

1. Introduction. The term "incipient infinite cluster" has been used, mostly in the physics literature, in reference to the large connected clusters that can be seen in numerical simulations of critical percolation. The name reflects the idea that as the bond (or site) density is raised above threshold, some of these large clusters connect and "give birth" to the infinite cluster; see Borgs, Chayes, Kesten and Spencer (2000).

A mathematically rigorous definition of what one might call the "inifnite cluster at criticality" was given by Kesten (1986). The definition, which we review in Section 1.2, is obtained by conditioning the critical percolation process to have an open path connecting the origin to the boundary of a large box, whose size increases to infinity. In the weak limit an infinite cluster containing the origin is obtained, which we call the IIC.

The following alternative definition was proposed by Aizenman (1997) and expected to be equivalent. Consider the set of sites in a large box that are connected (at the critical density) to both the left and right sides, and pick one uniformly, given the configuration. Translate this site to the origin and let the size of the box go to infinity. By its choice, the translated site is conditioned to have an open connection to a long distance, so we may expect the IIC to arise in the weak limit. We prove this in Theorem 1.

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In this paper we discusss two more natural procedures that produce the IIC and note that analogous results about invasion percolation were obtained in Járai (2000) which will be published in Járai (to appear).

A procedure useful for numerical simulations is to pick a random site (uniformly) from the largest cluster in the box where the simulation is performed. Again, we can ask whether the law of the cluster when viewed from the random site converges to the IIC. This is indeed the case, and more generally, we show this for the kth largest cluster where $k \geq 1$ is fixed.

The third setting we consider is the inhomogeneous model of Chayes, Chayes and Durrett (1987). In this model the probability that a bond at distance k from the origin is open is taken to be $p_c + k^{-\lambda}$. Assuming that the correlation length exponent ν exists, it can be shown that for $\lambda < 1/\nu$ there is an infinite cluster and this cluster was proposed in the above reference as an alternative to the IIC. Our aim in Section 4 below is to explore the relationship between the CCD cluster and the IIC.

The precise formulation of our main results are given in Section 1.3. Before stating our results we fix some notation and terminology in the next subsection.

For an approach to the incipient infinite cluster in high dimensions see Hara and Slade (2000a, b).

REMARK. We formulate results in the setting of bond percolation on $\mathbb{Z}^2$. This can be generalized to other common two-dimensional graphs, as long as the Russo–Seymour–Welsh Lemma and Theorems 1 and 2 of Kesten (1987) hold.

1.1. *Notation and terminology.* We consider bond percolation on the square lattice $\mathbb{Z}^2$. We denote by $\mathbb{E}^2$ the set of nearest neighbor bonds in $\mathbb{Z}^2$ and each bond $e \in \mathbb{E}^2$ is declared to be *open* with probability p and *closed* with probability $1 - p$, independently. We let P_p (resp. E_p) denote the corresponding product measure (resp. expectation) on the configuration space $(\Omega, \mathcal{F}) = (\{0,1\}^{\mathbb{E}^2}, \mathcal{F})$, where 0 means closed, 1 means open, and $\mathcal{F}$ is the usual σ-field on Ω. We say that $E \in \mathcal{F}$ is a *cylinder event*, if E is given by conditions on the states of finitely many edges only. If A is an event, $I[A]$ denotes its indicator function.

If G is a subgraph of $(\mathbb{Z}^2, \mathbb{E}^2)$, we write $E(G)$ for the set of edges of G and $v \in G$ means that v is a vertex of G. Sometimes we define G by specifying $E(G) \subset \mathbb{E}^2$ and it is understood that the vertex set consists of those $v \in \mathbb{Z}^2$ that are incident to an edge in $E(G)$. If $A, B \subset \mathbb{Z}^2$, we write $A \leftrightarrow B$ for the event that some vertex in A is connected by an open path to some vertex in B. We denote by $\mathcal{C}(v)$ the *open cluster* containing the vertex v,

$$\mathcal{C}(v) = \{w \in \mathbb{Z}^2 : v \leftrightarrow w\}$$

and write $\mathcal{C} = \mathcal{C}(0)$. For a countable set A we write $|A|$ for the cardinality of A. The *percolation probability* is defined by

$$\theta(p) = P_p(|\mathcal{C}| = \infty).$$

We also use the notation $\{0 \leftrightarrow \infty\}$ for the event $\{|\mathcal{C}| = \infty\}$. The *critical probability* is

$$p_c = \inf\{p \geq 0 : \theta(p) > 0\}.$$

The fact that $p_c = 1/2$ for the square lattice will not play a role; our arguments can be generalized to other standard two-dimensional lattices.

We introduce the norm

$$|v| = \max\{|v_1|, |v_2|\}$$

for $v = (v_1, v_2) \in \mathbb{Z}^2$. If A and B are two sets of vertices, we put $\mathrm{dist}(A, B) = \min\{|v - w| : v \in A, w \in B\}$. For a bond $e = \langle v, w\rangle \in \mathbb{E}^2$ we let $|e| = \max\{|v|, |w|\}$. The *box* of radius n centered at the vertex v is

$$B(n, v) = \{w \in \mathbb{Z}^2 : |v - w| \leq n\}$$

and we write $B(n) = B(n, 0)$. We use the notation

$$\mathrm{An}(n, m) = B(n) \setminus B(m) = \{v \in \mathbb{Z}^2 : m < |v| \leq n\}$$

for an annulus. The *boundary* of a lattice graph G is defined as

$$\partial G = \big\{v \in G : v \text{ is incident to an edge not belonging to } E(G)\big\},$$

thus $\partial B(n) = \{v \in \mathbb{Z}^2 : |v| = n\}$. Sometimes it will be convenient to work with

$$\Lambda_n = \{w \in \mathbb{Z}^2 : -n \leq w_i < n,\ i = 1, 2\}.$$

When we refer to $B(a)$, Λ_a, etc. where a is not an integer, we mean to replace a by its integer part $\lfloor a \rfloor$. The symbol $\lceil a \rceil$ will denote the smallest integer that is greater than or equal to a.

An important quantity for us is the *point-to-box connectivity*:

$$\pi(p, n) = P_p\big(0 \leftrightarrow \partial B(n)\big).$$

We write $\pi_n = \pi(n) = \pi(p_c, n)$. The quantity

$$s(n) = n^2 \pi_n$$

will often show up, since it represents the order of magnitude of the largest critical clusters in a box of linear size n; see Borgs, Chayes, Kesten and Spencer (1999). It is easy to see that for $p > 0$ we have

$$P_p(0 \leftrightarrow \partial \Lambda_n) \asymp \pi(p, n), \tag{1.1}$$

where the notation $a_n \asymp b_n$ means that there are constants $0 < C_1 < C_2 < \infty$, such that $C_1 a_n \leq b_n \leq C_2 a_n$. The constants implicitly present in (1.1) are also independent of p, as long as p is bounded away from 0.

The dual lattice of $(\mathbb{Z}^2, \mathbb{E}^2)$ is denoted by $(\mathbb{Z}^2_*, \mathbb{E}^2_*)$, where $\mathbb{Z}^2_* = (1/2, 1/2) + \mathbb{Z}^2$. For each $e \in \mathbb{E}^2$ there is a unique edge $e_* \in \mathbb{E}^2_*$ that intersects e. We call e_* open if and only if e is open.

We use the notation $f(n) = o(g(n))$ for $\lim_{n \to \infty} f(n)/g(n) = 0$, and $f(n) = O(g(n))$ for the fact that $f(n)/g(n)$ is bounded.

1.2. *Definition of the IIC.* It was shown by Kesten (1986) that for any cylinder event E the limit

$$\nu(E) = \lim_{n\to\infty} P_{p_c}\big(E \mid 0 \leftrightarrow \partial B(n)\big) \tag{1.2}$$

exists. It follows that ν has a unique extension to a probability measure on $\mathcal{F}$ and under the measure ν the cluster

$$\mathcal{C}(0) = \{v \in \mathbb{Z}^2 : v \leftrightarrow 0\}$$

is almost surely infinite. Following the terminology of Kesten (1986) we call the distribution of the cluster $\mathcal{C}(0)$ under ν the incipient infinite cluster (IIC).

1.3. *Main results.* All constants in this paper are strictly positive and finite. Constants denoted by C_i may have different meanings in different theorems. When we need to refer to a constant from a different theorem or proof, this will be done explicitly.

The first object we study is the union of *spanning clusters* in $B(n)$. Let $\mathrm{LS} = \{-n\} \times [-n, n]$ and $\mathrm{RS} = \{n\} \times [-n, n]$ denote the left and right sides of $B(n)$ and define

$$\mathrm{SC}_n = \{v \in B(n) : \mathrm{LS} \leftrightarrow v \leftrightarrow \mathrm{RS} \text{ inside } B(n)\}.$$

The following procedure has been proposed by Aizenman (1997). Let $p = p_c$, and choose a site I_n uniformly at random from SC_n (if not empty), and shift that site to the origin. In other words: look at SC_n from the perspective of one of its sites. We can expect that as $n \to \infty$ the law of the translated SC_n converges weakly to the IIC.

To formulate the statement precisely let τ_v denote translation by $v \in \mathbb{Z}^2$. Translations act on Ω by $\tau_v\omega(\langle u, x\rangle) = \omega(\langle u - v, x - v\rangle)$, and on events by $\tau_v A = \{\tau_v\omega : \omega \in A\}$.

THEOREM 1. *Let I_n denote a site of* SC_n *chosen uniformly at random, given the configuration ω, when* SC_n *is not empty. Then for any cylinder event $E \in \mathcal{F}$*

$$\lim_{n\to\infty} P_{p_c}(\tau_{I_n} E \mid \mathrm{SC}_n \neq \varnothing) = \nu(E).$$

REMARK. The following question was suggested to us by H. Kesten: what do we see if we choose the vertex from the *backbone* of the spanning cluster? The backbone is the set of vertices that have disjoint connections to the left and right sides of $B(n)$. We believe that the limit is

$$\widetilde{\nu}(E) = \lim_{n\to\infty} P_{p_c}\big(E \mid 0 \text{ has two disjoint connections to } \partial B(n)\big). \tag{1.3}$$

The existence of the limit in (1.3) can most likely be shown by the method of Kesten (1986). We note that $\widetilde{\nu}$ is singular with respect to ν. Due to the conditioning in (1.3), there are $\widetilde{\nu}$-a.s. two disjoint infinte open paths starting at 0, but ν-a.s. there is only one such path by the BK inequality [van den Berg and Kesten (1985)]. We believe that similar limiting measures exist in other cases, for example, if we condition the vertex to be the end-vertex of a bond that is pivotal for a connection between the left and right sides.

We have the following variation on Theorem 1 in which the site is nonrandom.

THEOREM 2. *Let $h(n) \to \infty$ in such a way that $h(n) \le n$. Then*

$$\lim_{\substack{n\to\infty \\ |v|\le n-h(n)}} P_{p_c}(\tau_v E \mid v \in \mathrm{SC}_n) = \nu(E).$$

We can also relate the large clusters seen at criticality to the IIC. Let $\mathcal{C}_n^{(1)}, \mathcal{C}_n^{(2)}, \dots$ denote the clusters inside $B(n)$, ordered by size, that is by the number of vertices each contains. In determining the clusters we use free boundary conditions, that is, for two vertices to belong to the same cluster they have to be connected inside $B(n)$. If there are clusters of equal size we use some lexicographic ordering between them; it turns out that this has no importance. The size distribution of $\mathcal{C}_n^{(1)}, \mathcal{C}_n^{(2)}, \dots$ both near and away from the critical point was studied extensively by Borgs, Chayes, Kesten and Spencer (1999, 2000). We prove the following theorem.

THEOREM 3. *Fix $k \ge 1$, and let I_n denote a vertex chosen uniformly at random from $\mathcal{C}_n^{(k)}$, the kth largest cluster in $B(n)$. Then for any cylinder event E,*

$$\lim_{n\to\infty} P_{p_c}(\tau_{I_n} E) = \nu(E).$$

REMARK. We need to be a bit more precise about what happens when there are fewer than k clusters altogether. The probability of this goes to 0, so the statement of the theorem is not affected by whatever we define I_n to be in this case.

An important step in the proof of Theorem 3 is to show the following proposition.

PROPOSITION 1. *Let $W_n^{(1)} \ge W_n^{(2)} \ge \cdots$ denote the sizes of the clusters in $B(n)$ in decreasing order. For any $k \ge 1$*

$$W_n^{(k)} - W_n^{(k+1)} \to \infty \qquad \text{in } P_{p_c}\text{-probability as } n \to \infty.$$

REMARK. We actually show a bit more: $W_n^{(k)} - W_n^{(k+1)}$ cannot be of smaller order than $\sqrt{s(n)} = \sqrt{n^2\pi_n}$. Results by Borgs, Chayes, Kesten and Spencer (2000) show that at p_c one has $W_n^{(k)} \asymp s(n)$ for any fixed k. This suggests that the true order of the difference should be $s(n)$.

The last setting we consider is the inhomogeneous model studied by Chayes, Chayes and Durrett (1987). In this model, each bond $e \in \mathbb{E}^2$ is open with a probability $q(e) \geq p_c$. We assume that $q(e) \to p_c$, as $|e| \to \infty$. We denote by $P_{\mathbf{q}}$ the corresponding probability measure. Recall that $\mathcal{C} = \{v \in \mathbb{Z}^2 : 0 \leftrightarrow v\}$ and let $\mathcal{C}_n = \mathcal{C} \cap B(n)$. By results of Chayes, Chayes and Durrett (1987), if the decay of $q(e)$ to p_c is sufficiently slow, then we have $P_{\mathbf{q}}(|\mathcal{C}| = \infty) > 0$. We call the distribution of $\mathcal{C}$ under $P_{\mathbf{q}}$ the Chayes–Chayes–Durrett cluster.

It can be expected that away from the origin $\mathcal{C}$ looks similar to a critical percolation cluster. The next theorem makes this precise.

THEOREM 4. *Assume that $q(e)$ is a function of $|e|$ only, and that it is decreasing in $|e|$. Let I_n denote a vertex chosen uniformly at random from $\mathcal{C}_n$. Then for any cylinder event E,*

$$\lim_{\substack{m,n\to\infty \\ m\geq n}} P_{\mathbf{q}}\big(\tau_{I_n} E \mid 0 \leftrightarrow \partial B(m)\big) = \nu(E). \tag{1.4}$$

In particular, if $P_{\mathbf{q}}(|\mathcal{C}| = \infty) > 0$, we have

$$\lim_{n\to\infty} P_{\mathbf{q}}(\tau_{I_n} E \mid 0 \leftrightarrow \infty) = \nu(E).$$

In the special case $q(e) \equiv p_c$ we obtain a result for the IIC itself, saying that a typical vertex of the IIC looks like the origin.

COROLLARY 1. *Let $q(e) \equiv p_c$. Then*

$$\lim_{n\to\infty} \nu(\tau_{I_n} E) = \nu(E).$$

A problem related to Theorem 4 is whether the Chayes–Chayes–Durrett cluster is singular with respect to the IIC. To discuss this assume that the limit

$$\mu_{\mathbf{q}}(E) = \lim_{n\to\infty} P_{\mathbf{q}}\big(E \mid 0 \leftrightarrow \partial B(n)\big) \tag{1.5}$$

exist for cylinder events. [This is in fact true for any choice of $q(e)$, as long as $q(e) \geq p_c$ and $d = 2$, by a generalization of (1.2); see (4.1).] We give a sufficient condition for $\mu_{\mathbf{q}}$ and ν to be singular. To formulate the result, we use the *finite-size scaling correlation length* introduced by Chayes, Chayes and Fröhlich (1985) and further studied by Kesten (1987). Let

$$\sigma(n_1, n_2, p) = P_p(\text{there is an open horizontal crossing of } [0, n_1] \times [0, n_2]),$$

and define for $p > p_c$

$$L(p, \varepsilon) = \min\{n : \sigma(n, n, p) \geq 1 - \varepsilon\}.$$

REMARK. In the definition of Kesten (1987) the meaning of $[0, n] \times [0, n]$, and hence of $L(p, \varepsilon)$ is slightly different. However, his results remain valid in our setting.

