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LYAPUNOV EXPONENTS OF LINEAR STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS. PART II. EXAMPLES AND CASE STUDIES

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We give several examples and examine case studies of linear stochastic functional differential equations. The examples fall into two broad classes: regular and singular, according to whether an underlying stochastic semiflow exists or not. In the singular case, we obtain upper and lower bounds on the maximal exponential growth rate $\overline{\lambda}_1(\sigma)$ of the trajectories expressed in terms of the noise variance σ. Roughly speaking we show that for small σ, $\overline{\lambda}_1(\sigma)$ behaves like $-\sigma^2/2$, while for large σ, it grows like $\log\sigma$. In the regular case, it is shown that a discrete Oseledec spectrum exists, and upper estimates on the top exponent λ_1 are provided. These estimates are sharp in the sense that they reduce to known estimates in the deterministic or nondelay cases.

1. Introduction and some preliminaries. Lyapunov exponents for linear stochastic ordinary differential equations (without memory) have been studied by many authors; compare, for example, [1], [2], [3], [17], [18], [7] and the references therein.

Issues of asymptotic stability of stochastic functional differential equations (sfde's) were treated by Kushner [9], Mizel and Trutzer [11], Mohammed [12]–[15], Mohammed and Scheutzow [16], Scheutzow [20] and Kolmanovskii and Nosov [8]. In [10], Chapter 5, several results are given concerning the top exponential growth rate for a class of stochastic delay differential equations driven by C-valued semimartingales. The results in [10] assume (among other things) that the second-order characteristics of the driving semimartingales are *time-dependent* and *decay to zero exponentially fast in time*, *uniformly in the space variable*.

In an earlier paper, we established the existence of stochastic flows and an Oseledec–Lyapunov spectrum for a large class of linear sfde's in Euclidean space [16]. The present article is a sequel to [16]. Our aim is to examine closely a variety of examples of linear sfde's. These examples fall into two distinct classes: *regular* and *singular*, depending on whether stochastic flows exist or not. We shall be concerned with whether our examples are regular

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or singular. For singular equations we obtain estimates on the *maximal exponential growth rate* $\overline{\lambda}_1$ as defined at the end of this section. In the regular case we shall study questions of existence of the stochastic semiflow and its Lyapunov spectrum. Furthermore, we obtain sharp upper bounds on the *top Lyapunov exponent* λ_1.

We begin by summarizing the main results from [16] that we will need in this article. In order to do so, we will adopt the following assumptions and notations.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space satisfying the usual conditions. Consider the following class of linear one-dimensional sfde's:

$$\text{(I)}\qquad \begin{aligned} dx(t) &= \left\{\int_{[-r,0]} x(t+s)\nu(t)(ds)\right\} dt \\ &\qquad + \int_{-r}^{0} K(t)(s)x(t+s)\,ds\,dN(t) + x(t)\,dL(t), \qquad t > 0, \\ x(0) &= v \in \mathrm{R}, \qquad x(s) = \eta(s),\ -r < s < 0,\ r \geq 0 \end{aligned}$$

driven by continuous one-dimensional semimartingales N and L on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ with stationary (ergodic) increments. The process ν is stationary, measure-valued and satisfies mild regularity hypotheses. The process K is stationary and $C^1([-r,0], \mathrm{R})$-valued. The reader may refer to Hypotheses (C) in [16] (Section 2, pages 74–77) for full details of the conditions needed. These hypotheses allow for a multidimensional version of (I), together with jumps in the process N and (the bounded-variation part of) L.

Let $M_2 := \mathrm{R} \times L^2([-r,0], \mathrm{R})$ denote the Delfour–Mitter Hilbert space with the norm

$$\|(v,\eta)\|_{M_2} := \left[v^2 + \int_{-r}^{0} \eta(s)^2\,ds\right]^{1/2}, \qquad v \in \mathrm{R},\ \ \eta \in L^2([-r,0], \mathrm{R}).$$

Define the *trajectory* of (I) by $(x(t), x_t)$, $t \geq 0$, where x_t denotes the segment

$$x_t(s) := x(t+s), \qquad s \in [-r,0],\ \ t \geq 0.$$

Recall the following definition of regularity introduced in [15].

DEFINITION. A linear sfde is said to be *regular* with respect to M_2 if its trajectory random field $\{(x(t), x_t)\colon (x(0), x_0) = (v,\eta) \in M_2,\ t \geq 0\}$ admits a (Borel $\mathrm{R}^+ \otimes$ Borel $M_2 \otimes \mathscr{F}$, Borel M_2)-measurable version $X\colon \mathrm{R}^+ \times M_2 \times \Omega \to M_2$ with a.a. sample functions continuous on $\mathrm{R}^+ \times M_2$. The sfde is said to be *singular* otherwise.

Theorem 4.2 in [16] identifies a large class of regular sfde's. Indeed, one gets a semiflow consisting of bounded linear operators $X(t, \cdot, \omega)$, $t \geq 0$, $\omega \in \Omega$, on M_2. Furthermore, for $t \geq r$, and a.a. $\omega \in \Omega$, these operators are compact.

Under fairly general moment and ergodicity conditions on the driving processes ν, K, N, L in (I), one can establish the regularity of (I) and the existence of a discrete nonrandom Lyapunov spectrum

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t,(v(\omega),\eta(\omega)),\omega)\|_{M_2} \quad \text{a.a. } \omega \in \Omega,\ (v,\eta) \in L^2(\Omega, M_2).$$

See Theorem 5.2 of [16] and its hypotheses. In particular, the above limit takes up a countable fixed set of values $\{\lambda_i\}_{i=1}^{\infty}$, called *Lyapunov exponents*. This Lyapunov spectrum is bounded above, and the top exponent λ_1 is given by

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log \|X(t,\cdot,\omega)\|_{L(M_2)}$$

for almost all $\omega \in \Omega$, where $\|\cdot\|_{L(M_2)}$ denotes the uniform operator norm on M_2. This result is proved in [16], using the compactness of $X(t,\cdot,\omega)\colon M_2 \to M_2$, $t \geq r$, together with an infinite-dimensional version of Oseledec's multiplicative ergodic theorem due to Ruelle [19]. In the white noise case, the result was previously established by the first author in [14] (Theorem 4). Note that the Lyapunov spectrum of (I) does not change if one uses the state space $D([-r,0],\mathrm{R})$ of cadlag functions, with the supremum norm $\|\cdot\|_\infty$ ([16], Remark following Theorem 5.3).

In Section 4 of this article, we derive upper bounds on λ_1 for various special cases of (I). These upper bounds are expressed in terms of the coefficients of the equation. See Theorems 4.1, 4.2, and 4.3. The upper bounds are sharp in the sense that they reduce to the corresponding well-known bounds in the deterministic and/or nondelay case.

In the singular case, however, there is no stochastic flow (Theorem 2.1) and we do not know whether a set of Lyapunov exponents

$$\lambda((v,\eta),\cdot) := \lim_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\|_{M_2}, \qquad (v,\eta) \in M_2$$

exists. The existence of Lyapunov exponents for singular equations seems to be a hard problem, even in the linear case. Nevertheless, we can still define the *maximal exponential growth rate*,

$$\overline{\lambda}_1 := \sup_{(v,\eta)\in M_2} \limsup_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\|_{M_2}$$

for the trajectory random field $\{(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\colon t \geq 0,\ (v,\eta) \in M_2\}$. In general $\overline{\lambda}_1$ may depend on $\omega \in \Omega$. Note however that $\overline{\lambda}_1 = \lambda_1$ in the regular case, as is evident from Proposition 1.1. In Section 2, we look at possibly singular equations of the form

$$dx(t) = \sigma \int_{[-r,0]} x(t+s)\, d\nu(s)\, dW(t), \qquad t > 0,$$
$$(x(0), x_0) \in M_2,$$

where W is a Wiener process and ν is a fixed finite Borel measure on $[-r,0]$. For the above equation we specify a growth condition on the Fourier coefficients of the measure ν under which the equation becomes singular (Theorem 2.1). Theorem 2.1 thus underscores the *extremely erratic dependence on the* *initial paths* of solutions of one-dimensional delay equations driven by white noise. In contrast, it is shown in Theorem 2.2 that for small noise variance, *uniform almost sure global asymptotic stability* still persists in spite of the erratic solution field. Indeed for small σ, one has $\overline{\lambda}_1 \le -\sigma^2/2 + o(\sigma^2)$ uniformly in the initial path [Theorem 2.2 and Remark (iv), p. 1223]. For large σ and $\nu = \delta_{-r}$, we show that $(1/2r)\log|\sigma| + o(\log|\sigma|) \le \overline{\lambda}_1 \le (1/r)\log|\sigma|$ [Theorem 2.3 and Remark (ii), p. 1228]. This result is also in sharp contrast with the nondelay case ($r = 0$), where one has $\lambda_1 = -\sigma^2/2$ *for all* values of σ. The proofs of Theorems 2.2 and 2.3 involve very delicate constructions of new types of Lyapunov functionals on the underlying state space.

In Section 3, we characterize the Lyapunov spectrum for delay equations driven by Poisson noise (Theorem 3.1). This class of equations is interesting because it is regular but does not satisfy the set-up in [16]. The characterization of the spectrum in this case study is effected *without using the Oseledec theorem*.

We end this section by stating a proposition that we will use frequently in the sequel.

PROPOSITION 1.1. *Suppose that the sfde* (I) *satisfies the hypotheses of Theorem* 5.2 *in [16], and let* λ_1 *be its top Lyapunov exponent. Suppose there exists a real number* β *such that*

$$P\left(\limsup_{t\to\infty}\frac{1}{t}\log\|(x(t,(v,\eta)),x_t(\cdot,(v,\eta)))\|_{M_2}\le\beta\right)=1$$

for all $(v,\eta)\in M_2$, *where* β *is independent of* (v,η). *Then* $\lambda_1\le\beta$. *In particular,* $\lambda_1=\overline{\lambda}_1$.

PROOF. Denote by X: $\mathrm{R}^+\times M_2\times\Omega\to M_2$ the stochastic semiflow of (I). Let $\{E_i(\omega)\}_{i=1}^{\infty}$ be the underlying Oseledec spaces in M_2, with $E_1(\omega)=M_2$ and

$$E_2(\omega)=\left\{(v,\eta)\in M_2\colon\ \limsup_{t\to\infty}\frac{1}{t}\log\|(X(t,(v,\eta),\omega)\|_{M_2}<\lambda_1\right\}$$

for almost all $\omega\in\Omega$ ([16], Theorem 5.2). Pick a countable dense subset H of M_2. Since $E_2(\omega)$ is a *closed proper* subspace of M_2 and H is dense, it follows that for almost every $\omega\in\Omega$ there exist $(v_0(\omega),\eta_0(\omega))\in H\cap E_2(\omega)^c$. Therefore

$$P\left(\lim_{t\to\infty}\frac{1}{t}\log\|(X(t,(v,\eta),\cdot))\|_{M_2}<\lambda_1 \text{ for all } (v,\eta)\in H\right)=0.$$

However, by our hypothesis,

$$P\left(\lim_{t\to\infty}\frac{1}{t}\log\|(X(t,(v,\eta),\cdot))\|_{M_2}\le\beta \text{ for all } (v,\eta)\in H\right)=1.$$

Therefore $\lambda_1\le\beta$. This proves the proposition. □

We now look at our examples in some detail.

2. Delay equations driven by white noise. Consider the one-dimensional stochastic linear delay equation

$$\text{(II)} \qquad \begin{aligned} dx(t) &= \sigma x(t-r)\,dW(t), \qquad t>0,\\ (x(0),x_0) &= (v,\eta)\in M_2 := \mathrm{R}\times L^2([-r,0],\mathrm{R}), \end{aligned}$$

driven by a standard Wiener process W, where $\sigma \in \mathrm{R}$ is fixed and r is a *positive* delay.

It is known that (II) is singular with respect to M_2 for all nonzero σ ([12], [13]).

Here we will examine the regularity of the more general one-dimensional linear sfde:

$$\text{(III)} \qquad \begin{aligned} dx(t) &= \int_{[-r,0]} x(t+s)\,d\nu(s)\,dW(t), \quad t>0,\\ (x(0),x_0) &\in M_2 := \mathrm{R}\times L^2([-r,0],\mathrm{R}), \end{aligned}$$

where W is a Wiener process and ν is a fixed finite real-valued signed Borel measure on $[-r,0]$.

In this section and throughout the paper, we shall denote by $(\mathscr{F}_t^W)_{t\geq 0}$ the filtration generated by W, namely, $\mathscr{F}_t^W$ is the σ-algebra generated by $\{W(u)\colon 0\leq u\leq t\}$ for each $t\geq 0$.

It follows from Theorem 4.2 of [16] that (III) is regular if ν has a C^1 density with respect to Lebesgue measure on $[-r,0]$. The following theorem gives conditions on the measure ν under which (III) is singular.

THEOREM 2.1. *Let $r>0$, and suppose that there exists $\varepsilon\in(0,r)$ such that* $\operatorname{supp}\nu\subset[-r,-\varepsilon]$. *Suppose $0<t_0\leq\varepsilon$. For each $k\geq 1$, set*

$$\nu_k := \sqrt{t_0}\left|\int_{[-r,0]}\exp\left(\frac{2\pi i k s}{t_0}\right)d\nu(s)\right|.$$

Assume that

$$\sum_{k=1}^{\infty}\nu_k x^{1/\nu_k^2} = \infty \tag{2.1}$$

for all $x\in(0,1)$. Let $Y\colon[0,\varepsilon]\times M_2\times\Omega\to\mathrm{R}$ be any Borel-measurable version of the solution field $\{x(t)\colon 0\leq t\leq\varepsilon,\ (x(0),x_0)=(v,\eta)\in M_2\}$ of (III)*. Then for a.a. $\omega\in\Omega$, the map $Y(t_0,\cdot,\omega)\colon M_2\to\mathrm{R}$ is unbounded in every neighborhood of every point in M_2, and (hence) non-linear.*

REMARK. (i) Condition (2.1) of the theorem is implied by

$$\lim_{k\to\infty}\nu_k\sqrt{\log k}=\infty.$$

(ii) For the delay equation (II), $\nu = \delta_{-r}$, $\varepsilon = r$. In this case condition (2.1) is satisfied for *every* $t_0 \in (0, r]$.

PROOF. This proof is joint work with Victor Mizel.

The main idea is to track the solution random field of (a complexified version of) (III) along the classical Fourier basis:

$$\eta_k(s) = \exp\left(\frac{2\pi iks}{t_0}\right), \qquad -r \le s \le 0,\ k \ge 1 \tag{2.2}$$

in $L^2([-r, 0], \mathrm{C})$. On this basis, the solution field gives an infinite family of independent Gaussian random variables. This allows us to show that no Borel measurable version of the solution field can be bounded with positive probability on an arbitrarily small neighborhood of 0 in M_2, and hence on any neighborhood of any point in M_2 (cf. [12], [13]). In order to simplify the computations, we shall complexify the state space in (III) by allowing (v, η) to belong to $M_2^C := \mathrm{C} \times L^2([-r, 0], \mathrm{C})$; that is, we consider the sfde

$$\begin{aligned} dx(t) &= \int_{[-r,0]} x(t+s)\, d\nu(s)\, dW(t), \qquad t > 0, \\ (x(0), x_0) &= (v, \eta) \in M_2^C, \end{aligned} \tag{III^C}$$

where $x(t) \in \mathrm{C}$, $t \ge -r$, and ν, W are real-valued.

We prove the theorem using a contradiction argument. Let Y: $[0, \varepsilon] \times M_2 \times \Omega \to \mathrm{R}$ be any Borel-measurable version of the solution field $\{x(t)$: $0 \le t \le \varepsilon$, $(x(0), x_0) = (v, \eta) \in M_2\}$ of (III). Suppose, if possible, that there exists a set $\Omega_0 \in \mathscr{F}$ of positive P-measure, $(v_0, \eta_0) \in M_2$ and a positive δ such that for all $\omega \in \Omega_0$, $Y(t_0, \cdot, \omega)$ is bounded on the open ball $B((v_0, \eta_0), \delta)$ in M_2 of center (v_0, η_0) and radius δ. Define the complexification $Z(\cdot, \omega)$: $M_2^C \to \mathrm{C}$ of $Y(t_0, \cdot, \omega)$: $M_2 \to \mathrm{R}$ by

$$Z(\xi_1 + i\xi_2, \omega) := Y(t_0, \xi_1, \omega) + i\, Y(t_0, \xi_2, \omega), \qquad i = \sqrt{-1},$$

for all ξ_1, $\xi_2 \in M_2$, $\omega \in \Omega$. Let $(v_0, \eta_0)^C$ denote the complexification $(v_0, \eta_0)^C := (v_0, \eta_0) + i(v_0, \eta_0)$. Clearly $Z(\cdot, \omega)$ is bounded on the complex ball $B((v_0, \eta_0)^C, \delta)$ in M_2^C for all $\omega \in \Omega_0$. Now define the sequence $\{Z_k\}_{k=1}^{\infty}$ of complex random variables by

$$Z_k(\omega) := Z((\eta_k(0), \eta_k), \omega) - \eta_k(0), \qquad \omega \in \Omega,\ k \ge 1.$$

Then

$$Z_k = \int_0^{t_0} \int_{[-r,-\varepsilon]} \eta_k(u+s)\, d\nu(s)\, dW(u), \qquad k \ge 1.$$

By standard properties of the Itô integral, together with Fubini's theorem, one has

$$EZ_k\overline{Z_l} = \int_{[-r,-\varepsilon]} \int_{[-r,-\varepsilon]} \int_0^{t_0} \eta_k(u+s)\overline{\eta_l(u+s')}\, du\, d\nu(s)\, d\nu(s') = 0$$

for $k \neq l$, because

$$\int_0^{t_0} \eta_k(u+s)\overline{\eta_l(u+s')}\,du = 0$$

whenever $k \neq l$, for all s, $s' \in [-r, 0]$. Furthermore

$$\int_0^{t_0} \eta_k(u+s)\overline{\eta_k(u+s')}\,du = t_0 \exp\left(\frac{2\pi i k(s-s')}{t_0}\right)$$

for all s, $s' \in [-r, 0]$. Hence

$$\begin{aligned} E|Z_k|^2 &= \int_{[-r,-\varepsilon]}\int_{[-r,-\varepsilon]} t_0 \exp\left(\frac{2\pi i k(s-s')}{t_0}\right) d\nu(s)\,d\nu(s') \\ &= t_0 \left|\int_{[-r,0]} \exp\left(\frac{2\pi i k s}{t_0}\right) d\nu(s)\right|^2 \\ &= \nu_k^2. \end{aligned}$$

Now $Z(\cdot, \omega)$: $M_2^C \to \mathrm{C}$ is bounded on $B((v_0, \eta_0)^C, \delta)$ for all $\omega \in \Omega_0$, and $\|(\eta_k(0), \eta_k)\| = \sqrt{r+1}$ for all $k \geq 1$. Hence by the linearity property

$$\begin{aligned} &Z\left((v_0, \eta_0)^C + \frac{\delta}{2\sqrt{r+1}}(\eta_k(0), \eta_k), \cdot\right) \\ &\quad = Z((v_0, \eta_0)^C, \cdot) + \frac{\delta}{2\sqrt{r+1}} Z((\eta_k(0), \eta_k), \cdot), \qquad k \geq 1, \end{aligned}$$

a.s., it follows that

$$P\left(\sup_{k\geq 1} |Z_k| < \infty\right) > 0. \tag{2.3}$$

It is easy to check that $\{\operatorname{Re} Z_k, \operatorname{Im} Z_k\colon k \geq 1\}$ are independent $\mathscr{N}(0, \nu_k^2/2)$-distributed Gaussian random variables. We now follow the computation in [4], page 317, in order to reach a contradiction to (2.3). More specifically, for each integer $N \geq 1$, we have

$$\begin{aligned} P\left(\sup_{k\geq 1} |Z_k| < N\right) &\leq \prod_{k\geq 1} P(|\operatorname{Re} Z_k| < N) \\ &= \prod_{k\geq 1}\left[1 - \frac{2}{\sqrt{2\pi}} \int_{(\sqrt{2}N)/\nu_k}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx\right] \qquad (2.4) \\ &\leq \exp\left\{-\frac{2}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \int_{(\sqrt{2}N)/\nu_k}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx\right\}. \end{aligned}$$

It is easy to see that there exists $N_0 > 1$ (independent of $k \geq 1$) such that

$$\int_{(\sqrt{2}N)/\nu_k}^{\infty} \exp\left(\frac{-x^2}{2}\right) dx \geq \frac{\nu_k}{2\sqrt{2}N} \exp\left(\frac{-N^2}{\nu_k^2}\right) \tag{2.5}$$

for all $N \geq N_0$ and all $k \geq 1$.

Now, combining (2.4) and (2.5) and using hypothesis (2.1) of the theorem immediately yields

$$P\Big(\sup_{k\geq 1}|Z_k| < N\Big) = 0$$

for all $N \geq N_0$. Hence

$$P\Big(\sup_{k\geq 1}|Z_k| < \infty\Big) = 0.$$

This clearly contradicts (2.3).

Since $Y(t_0, \cdot, \omega)$ is locally unbounded, it must be nonlinear because of Douady's theorem ([21], Part II, pages 155–160).

The proof of the theorem is now complete. □

Note that the pathological phenomenon in Theorem 2.1 is peculiar to the delay case $r > 0$. The proof of the theorem suggests that this pathology is due to the *Gaussian nature* of the Wiener process W coupled with the *infinite-dimensionality* of the state space M_2. Because of this, one may expect similar difficulties in certain types of linear stochastic partial differential equations driven by *multidimensional* white noise ([5]).

PROBLEM. Classify all finite signed measures ν on $[-r, 0]$ for which (III) is regular.

Note that (II) automatically satisfies the conditions of Theorem 2.1, and hence its trajectory field *explodes on every small neighborhood* of $0 \in M_2$. In view of the singular nature of (II), one notes the following striking fact when the variance σ of the noise is small. It is shown in Theorem 2.2 that the maximal exponential growth rate $\overline{\lambda}_1$ of (II) is *negative* for small σ and is bounded away from zero *independently of the choice of the initial path* in M_2.

THEOREM 2.2. *Let ν be a probability measure on $[-1, 0]$, and consider*

$$\text{(III}'\text{)}\qquad \begin{aligned} dx(t) &= \sigma\bigg(\int_{[-1,0]} x(t+s)\,d\nu(s)\bigg)\,dW(t), \qquad t \geq 0,\\ x_0 &= \eta, \end{aligned}$$

where $\sigma \in \mathrm{R}\backslash\{0\}$, $\eta \in C := C([-1, 0], \mathrm{R})$, $W(t)$, $t \geq 0$, standard one-dimensional Brownian motion, and $x(\cdot, \eta)$ is the solution of (III$'$) through η. Then there exists $\sigma_0 > 0$ and a continuous, strictly negative function ϕ: $(0, \sigma_0) \to \mathrm{R}^-$ such that

$$P\bigg(\limsup_{t\to\infty} \frac{1}{t} \log \|x_t(\cdot, \eta)\|_\infty \leq \phi(|\sigma|)\bigg) = 1$$

for all $\eta \in C$ and all σ with $0 < |\sigma| < \sigma_0$, where $\|\cdot\|_\infty$ denotes the sup norm *on C. The constant σ_0 and the function ϕ can be chosen independently of ν and of $\eta \in C$.*

REMARK. (i) The conclusion of the theorem is equivalent to the following statement:

$$P\left(\limsup_{t\to\infty}\frac{1}{t}\log\|(x(t,(v,\eta)),x_t(\cdot,(v,\eta)))\|_{M_2}\le\phi(|\sigma|)\right)=1$$

for all $(v,\eta)\in M_2$ and all σ with $0<|\sigma|<\sigma_0$. See the remark following Theorem 5.3 in [16].