We fix $\varepsilon = \varepsilon_0$, and let $L(p) = L(p, \varepsilon_0)$, where ε_0 is a small positive constant. We will not be concerned with how exactly ε_0 is chosen, only that $L(p)$ satisfies certain properties (see the beginning of Section 4.2).

THEOREM 5. *Assume that $q(e) \geq p_c$ is a function of $|e|$ only and that it is decreasing in $|e|$. Suppose that*

$$\lim_{|e|\to\infty} \frac{L(q(e))}{|e|} = 0.$$

[This holds whenever $P_{\mathbf{q}}(0 \leftrightarrow \infty) > 0$.] Let $\mu_{\mathbf{q}}$ be defined by (1.5). *Then $\mu_{\mathbf{q}}$ and ν are singular with respect to each other.*

The organization of the rest of the paper is the following. We summarize some preliminary results in Section 1.4. Then we prove Theorem 1 in Section 2. We start the proofs with the case of the spanning clusters because it is easiest to demonstrate our technique with this example. Later proofs will go along similar lines and will not always be spelled out completely. Our results about large clusters are proved in Section 3 and the ones about the Chayes–Chayes–Durrett cluster in Section 4.

1.4. *Preliminaries.* Recent breakthroughs by Smirnov (2001) and Lawler, Schramm and Werner (2002) give very precise information about certain percolation quantities, including those important for this paper. For site percolation on the triangular lattice Lawler, Schramm and Werner (2002) show that $\pi_n = n^{-5/48+o(1)}$ as $n \to \infty$. Smirnov and Werner (2001) show the existence of other critical exponents, on the triangular lattice, using the results of Kesten (1987). The proofs in this paper are largely model independent and can be based on properties of π_n that are known for other standard two-dimensional lattices as well. It might be useful for the reader to simply think of $\pi(p_c, n)$ as $n^{-5/48}$ throughout the paper, which makes some of the technical statements of Theorem 7 below more clear.

The statements of the following theorem are well-known consequences of the Russo–Seymour–Welsh Lemma [Russo (1978), Seymour and Welsh (1978)] and the FKG inequality [see Grimmett (1999)].

THEOREM 6 (RSW). (i) *For any $\kappa \geq 1$ there is a constant C_κ, such that for all $n \geq 1$ we have*

$$P_{p_c}(\textit{there is an open horizontal crossing of } [0, \kappa n] \times [0, n]) \geq C_\kappa. \tag{1.6}$$

(ii) *There are constants $C > 0$, and $\mu > 0$, such that for all $n > m \geq 1$ we have*

$$P_{p_c}(\textit{there is no open circuit in } \mathrm{An}(n, m)) \leq C\left(\frac{m}{n}\right)^{\mu}. \tag{1.7}$$

We are going to use the following properties of $\pi(n)$.

THEOREM 7. (i) *There are constants* $C_1 < C_2$ *such that*

$$C_1 P_{p_c}\big(|\mathcal{C}| \geq s(n)\big) \leq \pi(n) \leq C_2 P_{p_c}\big(|\mathcal{C}| \geq s(n)\big). \tag{1.8}$$

(ii) *There exists a constant* $D > 0$, *such that, for* $p \geq p_c$,

$$\frac{\pi(p,m)}{\pi(p,n)} \geq D\sqrt{\frac{n}{m}}, \qquad m \geq n \geq 1. \tag{1.9}$$

(iii) *There exists* $C_3 < C_4$, *such that, if* $n > N \geq 1$, *then*

$$C_3 \frac{\pi(n)}{\pi(N)} \leq P_{p_c}(\partial\Lambda_N \leftrightarrow \partial\Lambda_n) \leq C_4 \frac{\pi(n)}{\pi(N)}. \tag{1.10}$$

(iv) *There exist* C_5 *and* C_6 *such that, for* $L \geq 1$, $p \geq p_c$ *and* $\beta = 0, 1$, *we have*

$$\sum_{r=0}^{L} (r+1)^\beta \pi(p,r) \leq C_5 L^{\beta+1} \pi(p,L) \tag{1.11}$$

and

$$\sum_{r=0}^{L} (r+1)\pi(p,r)^2 \leq C_6 L^2 \pi(p,L)^2. \tag{1.12}$$

(v) *If* N *is fixed*, $L \geq N \geq 1$, $p \geq p_c$ *and* $\beta = 0, 1$ *then*

$$\sum_{r=0}^{L/N} (Nr+1)^\beta \frac{\pi(p,Nr)}{\pi(p,N)} \leq C_5 \frac{L^{\beta+1}\pi(p,L)}{N\pi(p,N)} \tag{1.13}$$

and

$$\sum_{r=0}^{L/N} (Nr+1)\left(\frac{\pi(p,Nr)}{\pi(p,N)}\right)^2 \leq C_6 \frac{L^2 \pi(p,L)^2}{N\pi(p,N)^2}. \tag{1.14}$$

REMARK. For the triangular lattice, the result of Lawler, Schramm and Werner (2002) and Theorems 1 and 2 of Kesten (1987) imply (ii) with $D(n/m)^{5/48+\varepsilon}$ on the right-hand side. The inequalities in (1.10) correspond to the fact that $P_{p_c}(\partial\Lambda_N \leftrightarrow \partial\Lambda_n)$ should scale as $(N/n)^{5/48}$. A slightly weaker form of this scaling is proved in (3.1), Lawler, Schramm and Werner (2002). Properties (iv) and (v) are straightforward, given a power law decay of π_n that is slower than n^{-1}.

PROOF OF THEOREM 7. The lower bound in (1.8) is a special case of Proposition 4.5 of Borgs, Chayes, Kesten and Spencer (1999). Note that their Assumption (II) is valid for $d = 2$ by (1.9). Likewise the upper bound is a special case of Proposition 5.2 of Borgs, Chayes, Kesten and Spencer (1999). Note that their Assumption (I) is well known for $d = 2$.

The bound (1.9) is a standard extension of Corollary 3.15 of van den Berg and Kesten (1985). Their proof also works for any $p \geq p_c$.

The bounds in (1.10) follow by a standard RSW argument.

Statement (iv) is a special case of (v).

The proof of (v) goes along the same lines as the proof of Lemma 4.4 of Borgs, Chayes, Kesten and Spencer (1999). Using (1.9) we have

$$\begin{aligned}
\sum_{r=0}^{L/N} (Nr+1)^{\beta} \frac{\pi(p, Nr)}{\pi(p, N)} &= \frac{\pi(p, L)}{\pi(p, N)} \sum_{r=0}^{L/N} (Nr+1)^{\beta} \frac{\pi(p, Nr)}{\pi(p, L)} \\
&\leq \frac{C_9}{D} \frac{\pi(p, L)}{\pi(p, N)} \sum_{r=0}^{L/N} (Nr+1)^{\beta} \left(\frac{L}{Nr+1} \right)^{1/2} \\
&= \frac{C_9}{D} \frac{\pi(p, L)}{\pi(p, N)} L^{1/2} \sum_{r=0}^{L/N} (Nr+1)^{\beta - 1/2} \\
&\leq C_5 \frac{L^{\beta+1} \pi(p, L)}{N \pi(p, N)}.
\end{aligned}$$

The proof of (1.14) is quite similar. $\square$

2. Spanning clusters. We start with a rough outline of the argument for Theorem 1. Recall that SC_n denotes the union of spanning clusters and I_n is a random vertex from SC_n. We can write

$$P_{p_c}(\tau_{I_n} E \mid \mathrm{SC}_n \neq \varnothing) = \sum_{v \in B(n)} E_{p_c}\left(\frac{I[\tau_v E, v \in \mathrm{SC}_n]}{|\mathrm{SC}_n|} \,\Big|\, \mathrm{SC}_n \neq \varnothing \right). \tag{2.1}$$

Consider a large box $B(N, v)$ centered at v, where $1 \ll N \ll n$. The event $v \in \mathrm{SC}_n$ implies that $v \leftrightarrow \partial B(N, v)$. The latter is the conditioning in Kesten's theorem, so we hope to apply (1.2) inside the box $B(N, v)$.

Some work is needed to decouple from what is outside the box $B(N, v)$. We put a thick annulus $B(M, v) \setminus B(N, v)$ around the box, where $N \ll M \ll n$. Assume that there is an open circuit in this annulus [here, and later always, we understand that the circuit surrounds the smaller box $B(N, v)$]. Consider the outermost such circuit. A well-known observation is that the outermost open circuit acts as a spatial analogue of a stopping time: the event that it equals a given circuit $\mathcal{D}$ only depends on the configuration outside $\mathcal{D}$. Therefore, if we condition on the outermost open circuit, we can apply Kesten's theorem inside the circuit.

Equation (1.7) suggests that the open circuit will exist with large probability if the annulus is thick enough. However, since the center of the annulus is random, we need some information about the size distribution of the set from which I_n is chosen. The result providing this information is discussed in Section 2.1 and the proof of Theorem 1 is given in Section 2.2.

2.1. *Tightness of* $|\mathrm{SC}_n|$. We prove the following result about $|\mathrm{SC}_n|$.

THEOREM 8. (i) *There are constants* C_1 *and* C_2 *such that, for* $n \geq 1$,

$$C_1 \leq \frac{E_{p_c}|\mathrm{SC}_n|}{s(n)} \leq C_2. \tag{2.2}$$

(ii) *We have*

$$\lim_{\varepsilon\to 0}\inf_{n\geq 1} P_{p_c}\left(\varepsilon < \frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} < \frac{1}{\varepsilon} \,\middle|\, \mathrm{SC}_n \neq \varnothing\right) = 1. \tag{2.3}$$

REMARK. It also holds that for any $t \geq 1$ we have $E_{p_c}|\mathrm{SC}_n|^t \asymp [s(n)]^t$. This extension can be shown using the idea of either Nguyen (1988) or Theorem 8 of Kesten (1986). Since we do not need this extension, we omit the proof.

PROOF OF THEOREM 8. (i) Define $\tilde{\pi}(n) = P_{p_c}(0 \leftrightarrow LS \text{ inside } B(n))$. It is easy to see that $\tilde{\pi}(n) \leq \pi(n) \leq 4\tilde{\pi}(n)$. We first show the lower bound in (2.2). For $v = (v_1, v_2) \in B(n/2)$ let $S(v) = [-n, n] \times [v_2 - n/2, v_2]$. Define the events

$$A_1(v) = \{\text{there is an open horizontal crossing in } S(v)\},$$
$$A_2(v) = \{v \leftrightarrow \mathrm{BS} \text{ inside } B(n/2, v)\},$$

where BS denotes the bottom side of $B(n/2, v)$, that is, $\mathrm{BS} = [v_1 - n/2, v_1 + n/2] \times \{v_2 - n/2\}$. Note that both events are increasing. Using the FKG inequality and the RSW theorem, we get

$$\begin{aligned} P_{p_c}(v \in \mathrm{SC}_n) &\geq P_{p_c}\big(A_1(v) \cap A_2(v)\big) \geq P_{p_c}(A_1(v))P_{p_c}(A_2(v)) \\ &\geq C_3\tilde{\pi}(n/2) \geq (C_3/4)\pi(n/2) \geq (C_3/4)\pi(n). \end{aligned}$$

Hence

$$E_{p_c}|\mathrm{SC}_n| \geq \sum_{v\in B(n/2)} P_{p_c}(v \in \mathrm{SC}_n) \geq C_4 n^2 \pi(n).$$

For the upper bound in (2.2) we use the inclusion of events

$$\{\mathrm{LS} \leftrightarrow v \leftrightarrow \mathrm{RS}\} \subset \{v \leftrightarrow \partial B(v, n)\}$$

to write

$$\begin{aligned} E_{p_c}|\mathrm{SC}_n| &= \sum_{v\in B(n)} P_{p_c}\big(\mathrm{LS} \leftrightarrow v \leftrightarrow \mathrm{RS} \text{ inside } B(n)\big) \\ &\leq \sum_{v\in B(n)} P_{p_c}\big(v \leftrightarrow \partial B(v, n)\big) = (2n+1)^2\pi_n \leq C_5 s(n). \end{aligned} \tag{2.4}$$

(ii) The proof of (2.3) can be broken up into two parts: showing an upper and a lower bound on $|\mathrm{SC}_n|$ in terms of $E_{p_c}|\mathrm{SC}_n|$.

For the upper bound we can use Markov's inequality:

$$\begin{aligned}P_{p_c}\left(\frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} \geq \frac{1}{\varepsilon} \,\Big|\, \mathrm{SC}_n \neq \varnothing\right) &= \frac{1}{P_{p_c}(\mathrm{SC}_n \neq \varnothing)} P_{p_c}\left(\frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} \geq \frac{1}{\varepsilon}\right) \\ &\leq \frac{\varepsilon}{P_{p_c}(\mathrm{SC}_n \neq \varnothing)}.\end{aligned} \tag{2.5}$$

By the RSW theorem $P_{p_c}(\mathrm{SC}_n \neq \varnothing) \geq C_6 > 0$, hence the right-hand side of (2.5) goes to zero uniformly in n, as $\varepsilon \to 0$.

To give a lower bound on $|\mathrm{SC}_n|$ we modify the proof of the second part of Theorem 8 of Kesten (1986). Let R denote the lowest open crossing in $B(n)$. Our plan is to show that the number of vertices above R that are connected to R is larger than $\varepsilon E_{p_c}|\mathrm{SC}_n|$ with large probability. We first need to show that R is not too close to the top side so that there is "enough space" for these vertices. For $0 < a < 1$ let $S_a = [-n, n] \times [-n, \lceil an \rceil]$. We need the following lemma:

LEMMA 1. *There are constants c_1, α such that*

$$P_{p_c}(R \text{ lies in } S_a \mid R \text{ exists}) \geq 1 - c_1(1-a)^{\alpha}.$$

PROOF. The proof is almost identical to the proof of a more general lemma we are going to state later, so we omit it. (See Lemma 4 in Section 3.1.)

REMARK. A stronger statement with $\alpha = 1$ can be proved by a method of Zhang (preprint).

We continue with the proof of tightness. Let $\delta > 0$ be fixed. It follows from Lemma 1 that we can choose a close enough to 1, so that

$$P_{p_c}(R \text{ lies in } S_a \mid R \text{ exists}) \geq 1 - \frac{\delta}{2}, \tag{2.6}$$

uniformly in n. Fix such an a.

We show that for ε small enough, for any n, and for any crossing r_0 of S_a we have

$$P_{p_c}\left(\frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} < \varepsilon \,\Big|\, R = r_0\right) \leq \frac{\delta}{2}. \tag{2.7}$$

This is enough because (2.6) and (2.7) imply

$$\begin{aligned}
&P_{p_c}\left(\frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} < \varepsilon \,\middle|\, \mathrm{SC}_n \neq \varnothing\right)\\
&\quad = P_{p_c}\left(\frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} < \varepsilon \,\middle|\, R \text{ exists}\right)\\
&\quad \le P_{p_c}(R \not\subset S_a \mid R \text{ exists}) + P_{p_c}\left(\frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} < \varepsilon \,\middle|\, R \subset S_a\right)\\
&\quad \le \frac{\delta}{2} + \sum_{r_0} P_{p_c}\left(\frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} < \varepsilon \,\middle|\, R = r_0\right) P_{p_c}(R = r_0 \mid R \subset S_a) \le \delta.
\end{aligned}$$

The sum is over all horizontal crossings r_0 of S_a. Note that it is enough to show (2.7) for n large enough.

Fix the path r_0. Let G denote the open region in $[-n, n]^2$ that lies above r_0, when r_0 is viewed as a curve in $\mathbb{R}^2$. We define the graph H by

$$E(H) \stackrel{\text{def}}{=} \{e \in \mathbb{E}^2 : e \text{ lies in } G \text{ apart maybe from its endpoints}\}.$$

Observe that the event $\{R = r_0\}$ only depends on the states of the edges on and below r_0, hence it is independent of the states of the edges of H. Thus, conditioned on $\{R = r_0\}$ we still have a Bernoulli percolation process on H.