(ii) If (III′) is regular, then Proposition 1.1 and Theorem 2.2 imply that the top Lyapunov exponent $\lambda_1\le\phi(|\sigma|)$ for $0<|\sigma|<\sigma_0$ and σ_0 sufficiently small.

PROOF OF THEOREM 2.2. Without loss of generality, we may and will assume throughout this proof that $\sigma>0$.

For $\eta\in C$ define $R(\eta):=\overline{\eta}-\underline{\eta}$, the diameter of the range of η, where $\overline{\eta}:=\sup_{-1\le s\le 0}\eta(s)$ and $\underline{\eta}:=\inf_{-1\le s\le 0}\eta(s)$. Fix $\alpha\in(0,1)$ and let $\beta>0$ be a parameter to be specified later. All future "constants" may depend on α. Define the continuous functional $V\colon C\to\mathbb{R}^+$ by

$$V(\eta):=(R(\eta)\vee|\eta(0)|)^{\alpha}+\beta R(\eta)^{\alpha},\qquad \eta\in C.$$

We will show that V is a Lyapunov functional for (III′) in the sense that

$$EV(x_1(\cdot,\eta))\le\delta(\sigma)V(\eta)\tag{2.6}$$

for some continuous function $\delta\colon(0,\sigma_0)\to[0,1)$ and all $\eta\in C$.

Let us first show that if (2.6) holds for $0<\sigma<\sigma_0$, then the assertion of the theorem follows. Suppress $\eta\in C$ and write $x:=x(\cdot,\eta)$. By the Markov property of $(x_n)_{n\ge0}$, (2.6) implies that $\delta(\sigma)^{-n}V(x_n)$, $n\ge0$, is a nonnegative $(\mathscr{F}_n^W)_{n\ge0}$-supermartingale. Therefore there exists a random variable Z taking values in $[0,\infty)$ such that

$$\delta(\sigma)^{-n}V(x_n)\to Z$$

as $n\to\infty$, almost surely. Hence $\limsup_{n\to\infty}(1/n)\log V(x_n)\le\log\delta(\sigma)$ a.s. Furthermore,

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|&\le\limsup_{n\to\infty}\frac{1}{n}\log(|x(n)|+R(x_n))\\&=\frac{1}{\alpha}\limsup_{n\to\infty}\frac{1}{n}\log V(x_n)\le\frac{1}{\alpha}\log\delta(\sigma)=:\phi(\sigma)<0.\end{aligned}$$

This implies the assertion of the theorem.

We now prove (2.6). Fix $\eta\in C$ and define

$$\|\eta\|_0:=\left[\int_0^1\left\{\eta(0)\nu((-s,0])+\int_{[-1,-s]}\eta(s+u)\,d\nu(u)\right\}^2ds\right]^{1/2}.$$

Let $Y(t):=x(t)-\eta(0)$, $0\le t\le1$. It follows from [12], Theorem 4.3, pages 151, 152, that $\sup_{0\le t\le1}E(Y(t)^{2n})<\infty$ for all positive integers n. By standard properties of the Itô integral, we get

$$
\begin{aligned}
\operatorname{var}(Y(1)) = E(Y(1)^2) &= \sigma^2 E\int_0^1 \left(\int_{[-1,0]} x(t+s)\,d\nu(s)\right)^2 dt \\
&= \sigma^2 \int_0^1 E\biggl(\int_{[-1,-t]} \eta(t+s)\,d\nu(s) \\
&\qquad\qquad + \eta(0)\nu((-t,0]) + \int_{(-t,0]} Y(t+s)\,d\nu(s)\biggr)^2 dt \\
&\geq \sigma^2 \|\eta\|_0^2
\end{aligned}
\tag{2.7}
$$

because $E\int_{(-t,0]} Y(t+s)\,d\nu(s) = 0$.

Next we want to show that $E(Y(1)^4) \leq D\sigma^4\|\eta\|_0^4$ where $D > 0$ is a constant (not depending on η, ν and $\sigma \in (0,1]$). A simple application of Itô's formula gives

$$
\begin{aligned}
EY^4(t) &= 6\sigma^2 \int_0^t E\left[Y^2(s)\left(\int_{[-1,0]} x(s+u)\,d\nu(u)\right)^2\right] ds \\
&\leq 12\sigma^2 \int_0^t EY^2(s)\left(\int_{[-1,-s]} \eta(s+u)\,d\nu(u) + \eta(0)\nu((-s,0])\right)^2 ds \\
&\quad + 12\sigma^2 \int_0^t E\left[Y^2(s)\left(\int_{(-s,0]} Y(s+u)\,d\nu(u)\right)^2\right] ds
\end{aligned}
\tag{2.8}
$$

for $0 \leq t \leq 1$.

Define the continuous functions g, h: $[0,1] \to \mathrm{R}^+$ by

$$
\begin{aligned}
g(s) &:= \left(\int_{[-1,-s]} \eta(s+u)d\nu(u) + \eta(0)\nu((-s,0])\right)^2, \qquad 0 \leq s \leq 1, \\
h(t) &:= \sup_{0\leq s\leq t} EY^4(s), \qquad 0 \leq t \leq 1.
\end{aligned}
$$

Then, using Hölder's inequality, (2.8) implies

$$
\begin{aligned}
h(t) &\leq 12\sigma^2 \int_0^t h(s)\,ds + 12\sigma^2 \int_0^t (h(s))^{1/2}\, g(s)\,ds \\
&\leq 12\sigma^2 \int_0^t h(s)\,ds + 12\sigma^2 (h(t))^{1/2} \|\eta\|_0^2
\end{aligned}
$$

for all $t \in [0,1]$. Dividing through by $(h(t))^{1/2}$ (when nonzero) and using the fact that $h(t)$ is nondecreasing in t, we obtain the inequality

$$
(h(t))^{1/2} \leq 12\sigma^2 \int_0^t (h(s))^{1/2}\,ds + 12\sigma^2 \|\eta\|_0^2
$$

for all $t \in [0,1]$. Hence applying Gronwall's lemma to the above inequality, we get

$$
\sup_{0\leq s\leq 1} EY^4(s) = h(1) \leq D\sigma^4 \|\eta\|_0^4
\tag{2.9}
$$

for all $\sigma \in (0,1]$, where D is a positive constant *independent of the choice of* $\eta \in C, \nu$ *and* $\sigma \in (0,1]$.

If $\|\eta\|_0 = 0$, then $x(t) = 0$ a.s. for all $t \geq 0$, and the assertion of the theorem follows trivially. Hence we will assume that $\|\eta\|_0 > 0$. Without loss of generality, we also assume that $\eta(0) \geq 0$. Define $q := (\eta(0)/\|\eta\|_0) \geq 0$.

We will devote the rest of this proof to establishing the inequality

$$E\left(\frac{R(Y_1)}{\|\eta\|_0} \vee \left|q + \frac{Y(1)}{\|\eta\|_0}\right|\right)^\alpha + \beta E\left(\frac{R(Y_1)}{\|\eta\|_0}\right)^\alpha \leq \delta(\sigma)\big((|1-q|^\alpha \vee q^\alpha) + \beta|1-q|^\alpha\big) \tag{2.10}$$

for all $q \geq 0$, and suitably chosen β, σ, $\delta(\sigma)$. It is not hard to see that

$$R(\eta) \geq \big|\|\eta\|_0 - |\eta(0)|\big|.$$

Using this, (2.10) clearly implies (2.6).

First, observe that $P(R(Y_1) \geq u) \leq P(\overline{Y}_1 \geq (u/2)) + P(\underline{Y}_1 \leq -(u/2))$ for all $u > 0$. Then, by Chebychev's and Doob's inequalities, we have

$$P\left(\overline{Y}_1 \geq \frac{u}{2}\right) \leq \frac{16E(\overline{Y}_1^4)}{u^4} \leq \tilde{D}\frac{E(Y(1)^4)}{u^4}$$

for all $u > 0$, where $\tilde{D} = (4^6/3^4)$. Similarly, $P(\underline{Y}_1 \leq -(u/2)) \leq \tilde{D}u^{-4}E(Y(1))^4$, $u > 0$. Hence

$$P(R(Y_1) \geq u) \leq 2\tilde{D}D\sigma^4\|\eta\|_0^4 u^{-4} \tag{2.11}$$

for all $u > 0$.

Next, observe that

$$\begin{aligned} E\left(\frac{R(Y_1)}{\|\eta\|_0}\right)^\alpha &= \int_0^\infty P\left(\frac{R(Y_1)}{\|\eta\|_0} \geq t^{1/\alpha}\right)dt \\ &\leq \int_0^\infty \left(\frac{2D\tilde{D}\sigma^4}{t^{4/\alpha}} \wedge 1\right)dt = D_1\sigma^\alpha, \end{aligned} \tag{2.12}$$

where $D_1 := (2D\tilde{D})^{\alpha/4}(1-(\alpha/4))^{-1}$.

Furthermore, since $R(Y_1) \geq |Y(1)|$, then

$$E\left(\frac{R(Y_1)}{\|\eta\|_0} \vee \left|q + \frac{Y(1)}{\|\eta\|_0}\right|\right)^\alpha \leq I_1 + I_2,$$

where

$$I_1 := \int_{(R(Y_1)/(\|\eta\|_0) \geq (q/2))} \left(q + \frac{R(Y_1)}{\|\eta\|_0}\right)^\alpha dP,$$

$$I_2 := \int_{(R(Y_1)/(\|\eta\|_0) < (q/2))} \left(q + \frac{Y(1)}{\|\eta\|_0}\right)^\alpha dP.$$

Using (2.12) we have

$$I_1 \leq \int_{(R(Y_1)/(\|\eta\|_0)\geq(q/2))} \left[3\frac{R(Y_1)}{\|\eta\|_0}\right]^\alpha dP \leq 3^\alpha D_1 \sigma^\alpha.$$

From (2.11) we also get

$$\begin{aligned} I_1 &= \int_0^\infty P\left(\frac{R(Y_1)}{\|\eta\|_0} \geq \left(\frac{q}{2}\right) \vee (t^{1/\alpha} - q)\right) dt \\ &\leq 2\tilde{D}D\sigma^4 \int_0^\infty \left[\left(\frac{q}{2}\right) \vee (t^{1/\alpha} - q)\right]^{-4} dt \\ &= 2\tilde{D}D\sigma^4 \left[\left(\frac{3q}{2}\right)^\alpha \left(\frac{q}{2}\right)^{-4} + \int_{\frac{3q}{2}}^\infty (u-q)^{-4} \alpha u^{\alpha-1}\, du\right] \\ &\leq 2\tilde{D}D\sigma^4 \left[q^{\alpha-4} 2^{4-\alpha} 3^\alpha + \int_{\frac{3q}{2}}^\infty \alpha(u-q)^{-4+\alpha-1}\, du\right] \\ &= \hat{D} q^{\alpha-4} \sigma^4, \end{aligned}$$

where $\hat{D} := 2^{5-\alpha}\tilde{D}D[3^\alpha + (\alpha/(4-\alpha))]$.

Next, we derive two estimates for I_2. The first one is the obvious inequality

$$I_2 \leq \left(\frac{3q}{2}\right)^\alpha.$$

The second estimate will be required in order to handle large values of q. To derive this estimate, we first observe that there exists $\bar{c} > 0$ (depending on α) such that

$$(1+y)^\alpha \leq 1 + \alpha y - \bar{c} y^2$$

for all $y \in [-1/2, 1/2]$.

Assume that $q > 0$. Setting $Z := Y(1)/(q\|\eta\|_0)$ and using (2.9), we have

$$\begin{aligned} &\int_{(|Z|\leq 1/2)} (1+Z)^\alpha\, dP \leq \int_{(|Z|\leq 1/2)} (1 + \alpha Z - \bar{c} Z^2)\, dP, \\ &\int_{(|Z|\leq 1/2)} (1+\alpha Z)\, dP = 1 - \int_{(|Z|>1/2)} (1+\alpha Z)\, dP \leq 1 + \int_{(|Z|>1/2)} Z^4\, dP \\ &\qquad\qquad \leq 1 + EZ^4 \leq 1 + D\sigma^4 q^{-4}; \end{aligned}$$

and, by (2.7),

$$\int_{(|Z|\leq 1/2)} Z^2\, dP = EZ^2 - \int_{(|Z|>1/2)} Z^2\, dP \geq \sigma^2 q^{-2} - 4EZ^4 \geq \sigma^2 q^{-2} - 4D\sigma^4 q^{-4}$$

for all $q > 0$. Therefore

$$I_2 \leq q^\alpha \left[1 + D\frac{\sigma^4}{q^4} - \bar{c}\left(\frac{\sigma^2}{q^2} - 4D\frac{\sigma^4}{q^4}\right)^+\right].$$

Next choose $D_2 > 3^{\alpha-1}\bar{c}$, $\beta = \beta(\sigma) := D_2\sigma^2$ and $\delta(\sigma) := (1 - \frac{1}{3}\bar{c}\sigma^2)$. Using (2.11), (2.12) and the above estimates for I_1 and I_2, (2.10) will follow once we have proved that there exists $\sigma_0 \in (0, 1)$ such that

$$
\begin{aligned}
&(3^\alpha D_1\sigma^\alpha) \wedge (\hat{D}q^{\alpha-4}\sigma^4) + \left(\frac{3q}{2}\right)^\alpha \wedge \left(q^\alpha\left[1 + D\frac{\sigma^4}{q^4} - \bar{c}\left(\frac{\sigma^2}{q^2} - 4D\frac{\sigma^4}{q^4}\right)^+\right]\right)\\
&\qquad + D_1D_2\sigma^{2+\alpha}\\
&\qquad \le \delta(\sigma)((|1-q|^\alpha \vee q^\alpha) + \beta|1-q|^\alpha)
\end{aligned}
\tag{2.13}
$$

for all $q \geq 0$ and $0 < \sigma < \sigma_0$.

We prove (2.13) by pure (elementary) analysis using a case-by-case argument for different ranges of values of q. In each case below, σ_0 is sufficiently small and independent of the choice of q in the respective range.

Case (i). $0 \leq q \leq 1/3$. The left-hand side of (2.13) is dominated by

$$
3^\alpha D_1\sigma^\alpha + \left(\frac{3q}{2}\right)^\alpha + D_1D_2\sigma^{2+\alpha} \le \left(1 - \frac{1}{3}\bar{c}\sigma^2\right)(1-q)^\alpha = \delta(\sigma)(1-q)^\alpha
$$

for all $\sigma \in (0, \sigma_0)$ where $\sigma_0 > 0$ is sufficiently small.

Case (ii). $1/3 < q < 3/2$. The left-hand side of (2.13) is dominated by

$$
\hat{D}q^{\alpha-4}\sigma^4 + q^\alpha\left(1 - \bar{c}\frac{\sigma^2}{q^2}\right) + q^{\alpha-4}D\sigma^4(1+4\bar{c}) + D_1D_2\sigma^{2+\alpha} \le \delta(\sigma)q^\alpha
$$

for all $\sigma \in (0, \sigma_0)$, and $\sigma_0 > 0$ sufficiently small.

Case (iii). $3/2 \leq q < \infty$. Observe that

$$
\begin{aligned}
\delta(\sigma)\left(1 + \frac{\beta(\sigma)}{3^\alpha}\right) &= \left(1 - \frac{1}{3}\bar{c}\sigma^2\right)\left(1 + \frac{D_2\sigma^2}{3^\alpha}\right)\\
&= 1 + \sigma^2\left(\frac{D_2}{3^\alpha} - \frac{\bar{c}}{3}\right) - \frac{\bar{c}D_2}{3^{\alpha+1}}\sigma^4.
\end{aligned}
\tag{2.14}
$$

Therefore, there is a positive constant $D_3 = D_3(\alpha)$ and a sufficiently small $\sigma_0 > 0$ (both independent of q), such that the left-hand side of (2.13) is dominated by

$$
\begin{aligned}
\hat{D}q^{\alpha-4}\sigma^4 + q^\alpha\left(1 + D\frac{\sigma^4}{q^4}\right) + D_1D_2\sigma^{2+\alpha} &\le D_3\sigma^4 + q^\alpha + D_1D_2\sigma^{2+\alpha}\\
&\le q^\alpha\delta(\sigma)\left(1 + \frac{\beta(\sigma)}{3^\alpha}\right)\\
&\le \delta(\sigma)[q^\alpha + \beta(\sigma)(q-1)^\alpha]
\end{aligned}
$$

for all $\sigma \in (0, \sigma_0)$ and sufficiently small $\sigma_0 > 0$. Note that the second inequality follows from (2.14) and the fact that $D_2 > 3^{\alpha-1}\bar{c}$. The last inequality holds because $q \geq \frac{3}{2}$.

This completes the proof of the theorem. $\square$

REMARK. (i) For any positive delay $r > 0$, we can rescale the equation

$$
\text{(III}''\text{)} \qquad \begin{aligned} dx(t) &= \sigma\left(\int_{[-r,0]} x(t+s)\, d\nu(s)\right) dW(t), \qquad t \geq 0, \\ x_0 &= \eta \end{aligned}
$$

by a linear time scale $t \mapsto t/r$. This reduces the question of stability of (III$''$) to the case $r = 1$. Therefore (III$''$) also satisfies the conclusion of Theorem 2.2.

Now consider the delay case $\nu = \delta_{-r}$, where $r > 0$ is fixed. Then (II) also satisfies Theorem 2.2. The choice of σ_0 depends on r. The above scaling argument has the effect of replacing r by 1 and σ by $\sigma\sqrt{r}$. In fact, if $\overline{\lambda}_1(r, \sigma)$ is the maximal exponential growth rate of (II), then it is easy to see that $\overline{\lambda}_1(r, \sigma) = (1/r)\overline{\lambda}_1(1, \sigma\sqrt{r})$. Hence σ_0 decreases (like $(1/\sqrt{r})$) as r increases. In particular, (for a fixed σ), a *small delay* r tends to *stabilize* equation (III$''$) or (II), while on the other hand (and in view of Theorem 2.3) a *large delay* in (II) has a *destabilizing* effect.

(ii) We conjecture that, even in the singular case, the following almost sure lim sup,

$$
\limsup_{t\to\infty} \frac{1}{t} \log \|(x(t, (v, \eta)), x_t(\cdot, (v, \eta)))\|_{M_2}
$$

is actually a *limit* whose value is nonrandom and independent of all $\eta \in C$ with $\|\eta\|_0 > 0$.

(iii) Using a Lyapunov function(al) argument, we will show in Theorem 2.3 that for sufficiently large σ, the singular delay equation (II) is unstable. This result is in sharp contrast with the corresponding result in the nondelay case $r = 0$, where

$$
\lim_{t\to\infty} \frac{1}{t} \log |x(t)| = -\sigma^2/2 < 0
$$

for *all* $\sigma \in \mathrm{R}$, even when σ is large.

(iv) The growth rate function ϕ in Theorem 2.2 satisfies

$$
\phi(\sigma) = -\sigma^2/2 + o(\sigma^2) \tag{2.15}
$$

as $\sigma \to 0^+$. This agrees with the nondelay case $r = 0$. In order to see (2.15), one may modify the proof of Theorem 2.2 by making the following choices:

$$
\begin{aligned} V(\eta) &:= [(MR(\eta)) \vee |\eta(0)|]^\alpha + \beta R(\eta)^\alpha, \qquad \eta \in C, \\ \overline{c} &:= (\alpha/2)(1-\alpha) - \varepsilon, \\ \delta(\sigma) &:= 1 - \overline{c}\sigma^2(1-\varepsilon), \end{aligned}
$$

for sufficiently small ε, $\alpha > 0$ and sufficiently large M.

THEOREM 2.3. *Consider the equation*

$$
\text{(II)} \qquad \begin{aligned} dx(t) &= \sigma x(t-r)\, dW(t), \qquad t > 0, \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathrm{R} \times L^2([-r, 0], \mathrm{R}), \end{aligned}
$$

driven by a standard Wiener process W with a positive delay r and $\sigma \in \mathbb{R}$. Then there exists a continuous function ψ: $(0,\infty) \to \mathbb{R}$ which is increasing to infinity such that

$$(2.16)\qquad P\left(\liminf_{t\to\infty} \frac{1}{t} \log \|(x(t,(v,\eta)), x_t(\cdot,(v,\eta)))\|_{M_2} \geq \psi(|\sigma|)\right) = 1,$$

for all $(v,\eta) \in M_2\backslash\{0\}$ and all $\sigma \neq 0$. The function ψ is independent of the choice of $(v,\eta) \in M_2\backslash\{0\}$.

REMARK. It is easy to see that $\|\cdot\|_{M_2}$ can be replaced by the sup norm on C.

PROOF OF THEOREM 2.3. We break the proof up into two main steps. In Step 1, we prove a discrete version of (2.16). In Step 2, we use an interpolation argument to obtain the continuous-time limit (2.16).

Step 1. We shall first prove that

$$(2.17)\qquad P\left(\liminf_{n\to\infty} \frac{1}{n} \log \|(x(n,(v,\eta)), x_n(\cdot,(v,\eta)))\|_{M_2} \geq \psi(|\sigma|)\right) = 1,$$

for all $(v,\eta) \in M_2\backslash\{0\}$ and all $\sigma \neq 0$.

In view of Remark (i), following the proof of Theorem 2.2, we will assume that $r = 1$, without loss of generality. We will also assume that $\sigma > 0$.

Define the continuous functional V: $M_2\backslash\{0\} \to [0,\infty)$ by

$$V((v,\eta)) = \left(v^2 + \sigma \int_{-1}^{0} \eta^2(s)\,ds\right)^{-1/4}$$

We will show that for $(v,\eta) \neq 0$

$$(2.18)\qquad EV((x(1,(v,\eta)), x_1(\cdot,(v,\eta)))) \leq K\sigma^{-1/4} V((v,\eta))$$

for all $\sigma > 0$, where K is a positive constant suitably chosen independently of $(v,\eta) \in M_2\backslash\{0\}$. We show first that (2.18) implies the assertion of the theorem. Set $X(t) := (x(t,(v,\eta)), x_t(\cdot,(v,\eta)))$, $t \geq 0$. Then (2.18) and the Markov property imply that $K^{-n}\sigma^{n/4}V(X(n))$, $n \geq 0$, is a nonnegative $(\mathscr{F}_n^W)_{n\geq 0}$-supermartingale, which therefore converges almost surely to a nonnegative random variable. Therefore $\limsup_{n\to\infty}(1/n)\log V(X(n)) \leq \log K - \frac{1}{4}\log\sigma$. Hence

$$\liminf_{n\to\infty} \frac{1}{n} \log \|X(n)\|_{M_2} \geq \tfrac{1}{2}\log\sigma - 2\log K$$

almost surely. This in turn implies (2.17) with $\psi(\sigma) := \frac{1}{2}\log\sigma - 2\log K$, $\sigma > 0$.