Let v be any vertex on the path r_0 with maximal second coordinate and fix this vertex v (see Figure 1). We assume that v is in the right half of $B(n)$, that is, $v_1 \ge 0$; the other case can be treated analogously. We consider an annulus $v + \mathrm{An}(7m, m)$, where m will later be chosen to be of order n. Let

$$A(m) = \{w \in \mathbb{Z}^2 : v_1 - 5m \le w_1 < v_1 - 3m, v_2 < w_2 \le v_2 + 5m\}.$$

(We suppress the dependence on v in our notation.) In Figure 1 the striped region is the set $A(m)$. We want to make sure that the rectangle $v + [-7m, 0] \times [0, 7m]$ is contained in H. Since $v_2 \le an$, for this we need to have

$$7m \le (1 - a)n. \tag{2.8}$$

Observe that since $v_1 \ge 0$, (2.8) is in fact enough to ensure that $v + [-7m, 0] \times [0, 7m] \subset H$.

We are going to estimate

$$Y(m) \stackrel{\text{def}}{=} \big|\{w \in A(m) : w \leftrightarrow r_0 \text{ inside } [v + \mathrm{An}(7m, m)] \cap B(n)\}\big|$$

and note that on $\{R = r_0\}$ we have $|\mathrm{SC}_n| \ge Y(m)$.

Define a new random configuration ω' in the following way: for $e \in E(H)$, that is, for edges above r_0, put $\omega'(e) = \omega(e)$. On the rest of the lattice let ω' be a new independent configuration with bond density p_c. Then ω' is Bernoulli percolation at p_c. We denote its law by P'_{p_c}.

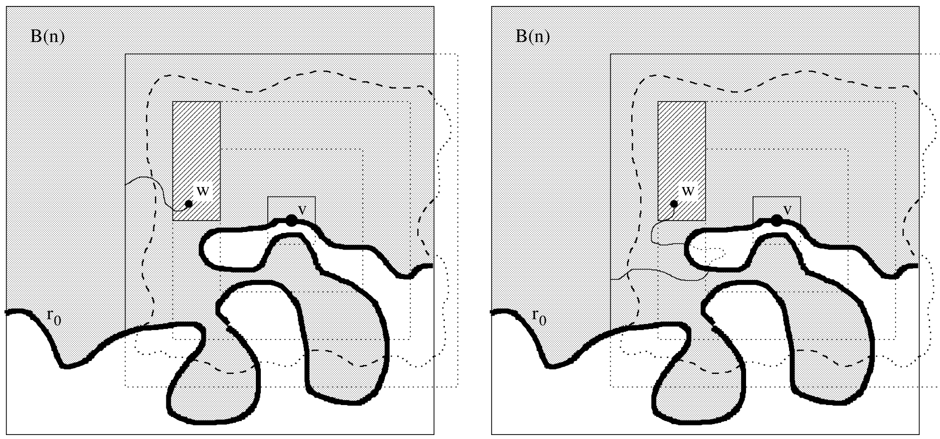

FIG. 1. *If in the modified configuration there is an open circuit in* $v + \mathrm{An}(7m, 5m)$ *and an open path from* w *to the left side of* $B(v, 7m)$, *then* w *is connected to* r_0, *and hence* $w \in \mathrm{SC}_n$ *in the old configuration. The two pictures show the two possible ways this could happen.*

We let

$$F'(m) = \{\text{there is an open circuit in } v + \mathrm{An}(7m, 5m) \text{ in the configuration } \omega'\}.$$

Now suppose that $w \in A(m)$ is connected to the left side of $B(v, 7m)$ inside $[v + \mathrm{An}(7m, m)] \cap B(n)$ in the configuration ω' and that $F'(m)$ occurs. (See Figure 1.) Then, by the geometry of the construction, w is connected to r_0 inside $[v + \mathrm{An}(7m, m)] \cap B(n)$ in the configuration ω. This gives us that for all $w \in A(m)$ we have

$$\begin{aligned} &P_{p_c}\big(w \leftrightarrow r_0 \text{ inside } [v + \mathrm{An}(7m, m)] \cap B(n)\big) \\ &\qquad \geq P'_{p_c}\big(F'(m), w \leftrightarrow \text{ left side of } B(v, 7m) \text{ inside } [v + \mathrm{An}(7m, m)] \cap B(n)\big). \end{aligned}$$

By the FKG inequality and the RSW Theorem the right-hand side is at least

$$C_7 P'_{p_c}\big(w \leftrightarrow \text{ left side of } B(v, 7m) \text{ inside } [v + \mathrm{An}(7m, m)] \cap B(n)\big). \tag{2.9}$$

By the same method as in part (i) of this theorem, we can show that (2.9) is at least $C_8 \tilde{\pi}(m) \geq (C_8/4)\pi(m)$. This implies that

$$E_{p_c} Y(m) \geq C_9 m^2 \pi(m). \tag{2.10}$$

The proof can now be completed by copying the argument of Theorem 8 in Kesten (1986). We have already noted that $|\mathrm{SC}_n| \geq Y(m)$. As in Kesten (1986), we can use (1.11) and (1.12) to prove the second moment bound

$$E_{p_c} Y(m)^2 \leq C_{10}\big(m^2 \pi(m)\big)^2.$$

Using the one-sided Chebyshev inequality [Exercise I.3.6 in Durrett (1996)] this implies that

$$P_{p_c}\big(Y(m) \geq \tfrac{1}{2} E_{p_c} Y(m)\big) \geq C_{11} > 0.$$

Then we repeat the construction with different values of m corresponding to disjoint annuli [so that the $Y(m)$'s are independent]. Namely, if k is the unique integer for which

$$7^k \le (1-a)n < 7^{k+1},$$

then let $m_\ell = 7^{k-\ell}$, with $\ell = 1, 2, \dots, j$, where j will be chosen later. Let ε be such that

$$\varepsilon \le \frac{(1-a)^2 C_9}{2 \cdot 7^{2j+2} \cdot C_2},$$

where C_2 is the constant appearing in part (i) of this theorem. For n large enough $1 \le 7^{k-j}$, so our construction makes sense for $m = m_1, \dots, m_j$. For $1 \le \ell \le j$ we have

$$\begin{aligned} \tfrac{1}{2} E_{p_c} Y(m_\ell) &\ge \tfrac{1}{2} C_9 m_\ell^2 \pi(m_\ell) \ge \tfrac{1}{2} C_9 7^{2(k-j)} \pi(n) \\ &\ge \tfrac{1}{2} C_9 (1-a)^2 7^{-2j-2} n^2 \pi(n) \ge \varepsilon C_2 n^2 \pi(n), \end{aligned} \tag{2.11}$$

where we used (2.10), the lowest possible value of m_ℓ and the fact $\pi(m_\ell) \ge \pi(n)$, the choice of k, and finally the choice of ε. Using (2.11) we obtain

$$\begin{aligned} &P_{p_c}\left(\frac{|\mathrm{SC}_n|}{E_{p_c}|\mathrm{SC}_n|} \ge \varepsilon \;\Big|\; R = r_0 \right) \\ &\quad \ge P_{p_c}\big(|\mathrm{SC}_n| \ge \varepsilon C_2 n^2 \pi(n) \mid R = r_0\big) \\ &\quad \ge P_{p_c}\left(\frac{Y(m_\ell)}{E_{p_c} Y(m_\ell)} \ge \frac{1}{2} \text{ for some } 1 \le \ell \le j \;\Big|\; R = r_0 \right) \\ &\quad = P_{p_c}\left(\frac{Y(m_\ell)}{E_{p_c} Y(m_\ell)} \ge \frac{1}{2} \text{ for some } 1 \le \ell \le j \right) \\ &\quad = 1 - \prod_{\ell=1}^{j} \left(1 - P_{p_c}\left(Y(m_\ell) \ge \frac{1}{2} E_{p_c} Y(m_\ell) \right) \right) \\ &\quad \ge 1 - (1 - C_{11})^j, \end{aligned} \tag{2.12}$$

where we used independence in the penultimate step. Choose j so that the right-hand side of (2.12) is greater than $1 - \delta/2$. Decreasing ε, if necessary, we get the statement of (2.7) for all n and this completes the proof. $\square$

2.2. *Spanning clusters look like the IIC.* We are ready to prove Theorem 1. We are going to write $A_n = \{\mathrm{SC}_n \ne \varnothing\}$ for short.

PROOF OF THEOREM 1. Recall that in (2.1) we have written $P_{p_c}(\tau_{I_n} E \mid A_n)$ as a sum over $v \in B(n)$. Let $\tau_v \mathrm{An}(M, N) = v + \mathrm{An}(M, N)$ be an annulus centered at v. We will take $1 \ll N \ll M \ll n$. In any case, we are going to take N large enough so that $B(N)$ contains all edges on which E depends. Let

$$F = F(M, N) = \{\text{there is an open circuit in } \mathrm{An}(M, N)\}. \tag{2.13}$$

The event that the annulus around the random vertex does not contain an open circuit is $\tau_{I_n} F^c$. In view of the RSW Theorem it is intuitive that if M/N is large then the probability of this event should be small. Our first goal is to prove this.

Let $\varepsilon > 0$. Similarly to (2.1) we write

$$\begin{aligned} P_{p_c}(\tau_{I_n} F^c \mid A_n) &= \sum_{v \in B(n)} E_{p_c}\biggl(\frac{I[v \in \mathrm{SC}_n, \tau_v F^c]}{|\mathrm{SC}_n|} \biggm| A_n \biggr) \\ &= \frac{1}{P_{p_c}(A_n)} \sum_{v \in B(n)} E_{p_c}\biggl(\frac{I[v \in \mathrm{SC}_n, \tau_v F^c]}{|\mathrm{SC}_n|}; A_n \biggr). \end{aligned} \tag{2.14}$$

By part (ii) of Theorem 8 it follows that for sufficiently small $x = x(\varepsilon) > 0$ we have

$$P_{p_c}(|\mathrm{SC}_n| < x E_{p_c} |\mathrm{SC}_n|, A_n) \le \frac{\varepsilon}{2} P_{p_c}(A_n). \tag{2.15}$$

Hence, noting that

$$\sum_{v \in B(n)} \frac{I[v \in \mathrm{SC}_n, \tau_v F^c]}{|\mathrm{SC}_n|} \le 1,$$

we get that the right-hand side of (2.14) is at most

$$\frac{\varepsilon}{2} + \frac{1}{P_{p_c}(A_n)} \sum_{v \in B(n)} E_{p_c}\biggl(\frac{I[\tau_v F^c] I[v \in \mathrm{SC}_n]}{x E_{p_c} |\mathrm{SC}_n|} \biggr). \tag{2.16}$$

The variable $I[\tau_v F^c]$ is decreasing and $I[v \in \mathrm{SC}_n]$ is increasing, so the FKG inequality implies that the summand in (2.16) is less than

$$\frac{1}{x E_{p_c} |\mathrm{SC}_n|} P_{p_c}(\tau_v F^c) P_{p_c}(v \in \mathrm{SC}_n) = \frac{P_{p_c}(F^c)}{x E_{p_c} |\mathrm{SC}_n|} P_{p_c}(v \in \mathrm{SC}_n).$$

Summing over v we get that the expression in (2.16) is less than

$$\frac{\varepsilon}{2} + \frac{P_{p_c}(F^c)}{x P_{p_c}(A_n)} \le \frac{\varepsilon}{2} + \frac{P_{p_c}(F^c)}{C_3 x},$$

where we used that $P_{p_c}(A_n) \ge C_3 > 0$ for some constant C_3. By part (ii) of the RSW Theorem, if $M/N \ge n_1(\varepsilon)$ then $P_{p_c}(F^c) \le C_3 x \varepsilon / 2$. Hence,

$$P_{p_c}(\tau_{I_n} F^c \mid A_n) \le \varepsilon. \tag{2.17}$$

The estimate (2.17) shows that up to an additive error ε we can write

$$P_{p_c}(\tau_{I_n}E \mid A_n) \approx P_{p_c}(\tau_{I_n}E \cap \tau_{I_n}F \mid A_n) \\ = \sum_{v\in B(n)} E_{p_c}\biggl(\frac{I[\tau_v E, v \in \mathrm{SC}_n, \tau_v F]}{|\mathrm{SC}_n|} \biggm| A_n\biggr). \tag{2.18}$$

For technical reasons we will need to restrict the sum over v in (2.18) to sites that are not too close to the boundary. For this we show that for a suitable function $f(n)$ the event $G_n = \{I_n \in B(n - f(n))\}$ occurs with large probability. We write

$$P_{p_c}(G_n^c \mid A_n) = \sum_{n-f(n)<|v|\le n} E_{p_c}\biggl(\frac{I[v \in \mathrm{SC}_n]}{|\mathrm{SC}_n|} \biggm| A_n\biggr).$$

Similarly to the calculation for $\tau_{I_n}F^c$, we use (2.15) to show that the right-hand side is less than

$$\begin{aligned} &\frac{\varepsilon}{2} + \sum_{n-f(n)<|v|\le n} E_{p_c}\biggl(\frac{I[v \in \mathrm{SC}_n]}{xE_{p_c}|\mathrm{SC}_n|} \biggm| A_n\biggr) \\ &\le \frac{\varepsilon}{2} + \sum_{n-f(n)<|v|\le n} \frac{1}{xE_{p_c}|\mathrm{SC}_n|} \\ &\le \frac{\varepsilon}{2} + \frac{C_4 n f(n)}{xE_{p_c}|\mathrm{SC}_n|}. \end{aligned} \tag{2.19}$$

By Theorem 8 we have $E_{p_c}|\mathrm{SC}_n| \ge C_1 n^2 \pi(n)$. Therefore, if $f(n) = o(n\pi(n))$, the second term on right-hand side of (2.19) will be $\le \varepsilon/2$ for large n. To end this step we note that the condition on $f(n)$ allows $f(n) \to \infty$. This follows from the fact that $\pi(n) \ge D'/\sqrt{n}$, a consequence of (1.9). Fix $f(n)$ with the above properties. Then for $n \ge n_2(\varepsilon)$ we have

$$P_{p_c}(G_n^c \mid A_n) \le \varepsilon. \tag{2.20}$$

The next step is to decompose F according to the outermost open circuit. Define

$$F(\mathcal{D}) = \{\mathcal{D} \text{ is the outermost open circuit in } \mathrm{An}(M, N)\}; \tag{2.21}$$

thus F is the disjoint union of the events $F(\mathcal{D})$ where $\mathcal{D}$ runs over all circuits in $\mathrm{An}(M, N)$. Likewise we have $\tau_v F = \bigcup_{\mathcal{D}} \tau_v F(\mathcal{D})$. Then using the bounds (2.17) and (2.20) we have

$$\begin{aligned} &P_{p_c}(\tau_{I_n}E \mid A_n) \\ &\quad\le 2\varepsilon + \frac{1}{P(A_n)} \sum_{|v|\le n-f(n)} \sum_{\mathcal{D}} E_{p_c}\biggl(\frac{I[\tau_v E, v \in \mathrm{SC}_n, \tau_v F(\mathcal{D})]}{|\mathrm{SC}_n|}; A_n\biggr) \\ &\quad\le 2\varepsilon + P_{p_c}(\tau_{I_n}E \mid A_n), \end{aligned} \tag{2.22}$$

if n and M/N are large enough.