Let us now show that (2.18) holds for $(v,\eta) \neq 0$. Define $\|\eta\|_2 := (\int_{-1}^{0} \eta^2(s)\,ds)^{1/2}$. If $\|\eta\|_2 = 0$, then (2.18) holds with $K = 1$. Without loss of generality, we shall assume that $v \geq 0$ and $\|\eta\|_2 > 0$. Define $\mu := v/(\sigma\|\eta\|_2)$. Then (2.18) is equivalent to

$$E\biggl(\sigma\int_0^1\biggl(\mu+\frac{1}{\|\eta\|_2}\int_0^t \eta(u-1)\,dW(u)\biggr)^2 dt+(\mu+B)^2\biggr)^{-1/4} \leq K(1+\sigma\mu^2)^{-1/4} \tag{2.19}$$

for all $\mu \geq 0$, where $B = (1/\|\eta\|_2)\int_0^1 \eta(u-1)\,dW(u)$. Note that B is an $\mathscr{N}(0,1)$-Gaussian random variable.

Case (i). $\sigma\mu^2 \leq 1$. The left-hand side of (2.19) is dominated by

$$\begin{aligned}E|\mu+B|^{-1/2} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|\mu+x|^{-1/2}\exp(-x^2/2)\,dx\\ &\leq 2^{-1/4}\tilde{K} \leq \tilde{K}(1+\sigma\mu^2)^{-1/4},\end{aligned}$$

where

$$\tilde{K} := \frac{2^{1/4}}{\sqrt{2\pi}}\sup_{y\in \mathrm{R}^+}\int_{-\infty}^{\infty}|y+x|^{-1/2}\exp(-x^2/2)\,dx < \infty.$$

Case (ii). $\sigma\mu^2 > 1$. It suffices to show that there exists $K' < \infty$ such that

$$E\biggl[\sigma\int_0^1\biggl(\mu+\frac{1}{\|\eta\|_2}\int_0^t \eta(u-1)\,dW(u)\biggr)^2 dt\biggr]^{-1/4} \leq K'(\sigma\mu^2)^{-1/4}. \tag{2.20}$$

That is,

$$\sup E\biggl[\int_0^1\biggl(1+\frac{1}{\mu\|\eta\|_2}\int_0^t \eta(u-1)\,dW(u)\biggr)^2 dt\biggr]^{-1/4} < \infty,$$

where the supremum is taken over all $\eta \in L^2([-1,0],\mathrm{R})\backslash\{0\}$, and $\mu > 0$. Set

$$Y(t) := 1+\frac{1}{\mu\|\eta\|_2}\int_0^t \eta(u-1)\,dW(u), \qquad t \geq 0.$$

Then $Y(t)$ is Gaussian with mean 1. It is easy to see that there exists a positive constant β independent of t [and the variance of $Y(t)$] such that

$$P(|Y(t)| \leq u) \leq \beta u$$

for all $0 < u \leq 1/2$. Hence $P(Y^2(t) \leq u) \leq \beta\sqrt{u}$ for $0 \leq u \leq 1/4$. Now

$$E\biggl(\int_0^1 Y^2(t)\,dt\biggr)^{-1/4} = \int_0^{\infty} P\biggl(\int_0^1 Y^2(t)\,dt \leq u^{-4}\biggr)\,du$$

and

$$
\begin{aligned}
P\biggl(\int_0^1 Y^2(t)\,dt \le z\biggr) &\le P(l(s\colon Y^2(s) \le 2z) \ge 1/2)\\
&\le 2(l \otimes P)\{(s,\omega)\colon Y^2(s,\omega) \le 2z\}\\
&= 2\int_0^1 P(Y^2(s) \le 2z)\,ds \le 2\sqrt{2}\beta\sqrt{z}
\end{aligned}
$$

for all $0 \le z \le 1/8$. The symbol l denotes Lebesgue measure on $[0,1]$.

Therefore

$$
\begin{aligned}
E\biggl(\int_0^1 Y^2(t)\,dt\biggr)^{-1/4} &\le 8^{1/4} + \int_{8^{1/4}}^{\infty} 2\sqrt{2}\beta\sqrt{u^{-4}}\,du\\
&= 8^{1/4} + \int_{8^{1/4}}^{\infty} 2\sqrt{2}\beta u^{-2}\,du =: K' < \infty.
\end{aligned}
$$

This proves (2.17).

Step 2. We now show (2.16). In view of (2.17), it suffices to show that a.s.

$$
\limsup_{n\to\infty} \frac{1}{n} Y_n \le 0, \tag{2.21}
$$

where

$$
Y_n := -\log \inf_{n\le t\le n+1} \frac{\|X(t)\|_{M_2}}{\|X(n)\|_{M_2}}.
$$

For $(v,\eta) \in M_2 \setminus \{0\}$ define

$$
Z(v,\eta) := -\inf_{0\le t\le 1} \log \frac{\|X(t)\|_{M_2}}{\|(v,\eta)\|_{M_2}}.
$$

Fix $\beta > 1$. We will show that

$$
\sup_{(v,\eta)\in M_2\setminus\{0\}} E\bigl(Z(v,\eta)^{\beta}\bigr) < \infty. \tag{2.22}
$$

Then by (2.22) and the Markov property, we get

$$
\sum_{n=1}^{\infty} P(Y_n \ge \alpha n) \le \sum_{n=1}^{\infty} \frac{E(Y_n^{\beta})}{(\alpha n)^{\beta}} \le \sup_{(v,\eta)\in M_2\setminus\{0\}} E(Z(v,\eta)^{\beta}) \sum_{n=1}^{\infty} (\alpha n)^{-\beta} < \infty
$$

for every $\alpha > 0$. Therefore if we prove (2.22), then (2.21) will follow from the above relation and the Borel–Cantelli lemma.

We now prove (2.22). Consider

$$
E(Z(v,\eta)^{\beta}) \le E\biggl(-\log \inf_{0\le t\le 1} \frac{[v + \sigma\int_0^t \eta(s-1)\,dW(s)]^2 + \int_{t-1}^0 \eta^2(s)\,ds}{v^2 + \int_{-1}^0 \eta^2(s)\,ds}\biggr)^{\beta}.
$$

When $\eta = 0$, the right-hand side of the above inequality is equal to 0. So we will assume that $\eta \neq 0$. Defining μ as before, the right-hand side of the above inequality is equal to

$$E\biggl(-\log \inf_{0\le t\le 1} \frac{[\mu + (1/\|\eta\|_2)\int_0^t \eta(s-1)\,dW(s)]^2 + (1/(\sigma^2\|\eta\|_2^2))\int_{t-1}^0 \eta^2(s)\,ds}{\mu^2 + (1/\sigma^2)}\biggr)^\beta. \tag{2.23}$$

For different σ_1 and σ_2 the ratio of the corresponding terms in the above expression is bounded above and below uniformly in μ, η. Therefore for the rest of this proof we will assume without loss of generality that $\sigma = 1$. We will also assume that $\mu \ge 0$.

By a time-change argument, (2.23) equals

$$E\biggl(-\log \inf_{0\le t\le 1}\biggl[\frac{(\mu + B(t))^2 + 1 - t}{\mu^2 + 1}\biggr]\biggr)^\beta,$$

where $B(t)$, $0 \le t \le 1$, is an $(\mathscr{F}_t^W)_{0\le t\le 1}$-standard Brownian motion. Now $M(t) := (\mu + B(t))^2 + 1 - t$, $t \in [0,1]$, is a continuous nonnegative martingale. Let $M_* := \inf_{0\le t\le 1} M(t)$. Using a stopping-time argument and the martingale property, it follows that

$$P(M_* \le x) \le 2P(M(1) \le 2x)$$

for all $x > 0$. Therefore

$$\begin{aligned} EZ(v,\eta)^\beta &\le \int_0^\infty P\biggl(\biggl[-\log\biggl(\frac{M_*}{\mu^2+1}\biggr)\biggr]^\beta \ge x\biggr)\,dx \\ &= \int_0^\infty \mathbb{P}\bigl(M_* \le (\mu^2+1)\exp(-x^{1/\beta})\bigr)\,dx, \\ &\le 2\int_0^\infty \Bigl[\Phi\bigl(-\mu + [2(\mu^2+1)]^{1/2}\exp(-x^{1/\beta}/2)\bigr) \\ &\qquad - \Phi\bigl(-\mu - [2(\mu^2+1)]^{1/2}\exp(-x^{1/\beta}/2)\bigr)\Bigr]\,dx, \end{aligned} \tag{2.24}$$

where $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^x \exp(-y^2/2)\,dy$. Now fix $x_0 > 0$ such that $4\exp(-x_0^{1/\beta}) \le 1$. Then the integrand in (2.24) is dominated by $K\exp(-x^{1/\beta}/2)$ for some $K > 0$, uniformly for all $\mu \ge 0$ and all $x \ge x_0$. Therefore (2.24) implies that

$$\sup_{(v,\eta)\in M_2\setminus\{0\}} E\bigl(Z(v,\eta)^\beta\bigr) < \infty.$$

This completes the proof of the theorem. □

REMARKS. (i) Observe that the functional V in the above proof does not satisfy an inequality of the type

$$\lim_{t\to 0+} \frac{1}{t}[EV((x(t), x_t)) - V((v,\eta))] \le -KV(v,\eta)$$

for some $K > 0$ and all $(v,\eta) \in M_2\setminus\{0\}$. To see this, simply pick $v = 0$, $\eta \equiv 0$ on $[-1,-2/3] \cup [-1/3, 0]$ and $\eta \neq 0$ on $(-2/3, -1/3)$.

(ii) For the singular equation (II), one can show that

$$\overline{\lambda}_1 \le \inf_{\delta \in \mathrm{R}} \{-\delta + (\delta + \tfrac{1}{2}\sigma^2 \exp(2\delta r))^+\}, \tag{2.25}$$

where $\overline{\lambda}_1$ is its maximal exponential growth rate.

To see (2.25), we modify a technique due to Kushner [9]. Consider the equation

$$\begin{aligned} dx(t) &= \mu x(t)\,dt + \sigma x(t-r)\,dW(t), \qquad t > 0, \\ (x(0), x_0) &= (v, \eta) \in M_2 := \mathrm{R} \times L^2([-r, 0], \mathrm{R}). \end{aligned} \tag{II$'$}$$

Define the functional $V^*\colon M_2 \to \mathrm{R}^+$ by

$$V^*((v, \eta)) := v^2 + \sigma^2 \int_{-r}^0 \eta(s)^2\,ds, \qquad (v, \eta) \in M_2.$$

Set $x(t) = x(t, (v, \eta), \cdot)$, $t \ge -r$. A simple application of Itô's formula gives

$$EV^*((x(t), x_t)) = V^*((v, \eta)) + \int_0^t EQ(x(u))\,du, \qquad t > 0,$$

where $Q(z) := (2\mu + \sigma^2)z^2$, $z \in \mathrm{R}$. The above relation, together with the Markov property, implies that the process $\{\exp(-(2\mu+\sigma^2)t)V^*((x(t), x_t))\colon t \ge 0\}$ is a nonnegative $(\mathscr{F}_t^W)_{t\ge 0}$-supermartingale. Therefore the maximal exponential growth rate for (II$'$) is less than or equal to $[\mu + (\sigma^2/2)]^+$. Using an exponential shift (cf. Section 4) and setting $\mu = 0$ immediately gives (2.25). Now put $\delta = -(1/r)\log|\sigma|$ in (2.25). This gives $\overline{\lambda}_1 \le (1/r)\log|\sigma|$ for large $|\sigma|$. On the other hand, one can show that a.s.

$$\liminf_{|\sigma|\to\infty} \frac{\liminf_{t\to\infty} (1/t)\log\|(x^\sigma(t), x_t^\sigma)\|}{\log|\sigma|} \ge 1/(2r),$$

where x^σ is the solution of (II) for a given $\sigma \in \mathrm{R}$. This follows from the proof of Theorem 2.3. The above discussion shows that $(1/2r)\log|\sigma| + o(\log|\sigma|) \le \overline{\lambda}_1 \le (1/r)\log|\sigma|$ for large $|\sigma|$. It is interesting to observe that the estimate (2.25) does *not* yield the stability of (II) for small σ that was demonstrated by the conclusion of Theorem 2.2.

3. Delay equations with Poisson noise. This class of equations was studied in [20].

Consider the one-dimensional linear delay equation

$$\begin{aligned} dx(t) &= x((t-1)-)\,dN(t), \qquad t > 0, \\ x_0 &= \eta \in D := D([-1, 0], \mathrm{R}). \end{aligned} \tag{IV}$$

The process $N(t) \in \mathrm{R}$ is a Poisson process with i.i.d. interarrival times $\{T_i\}_{i=1}^\infty$ which are exponentially distributed with the same parameter μ. The jumps $\{Y_i\}_{i=1}^\infty$ of N are i.i.d. and independent of all the T_i's. Let

$$j(t) := \sup\left\{ j \ge 0\colon \sum_{i=1}^{j} T_i \le t \right\}.$$

Then

$$N(t) = \sum_{i=1}^{j(t)} Y_i.$$

Equation (IV) can be solved a.s. in forward steps of lengths 1, using the relation

$$x^\eta(t) = \eta(0) + \sum_{i=1}^{j(t)} Y_i x\left(\left(\sum_{j=1}^{i} T_j - 1\right)-\right) \quad \text{a.s.}$$

It is easy to see that the trajectory $\{x_t : t \geq 0\}$ of (IV) is a Markov process in the state space D (with the supremum norm $\|\cdot\|_\infty$). It is interesting to observe that equation (IV) does not satisfy the general set-up in [16]. In spite of this, the above relation implies that (IV) is regular in D; that is, it admits a measurable flow X: $\mathrm{R}^+ \times D \times \Omega \to D$ with $X(t, \cdot, \omega)$ continuous linear for all $t \geq 0$ and a.a. $\omega \in \Omega$. It is interesting to compare this with the singular equation (II) of Section 2. Furthermore, we will show that (IV) has an a.s. Lyapunov spectrum which can be characterized directly without appealing to the Oseledec theorem. This can be done by interpolating between the sequence of random times:

$$\tau_0(\omega) := 0,$$

$$\tau_1(\omega) := \inf\left\{n \geq 1\colon \sum_{j=1}^{k} T_j \notin [n-1, n] \text{ for all } k \geq 1\right\},$$

$$\tau_{i+1}(\omega) := \inf\left\{n > \tau_i(\omega)\colon \sum_{j=1}^{k} T_j \notin [n-1, n] \text{ for all } k \geq 1\right\}, \qquad i \geq 1.$$

It is easy to see that $\{\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots\}$ are i.i.d. Furthermore $E\tau_1 = \mathrm{e}^\mu$.

THEOREM 3.1. *Let $\xi \in D$ stand for the constant path $\xi(s) = 1$ for all $s \in [-1, 0]$. Suppose $E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty$ exists (possibly $= +\infty$ or $-\infty$). Then the a.s. Lyapunov spectrum*

$$\lambda(\eta) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty, \quad \eta \in D, \ \omega \in \Omega$$

of (IV) is $\{-\infty, \lambda_1\}$ where

$$\lambda_1 = e^{-\mu} E \log \|X(\tau_1(\cdot), \xi, \cdot)\|_\infty.$$

In fact,

$$\lim_{t\to\infty} \frac{1}{t} \log \|X(t, \eta, \omega)\|_\infty = \begin{cases} \lambda_1, & \eta \notin \operatorname{Ker} X(\tau_1(\omega), \cdot, \omega), \\ -\infty, & \eta \in \operatorname{Ker} X(\tau_1(\omega), \cdot, \omega). \end{cases}$$

PROOF. For simplicity of notation, we shall denote the sup norm $\|\cdot\|_\infty$ by $\|\cdot\|$ throughout this proof.

If $P(X(\tau_1, \xi, \cdot) \equiv 0) > 0$, then it is easy to see that

$$\lim_{t\to\infty} (1/t) \log \|(X(t, \eta, \cdot))\| = -\infty,$$

for all $\eta \in D$, a.s. So we will assume that

$$P(X(\tau_1, \xi, \cdot) \equiv 0) = 0.$$

Define the i.i.d. sequence of random variables $\{S_i\}_{i=1}^{\infty}$ by

$$S_i := \frac{\|(X(\tau_i, \xi, \cdot))\|}{\|(X(\tau_{i-1}, \xi, \cdot))\|}, \qquad i = 1, 2, \dots$$

with $\tau_0 := 0$.

The law of large numbers implies that

$$\lim_{n\to\infty} \frac{1}{\tau_n} \log \|(X(\tau_n, \xi, \omega))\| = e^{-\mu}(E \log S_1)$$

for a.a. $\omega \in \Omega$.

For $t \geq 0$ and $\omega \in \Omega$, let $n = n(\omega, t)$ satisfy $\tau_n \leq t < \tau_{n+1}$. Then

$$\frac{1}{t} \log \|(X(t, \xi, \omega))\| = \frac{1}{t} \log \|(X(\tau_n, \xi, \omega))\| + \frac{1}{t} \log \frac{\|(X(t, \xi, \omega))\|}{\|(X(\tau_n, \xi, \omega))\|}.$$

Since $(\tau_n/t) \longrightarrow 1$ as $t \to \infty$, the term $(1/t) \log \|(X(\tau_n, \xi, \omega))\|$ converges to $e^{-\mu}(E \log S_1)$ for a.a. $\omega \in \Omega$ by what we showed above.

Let us now assume that $E|\log S_1| < \infty$. We will show that, for a.a. $\omega \in \Omega$,

$$\lim_{t\to\infty} \frac{1}{t} \left| \log \frac{\|(X(t, \xi, \omega))\|}{\|(X(\tau_n, \xi, \omega))\|} \right| = 0. \tag{3.1}$$

Let

$$M(\omega) := \sup_{0\leq s\leq \tau_1} |X(s, \xi, \omega)(0)|, \qquad m := \inf_{0\leq s\leq \tau_1} |X(s, \xi, \omega)(0)|, \qquad \omega \in \Omega.$$

Then there is a positive number q such that

$$P(S_1 > \alpha \,|\, M > \alpha) \geq q > 0$$

for all $\alpha \geq 0$. This holds because after a jump to a value greater than α in absolute value, the probability of staying in that state for another two units of time is positive (independently of α). Using this and the fact that $E \log S_1 < \infty$, implies that $E \log M < \infty$. Similarly it follows that $E \log m > -\infty$. Now by the first Borel–Cantelli lemma, we get

$$\lim_{n\to\infty} \frac{1}{n} \sup_{\tau_n \leq t < \tau_{n+1}} \left| \log \frac{\|(X(t, \xi, \omega))\|}{\|(X(\tau_n, \xi, \omega))\|} \right| = 0$$

for a.a. $\omega \in \Omega$. Hence (3.1) follows.

The cases $E \log S_1 = \infty$ or $(-\infty)$, respectively, can be treated similarly.

The remaining assertions of the theorem can now easily be proved. □

REMARKS. (i) Let p: $D \to [0, \infty)$ be a measurable functional, satisfying (a) $p(\lambda\eta) = |\lambda| p(\eta)$ for all $\eta \in D$ (homogeneity). (b) There exist $\gamma_1, \gamma_2 > 0$ such that

$$\gamma_1 \inf_{-1\le s\le 0} |\eta(s)| \le p(\eta) \le \gamma_2 \sup_{-1\le s\le 0} |\eta(s)|$$

for all $\eta \in D$.

For example, p can be any L^p-norm $(1 \le p < \infty)$, but also, for example, the functional $p(\eta) := \inf_{-1\le s\le 0} |\eta(s)|$. Using the same arguments as in the proof of the theorem, it can be shown that the Lyapunov spectrum

$$\lim_{t\to\infty} \frac{1}{t} \log p(X(t, \eta, \omega))$$

with respect to the functional p satisfies the conclusion of Theorem 3.1. In particular, the above spectrum is nonrandom and is independent of the choice of the functional p satisfying (a) and (b).

(ii) The same methods work (with a few obvious changes) for

$$dx(t) = \left(\int_{[-r,0]} x((t+s)-)\, d\mu(s) \right) dN(t)$$

for any finite signed measure μ whose support is bounded away from zero.

(iii) By the same methods we can also treat the equation

$$dx(t) = \alpha x(t-)\, dt + x((t-1)-)\, dN(t)$$

for a fixed $\alpha \in$ R. Just define $z(t) := e^{-\alpha t} x(t)$ and apply Itô's formula to get

$$dz(t) = z((t-1)-)\, e^{-\alpha}\, dN(t).$$

The above equation can be analyzed using the method of proof of Theorem 3.1. Let p: $D \to$ R be as in Theorem 3.1, Remark (i). Suppose $\eta \in D$ and define $\tilde\eta \in D$ by $\tilde\eta(s) = e^{-\alpha s}\eta(s)$, $s \in [-1, 0]$. Let $Z(t, \eta, \omega) := z_t(\cdot, \eta, \omega)$. Then

$$\lim_{t\to\infty} \frac{1}{t} \log p(X(t, \eta, \omega)) = \alpha + \lim_{t\to\infty} \frac{1}{t} \log p(Z(t, \tilde\eta, \omega)).$$

Hence the Lyapunov spectrum of X is that of Z shifted by α.

4. A class of regular equations.

In this section we shall outline a general scheme to obtain estimates on the top Lyapunov exponent for a class of one-dimensional regular linear stochastic functional differential equations. After outlining our scheme, we will apply it to several specific examples within the above class.

Our scheme can be applied to multidimensional linear equations with multiple delays.

The reader may note that the approach in [9] yields strictly weaker estimates than ours in all cases.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space satisfying the usual conditions. Consider the following class of one-dimensional linear sfde's:

$$\text{(V)} \qquad \begin{aligned} dx(t) &= \left\{ \nu_1 x(t) + \mu_1 x(t-r) + \int_{-r}^{0} x(t+s)\sigma_1(s)\,ds \right\} dt \\ &\quad + \left\{ \nu_2 x(t) + \int_{-r}^{0} x(t+s)\sigma_2(s)\,ds \right\} dM(t), \end{aligned}$$

where $r > 0$, $\sigma_1, \sigma_2 \in C^1([-r, 0], \mathrm{R})$, and M is a continuous $(\mathscr{F}_t)_{t\geq 0}$-local martingale with (stationary) ergodic increments. By the ergodic theorem, we have the a.s. deterministic limit $\beta := \lim_{t\to\infty} \langle M\rangle(t)/t$. Assume that $\beta < \infty$ and $\langle M\rangle(1) \in L^\infty(\Omega, \mathrm{R})$.

It is easy to see that equation (V) is regular with respect to M_2 because of Theorem 4.2 of [16]. Hence (V) has a sample-continuous stochastic semiflow X: $\mathrm{R}^+ \times M_2 \times \Omega \to M_2$. Furthermore, equation (V) satisfies all the hypotheses of Theorem 5.2 of [16]. Therefore the stochastic semiflow X has a fixed (nonrandom) Lyapunov spectrum. Let λ_1 be its top exponent. We wish to develop an upper bound for λ_1. In view of the proof of Theorem 5.2 in [16], there is a shift-invariant set $\Omega^* \in \mathscr{F}$ of full P-measure and a measurable random field λ: $M_2 \times \Omega \to \mathrm{R} \cup \{-\infty\}$,

$$\text{(4.1)} \qquad \lambda((v, \eta), \omega) := \lim_{t\to\infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_{M_2}, \quad (v, \eta) \in M_2,\ \omega \in \Omega^*,$$

giving the Lyapunov spectrum of (V).