Fix v and $\mathcal{D}$, and assume that $f(n) > M$. On the event $\tau_v F(\mathcal{D})$ we have that $\{v \in \mathrm{SC}_n\}$ occurs if and only if $\{v \leftrightarrow \tau_v \mathcal{D}\}$ and $\{\tau_v \mathcal{D} \subset \mathrm{SC}_n\}$ both occur. Therefore, the numerator inside the expectation in (2.22) factors as

$$I[\tau_v E,\ v \leftrightarrow \tau_v \mathcal{D}] I[\tau_v F(\mathcal{D}), \tau_v \mathcal{D} \subset \mathrm{SC}_n]. \tag{2.23}$$

Let $\mathrm{ext}(\tau_v \mathcal{D})$ denote the exterior of $\tau_v \mathcal{D}$, that is the graph consisting of the set of edges outside $\tau_v \mathcal{D}$ or on the circuit $\tau_v \mathcal{D}$. The bulk of the denominator $|\mathrm{SC}_n|$ comes from $\mathrm{ext}(\tau_v \mathcal{D})$, so we would like to group it with the second factor in (2.23). We let

$$W_n(\tau_v \mathcal{D}) = |\{w \in \mathrm{ext}(\tau_v \mathcal{D}) : w \in \mathrm{SC}_n\}|. \tag{2.24}$$

For $n \geq n_3(M, \varepsilon)$ we have the inequalities

$$W_n(\tau_v \mathcal{D}) \leq |\mathrm{SC}_n| \leq (1+\varepsilon) W_n(\tau_v \mathcal{D}) \tag{2.25}$$

for all v and $\mathcal{D}$. (We can choose n_3 so that it also takes care of the previously mentioned condition $f(n) > M$.) Let

$$X_{\mathcal{D},v,E} = I[\tau_v E,\ v \leftrightarrow \tau_v \mathcal{D}],$$

$$Y_{\mathcal{D},v,n} = \frac{I[\tau_v F(\mathcal{D}),\ \tau_v \mathcal{D} \subset \mathrm{SC}_n]}{W_n(\tau_v \mathcal{D})}.$$

(It is understood that $Y = 0$, when of the form $0/0$.) Denoting the expectation in (2.22) by $E(\mathcal{D}, v, n, E)$, equation (2.25) yields

$$E(\mathcal{D}, v, n, E) \leq E_{p_c} X_{\mathcal{D},v,E} Y_{\mathcal{D},v,n} \leq (1+\varepsilon) E(\mathcal{D}, v, n, E), \tag{2.26}$$

provided $n \geq n_3$.

It is straightforward to check that X and Y are independent. By translation invariance we have $E_{p_c} X_{\mathcal{D},v,E} = P_{p_c}(E, 0 \leftrightarrow \mathcal{D})$. A slight generalization of (1.2) yields that

$$\lim_{\substack{N\to\infty \\ \mathcal{D} \text{ surrounds } B(N)}} P_{p_c}(E \mid 0 \leftrightarrow \mathcal{D}) = \nu(E); \tag{2.27}$$

see Remark after Theorem 3 in Kesten (1986). Therefore, if $N \geq n_4(\varepsilon, E)$ then

$$\frac{1}{1+\varepsilon} P_{p_c}(E, 0 \leftrightarrow \mathcal{D}) \leq \nu(E) P_{p_c}(0 \leftrightarrow \mathcal{D}) \leq (1+\varepsilon) P_{p_c}(E, 0 \leftrightarrow \mathcal{D}). \tag{2.28}$$

[For this we need to suppose $\nu(E) > 0$. This is not restrictive, since otherwise we consider the event E^c.]

We choose the different quantities in the proof in the following order. Given $\varepsilon > 0$ we choose $N \geq n_4(\varepsilon)$ so that (2.28) is satisfied. Next we choose $M \geq N n_1(\varepsilon)$. For $n \geq \max\{n_2(\varepsilon), n_3(M, \varepsilon)\}$ we also have (2.22) and (2.26). Now replace E by the sure event Ω in our argument. Then, increasing n if necessary, we have (2.22), (2.26) and (2.28) with E replaced by Ω. Combining these inequalities with the similar inequalities for E we get

$$\begin{aligned}
P_{p_c}(\tau_{I_n} E \mid A_n) &\le 2\varepsilon + \frac{1}{P(A_n)} \sum_v \sum_{\mathcal{D}} E(\mathcal{D}, v, n, E) \\
&\le 2\varepsilon + \frac{1}{P(A_n)} \sum_v \sum_{\mathcal{D}} E_{p_c} X_{\mathcal{D},v,E} Y_{\mathcal{D},v,n} \\
&\le 2\varepsilon + \frac{(1+\varepsilon)\nu(E)}{P(A_n)} \sum_v \sum_{\mathcal{D}} E_{p_c} X_{\mathcal{D},v,\Omega} Y_{\mathcal{D},v,n} \\
&\le 2\varepsilon + \frac{(1+\varepsilon)^2\nu(E)}{P(A_n)} \sum_v \sum_{\mathcal{D}} E(\mathcal{D}, v, n, \Omega) \\
&\le 2\varepsilon + (1+\varepsilon)^2 \nu(E)
\end{aligned}$$

for n large enough and quite similarly

$$P_{p_c}(\tau_{I_n} E \mid A_n) \ge -2\varepsilon + \frac{1}{(1+\varepsilon)^2} \nu(E).$$

This completes the proof. $\square$

PROOF OF THEOREM 2. The proof is very similar to that of Theorem 1, but much simpler. By the FKG inequality we have $P_{p_c}(\tau_v F^c \mid v \in \mathrm{SC}_n) \le \varepsilon$ uniformly in $v \in B(n)$, if M/N is large enough. Therefore

$$P_{p_c}(\tau_v E \mid v \in \mathrm{SC}_n) \approx \frac{1}{P_{p_c}(v \in \mathrm{SC}_n)} \sum_{\mathcal{D}} P_{p_c}\bigl(\tau_v E, \tau_v F(\mathcal{D}), v \in \mathrm{SC}_n\bigr).$$

If $h(n) > M$ and $|v| \le n - h(n)$, equation (2.23) implies that the summand is

$$P_{p_c}(E, 0 \leftrightarrow \mathcal{D}) P_{p_c}\bigl(\tau_v F(\mathcal{D}), \tau_v \mathcal{D} \subset \mathrm{SC}_n\bigr).$$

For the first factor, (2.28) holds if N is large enough. Therefore choosing first N large, then M large, then n large, we get the claim as in Theorem 1. $\square$

3. Large clusters. The goal of this section is to show that if we pick a site uniformly at random from $\mathcal{C}_n^{(k)}$, the kth largest cluster in $B(n)$, then, again the IIC arises in the limit as $n \to \infty$. The main difficulty in applying the argument of the previous section is that for a fixed site v and circuit $\mathcal{D}$ we need to break up the event $\{v \in \mathcal{C}_n^{(k)}\}$ into two parts, one of which only depends on edges in $\mathrm{ext}(\tau_v \mathcal{D})$. For this we need to be able to identify the kth largest cluster in terms of the configuration in $\mathrm{ext}(\tau_v \mathcal{D})$ only. This is only possible if the gap between the sizes of $\mathcal{C}_n^{(k)}$ and $\mathcal{C}_n^{(k+1)}$ is typically large compared to $|\mathcal{C}_n^{(k)} \cap \mathrm{int}(\tau_v \mathcal{D})|$, where $\mathrm{int}(\tau_v \mathcal{D})$ denotes the interior of $\tau_v \mathcal{D}$, that is, the graph complement of $\mathrm{ext}(\tau_v \mathcal{D})$.

We show that the gap is indeed large by proving Proposition 1 in Section 3.1. Theorem 3 is proved in Section 3.2 and its proof can be read without the proof of Proposition 1.

The following result of Borgs, Chayes, Kesten and Spencer (2000) will be used throughout this section. It provides basic information about the size distribution of large clusters near p_c and we state it here for convenience. For the proof see Theorem 3.1 (i), Theorem 3.3 (i) and Section 6 in the cited reference.

THEOREM 9 (Borgs, Chayes, Kesten and Spencer). *For all $i \geq 1$ we have*

$$\liminf_{n\to\infty} P_{p_c}\left(K^{-1} \leq \frac{W_n^{(i)}}{s(n)} \leq K\right) \to 1 \qquad \text{as } K \to \infty. \tag{3.1}$$

3.1. *Gaps between cluster sizes.* The proof of Proposition 1 is based on a block argument. We briefly explain the argument before making the necessary definitions. Divide $B(n)$ into boxes with linear size $N \ll n$. For each box determine whether it contains an open circuit (surrounding a small box of half the size), and if it does, consider the outermost such circuit. Large clusters typically will contain many of these open circuits. Condition on the outermost circuit in each box and on the configuration outside the circuits (i.e., we condition on the states of all bonds that are not in the interior of any of the circuits). Then the configurations in the interiors of circuits corresponding to different boxes are conditionally independent. The idea is that this creates sufficient randomness to prevent the cluster sizes from being too close.

We spell out the proof for $k = 1$; the modifications for $k \geq 2$ are straightforward. We need some notation and definitions for the block argument. Divide $B(n)$ into smaller boxes congruent to Λ_N, where $N \geq 8$ will be a fixed power of 2 for the entire proof. The boxes are of the form $b(\mathbf{k}) = \Lambda_N + 2N\mathbf{k}$, where $\mathbf{k} \in \mathbb{Z}^2$. Whenever we say "box" during this proof we refer to one of these.

For each $\mathbf{k}$ determine if the annulus $\mathrm{an}(\mathbf{k}) = (\Lambda_N \setminus \Lambda_{N/2}) + 2N\mathbf{k}$ contains an open circuit and consider the outermost open circuit $\mathcal{D}_\mathbf{k}$ when there is one. Do this only for the boxes that are contained in $B(n)$. We define the graph $G_\mathbf{k}$ by $E(G_\mathbf{k}) = E(\operatorname{ext}\mathcal{D}_\mathbf{k}) \cap E(b(\mathbf{k}))$ [i.e., $G_\mathbf{k}$ is the part of $b(\mathbf{k})$ that lies outside $\mathcal{D}_\mathbf{k}$]. It is understood that $G_\mathbf{k} = b(\mathbf{k})$, if $\mathcal{D}_\mathbf{k}$ does not exist and $G_\mathbf{k} = b(\mathbf{k}) \cap B(n)$, if $b(\mathbf{k})$ intersects $B(n)$ but it is not entirely contained in $B(n)$. Given the configuration in $G_\mathbf{k}$ the configuration in $\operatorname{int}(\mathcal{D}_\mathbf{k})$ is an independent percolation process.

Fix an ordering of the boxes. Let $M \geq 1$ be an integer that we are going to choose later. We build the configuration in $B(n)$ in three steps.

(A) For each $\mathbf{k}$ pick the configuration in $G_\mathbf{k}$. This provides a configuration in the graph $\mathbf{G} = \bigcup_\mathbf{k} G_\mathbf{k}$.
(B) We mark certain boxes. Let $\mathcal{E}$ be an open cluster in the graph $\mathbf{G}$. Consider those $b(\mathbf{k})$ for which $\mathcal{D}_\mathbf{k}$ exists and is contained in $\mathcal{E}$, and mark the first M

such boxes (according to the fixed ordering of boxes). If there are not enough boxes available, mark as many as there are. Do the above marking procedure for all open clusters $\mathcal{E}$ in the graph $\mathbf{G}$. Now "fill in" the unmarked boxes, that is, pick the configuration in the set

$$\bigcup_{\substack{\mathbf{k}:\, b(\mathbf{k}) \text{ is} \\ \text{not marked}}} E(b(\mathbf{k})) \setminus E(G_{\mathbf{k}}).$$

At this stage we have picked the states of the edges of a graph $A_n(\omega)$, where we can define the graph $A_n(\omega)$ in the following way:

$$E(A_n(\omega)) \stackrel{\text{def}}{=} E(B(n)) \setminus \bigcup_{\substack{\mathbf{k}:\, b(\mathbf{k}) \text{ is} \\ \text{marked}}} E(\operatorname{int} \mathcal{D}_{\mathbf{k}}).$$

It follows from the construction that if G is any (nonrandom) subgraph of $B(n)$, then the event $K_G \stackrel{\text{def}}{=} \{A_n = G\}$ depends only on the edges in G.

(C) Let $\widetilde{\mathcal{C}}_n^{(1)}, \widetilde{\mathcal{C}}_n^{(2)}, \dots$ denote the open clusters in $A_n(\omega)$ ordered by size and let $\widetilde{W}_n^{(1)} \geq \widetilde{W}_n^{(2)} \geq \cdots$ denote their sizes, respectively. To each $\widetilde{\mathcal{C}}_n^{(j)}$ there correspond at most M marked boxes. Use the ordering of the boxes to label the open circuits in these marked boxes by $\mathcal{D}_1^{(j)}, \mathcal{D}_2^{(j)}, \dots$. Our final step is to "fill in" the interiors of the circuits $\mathcal{D}_t^{(j)}$, that is, pick the configuration inside these circuits.

Let

$$X_t^{(j)} = \big|\{w \in \operatorname{int}(\mathcal{D}_t^{(j)}) : w \leftrightarrow \mathcal{D}_t^{(j)}\}\big|, \qquad t = 1, \dots, M; \ j = 1, 2, \dots,$$

where we define $X_t^{(j)} = 0$ if $\mathcal{D}_t^{(j)}$ does not exist. Observe that the $X_t^{(j)}$ are conditionally independent, given the configuration obtained in step (B). If M is large we can hope that the distribution of $\sum_t X_t^{(j)}$ will be sufficiently spread out to give the result of the proposition. For $j = 1, 2, \dots$ let

$$Z_n^{(j)} = \widetilde{W}_n^{(j)} + \sum_{t=1}^{M} X_t^{(j)}. \tag{3.2}$$

That is, $Z_n^{(j)}$ is the size of the open cluster in $B(n)$ that contains $\widetilde{\mathcal{C}}_n^{(j)}$.

We prove Proposition 1 via three lemmas that correspond to the following three steps.

STEP 1. For some j_0, with large probability, $W_n^{(1)}$ and $W_n^{(2)}$ appear among the $Z_n^{(j)}$ with $1 \leq j \leq j_0$. This means we only have to care about a fixed number of $Z_n^{(j)}$'s.

STEP 2. For n large enough there are at least M marked boxes for $\widetilde{\mathcal{C}}_n^{(j)}$, $1 \leq j \leq j_0$, with large probability.

STEP 3. For a given r, the probability that $|Z_n^{(j')} - Z_n^{(j'')}| \le r$ for some $1 \le j' < j'' \le j_0$ is small, if M is large enough.

The following lemma takes care of Step 1.

LEMMA 2. *Consider the events*

$$A_1(i) = \bigcap_{1 \le j \le i} \{Z_n^{(j)} \ne W_n^{(1)}\},$$

$$A_2(i) = \bigcap_{1 \le j \le i} \{Z_n^{(j)} \ne W_n^{(2)}\}.$$

For any $\varepsilon > 0$ there is an $i = i(\varepsilon)$ such that, for all $M \ge 1$, there exists $n_0 = n_0(\varepsilon, M, N)$ such that, for all $n \ge n_0$, we have

$$P_{p_c}\big(A_1(i) \cup A_2(i)\big) \le \varepsilon. \tag{3.3}$$

PROOF. Write $P_{\ge t} = P_{p_c}(|\mathcal{C}(0)| \ge t)$, where $\mathcal{C}(0)$ denotes the cluster of the origin. Let $\widetilde{N}_n(t) = |\{j : t \le \widetilde{W}_n^{(j)}\}|$. We can bound $E_{p_c}\widetilde{N}_n(t)$ by an idea of Borgs, Chayes, Kesten and Spencer (2000). Namely, if $\mathcal{C}_n(v)$ and $\mathcal{C}(v)$ denote the connected component of the vertex v in $B(n)$ and in $\mathbb{Z}^2$, respectively, we have

$$\begin{aligned}
E_{p_c}\widetilde{N}_n(s(m)) &= \sum_{t=s(m)}^{\infty} \sum_{v \in B(n)} \frac{1}{t} P_{p_c}\big(v \in \widetilde{\mathcal{C}}_n^{(j)}, \widetilde{W}_n^{(j)} = t\big) \\
&\le \frac{1}{s(m)} \sum_{v \in B(n)} P_{p_c}\big(v \in \widetilde{\mathcal{C}}_n^{(j)}, \widetilde{W}_n^{(j)} \ge s(m)\big) \\
&\le \frac{1}{s(m)} \sum_{v \in B(n)} P_{p_c}\big(|\mathcal{C}_n(v)| \ge s(m)\big) \\
&\le \frac{1}{s(m)} \sum_{v \in B(n)} P_{p_c}\big(|\mathcal{C}(v)| \ge s(m)\big) \\
&= \frac{(2n+1)^2 P_{\ge s(m)}}{s(m)} \\
&\le C_1 \left(\frac{n}{m}\right)^2.
\end{aligned}$$

In the last inequality we used (1.8).