We introduce the following family of equivalent norms:

$$\text{(4.2)} \qquad \|(v, \eta)\|_\alpha := \left\{ \alpha v^2 + \int_{-r}^{0} \eta(s)^2\,ds \right\}^{1/2}, \qquad (v, \eta) \in M_2,\ \alpha > 0,$$

on M_2. Clearly

$$\text{(4.3)} \qquad \lambda((v, \eta), \omega) = \lim_{t\to\infty} \frac{1}{t} \log \|X(t, (v, \eta), \omega)\|_\alpha, \qquad (v, \eta) \in M_2,\ \omega \in \Omega^*$$

for each $\alpha > 0$; that is, the Lyapunov spectrum of (V) with respect to $\|\cdot\|_\alpha$ is independent of $\alpha > 0$.

Let x be the solution of (V) starting at $(v, \eta) \in M_2$, and define the family of processes ρ_α, $\alpha > 0$, by

$$\text{(4.4)} \qquad \rho_\alpha(t)^2 := \|X(t)\|_\alpha^2 = \alpha x(t)^2 + \int_{t-r}^{t} x(u)^2\,du, \quad t > 0,\ \alpha > 0.$$

For each fixed $(v, \eta) \in M_2$, define the set $\Omega_0 \in \mathscr{F}$ by $\Omega_0 := \{\omega \in \Omega$: $\rho_\alpha(t, \omega) \neq 0$ for all $t > 0\}$. If $P(\Omega_0) = 0$, then by uniqueness there is a random time τ_0 such that a.s. $X(t, (v, \eta), \cdot) = 0$ for all $t \geq \tau_0$. This implies that $\lambda_1 = -\infty$. Hence we may suppose that $P(\Omega_0) > 0$. Using Itô's formula and a simple stopping-time argument gives

$$\begin{aligned}\log \rho_\alpha(t) = \log \rho_\alpha(0) &+ \int_0^t Q_\alpha(a(u), b(u), I_1(u))\, du \\ &+ \int_0^t \tilde{Q}_\alpha(a(u), I_2(u))\, d\langle M\rangle(u) \\ &+ \int_0^t R_\alpha(a(u), I_2(u))\, dM(u), \qquad t > 0,\end{aligned} \tag{4.5}$$

a.s. on Ω_0, where

$$\begin{aligned} Q_\alpha(z_1, z_2, z_3) &:= \nu_1 z_1^2 + \sqrt{\alpha}\mu_1 z_1 z_2 + \sqrt{\alpha}\, z_1 z_3 + \frac{1}{2}\frac{z_1^2}{\alpha} - \frac{1}{2} z_2^2, \\ \tilde{Q}_\alpha(z_1, z_3') &:= \alpha\left(\frac{1}{2} - z_1^2\right)\left(\frac{\nu_2}{\sqrt{\alpha}} z_1 + z_3'\right)^2, \\ R_\alpha(z_1, z_3') &:= \nu_2 z_1^2 + \sqrt{\alpha} z_1 z_3', \\ \|\sigma_i\|_2 &:= \left\{\int_{-r}^0 \sigma_i(s)^2\, ds\right\}^{1/2}, \qquad i = 1, 2 \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} a(t) &:= \frac{\sqrt{\alpha} x(t)}{\rho_\alpha(t)}, \qquad b(t) := \frac{x(t-r)}{\rho_\alpha(t)}, \\ I_i(t) &:= \frac{\int_{-r}^0 x(t+s)\sigma_i(s)\, ds}{\rho_\alpha(t)} \end{aligned} \tag{4.7}$$

for $i = 1, 2$, $t > 0$, a.s. on Ω_0.

In view of the estimate

$$|I_i(t)| \le \frac{1}{\rho_\alpha(t)}\left(\int_{-r}^0 x(t+s)^2\, ds\right)^{1/2} \|\sigma_i\|_2 = \sqrt{1 - a^2(t)}\|\sigma_i\|_2, \qquad i = 1, 2, \text{ a.s. on } \Omega_0,$$

the variables z_1, z_2, z_3, z_3' appearing in (4.6) must satisfy

$$|z_1| \le 1, \qquad z_2 \in \mathrm{R}, \qquad |z_3|^2 \le (1 - z_1^2)\|\sigma_1\|_2^2, \qquad |z_3'|^2 \le (1 - z_1^2)\|\sigma_2\|_2^2.$$

Let $\tau_1 := \inf\{t > 0 : \rho_\alpha(t) = 0\}$. Then the local martingale

$$\int_0^{t\wedge\tau_1} R_\alpha(a(u), I_2(u))\, dM(u), \qquad t > 0$$

is a time-changed (possibly stopped) Brownian motion. In view of this and the fact that a.s. $R_\alpha(a(u), I_2(u)) \le |\nu_2| + \sqrt{\alpha}\|\sigma_2\|_2$ for all $u \in [0, \tau_1)$, it follows that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^{t\wedge\tau_1} R_\alpha(a(u), I_2(u))\, dM(u) = 0 \quad \text{a.s.} \tag{4.8}$$

Now divide (4.5) by t, let $t \to \infty$, and use the above relation to obtain

$$
\begin{aligned}
\lambda((v,\eta),\omega) \le \limsup_{t\to\infty} \frac{1}{t}\int_0^t Q_\alpha(a(u), b(u), I_1(u))\,du \\
+ \limsup_{t\to\infty} \frac{1}{t}\int_0^t \tilde{Q}_\alpha(a(u), I_2(u))\,d\langle M\rangle(u)
\end{aligned}
\tag{4.9}
$$

a.s. on Ω_0, for all $\alpha > 0$.

SCHEME 1. If $\varepsilon_1 \le d\langle M\rangle(t)/dt \le \varepsilon_2$ a.s. for all t, then maximize $Q_\alpha(z_1, z_2, z_3) + \gamma\tilde{Q}_\alpha(z_1, z_3')$ over all $|z_1| \le 1$, $z_2 \in \mathrm{R}$, $|z_3| \le \sqrt{1-z_1^2}\|\sigma_1\|_2$, $|z_3'| \le \sqrt{1-z_1^2}\|\sigma_2\|_2$ and $\varepsilon_1 \le \gamma \le \varepsilon_2$ for *fixed* $\alpha > 0$. Hence for a.a. $\omega \in \Omega_0$, (4.9) gives an upper bound $Q_\alpha^*(\nu_1, \nu_2, \mu_1, \sigma_1, \sigma_2)$ for $\lambda((v,\eta),\omega)$. It is easy to see (from the cocycle property) that $\lambda((v,\eta),\omega) = -\infty$ for all $\omega \in \Omega_0^c \cap \Omega^*$. Therefore one has $\lambda((v,\eta),\omega) \le Q_\alpha^*(\nu_1, \nu_2, \mu_1, \sigma_1, \sigma_2)$ for a.a. $\omega \in \Omega$. Since the upper bound $Q_\alpha^*(\nu_1, \nu_2, \mu_1, \sigma_1, \sigma_2)$ is *independent* of $(v,\eta) \in M_2$ and $\omega \in \Omega$, it follows from Proposition 1.1 that $Q_\alpha^*(\nu_1, \nu_2, \mu_1, \sigma_1, \sigma_2)$ is also an upper bound for the top exponent λ_1 of (V) for each $\alpha > 0$. Hence $\lambda_1 \le \inf_{\alpha>0} Q_\alpha^*(\nu_1, \nu_2, \mu_1, \sigma_1, \sigma_2)$. This upper bound can be refined further by applying an exponential shift of the Lyapunov spectrum by an arbitrary amount δ and then minimizing the resulting upper bound over $\delta \in \mathrm{R}$ [cf. proofs of Theorems (4.1)–(4.3)]. The special case $\varepsilon_1 = \varepsilon_2 = 1$ corresponds to M being a standard Brownian motion.

SCHEME 2. Compute the maximal values of Q_α and $\tilde{Q}_\alpha$ for fixed $\alpha > 0$ separately over $|z_1| \le 1$, $z_2 \in \mathrm{R}$, z_3, z_3' as above. Let the maximal values of Q_α, $\tilde{Q}_\alpha$ thus obtained be denoted by q_α and $\tilde{q}_\alpha$, respectively. Then, as in Scheme 1, we have

$$\lambda_1 \le \inf_{\alpha>0}\{q_\alpha + \tilde{q}_\alpha\beta\}.$$

Again use the exponential shift by δ and then minimize over $\delta \in \mathrm{R}$, in order to further refine the above estimate (cf. proof of Theorem 4.3).

The above two schemes will now be used to develop upper bounds on λ_1 in the following special cases.

Consider the one-dimensional linear sfde:

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\,dt + \left\{\int_{-r}^0 x(t+s)\sigma_2(s)\,ds\right\}dW(t), \quad t > 0 \tag{VI}$$

with real constants ν_1, μ_1 and $\sigma_2 \in C^1([-r, 0], \mathrm{R})$. It is a special case of (V). Hence (VI) is regular with respect to M_2. Observe that the process $\int_{-r}^0 x(t+s)\sigma_2(s)\,ds$ has C^1 paths in t, and so the stochastic differential dW with respect to the one-dimensional Brownian motion W in (VI) may be interpreted in the Itô or Stratonovich sense *without changing the solution* x.

THEOREM 4.1. *Suppose λ_1 is the top a.s. Lyapunov exponent of (VI). Define the function*

$$\theta(\delta,\alpha) := -\delta + \left(\nu_1 + \delta + \frac{1}{2}\alpha\mu_1^2 e^{2\delta r} + \frac{1}{2\alpha}\right) \vee \left(\frac{\alpha}{2}\|\sigma_2\|_2^2 e^{2\delta^+ r}\right)$$

for all $\alpha \in \mathrm{R}^+$, $\delta \in \mathrm{R}$, where $\delta^+ := \max\{\delta, 0\}$.

Then

$$\lambda_1 \le \inf\{\theta(\delta,\alpha)\colon \delta \in \mathrm{R},\ \alpha \in \mathrm{R}^+\}. \tag{4.10}$$

PROOF. We follow Scheme 1. Putting $\nu_2 = \sigma_1 = 0$ in (V), it is easy to see from (4.6) that

$$\begin{aligned} Q_\alpha(z_1, z_2, 0) + \tilde{Q}_\alpha(z_1, z_3') &\le \left(\nu_1 + \frac{1}{2\alpha}\right) z_1^2 + \sqrt{\alpha}\,\mu_1 z_1 z_2 - \frac{1}{2} z_2^2 \\ &\quad + \alpha\left(\frac{1}{2} - z_1^2\right)^+ (1 - z_1^2)\|\sigma_2\|_2^2 \end{aligned} \tag{4.11}$$

for all $z_1 \in [-1,1]$, $z_2 \in \mathrm{R}$, $|z_3'|^2 \le (1 - z_1^2)\|\sigma_2\|_2^2$.