Now choose $C_2 = C_2(\varepsilon) > 0$ in such a way that

$$P_{p_c}\big(W_n^{(2)} \ge C_2 s(n)\big) \ge 1 - \frac{\varepsilon}{2}, \tag{3.4}$$

which is possible by (3.1). Let m be the largest integer such that $s(m) \le (C_2/2)s(n)$. Note that if n is large enough there exists such m by (1.9) and if we choose the largest one, we have $n/m \le C_3$ for some constant C_3. Let $i = i(\varepsilon)$ be an integer such that

$$i + 1 \ge \frac{2C_3^2 C_1}{\varepsilon}.$$

Then

$$\begin{aligned} P_{p_c}\biggl(\widetilde{W}_n^{(i+1)} \ge \frac{C_2}{2} s(n)\biggr) &\le P_{p_c}\bigl(\widetilde{W}_n^{(i+1)} \ge s(m)\bigr) \le P_{p_c}\bigl(\widetilde{N}_n(s(m)) \ge i+1\bigr) \\ &\le \frac{E_{p_c}\widetilde{N}_n(s(m))}{i+1} \le \frac{C_1(n/m)^2}{i+1} \le \frac{\varepsilon}{2}. \end{aligned} \tag{3.5}$$

For indices $j > i$, and on the complement of the event on the left-hand side of (3.5) we have

$$\begin{aligned} Z_n^{(j)} &\le \widetilde{W}_n^{(j)} + M(2N)^2 \le \widetilde{W}_n^{(i+1)} + M(2N)^2 \\ &\le \frac{C_2}{2} s(n) + M(2N)^2, \end{aligned} \tag{3.6}$$

where we used the trivial bound $X_t^{(j)} \le (2N)^2$.

Let n be so large that $M(2N)^2 < C_2 s(n)/2$. Then on the event on the left hand side of (3.4) we have that all clusters that are entirely contained in the interiors of the circuits [i.e., in $B(n) \setminus A_n(\omega)$] have size strictly less than $W_n^{(2)}$. Also, on this event, (3.6) implies

$$Z_n^{(j)} < W_n^{(2)},$$

for $j > i$. Therefore $W_n^{(1)}$ and $W_n^{(2)}$ have to occur among $Z_n^{(1)}, \dots, Z_n^{(i)}$ with probability at least $1 - \varepsilon$. This completes the proof of Lemma 2. □

REMARK. Note that we can in fact let M grow with n, as long as it grows slower than $s(n)$ (recall that N is fixed). This is important in proving the extension mentioned in the remark after Proposition 1.

Before stating the lemma that corresponds to Step 2, we introduce some terminology. If $\mathcal{E}$ is a connected subgraph of $B(n)$, which may or may not be a cluster, we say that a box $b(\mathbf{k})$ is *good* for $\mathcal{E}$, if there is an open circuit in the annulus $\mathrm{an}(\mathbf{k})$ and the outermost open circuit is a subset of $\mathcal{E}$.

LEMMA 3. *For any $\varepsilon > 0$, $i < \infty$, $M > 0$, there is an $n_1 = n_1(\varepsilon, i, M, N)$ such that, for $n \ge n_1$, we have*

$$P_{p_c}\bigl(\text{there are at least } M \text{ good boxes for } \widetilde{\mathcal{C}}_n^{(j)}, 1 \le j \le i\bigr) \ge 1 - \varepsilon. \tag{3.7}$$

The idea of the proof is to show that $\widetilde{\mathcal{C}}_n^{(j)}$ crosses some rectangle of linear size δn with large probability for some small $\delta > 0$. This is achieved by showing that its size $\widetilde{W}_n^{(i)}$ is of order $s(n)$, hence its diameter has to be of order n. Then we apply a block-version of the argument of Theorem 8 to show that spanning clusters have many good boxes.

PROOF OF LEMMA 3. We first want to show that there exists a C_1 such that $\widetilde{W}_n^{(i)} \ge C_1 s(n)$ holds with large probability. To this end we first prove that for any j we have

$$W_n^{(j)} - M(2N)^2 \le \widetilde{W}_n^{(j)} \le W_n^{(j)} \tag{3.8}$$

(whenever $\widetilde{\mathcal{C}}_n^{(j)}$ exists). Start by showing the second inequality. If we had $\widetilde{W}_n^{(j)} > W_n^{(j)}$ for some j, then for all $\ell \le j$ we would have

$$Z_n^{(\ell)} \ge \widetilde{W}_n^{(\ell)} \ge \widetilde{W}_n^{(j)} > W_n^{(j)},$$

which implies that

$$\left|\{\ell : Z_n^{(\ell)} > W_n^{(j)}\}\right| \ge j.$$

Since all the $Z_n^{(\ell)}$'s are cluster sizes, this is a contradiction.

Suppose now that the first inequality in (3.8) does not hold. Then for $\ell \ge j$ we have

$$Z_n^{(\ell)} \le \widetilde{W}_n^{(\ell)} + M(2N)^2 \le \widetilde{W}_n^{(j)} + M(2N)^2 < W_n^{(j)}.$$

We also have $W_n^{(1)} \ge \cdots \ge W_n^{(j)} > M(2N)^2$. These two facts together imply that the only candidates for $W_n^{(1)}, \dots, W_n^{(j)}$ are $Z_n^{(1)}, \dots, Z_n^{(j-1)}$, a contradiction.

Next use (3.1) to choose $C_1 = C_1(i, \varepsilon)$, such that (for large n)

$$P_{p_c}\big(W_n^{(i)} \ge 2C_1 s(n)\big) \ge 1 - \frac{\varepsilon}{2}.$$

Then by (3.8) there is an $n_2 = n_2(C_1, M)$, such that for $n \ge n_2$ we have

$$P_{p_c}\big(\widetilde{W}_n^{(i)} \ge C_1 s(n)\big) \ge 1 - \frac{\varepsilon}{2}.$$

Note that if there are fewer than M good boxes for $\widetilde{\mathcal{C}}_n^{(j)}$ then the cluster $\mathcal{E}$ containing $\widetilde{\mathcal{C}}_n^{(j)}$ has fewer than M good boxes. Therefore, to prove (3.7) it is sufficient to show that for n large enough

$$P_{p_c}\begin{pmatrix}\text{there is an open cluster } \mathcal{E} \text{ in } B(n) \text{ with } |\mathcal{E}| \ge C_1 s(n), \\ \text{but there are fewer than } M \text{ good boxes for } \mathcal{E}\end{pmatrix} \le \frac{\varepsilon}{2}. \tag{3.9}$$

As we said our plan now is to use that a cluster of order $s(n)$ has diameter comparable to n and that it necessarily crosses some rectangle whose sides are comparable to n.

REMARK. The reduction to spanning clusters seems to be a detour here; however, we were unable to use the method of Theorem 3.3 of Borgs, Chayes, Kesten and Spencer (2000) directly.

By Remark (xiii) of Borgs, Chayes, Kesten and Spencer (1999) we have constants c and d such that for $x > 0$, $n \geq 1$, $4/n \leq y \leq 1$,

$$P_{p_c}\big(\text{there is a cluster } \mathcal{E} \subset B(n) \text{ with } \operatorname{diam}(\mathcal{E}) \leq yn \text{ but } |\mathcal{E}| \geq xs(n)\big) \\ \leq C_2 y^{-2} \exp[-C_3 x/y].$$

This implies that there is a $\delta = \delta(C_1, \varepsilon)$ such that for n large enough

$$P_{p_c}\big(\text{there is a cluster } \mathcal{E} \subset B(n) \text{ with } |\mathcal{E}| \geq C_1 s(n), \text{ but } \operatorname{diam}(\mathcal{E}) \leq \delta n\big) \\ \leq \frac{\varepsilon}{4}. \tag{3.10}$$

Given that $\operatorname{diam}(\mathcal{E}) \geq \delta n$ we can find a rectangle that is crossed by $\mathcal{E}$ by the following argument. One can cover $B(n)$ by a family $\mathcal{U}$ of rectangles that consists of translates of $[0, \delta n/4] \times [0, \delta n/2]$ and $[0, \delta n/2] \times [0, \delta n/4]$ in such a way that:

(a) Each $Q \in \mathcal{U}$ is contained in $B(n)$.

(b) If $\mathcal{E}$ is a connected subgraph of $B(n)$ and $\operatorname{diam}(\mathcal{E}) > \delta n$ then there is a $Q \in \mathcal{U}$ that is spanned by $\mathcal{E}$ in the short direction.

(c) $|\mathcal{U}| \leq C_7/\delta^2$.

We fix such a family $\mathcal{U}$. Then (3.9) will follow from (3.10) and the following: for each $Q \in \mathcal{U}$ (and n large enough)

$$P_{p_c}\left(\begin{array}{c}\text{there is a cluster } \mathcal{E} \text{ such that } \mathcal{E} \text{ spans } Q \text{ in the short direction,}\\ \text{but there are fewer than } M \text{ good boxes for } \mathcal{E}\end{array}\right) \\ \leq \frac{\varepsilon}{4|\mathcal{U}|}. \tag{3.11}$$

Let $Q_0 = [0, \delta n/4] \times [0, \delta n/2]$. Then (3.11) follows if we show the bound

$$P_{p_c}\left(\begin{array}{c}\text{there is a cluster } \mathcal{E}_0 \text{ in } Q_0 \text{ spanning } Q_0 \text{ horizontally,}\\ \text{but there are fewer than } M \text{ good boxes for } \mathcal{E}_0\end{array}\right) \leq \frac{\varepsilon\delta^2}{4C_7}, \tag{3.12}$$

where we used the bound (c) to replace $|\mathcal{U}|$ by a constant. We note that the event in (3.12) depends only on the configuration in Q_0.

Since $P_{p_c}(\text{there is an open horizontal crossing in } Q_0) \leq 1 - \eta < 1$, the BK inequality, van den Berg and Kesten (1985), implies that there is an $\ell_0 = \ell_0(\varepsilon, \delta)$ such that

$$P_{p_c}(\text{there are more than } \ell_0 \text{ spanning clusters in } Q_0) \leq \frac{\varepsilon\delta^2}{8C_7}.$$

So we need

$$(3.13)\quad P_{p_c}\left(\begin{array}{c}\mathcal{E}_0 \text{ is the } \ell\text{th lowest spanning cluster}\\ \text{and } \mathcal{E}_0 \text{ has fewer than } M \text{ good boxes}\end{array}\right) \le \frac{\varepsilon\delta^2}{8C_7\ell_0}, \qquad 1 \le \ell \le \ell_0.$$

We are going to prove this along the same lines as the lower bound in part (ii) of Theorem 8. The first step, again, is to ensure that there is "enough space" above $\mathcal{E}_0$.

Put $n' = \lfloor \delta n/4 \rfloor$. We use the following generalization of Lemma 1.

LEMMA 4. *There are constants $c_1, \alpha > 0$, such that, for any $\ell \ge 1$,*

$$P_{p_c}\left(\begin{array}{c}\textit{the } \ell\textit{th lowest spanning cluster of } Q_0 \textit{ exists}\\ \textit{but it is not contained in } [0, n'] \times [0, (1+a)n']\end{array}\right) \le c_1(1-a)^\alpha.$$

We defer the proof to the end of this subsection. By Lemma 4 we have an $a = a(\varepsilon, \delta)$ satisfying $1/2 < a < 1$, such that

$$(3.14)\quad P_{p_c}\left(\begin{array}{c}\text{there are at least } \ell \text{ spanning clusters and}\\ \text{the } \ell\text{th one is not contained in } [0, n'] \times [0, (1+a)n']\end{array}\right) \le \frac{\varepsilon\delta^2}{16C_7\ell_0}$$

uniformly in n.

Suppose there are at least ℓ (horizontal) spanning clusters. Then there exist open horizontal crossings R_i of $[0, n'] \times [0, 2n']$ $(1 \le i \le \ell)$, such that:

(i) R_1 is the lowest open horizontal crossing,

(ii) R_i is the lowest open horizontal crossing disjoint from the cluster containing R_{i-1} (with free boundary conditions).

We are going to condition on $\{R_\ell = r_0\}$, where r_0 is a horizontal crossing inside $[0, n'] \times [0, (1+a)n']$. This event only depends on edges on and below r_0. We define the graph H (the region above r_0), the highest vertex v of r_0 and the set $A(m)$ as in the proof of Theorem 8. We are going to estimate the number of boxes inside $A(m)$ that are good for R_ℓ. We assume that v is in the right half of Q_0; the other case is analogous.

Because $a > 1/2$, requiring $7m \le (1-a)n'$ ensures that the rectangle $v + [-7m, 0] \times [0, 7m]$ lies in H.

Define

$$V(m) = \{b(\mathbf{k}) \subset A(m) : \mathcal{D}_{\mathbf{k}} \text{ exists and } \mathcal{D}_{\mathbf{k}} \leftrightarrow R_\ell \text{ inside } v + \mathrm{An}(7m, m)\},$$

$$Y(m) = |V(m)|.$$

The number of good boxes for the ℓth lowest spanning cluster is at least $Y(m)$. We next estimate the moments of $Y(m)$.

Lower bound. We use the modified configuration ω', just as in the proof of Theorem 8. Recall that in H, ω' is the same as ω and it is a new independent configuration everywhere else.

For a box $b(\mathbf{k})$ to be in $V(m)$ it is sufficient that in the configuration ω' the following three events occur:

(i) $B_1' = \{\text{there is an open circuit } \mathcal{D} \text{ in } \mathrm{an}(\mathbf{k})\}$.

(ii) $B_2' = \left\{ \begin{array}{c} \partial(\Lambda_{N/2} + 2N\mathbf{k}) \leftrightarrow \text{ left side of } B(v, 7m) \\ \text{inside } (v + \mathrm{An}(7m, m)) \cap Q_0 \end{array} \right\}$.

(iii) $F' = F'(m) = \{\text{there is an open circuit in } v + \mathrm{An}(7m, 5m)\}$.

Applying the FKG inequality, the RSW theorem and (1.10) we get

$$\begin{aligned} P_{p_c}\big(b(\mathbf{k}) \in V(m)\big) &\geq P'_{p_c}(B_1' \cap B_2' \cap F') \geq P'_{p_c}(B_1')P'_{p_c}(B_2')P'_{p_c}(F') \\ &\geq C_8 \frac{\pi(m)}{\pi(N)}. \end{aligned}$$

This implies the lower bound

$$E_{p_c} Y(m) \geq C_9 \frac{m^2 \pi(m)}{N^2 \pi(N)}.$$

Upper bound. We note that if $b(\mathbf{k}) \subset A(m)$ and k_1 is the largest integer multiple of N for which $\Lambda_{k_1} + 2N\mathbf{k}$ does not touch $\partial A(m)$, then

$$P_{p_c}\big(b(\mathbf{k}) \in V(m)\big) \leq P_{p_c}\big(\partial b(\mathbf{k}) \leftrightarrow \partial(\Lambda_{k_1} + 2N\mathbf{k})\big) \leq C_{10} \frac{\pi(k_1)}{\pi(N)},$$

by (1.10). For a given integer r the number of boxes for which $k_1 = rN$ is bounded by $C_{11} m/N$. Therefore, an application of (1.13) with $\beta = 0$ yields

$$E_{p_c} Y(m) \leq \sum_{r=0}^{m/N} C_{11} C_{10} \frac{m}{N} \frac{\pi(rN)}{\pi(N)} \leq C_{12} \frac{m^2 \pi(m)}{N^2 \pi(N)}.$$

The upper bound

$$E_{p_c} Y^2(m) \leq C_{13} \left(\frac{m^2 \pi(m)}{N^2 \pi(N)} \right)^2$$

follows similarly using (1.13) and (1.14).

As in the proof of Theorem 8 we conclude that there is an $\eta = \eta(\varepsilon, a, \delta, N) > 0$ such that, if $G_n(\ell)$ denotes the set of good boxes for the ℓth lowest spanning cluster, then uniformly in r_0 we have

$$P_{p_c}\left(|G_n(\ell)| < \eta \frac{s(n)}{s(N)} \;\middle|\; R_\ell = r_0 \right) \leq \frac{\varepsilon \delta^2}{16 C_7 \ell_0}. \tag{3.15}$$

For n large enough $\eta s(n)/s(N) > M$, hence (3.15) and (3.14) imply (3.13). This completes the proof of Lemma 3. $\square$

REMARK. Again, the dependence of n_1 on M in Lemma 3 can be replaced by the condition that M grows slower than $s(n)$.

In establishing Step 3 we are going to apply the Kolmogorov–Rogozin inequality [see Esseen (1966)] in the following form.

LEMMA 5. *For a random variable Y define its* concentration function *by*

$$Q(Y;\lambda)=\sup_x P(x\le Y\le x+\lambda),\qquad \lambda\ge 0.$$

Let $Y_1,\dots,Y_n$ be independent, $S_n=\sum_{s=1}^n Y_s$. Then there is a universal constant C such that, for any real number $0<\lambda\le L$, one has

$$Q(S_n;L)\le C\frac{L}{\lambda}\left\{\sum_{s=1}^n\bigl(1-Q(Y_s;\lambda)\bigr)\right\}^{-1/2}.$$

The following lemma takes care of Step 3.

LEMMA 6. *Let $X_s^{(j)}$ $(1\le j\le i, 1\le s\le M)$ be independent random variables with distribution*

$$X_s^{(j)}\stackrel{d}{=}\bigl|\bigl\{v\in\operatorname{int}\bigl(\mathcal{D}(j,s)\bigr):v\leftrightarrow\mathcal{D}(j,s)\bigr\}\bigr|,$$

where $\mathcal{D}(j,s)$ are arbitrary (nonrandom) circuits inside $\Lambda_N\setminus\Lambda_{N/2}$. Also, let $a_1,\dots,a_i$ be arbitrary integers. Then for any $\varepsilon>0$, $r<\infty$ and positive integer i there is an integer $M=M(\varepsilon,i,N,r)$ such that

$$P_{p_c}\left(\left|a_{j'}+\sum_{s=1}^M X_s^{(j')}-a_{j''}-\sum_{s=1}^M X_s^{(j'')}\right|\le r\text{ for some }1\le j'<j''\le i\right)\le\varepsilon. \tag{3.16}$$

PROOF. It is sufficient to show that if $Y_s=X_s^{(j')}-X_s^{(j'')}$, then $Q(S_M;2r)\le\varepsilon/\binom{i}{2}$. Since Y_s is integer valued and nonconstant, taking $\lambda=1/2$ we have $Q(Y_s;1/2)<1$. To get an estimate which is uniform over the circuits we note that there are finitely many choices for the circuits $\mathcal{D}(j',s)$ and $\mathcal{D}(j'',s)$ and the set of all possible circuits does not depend on s, only on N. Therefore, we have

$$\delta=\delta(N)=\sup_s\sup_{\substack{\mathcal{D}(j',s)\\ \mathcal{D}(j'',s)}}Q(Y_s;1/2)=\sup_{\substack{\mathcal{D}(j',1)\\ \mathcal{D}(j'',1)}}Q(Y_1;1/2)<1.$$

This gives us

$$Q(S_M;2r)\le C4r[M(1-\delta)]^{-1/2}\stackrel{M\to\infty}{\to}0.$$

This proves that there is an M, such that (3.16) holds, and Lemma 6 is proved. $\square$

REMARK. The dependence of M on r can be replaced by the requirement that $r/\sqrt{M}$ is sufficiently small.

Now we are ready to assemble the proof of Proposition 1.

PROOF OF PROPOSITION 1. Fix $\varepsilon > 0$ and $r < \infty$. Choose i so that (3.3) in Lemma 2 is satisfied. Then choose M so that (3.16) in Lemma 6 is satisfied.

Using the notation of Lemma 2 let

$$\begin{aligned} A &= A_1^c(i) \cap A_2^c(i) \\ &= \{W_n^{(1)} = Z_n^{(j')} \text{ and } W_n^{(2)} = Z_n^{(j'')} \text{ for some } 1 \le j', j'' \le i\}. \end{aligned}$$

Let

$$B = \{\text{for } 1 \le j \le i \text{ there are at least } M \text{ good boxes for } \widetilde{\mathcal{C}}_n^{(j)}\}$$

denote the event in the statement of Lemma 3, and let

$$C = \{|Z_n^{(j')} - Z_n^{(j'')}| \le r \text{ for some } 1 \le j' < j'' \le i\}.$$

We recall that the way the graph $A_n(\omega)$ was defined implies that the event $K_G = \{A_n = G\}$ depends only on the states of the edges of G and so does B. Let σ denote a configuration on G for which K_G occurs. Then given K_G, B and the configuration σ, the variables $X_s^{(j)}$ are conditionally independent and are determined by Bernoulli percolation processes in the interiors of the corresponding circuits. By Lemma 6 we have

$$P_{p_c}(C \mid B, K_G, \sigma) \le \varepsilon,$$

so

$$P_{p_c}(C \cap B) = P_{p_c}(C \mid B) P_{p_c}(B) \le \varepsilon P_{p_c}(B) \le \varepsilon.$$

Choosing n large enough we have $P_{p_c}(B^c) \le \varepsilon$ by Lemma 3 and $P_{p_c}(A^c) \le \varepsilon$ by Lemma 2. Therefore,

$$\begin{aligned} P_{p_c}(W_n^{(1)} - W_n^{(2)} \le r) &\le P_{p_c}(A^c) + P_{p_c}(C) \le P_{p_c}(A^c) + P_{p_c}(B^c) + P_{p_c}(B \cap C) \\ &\le 3\varepsilon \end{aligned}$$

for n large enough, which proves Proposition 1. $\square$

REMARK. Our remarks after Lemmas 2, 3 and 6 prove the remark after Proposition 1.

PROOF OF LEMMA 4. We assume $\ell \ge 2$. There are only slight changes when $\ell = 1$. Define inductively the open horizontal crossings R_i of $[0, n'] \times [0, 2n']$ by:

(i) R_1 is the lowest open crossing,
(ii) R_i is the lowest open crossing disjoint from the cluster containing R_{i-1}.

Let r_0 be a path that crosses $[0, n'] \times [0, 2n']$ horizontally. We condition on the event $\{R_{\ell-1} = r_0\}$, and show that uniformly in r_0 we have

$$P_{p_c}\big(R_\ell \text{ exists, but } R_\ell \not\subset [0, n'] \times [0, (1+a)n'] \mid R_{\ell-1} = r_0\big) \\ \le c_1(1-a)^\alpha, \tag{3.17}$$

which implies the lemma.

First assume that r_0 is contained in $S_a = [0, n'] \times [0, (1+a)n']$. Let Q_0^* and S_a^* denote the following dual graphs

$$\begin{aligned} Q_0^* &= \text{part of } \big[\tfrac{1}{2}, n' - \tfrac{1}{2}\big] \times \big[-\tfrac{1}{2}, 2n' + \tfrac{1}{2}\big] \text{ lying above } r_0, \\ S_a^* &= \text{part of } \big[\tfrac{1}{2}, n' - \tfrac{1}{2}\big] \times \big[-\tfrac{1}{2}, \lceil (1+a)n' \rceil + \tfrac{1}{2}\big] \text{ lying above } r_0. \end{aligned} \tag{3.18}$$

Let U^* denote the top side of Q_0^*.

If R_ℓ exists but is not contained in S_a, then there exists a closed vertical dual crossing of S_a^*. We show that with large probability this vertical crossing can be extended to U^*. Let T (resp. T') denote the leftmost (resp. rightmost) vertical dual crossing of S_a^*. Let v (resp. v') denote the top vertex of T (resp. T'). When T and T' exist, at least one of the events $\{v_1 \le 0\}$, $\{v'_1 \ge 0\}$ has to occur. Therefore we have

$$\begin{aligned} &P_{p_c}(R_\ell \text{ exists}, \ R \not\subset S_a \mid R_{\ell-1} = r_0) \\ &\quad \le P_{p_c}(R_\ell \text{ exists}, \ T \text{ and } T' \text{ exist} \mid R_{\ell-1} = r_0) \\ &\quad \le P_{p_c}(R_\ell \text{ exists}, \ T \text{ exists}, \ v_1 \le 0 \mid R_{\ell-1} = r_0) \\ &\qquad + P_{p_c}(R_\ell \text{ exists}, \ T' \text{ exists}, v'_1 \ge 0 \mid R_{\ell-1} = r_0). \end{aligned} \tag{3.19}$$

We are going to bound the first term on the right-hand side; by symmetry of the argument the second term will have the same bound.

When R_ℓ exists, there can be no closed dual path from T to U^* inside Q_0^*. This means that the first term on the right-hand side of (3.19) is at most

$$P_{p_c}(T \text{ exists}, \ v_1 \le 0, \ T \text{ is not connected to } U^* \text{ by a closed path inside } Q_0^*).$$

We break up the event $\{ T \text{ exists}, \ v_1 \le 0\}$ as a disjoint union of the events $\{T = t_0\}$ over paths t_0 that have their top vertex in the left half of S_a^* and show

$$P_{p_c}(T \text{ is not connected to } U^* \text{ inside } Q_0^* \mid T = t_0) \le c_1(1-a)^\alpha, \tag{3.20}$$

where by connected we mean connected by a closed dual path. Let $G = G_{t_0}$ denote the part of S_a^* to the left of t_0 (including the vertices and edges of t_0), and let $H = H_{t_0} = Q_0^* \setminus G$. Consider the sets

$$A_k = \big(B(v, 3^{k+1}) \setminus B(v, 3^k)\big) \cap H$$

for values of k that satisfy

$$n'(1-a) \le 3^k \le n'/6. \tag{3.21}$$

Let

$$F(k) = \left\{ \begin{array}{c} \text{there is a closed dual path } r \\ \text{connecting } t_0 \text{ to the top side of } Q_0^* \text{ inside } A_k \end{array} \right\}$$

The sets A_k have been chosen in such a way that

$$\begin{aligned} &P_{p_c}(F(k) \mid T = t_0) \\ &\quad \ge P_{p_c}\big(\text{there is a closed dual circuit in } B(3^{k+1}) \setminus B(3^k)\big) \ge c_2 > 0. \end{aligned}$$

The number of integers k that satisfy (3.21) is at least $-c_3 \log_3 3(1-a)$, where $c_3 > 0$. Since the events $F(k)$ are conditionally independent given $T = t_0$, the probability on the left-hand side of (3.20) is at most

$$P_{p_c}\left(\bigcap_k F^c(k) \,\middle|\, T = t_0\right) = \prod_k (1-c_2) \le c_4(1-a)^\alpha,$$

where $\alpha = -\log_3(1-c_2)$. Equations (3.19) and (3.20) imply (3.17) in the case r_0 is contained in S_a.

When r_0 is not contained in S_a, we show that

$$P_{p_c}(R_\ell \text{ exists} \mid R_{\ell-1} = r_0) \le c_1(1-a)^\alpha. \tag{3.22}$$

For this we show that with high conditional probability r_0 is connected by a closed dual path to the top side of Q_0, hence there is no open horizontal crossing above $R_{\ell-1}$. Let u be a vertex of r_0 with second coordinate larger than $(1+a)n'$. The same way we proved (3.20) (using dual circuits in annuli centered at u this time) we can show that

$$\begin{aligned} &P_{p_c}(\text{there is no closed dual path from } r_0 \text{ to the top side of } Q_0 \mid R_{\ell-1} = r_0) \\ &\quad \le c_1(1-a)^\alpha. \end{aligned}$$

This justifies (3.22) and completes the proof of the lemma. □

3.2. *Large clusters look like the IIC.* We are ready to prove Theorem 3, the principal result of this section. The argument goes along similar lines as the proof of Theorem 1.

PROOF OF THEOREM 3. Let $\varepsilon > 0$ be given. Choose $N = N(\varepsilon, E)$, as in (2.28), such that

$$(1+\varepsilon)^{-1}\nu(E) \le \frac{P_{p_c}(E, 0 \leftrightarrow \mathcal{D})}{P_{p_c}(0 \leftrightarrow \mathcal{D})} \le (1+\varepsilon)\nu(E) \tag{3.23}$$

for any circuit $\mathcal{D}$ surrounding $B(N)$. As in (2.13) and (2.21) we define the events F and $F(\mathcal{D})$:

$$\begin{aligned} F = F(M,N) &= \{\text{there is an open circuit in } \mathrm{An}(M,N)\} \\ &= \bigcup_{\mathcal{D}} F(\mathcal{D}). \end{aligned}$$

We choose M later. By (3.1) there is an $x = x(\varepsilon)$ such that

$$P_{p_c}\big(W_n^{(k)} \ge xs(n)\big) \ge 1 - \frac{\varepsilon}{2}.$$

Then we have

$$\begin{aligned} P_{p_c}(\tau_{I_n} F^c) &= \sum_{v \in B(n)} E_{p_c}\biggl(\frac{I[\tau_v F^c, v \in \mathcal{C}_n^{(k)}]}{W_n^{(k)}}\biggr) \\ &\le \frac{\varepsilon}{2} + \sum_{v \in B(n)} E_{p_c}\biggl(\frac{I[\tau_v F^c, v \in \mathcal{C}_n^{(k)}, W_n^{(k)} \ge xs(n)]}{W_n^{(k)}}\biggr) \\ &\le \frac{\varepsilon}{2} + \frac{1}{xs(n)} \sum_{v \in B(n)} P_{p_c}\big(\tau_v F^c, |\mathcal{C}(v)| \ge xs(n)\big) \\ &\le \frac{\varepsilon}{2} + \frac{P_{p_c}(F^c)}{xs(n)} \sum_{v \in B(n)} P_{p_c}\big(|\mathcal{C}(v)| \ge xs(n)\big), \end{aligned} \tag{3.24}$$

where in the last step we used the FKG inequality. To bound the sum let m be the largest integer such that $xs(n) \ge s(m)$. For n large enough such m exists by (1.9) and we have $n/m \le C_1 = C_1(x)$. By (1.8) we have

$$P_{p_c}\big(|\mathcal{C}(v)| \ge xs(n)\big) \le P_{p_c}\big(|\mathcal{C}(0)| \ge s(m)\big) \le C_2 \pi(m).$$

Thus the right-hand side of (3.24) is bounded by

$$\frac{\varepsilon}{2} + \frac{P_{p_c}(F^c) C_2 (2n+1)^2 \pi(m)}{xs(n)} \le \frac{\varepsilon}{2} + P_{p_c}(F^c) C_3(x), \tag{3.25}$$

by virtue of (1.9). The second term in (3.25) can be made less than $\varepsilon/2$ by choosing M large. This shows that with high probability the random vertex I_n is surrounded by an open circuit.

We define $G_n = \{I_n \in B(n - f(n))\}$, where $f(n) \to \infty$, and $f(n) = o(n\pi(n))$, as in the proof of Theorem 1. Using the tightness of $W_n^{(k)}$ again, it follows that $P_{p_c}(G_n^c) \le \varepsilon$ for n large.

For a circuit $\mathcal{D}$ let

$$\widehat{\mathcal{C}}_n^{(k)}(\mathcal{D}) = k\text{th largest cluster in } B(n) \setminus \mathrm{int}(\mathcal{D}).$$

We define an event B_n on which the cluster sizes are well-behaved and the decoupling works. Let $g(n)$ be a function for which $g(n) \to \infty$ in such a way that $g(n) = o(s(n))$. Put

$$B_n = \{W_n^{(1)} - W_n^{(2)} > |B(M)|, W_n^{(1)} > g(n)\} \qquad \text{if } k = 1$$

and

$$B_n = \left\{ \begin{matrix} W_n^{(k)} - W_n^{(k+1)} > |B(M)|, W_n^{(k)} > g(n) \\ W_n^{(k-1)} - W_n^{(k)} > |B(M)| \end{matrix} \right\} \qquad \text{if } k \geq 2.$$

Since M is fixed, we have $P_{p_c}(B_n) \to 1$ by Proposition 1 and (3.1).