For fixed $z_1 \in [-1,1]$, the right-hand side of (4.11) attains its absolute maximum when $z_2 = \sqrt{\alpha}\mu z_1$. Hence

$$\begin{aligned} &\sup_{(z_2, z_3') \in A} [Q_\alpha(z_1, z_2, 0) + \tilde{Q}_\alpha(z_1, z_3')] \\ &\qquad \le z_1^2\left(\nu_1 + \frac{1}{2}\alpha\mu_1^2 + \frac{1}{2\alpha}\right) + \alpha\left(\frac{1}{2} - z_1^2\right)^+ (1 - z_1^2)\|\sigma_2\|_2^2, \end{aligned} \tag{4.12}$$

where $A := \{(z_2, z_3')\colon z_2 \in \mathrm{R}, |z_3'|^2 \le (1 - z_1^2)\|\sigma_2\|_2^2\}$. We further maximize the right-hand side of (4.12) over $z_1^2 \in [0,1]$. This gives

$$\sup_{(z_1, z_2, z_3') \in B} [Q_\alpha(z_1, z_2, 0) + \tilde{Q}_\alpha(z_1, z_3')] \le \left(\nu_1 + \frac{1}{2}\alpha\mu_1^2 + \frac{1}{2\alpha}\right) \vee \left(\frac{\alpha}{2}\|\sigma_2\|_2^2\right), \tag{4.13}$$

where the supremum is taken over the set

$$B := \{(z_1, z_2, z_3')\colon |z_1| \le 1, z_2 \in \mathrm{R}, |z_3'|^2 \le (1 - z_1^2)\|\sigma_2\|_2^2\}.$$

The inequality (4.13) implies that

$$\lambda_1 \le \left(\nu_1 + \frac{1}{2}\alpha\mu_1^2 + \frac{1}{2\alpha}\right) \vee \left(\frac{\alpha}{2}\|\sigma_2\|_2^2\right) \tag{4.14}$$

for every $\alpha, \nu_1, \mu_1, \sigma_2$. Finally we apply an exponential shift to (VI) by an arbitrary amount $\delta \in \mathrm{R}$. Shift the Lyapunov spectrum $\sum = \{\lambda_i\}_{i=1}^\infty$ of (VI) by setting $y(t) := \mathrm{e}^{\delta t} x(t)$, $t \ge -r$, for a fixed $\delta \in \mathrm{R}$. Then y solves the linear sfde

$$dy(t) = \{(\nu_1 + \delta) y(t) + \mu_1 e^{\delta r} y(t - r)\}\,dt + \int_{-r}^0 y(t+s)\hat{\sigma}_2(s)\,ds\,dW(t), \tag{VI*}$$

where $\hat{\sigma}_2 \in C^1([-r, 0], \mathrm{R})$ is given by

$$\hat{\sigma}_2(s) := e^{-\delta s}\sigma_2(s), \qquad s \in [-r, 0].$$

Let $\sum^* := \{\lambda_i^*\}_{i=1}^{\infty}$ denote the Lyapunov spectrum of (VI*). Then it is easy to see that $\lambda_i^* = \lambda_i + \delta$ for all $i \geq 1$. Furthermore, applying (4.14) to (VI*), and noting that $\|\hat{\sigma}_2\|_2^2 \leq \|\sigma_2\|_2^2 e^{2\delta^+ r}$, we immediately get (4.10). This completes the proof of the theorem. □

The following corollary shows that the estimate in Theorem 4.1 reduces to a well-known estimate in the deterministic case $\sigma_2 \equiv 0$ ([6], pages 17, 18).

COROLLARY 4.1. *In* (VI)*, suppose $\mu_1 \neq 0$ and let δ_0 be the unique real solution of the transcendental equation*

$$\nu_1 + \delta + |\mu_1| e^{\delta r} = 0. \tag{4.15}$$

Then

$$\lambda_1 \leq -\delta_0 + \frac{1}{2} \frac{\|\sigma_2\|_2^2}{|\mu_1|} e^{|\delta_0| r}. \tag{4.16}$$

If $\mu_1 = 0$ and $\nu_1 \geq 0$, then $\lambda_1 \leq \frac{1}{2}(\nu_1 + \sqrt{\nu_1^2 + \|\sigma_2\|_2^2})$. If $\mu_1 = 0$ and $\nu_1 < 0$, then $\lambda_1 \leq \nu_1 + \frac{1}{2}\|\sigma_2\|_2 e^{-\nu_1 r}$.

PROOF. Suppose $\mu_1 \neq 0$. Denote by $f(\delta)$, $\delta \in \mathrm{R}$, the left-hand side of (4.15). Then $f(\delta)$ is an increasing function of δ. A simple application of Rolle's theorem shows that f has a unique real zero δ_0. Using (4.10), we may put $\delta = \delta_0$ and $\alpha = |\mu_1|^{-1} e^{-\delta_0 r}$ in the expression for $\theta(\delta, \alpha)$. This immediately yields (4.16).

Now suppose $\mu_1 = 0$. Put $\delta = (-\nu_1)^+$ in $\theta(\delta, \alpha)$ and minimize the resulting expression over all $\alpha > 0$. This proves the last two assertions of the corollary. □

REMARKS. (i) Although the upper bounds for λ_1 in Theorem 4.1 and its corollary agree with the corresponding bounds in the deterministic case (for $\mu_1 \geq 0$), they still do not yield the optimal bound when $\mu_1 = 0$ and σ_2 is strictly positive and sufficiently small; compare Theorem 2.2, Remark (ii).

(ii) It would be interesting to study the asymptotics of λ_1 for small delays $r \downarrow 0$.

Our second example is the stochastic delay equation

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\, dt + x(t)\, dM(t), \quad t > 0, \tag{VII}$$

where M is the local martingale appearing in (V) and satisfying the conditions therein. In particular, (VII) is regular with respect to M_2. Furthermore, we have the following estimate on its top exponent.

THEOREM 4.2. *In* (VII) *define δ_0 as in Corollary* 4.1. *Then the top a.s. Lyapunov exponent λ_1 of* (VII) *satisfies*

$$\lambda_1 \leq -\delta_0 + \frac{\beta}{16}. \tag{4.17}$$

PROOF. We follow Scheme 2. Here, put $\sigma_1 = \sigma_2 \equiv 0$, $\nu_2 = 1$ in (V). Therefore

$$\lambda_1 \leq \inf_{\alpha>0}\{q_\alpha + \tilde{q}_\alpha \beta\}$$

where

$$q_\alpha := \sup_{|z_1|\leq 1,\, z_2 \in \mathrm{R}} Q_\alpha(z_1, z_2), \qquad \tilde{q}_\alpha := \sup_{|z_1|\leq 1} \tilde{Q}_\alpha(z_1),$$

and

$$Q_\alpha(z_1, z_2) := \nu_1 z_1^2 + \sqrt{\alpha}\, \mu_1 z_1 z_2 + \frac{1}{2}\frac{z_1^2}{\alpha} - \frac{1}{2} z_2^2,$$

$$\tilde{Q}_\alpha(z_1) := \left(\frac{1}{2} - z_1^2\right) z_1^2, \qquad |z_1| \leq 1,\ z_2 \in \mathrm{R}.$$

An elementary computation shows that

$$q_\alpha = \left(\nu_1 + \frac{1}{2}\alpha\mu_1^2 + \frac{1}{2\alpha}\right)^+ \quad \text{and} \quad \tilde{q}_\alpha = \frac{1}{16}.$$

Using an exponential shift, as in the proof of Theorem 4.1, gives

$$\lambda_1 \leq \frac{\beta}{16} + \inf_{\alpha\in\mathrm{R}^+}\left[-\delta + \left(\nu_1 + \delta + \frac{1}{2}\alpha\mu_1^2 e^{2\delta r} + \frac{1}{2\alpha}\right)^+\right]$$

for all $\delta \in \mathrm{R}$. For fixed $\delta \in \mathrm{R}$, the infimum on the right-hand side of the above inequality is attained when $\alpha = (e^{-\delta r})/|\mu_1|$. Therefore

$$\lambda_1 \leq \frac{\beta}{16} + \inf_{\delta\in\mathrm{R}}\left[-\delta + \left(\nu_1 + \delta + |\mu_1| e^{\delta r}\right)^+\right].$$

The above minimum is attained if δ solves the transcendental equation (4.15). Hence the conclusion of the theorem. □

REMARK. The above estimate for λ_1 is sharp in the deterministic case $\beta = 0$ and $\mu_1 \geq 0$, but is not sharp when $\beta \neq 0$; for example, $M = W$, one-dimensional standard Brownian motion in the nondelay case ($\mu_1 = 0$). On the other hand, when $M = \nu_2 W$ for a fixed real ν_2, the above bound may be considerably sharpened as in the following theorem.

THEOREM 4.3. *For the equation*

$$dx(t) = \{\nu_1 x(t) + \mu_1 x(t-r)\}\, dt + \nu_2 x(t)\, dW(t) \tag{VIII}$$

set

$$\phi(\delta) := -\delta + \frac{1}{4\nu_2^2}\left[\left(|\mu_1| e^{\delta r} + \nu_1 + \delta + \frac{1}{2}\nu_2^2\right)^+\right]^2, \tag{4.18}$$

for $\nu_2 \neq 0$. *Then*

$$\lambda_1 \leq \inf_{\delta \in \mathrm{R}} \phi(\delta). \tag{4.19}$$

In particular, if δ_0 *is the unique solution of the equation*

$$\nu_1 + \delta + |\mu_1| e^{\delta r} + \tfrac{1}{2}\nu_2^2 = 0, \tag{4.20}$$

then $\lambda_1 \leq -\delta_0$.

PROOF. We use Scheme 1. Take $\sigma_1 = \sigma_2 = 0$, $M = W$ in (V). Then (4.6) implies that

$$\begin{aligned} &Q_\alpha(z_1, z_2, 0) + \tilde{Q}_\alpha(z_1, 0) \\ &\quad = \left(\nu_1 + \frac{1}{2\alpha} + \frac{\nu_2^2}{2}\right) z_1^2 + \sqrt{\alpha}\,\mu_1 z_1 z_2 - \frac{1}{2} z_2^2 - \nu_2^2 z_1^4 \end{aligned} \tag{4.21}$$

for $|z_1| \leq 1$, $z_2 \in \mathrm{R}$. For fixed $z_1 \in [-1, 1]$, the right-hand side of (4.21) attains its maximum when $z_2 = \sqrt{\alpha}\,\mu_1 z_1$. Therefore

$$\begin{aligned} \lambda_1 &\leq \sup_{|z_1| \leq 1} \{Q_\alpha(z_1, \sqrt{\alpha}\,\mu_1 z_1, 0) + \tilde{Q}_\alpha(z_1, 0)\} \\ &= \sup_{|z_1| \leq 1} \left[\left(\nu_1 + \frac{1}{2\alpha} + \frac{\mu_1^2}{2}\alpha + \frac{\nu_2^2}{2}\right) z_1^2 - \nu_2^2 z_1^4\right] \\ &\leq \frac{1}{16\nu_2^2}\left\{\left(2\nu_1 + \frac{1}{\alpha} + \mu_1^2 \alpha + \nu_2^2\right)^+\right\}^2 \end{aligned} \tag{4.22}$$

for all $\alpha \in \mathrm{R}^+$. Now set

$$h(\alpha) := 2\nu_1 + \frac{1}{\alpha} + \mu_1^2 \alpha + \nu_2^2, \qquad \alpha > 0.$$

Note that

$$\inf_{\alpha > 0} h(\alpha) = h(|\mu_1|^{-1}) = 2\nu_1 + 2|\mu_1| + \nu_2^2.$$

Hence

$$\lambda_1 \leq \frac{1}{16\nu_2^2}\left[(2\nu_1 + 2|\mu_1| + \nu_2^2)^+\right]^2. \tag{4.23}$$

The first assertion of the theorem follows by applying an exponential shift to (VIII) and using (4.23). The last assertion of the theorem is obvious. This completes the proof of the theorem. □

REMARK. The reader may check that the estimate in Theorem 4.3 agrees with the nondelay case $\mu_1 = 0$ whereby $\lambda_1 = \nu_1 - \frac{1}{2}\nu_2^2 = \inf_{\delta \in \mathrm{R}} \phi(\delta)$. See also [2], [3] and [1].

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