It follows that for n large enough

$$\begin{aligned} P_{p_c}(\tau_{I_n} E) &\leq 3\varepsilon + P_{p_c}(\tau_{I_n} E \cap \tau_{I_n} F \cap B_n \cap G_n) \\ &= 3\varepsilon + \sum_{|v| \leq n - f(n)} E_{p_c}\left(\frac{I[\tau_v E, v \in \mathcal{C}_n^{(k)}, \tau_v F] I[B_n]}{W_n^{(k)}} \right) \\ &= 3\varepsilon + \sum_{|v| \leq n - f(n)} \sum_{\mathcal{D}} E_{p_c}\left(\frac{I[\tau_v E, v \in \mathcal{C}_n^{(k)}, \tau_v F(\mathcal{D})] I[B_n]}{W_n^{(k)}} \right). \end{aligned} \tag{3.26}$$

Now fix v and $\mathcal{D}$, and consider the indicators inside the expectation on the right-hand side. Suppose that

$$\text{the events } \{v \in \mathcal{C}_n^{(k)}\}, \tau_v F(\mathcal{D}) \text{ and } B_n \text{ occur.} \tag{$*$}$$

Then we show that (for n large) $(*)$ implies

$$\mathcal{C}_n^{(k)} \setminus \operatorname{int}(\tau_v \mathcal{D}) = \widehat{\mathcal{C}}_n^{(k)}(\tau_v \mathcal{D}) \text{ and } \{v \leftrightarrow \tau_v \mathcal{D}\} \text{ and } \{\tau_v \mathcal{D} \subset \widehat{\mathcal{C}}_n^{(k)}\} \text{ occur.} \tag{3.27}$$

This is almost obvious, except that we need to rule out the absurd possibility

$$\widehat{\mathcal{C}}_n^{(k)}(\tau_v \mathcal{D}) \neq \mathcal{C}_n^{(k)} \setminus \operatorname{int}(\tau_v \mathcal{D}).$$

For this, first note that the inequalities

$$W_n^{(1)} \geq \cdots \geq W_n^{(k)} > g(n) > |B(M)|$$

imply that the $k-1$ largest clusters all lie entirely in $\operatorname{ext}(\tau_v \mathcal{D})$, and

$$\widehat{\mathcal{C}}_n^{(1)}(\tau_v \mathcal{D}) = \mathcal{C}_n^{(1)}, \ldots, \widehat{\mathcal{C}}_n^{(k-1)}(\tau_v \mathcal{D}) = \mathcal{C}_n^{(k-1)}.$$

The inequalities

$$\left|\mathcal{C}_n^{(k)} \setminus \operatorname{int}(\tau_v \mathcal{D})\right| \geq W_n^{(k)} - |B(M)| > W_n^{(k+1)} \geq \left|\widehat{\mathcal{C}}_n^{(k+1)}(\tau_v \mathcal{D})\right|$$

now imply the equality in (3.27) and the rest of (3.27) follows. On the events in $(*)$ the event $\widehat{B}_n$ defined below also occurs:

$$\widehat{B}_n = \{|\widehat{\mathcal{C}}_n^{(1)}| > g(n) - |B(M)|\} \qquad \text{if } k = 1$$

and

$$\widehat{B}_n = \{|\widehat{\mathcal{C}}_n^{(k)}| > g(n) - |B(M)|, |\widehat{\mathcal{C}}_n^{(k-1)}| - |\widehat{\mathcal{C}}_n^{(k)}| > |B(M)|\} \qquad \text{if } k \geq 2.$$

Note that $\widehat{B}_n$ only depends on the configuration outside $\tau_v\mathcal{D}$. Equation (3.27) also implies that $W_n^{(k)} \geq |\widehat{\mathcal{C}}_n^{(k)}|$. Putting these observations together we get that

$$\begin{aligned} &\frac{I[\tau_v E, v \in \mathcal{C}_n^{(k)}, \tau_v F(\mathcal{D})] I[B_n]}{W_n^{(k)}} \\ &\qquad \leq I[\tau_v E, v \leftrightarrow \tau_v \mathcal{D}] \frac{I[\tau_v F(\mathcal{D}), \tau_v \mathcal{D} \subset \widehat{\mathcal{C}}_n^{(k)}] I[\widehat{B}_n]}{|\widehat{\mathcal{C}}_n^{(k)}|}. \end{aligned} \tag{3.28}$$

Put

$$\begin{aligned} X_{\mathcal{D},v,E} &= I[\tau_v E, v \leftrightarrow \tau_v \mathcal{D}], \\ Y_{\mathcal{D},v,n} &= \frac{I[\tau_v F(\mathcal{D}), \tau_v \mathcal{D} \subset \widehat{\mathcal{C}}_n^{(k)}] I[\widehat{B}_n]}{|\widehat{\mathcal{C}}_n^{(k)}|}. \end{aligned} \tag{3.29}$$

Then (3.26), (3.28) and (3.29), the independence of X and Y and (3.23) imply that

$$\begin{aligned} P_{p_c}(\tau_{I_n} E) &\leq 3\varepsilon + \sum_{|v| \leq n - f(n)} \sum_{\mathcal{D}} E_{p_c}(X_{\mathcal{D},v,E} Y_{\mathcal{D},v,n}) \\ &= 3\varepsilon + \sum_{|v| \leq n - f(n)} \sum_{\mathcal{D}} P_{p_c}(E, 0 \leftrightarrow \mathcal{D}) E_{p_c}(Y_{\mathcal{D},v,n}) \\ &\leq 3\varepsilon + \nu(E)(1+\varepsilon) \sum_{|v| \leq n - f(n)} \sum_{\mathcal{D}} P_{p_c}(0 \leftrightarrow \mathcal{D}) E_{p_c}(Y_{\mathcal{D},v,n}). \end{aligned} \tag{3.30}$$

We show an analogous lower bound for $P_{p_c}(\tau_{I_n} E)$. First, by (3.23) we have

$$\begin{aligned} &\nu(E) \sum_{|v| \leq n - f(n)} \sum_{\mathcal{D}} P_{p_c}(0 \leftrightarrow \mathcal{D}) E_{p_c}(Y_{\mathcal{D},v,n}) \\ &\qquad \leq (1+\varepsilon) \sum_{|v| \leq n - f(n)} \sum_{\mathcal{D}} E_{p_c}(X_{\mathcal{D},v,E} Y_{\mathcal{D},v,n}). \end{aligned} \tag{3.31}$$

Now we start with the expression on the right-hand side and work our way back to $P_{p_c}(\tau_{I_n} E)$. On $\widehat{B}_n$ the events $v \leftrightarrow \tau_v \mathcal{D} \subset \widehat{\mathcal{C}}_n^{(k)}$ imply that $v \in \mathcal{C}_n^{(k)}$. Therefore, the inequality $|\widehat{\mathcal{C}}_n^{(k)}| > g(n) - |B(M)|$ implies that for n large we have

$$|\widehat{\mathcal{C}}_n^{(k)}(\tau_v \mathcal{D})| \geq (1+\varepsilon)^{-1} W_n^{(k)}. \tag{3.32}$$

This implies that we have

$$X_{\mathcal{D},v,E} Y_{\mathcal{D},v,n} \leq (1+\varepsilon) \frac{I[\tau_v E, v \in \mathcal{C}_n^{(k)}, \tau_v F(\mathcal{D})]}{W_n^{(k)}}. \tag{3.33}$$

Then (3.31), (3.32) and (3.33) imply that

$$
\begin{aligned}
&\nu(E) \sum_{|v|\le n-f(n)} \sum_{\mathcal{D}} P_{p_c}(0 \leftrightarrow \mathcal{D}) E_{p_c}(Y_{\mathcal{D},v,n}) \\
&\quad \le (1+\varepsilon)^2 \sum_{|v|\le n-f(n)} \sum_{\mathcal{D}} E_{p_c}\left(\frac{I[\tau_v E, v \in \mathcal{C}_n^{(k)}, \tau_v F(\mathcal{D})]}{W_n^{(k)}}\right) \\
&\quad \le (1+\varepsilon)^2 \sum_{|v|\le n-f(n)} E_{p_c}\left(\frac{I[\tau_v E, v \in \mathcal{C}_n^{(k)}, \tau_v F]}{W_n^{(k)}}\right) \\
&\quad \le (1+\varepsilon)^2 P_{p_c}(\tau_{I_n} E).
\end{aligned}
\tag{3.34}
$$

The bounds (3.30) and (3.34) also hold with E replaced by the sure event Ω, so we have

$$
1 = P_{p_c}(\tau_{I_n}\Omega) \le 3\varepsilon + (1+\varepsilon) \sum_{|v|\le n-f(n)} \sum_{\mathcal{D}} P_{p_c}(0 \leftrightarrow \mathcal{D}) E_{p_c}(Y_{\mathcal{D},v,n}) \tag{3.35}
$$

and

$$
\sum_{|v|\le n-f(n)} \sum_{\mathcal{D}} P_{p_c}(0 \leftrightarrow \mathcal{D}) E_{p_c}(Y_{\mathcal{D},v,n}) \le (1+\varepsilon)^2. \tag{3.36}
$$

Combining (3.30) with (3.36) and letting $\varepsilon \to 0$ we get

$$
\limsup_{n\to\infty} P_{p_c}(\tau_{I_n} E) \le \nu(E).
$$

Similarly, (3.34) and (3.35) imply that

$$
\liminf_{n\to\infty} P_{p_c}(\tau_{I_n} E) \ge \nu(E).
$$

4. Chayes–Chayes–Durrett cluster. In this section we consider the inhomogeneous model studied by Chayes, Chayes and Durrett (1987). In Section 4.1 we prove the analogues of Theorems 1 and 3 for this setting. In Section 4.2 we give sufficient conditions for the singularity of the CCD and IIC measures.

Recall that in the CCD model the bond e is open with probability $q(e)$ and $P_{\mathbf{q}}$ (resp. $E_{\mathbf{q}}$) denotes the underlying probability measure (resp. expectation). The following simple fact will often be used. If $q_1(e) \le q_2(e)$ for all e, and X is an increasing random variable, then $E_{\mathbf{q}_1} X \le E_{\mathbf{q}_2} X$. We also need a slight generalization of Kesten's theorem (1.2) to inhomogeneous edge probabilities.

THEOREM 10. *Assume $d = 2$, let $\mathbf{q} : \mathbb{E}^2 \to [p_c, 1]$, and let E be a cylinder event. Then*

$$
\mu_{\mathbf{q}}(E) \stackrel{def}{=} \lim_{n\to\infty} P_{\mathbf{q}}\big(E \mid 0 \leftrightarrow \partial B(n)\big) \tag{4.1}
$$

exists uniformly in $\mathbf{q}$. *Consequently, if* $q_k(e) \to p_c$ *for all* e, *then*

$$\lim_{\substack{k\to\infty \\ n\to\infty}} P_{\mathbf{q}_k}\big(E \mid 0 \leftrightarrow \partial B(n)\big) = \nu(E).$$

PROOF. The proof of Theorem 3 in Kesten (1986) can be applied, since it makes no use of the fact that the probabilities are homogeneous and all estimates are uniform in $p \ge p_c$. $\square$

The only extra assumption we need to make on the edge probabilities is

$$q(e) = f(|e|), \text{ where } f \text{ is decreasing and } f(r) \searrow p_c \text{ as } r \to \infty. \tag{4.2}$$

We fix $q(e)$ that satisfies this requirement.

4.1. *The CCD cluster and the IIC.* The approach for the proof of Theorem 4 is the same as in the previous two sections. We prove that there is an open circuit in the annulus $\mathrm{An}(M, N) + v$, where v is the random vertex. One difference is that this time we do not have a tightness result, only a slightly weaker statement. This forces us to increase M a bit further than before. We set up the stage for the proof up to a point where it becomes clear that the method of Theorem 1 works.

We start by proving some preliminary results about the size of the CCD cluster. Recall that $\mathcal{C} = \{v \in \mathbb{Z}^2 : 0 \leftrightarrow v\}$ and $\mathcal{C}_n = \mathcal{C} \cap B(n)$. Write $A_k = \mathrm{An}(2^k, 2^{k-1})$ for short, with the convention $A_0 = B(1)$. Let $W_n = |\mathcal{C}_n|$ and let

$$X_k = |\mathcal{C} \cap A_k| = |\{v \in A_k : 0 \leftrightarrow v\}|.$$

Let K denote the integer for which $2^K \le n < 2^{K+1}$. We have

$$W_n = X_0 + \cdots + X_K + \widehat{X}_n, \tag{4.3}$$

where $\widehat{X}_n = |\mathcal{C}_n \cap \mathrm{An}(n, 2^K)|$. Let $p_k = f(2^k)$ and let $a_k = 2^{2k}\pi(p_k, 2^k)$. We show that there are constants C_1, C_2, such that for $0 \le k \le K$ and $n \le m$ we have

$$C_1 a_k \le E_{\mathbf{q}}\big(X_k \mid 0 \leftrightarrow \partial B(m)\big) \le C_2 a_{k-1} \tag{4.4}$$

and

$$E_{\mathbf{q}}\big(\hat{X}_n \mid 0 \leftrightarrow \partial B(m)\big) \le C_2 a_K. \tag{4.5}$$

(We set $a_{-1} = 1$.) In proving the bounds we may assume $k \ge 3$, and for such k we let

$$\begin{aligned} A_k^0 &= \mathrm{An}\big((7/4)2^{k-1}, (5/4)2^{k-1}\big), \\ A_k' &= \mathrm{An}\big(2^k, (7/4)2^{k-1}\big), \\ A_k'' &= \mathrm{An}\big((5/4)2^{k-1}, 2^{k-1}\big). \end{aligned} \tag{4.6}$$

For the lower bound in (4.4) let G denote the event that there is an open circuit in A'_k. Since in A_k we have $q(e) \geq p_k$, we have, for any $v \in A_k^0$,

$$\begin{aligned}
&P_{\mathbf{q}}\big(v \in \mathcal{C}_n \mid 0 \leftrightarrow \partial B(m)\big) \\
&\qquad \geq P_{\mathbf{q}}\big(v \leftrightarrow \partial B(2^k), G \mid 0 \leftrightarrow \partial B(m)\big) \\
&\qquad \geq P_{\mathbf{q}}\big(v \leftrightarrow \partial B(2^k), G\big) \geq P_{p_k}\big(v \leftrightarrow \partial B(2^k), G\big) \\
&\qquad \geq P_{p_k}(G) P_{p_k}\big(v \leftrightarrow \partial B(2^k)\big) \geq P_{p_k}(G) P_{p_k}\big(v \leftrightarrow \partial B(2^{k+1}, v)\big) \\
&\qquad \geq C_3 \pi(p_k, 2^k),
\end{aligned}$$

where in the successive steps we used: inclusion of events, the FKG inequality, monotonicity of measures, the FKG inequality, inclusion of events, the RSW theorem and (1.9). Summing over v yields the lower bound in (4.4).

For the upper bound let $v \in A_k$ and let $r = r(v)$ be the radius of the largest box around v that is contained in A_k. Assume that v is such that $r \geq 4$, and let J be the event that there is an open circuit in $B(r, v) \setminus B(r/2, v)$. By the FKG inequality and the RSW Theorem

$$P_{\mathbf{q}}\big(v \leftrightarrow 0, J, 0 \leftrightarrow \partial B(m)\big) \geq C_4 P_{\mathbf{q}}\big(v \in \mathcal{C}_n, 0 \leftrightarrow \partial B(m)\big).$$

Observe that if v is connected to 0 then v has to be connected to $\partial B(r/2, v)$. Also, on the event J, if 0 is connected to $\partial B(m)$ then it is also connected to it outside $B(r/2, v)$. This is because if ρ is an open path from 0 to $\partial B(m)$, and $\mathcal{D}$ is an open circuit in $B(r, v) \setminus B(r/2, v)$, then we can replace any segment of ρ falling inside $\operatorname{int}(\mathcal{D})$ by a piece of $\mathcal{D}$. Putting these facts together we have

$$\begin{aligned}
&P_{\mathbf{q}}\big(v \in \mathcal{C}_n, 0 \leftrightarrow \partial B(m)\big) \\
&\qquad \leq \frac{1}{C_4} P_{\mathbf{q}}\big(v \leftrightarrow 0, J, 0 \leftrightarrow \partial B(m)\big) \\
&\qquad \leq \frac{1}{C_4} P_{\mathbf{q}}\big(v \leftrightarrow \partial B(r/2, v), 0 \leftrightarrow \partial B(m) \text{ outside } B(r/2, v)\big) \qquad (4.7)\\
&\qquad = \frac{1}{C_4} P_{\mathbf{q}}\big(v \leftrightarrow \partial B(r/2, v)\big) P_{\mathbf{q}}\big(0 \leftrightarrow \partial B(m) \text{ outside } B(r/2, v)\big) \\
&\qquad \leq \frac{1}{C_4} \pi(p_{k-1}, r/2) P_{\mathbf{q}}\big(0 \leftrightarrow \partial B(m)\big),
\end{aligned}$$

where we used independence at the equality sign and the fact that $q(e) \leq p_{k-1}$ in $B(r/2, v)$ and an inclusion of events at the last step. Sum the bound in (4.7) over v as we did in the proof of Theorem 8. Using (1.11) we get the upper bound in (4.4). The bound in (4.5) is proved similarly.

Although we cannot prove tightness of W_n in general, we show a weaker statement, (4.12) below, that will be sufficient for our needs. Let

$$\begin{aligned}
Y_k &= \big|\{v \in A_k^0 : v \leftrightarrow \partial A_k\}\big|, \\
G_k &= \{\text{there are open circuits } \mathcal{D}' \text{ and } \mathcal{D}'' \text{ in } A'_k \text{ and } A''_k\}.
\end{aligned}$$

Let $k^* = k^*(K)$ be the index that maximizes a_k for $0 \le k \le K$. As in the proof of Theorem 8, we get that there are constants C_5, C_6, C_7, C_8, such that

$$C_5 a_k \le E_{p_k} Y_k \le C_6 a_k, \tag{4.8}$$

$$C_7 a_k^2 \le E_{p_k} Y_k^2 \le C_8 a_k^2. \tag{4.9}$$

We show that for suitable ℓ and large n, with high probability at least one of the quantities $X_{k^*}, X_{k^*-1}, \dots, X_{k^*-\ell}$ is larger than some small multiple of a_{k^*}.

By monotonicity of p_k and π we have

$$a_{k^*} \le 4a_{k^*-1} \le \cdots \le 4^\ell a_{k^*-\ell}. \tag{4.10}$$

Let $x = 4^{-\ell} C_5/2$ and define the events

$$H_k = \left\{ Y_k \ge \frac{C_5}{2} a_k \right\} \cap G_k.$$

If $k^* - \ell \le k \le k^*$, then on the event $H_k \cap \{0 \leftrightarrow \partial B(m)\}$ we have $W_n \ge C_5 a_k/2 \ge x a_{k^*}$, by (4.10). Therefore, using the FKG inequality and independence of the events H_k, we get

$$\begin{aligned}
&P_{\mathbf{q}}\big(W_n < x a_{k^*} \mid 0 \leftrightarrow \partial B(m)\big) \\
&\quad \le P_{\mathbf{q}}\left(\bigcap_{k=k^*-\ell}^{k^*} H_k^c \,\middle|\, 0 \leftrightarrow \partial B(m) \right) \qquad (4.11)\\
&\quad \le P_{\mathbf{q}}\left(\bigcap_{k=k^*-\ell}^{k^*} H_k^c \right) = \prod_{k=k^*-\ell}^{k^*} P_{\mathbf{q}}(H_k^c).
\end{aligned}$$

Using that H_k is increasing, that $q(e) \ge p_k$ in A_k, the FKG inequality, the RSW Theorem and the second moment bounds (4.9) we get

$$\begin{aligned}
P_{\mathbf{q}}(H_k) &\ge P_{p_k}(H_k) \ge P_{p_k}(G_k) P_{p_k}\left(Y_k \ge \frac{C_5}{2} a_k \right) \\
&\ge P_{p_k}(G_k) P_{p_k}\left(Y_k \ge \frac{1}{2} E_{p_k} Y_k \right) \ge C_9 > 0.
\end{aligned}$$

This implies that the right-hand side of (4.11) is less than $(1 - C_9)^{\ell+1}$. Given any $\varepsilon > 0$ we can choose ℓ so large that $(1 - C_9)^{\ell+1} < \varepsilon$. For n large enough we have $k^* > \ell$, so the above argument gives that, for large n,

$$P_{\mathbf{q}}\big(W_n < x a_{k^*} \mid 0 \leftrightarrow \partial B(m)\big) \le \varepsilon. \tag{4.12}$$

We are ready to prove Theorem 4.

PROOF OF THEOREM 4. We only need to prove the first statement, since the second one then follows easily. Let $\varepsilon > 0$ be given. By (4.1) there exist

$N = N(\varepsilon, E)$ and $p_0 = p_0(\varepsilon) > p_c$, such that for any (nonrandom) circuit $\mathcal{D}$ surrounding $B(N)$ such that $q(e) \le p_0$ for $e \in E(\mathrm{int}(\mathcal{D}))$ we have

$$(1+\varepsilon)^{-1}\nu(E) \le \frac{P_{\mathbf{q}}(E, 0 \leftrightarrow \mathcal{D})}{P_{\mathbf{q}}(0 \leftrightarrow \mathcal{D})} \le (1+\varepsilon)\nu(E). \tag{4.13}$$

[We may assume $\nu(E) > 0$.] Let

$$F = F(M, N) = \{\text{there is an open circuit in } \mathrm{An}(M, N)\}.$$

Using (4.12) and the FKG inequality, we have

$$\begin{aligned} &P_{\mathbf{q}}\big(\tau_{I_n} F^c \mid 0 \leftrightarrow \partial B(m)\big) \\ &\quad = \sum_{v \in B(n)} E_{\mathbf{q}}\bigg(\frac{I[\tau_v F^c, v \in \mathcal{C}_n]}{W_n} \,\bigg|\, 0 \leftrightarrow \partial B(m)\bigg) \\ &\quad \le \varepsilon + \frac{1}{x a_{k^*}} \sum_{v \in B(n)} P_{\mathbf{q}}(\tau_v F^c) P_{\mathbf{q}}\big(v \in \mathcal{C}_n \mid 0 \leftrightarrow \partial B(m)\big). \end{aligned} \tag{4.14}$$

We can bound $P_{\mathbf{q}}(\tau_v F^c)$ uniformly in v by $P_{p_c}(F^c)$, since $q(e) \ge p_c$. This gives that the right-hand side of (4.14) is less than

$$\varepsilon + \frac{P_{p_c}(F^c)}{x a_{k^*}} E_{\mathbf{q}}\big(W_n \mid 0 \leftrightarrow \partial B(m)\big). \tag{4.15}$$

Observe that $a_{k^*} \ge (a_0 + \cdots + a_K)/(K+1)$, where $K \le \log n / \log 2$. By equations (4.3), (4.4) and (4.5) we have

$$E_{\mathbf{q}}\big(W_n \mid 0 \leftrightarrow \partial B(m)\big) \le C_9(a_0 + \cdots + a_K).$$

This gives that the right-hand side of (4.15) is less than

$$\varepsilon + P_{p_c}(F^c)\frac{C_{10} \log n}{x}.$$

By the RSW Theorem, for a suitable $C_{11} = C_{11}(N, \varepsilon, x)$ and $M = C_{11}(\log n)^{1/\mu}$ [where μ is the exponent appearing in (1.7)], we have

$$P_{\mathbf{q}}\big(\tau_{I_n} F^c \mid 0 \leftrightarrow \partial B(m)\big) \le 2\varepsilon.$$

In order to use (4.13) we need that $q(e) \le p_0$ in $B(M, I_n)$. We show that this occurs with high probability. Find an integer k_0, such that $p_k \le p_0$, whenever $k \ge k_0$. We show that if n is large enough then

$$P_{\mathbf{q}}\big(I_n \in B(2^{k_0 - 1} + M) \mid 0 \leftrightarrow \partial B(m)\big) \le 2\varepsilon. \tag{4.16}$$

The number of vertices in $B(2^{k_0-1}+M)$ is $O((\log n)^{1/\mu})$. Using (4.12), we get

$$\begin{aligned}
&P_{\mathbf{q}}\big(I_n \in B(2^{k_0-1}+M) \mid 0 \leftrightarrow \partial B(m)\big) \\
&\quad = \sum_{v \in B(2^{k_0-1}+M)} P_{\mathbf{q}}\biggl(\frac{I[v \in \mathcal{C}_n]}{W_n} \Bigm| 0 \leftrightarrow \partial B(m)\biggr) \\
&\quad \le \varepsilon + \frac{O((\log n)^{1/\mu})}{x a_{k^*}}.
\end{aligned}$$

We have $a_{k^*} \ge C_{12}(n^2\pi(n))/(K+1) \ge C_{13}n^{3/2}/\log n$, using (1.9). Thus (4.16) holds for large n. One can similarly show that $B(M, I_n) \subset B(n)$ with large probability.

As in (2.21) we write $F = \bigcup_{\mathcal{D}} F(\mathcal{D})$. Note that if $0 \notin \tau_v B(M)$ and $\tau_v \mathcal{D}$ is a circuit in $\tau_v \mathrm{An}(M, N) \subset B(n)$, then

$$\begin{aligned}
&I[\tau_v E, v \in \mathcal{C}_n, \tau_v F(\mathcal{D}), 0 \leftrightarrow \partial B(m)] \\
&\quad = I[\tau_v E, v \leftrightarrow \tau_v \mathcal{D}, \tau_v F(\mathcal{D}), \tau_v \mathcal{D} \leftrightarrow 0, 0 \leftrightarrow \partial B(m)] \\
&\quad = I[\tau_v E, v \leftrightarrow \tau_v \mathcal{D}] I[\tau_v F(\mathcal{D}), \tau_v \mathcal{D} \leftrightarrow 0, 0 \leftrightarrow \partial B(m)].
\end{aligned}$$

The two indicators in the last expression are independent. From here the proof is completely analogous to the case of the spanning clusters. $\square$

PROOF OF COROLLARY 1. In (1.4) take $m \to \infty$ first. Then, using Kesten's result (1.2), we get

$$\nu(E) = \lim_{n\to\infty} \lim_{m\to\infty} P_{p_c}\big(\tau_{I_n} E \mid 0 \leftrightarrow \partial B(m)\big) = \lim_{n\to\infty} \nu(\tau_{I_n} E).$$

4.2. *Singularity of the CCD and IIC measures.* The main tool for the proof of Theorem 5 is the finite-size scaling correlation length $L(p)$ whose definition we gave in Section 1.3. We are going to use the following statement from Kesten (1987).

THEOREM 11 (Kesten). *There is a constant C_1 (that only depends on ε_0) such that for $p > p_c$,*

$$\pi\big(p_c, L(p)\big) \le C_1 \theta(p). \tag{4.17}$$

Recall that $f(\ell)$ denotes the value of $q(e)$ where $|e| = \ell$.

PROOF OF THEOREM 5. We start by showing that if $P_{\mathbf{q}}(0 \leftrightarrow \infty) > 0$, then the condition $L(q(e))/|e| \to 0$ is satisfied. For suppose that $\ell_k \to \infty$ is such that $L(f(\ell_k)) > \delta \ell_k$. Then this implies that

$$\sigma\big(\delta\ell_k, \delta\ell_k, f(\ell_k)\big) < 1 - \varepsilon_0,$$

so

$$P_{f(\ell_k)}\left(\begin{array}{c}\text{there is a closed vertical dual crossing}\\ \text{in } \left[\frac{1}{2}, \delta\ell_k - \frac{1}{2}\right] \times \left[-\frac{1}{2}, \delta\ell_k + \frac{1}{2}\right]\end{array}\right) > \varepsilon_0.$$

It follows that there is a constant C_2, such that

$$P_{\mathbf{q}}\big(\text{there is a closed dual circuit in } \mathrm{An}_*(2\ell_k - 1/2, \ell_k + 1/2)\big) \geq C_2 > 0, \tag{4.18}$$

where An_* denotes an annulus on the dual lattice. We may assume that the annuli in (4.18) are disjoint for different k. Then by independence we get that there are infinitely many dual circuits disconnecting the origin from infinity, so that $P_{\mathbf{q}}(0 \leftrightarrow \infty) = 0$.

To prove the main result of Theorem 5 we show that W_n is asymptotically of larger order under $\mu_{\mathbf{q}}$ than under ν. Let $m > n$ be fixed and let K be the integer for which $2^K \leq n < 2^{K+1}$. We define a_k, W_n, A_k, X_k, Y_k, G_k, H_k and x as in the discussion preceeding the proof of Theorem 4. The bounds in (4.8) and (4.9) hold with $p_k = f(2^k)$, that is,

$$C_3 a_k \leq E_{p_k} Y_k \leq C_4 a_k$$

and

$$C_5 a_k^2 \leq E_{p_k} Y_k^2 \leq C_6 a_k^2.$$

For $K - \ell \leq k \leq K$ on the event $H_k \cap \{0 \leftrightarrow \partial B(m)\}$ we have $W_n \geq C_3 a_k/2 \geq x a_K$. Therefore, similarly to our findings in the previous subsection, noting that $q(e) \geq p_k$ in the annulus A_k, we get

$$\begin{aligned}
&P_{\mathbf{q}}\big(W_n < x a_K \mid 0 \leftrightarrow \partial B(m)\big)\\
&\quad \leq P_{\mathbf{q}}\left(\bigcap_{k=K-\ell}^{K} H_k^c \,\middle|\, 0 \leftrightarrow \partial B(m)\right) \qquad (4.19)\\
&\quad \leq \prod_{k=K-\ell}^{K} P_{\mathbf{q}}(H_k^c) \leq \prod_{k=K-\ell}^{K} P_{p_k}(H_k^c) \leq (1 - C_7)^{\ell+1}.
\end{aligned}$$

Letting $m \to \infty$ we obtain that for any $\varepsilon > 0$ there is an $x > 0$, such that for n large enough $\mu_{\mathbf{q}}(W_n < x a_K) \leq \varepsilon$.

On the other hand, there are constants $C_8, \lambda > 0$, such that for any $1 \leq j \leq n$

$$\pi(p_c, j) \geq C_8 \pi(p_c, n)\left(\frac{n}{j}\right)^{\lambda}. \tag{4.20}$$

This follows by a simple modification of Lemma 8.5 in Kesten (1982) or of Theorem 11.89 in Grimmett (1999). An application of (4.17) and (4.20) yields

$$\begin{aligned}
a_K &= 2^{2K}\pi(p_K, 2^K) \geq 2^{2K}\theta(p_K) \geq \frac{1}{C_1} 2^{2K}\pi\big(p_c, L(p_K)\big)\\
&\geq \frac{C_8}{C_1} 2^{2K}\pi(p_c, 2^K)\left(\frac{2^K}{L(p_K)}\right)^{\lambda} \geq C_9 n^2 \pi(p_c, n)\left(\frac{2^K}{L(p_K)}\right)^{\lambda}.
\end{aligned}$$

Since $2^K/L(p_K) \to \infty$, we can choose a function $g(n) \to \infty$, such that

$$\frac{1}{g(n)}\left(\frac{2^K}{L(p_K)}\right)^{\lambda} \to \infty.$$

By the observation following (4.19) we have

$$\frac{W_n}{n^2\pi(p_c,n)g(n)} \to \infty \qquad \text{in } \mu_{\mathbf{q}}\text{-measure.}$$

Also, by Theorem 8 in Kesten (1986), we have

$$\frac{W_n}{n^2\pi(p_c,n)g(n)} \to 0 \qquad \text{in } \nu\text{-measure.}$$

Thus, along a subsequence n_k we have

$$\limsup_{k\to\infty} \frac{W_{n_k}}{n_k^2\pi(p_c,n_k)g(n_k)} = \infty, \qquad \mu_{\mathbf{q}}\text{-a.s,}$$

and

$$\limsup_{k\to\infty} \frac{W_{n_k}}{n_k^2\pi(p_c,n_k)g(n_k)} = 0, \qquad \nu\text{-a.s.}$$

This can only happen if $\mu_{\mathbf{q}}$ and ν are singular with respect to each other. $\square$

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