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RIGHT INVERSES OF NONSYMMETRIC LÉVY PROCESSES[1]

BY MATTHIAS WINKEL

Université Pierre et Marie Curie

We analyze the existence and properties of right inverses K for nonsymmetric Lévy processes X, extending recent work of Evans [7] in the symmetric setting. First, both X and $-X$ have right inverses if and only if X is recurrent and has a nontrivial Gaussian component. Our main result is then a description of the excursion measure n^Z of the strong Markov process $Z = X - L$ (reflected process) where $L_t = \inf\{x > 0 : K_x > t\}$. Specifically, n^Z is essentially the restriction of n^X to the "excursions starting negative." Second, when only asking for right inverses of X, a certain "strength of asymmetry" is needed. Millar's [9] notion of creeping turns out necessary but not sufficient for the existence of right inverses. We analyze this both in the bounded and unbounded variation case with a particular emphasis on results in terms of the Lévy–Khintchine characteristics.

1. Introduction. Consider a real-valued Lévy process, that is, a continuous time process with stationary independent increments and càdlàg paths. Right inverses of Lévy processes have first been studied in the symmetric setting by Evans [7]. He defines an increasing process K to be a right inverse of a Lévy process X if it satisfies $X(K_x) = x$ for all $x \geq 0$. The minimal such K turns out to be a subordinator (i.e., an increasing Lévy process).

Specifically, Evans characterized the symmetric Lévy processes that have right inverses. He introduced $L_t = \inf\{x \geq 0 : K_x > t\}$ and the reflected process $Z = X - L$ and showed that Z is strong Markov with local time L at zero, he gave some fluctuation type identities for Z and L and proved formulae for the entrance laws to the excursion measure n^Z of Z. Evans also showed that K is distributed like a linear time change of the inverse local time of X at zero.

In this work we basically answer three questions left open by Evans.

First, what happens when symmetry fails? The only result of Evans that is clearly based on symmetry is the coincidence of the laws of K and the inverse local time of X in zero. Also, the class of processes possessing inverses is not restricted to those having a positive Gaussian component (in the symmetric setting, this is Evans's characterization result), but those form again an important class. We provide a further characterization in the bounded variation case and we also point

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out a large class of unbounded variation processes with zero Gaussian component that have right inverses.

Second, Evans remarked that in the Brownian case, n^Z is the excursion measure of Brownian motion on the negative half-line reflected at zero. How can we give an illustrative description of the excursion measure of Z in a less specific situation? Our answer is as follows. Evans's remark can be reformulated to: n^Z is a multiple of n^X restricted to negative excursions. We show, that whenever the Gaussian coefficient is positive, n^Z is a multiple of n^X restricted to "excursions starting negative."

Third, in the symmetric setting the existence of right inverses is equivalent to the possibility of X to creep across levels (i.e., enter $[x, \infty)$ continuously). Does this generalize to the nonsymmetric setting? No. We show that creeping is necessary for the existence of right inverses, but our characterization in the bounded variation case as well as a class of unbounded variation processes show that creeping is not sufficient.

The rest of the paper is organized as follows: a preliminary section both serves notational purposes and discusses the impact of the results and techniques of Evans [7], and some connections to Simon's [13] notion of subordination in the wide sense. Section 3 generalizes Evans's existence characterization to the nonsymmetric setting, gives further existence results in the unbounded variation case and discusses the relation to creeping. Section 4 provides results on the initial behavior of excursions of X leading to a description of the excursion measure n^Z when the Gaussian coefficient is positive. In Section 5 we deal with the bounded variation case using different methods from what has been presented before. A technical proof, that does not give much insight into the actual topic, was postponed to an Appendix concluding the article.

2. Preliminaries.

The concepts and the previous work needed in the sequel consist of generalities on Lévy processes, particularly concerning path properties, and the specific work of Evans [7] that provided the main motivation to this article.

2.1. *Classification and properties of Lévy processes.*

We fix our notation by briefly introducing into the theory of Lévy processes. We refer to Bertoin [2] and Sato [12] as standard references on the topic.

Lévy–Khintchine formula. Let X be a real-valued *Lévy process*, that is, a continuous time process with stationary independent increments and càdlàg paths. We express its distribution by the Lévy–Khintchine representation of its characteristic exponent

$$\begin{aligned}\psi(\lambda) &= -\log\bigl(E\bigl(\exp(i\lambda X_t)\bigr)\bigr)/t \\ &= ia\lambda + \tfrac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} (1 - e^{i\lambda x} + i\lambda x 1_{\{|x|\le 1\}})\Pi(dx)\end{aligned}$$

where $a \in \mathbb{R}$, the *Gaussian coefficient* $\sigma^2 \geq 0$ and the *Lévy measure* Π satisfying $\Pi(\{0\}) = 0$ and the integrability condition $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$ are called the *characteristics of* X. The Lévy measure Π is the intensity measure of the Poisson point process ΔX of jumps of X. X is called *spectrally negative* (respectively positive) if $\Pi((0,\infty)) = 0$ [respectively if $\Pi((-\infty, 0)) = 0$].

If $\sigma^2 = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|)\Pi(dx) < \infty$, we express ψ by

$$\psi(\lambda) = -ib\lambda + \int_{\mathbb{R}}(1 - e^{i\lambda x})\Pi(dx)$$

where $b \in \mathbb{R}$ is called the *drift coefficient*. X has then *bounded variation* and can be expressed as $X_t = bt + X_t^+ - X_t^-$ where X^+ and X^- are independent *subordinators* (i.e., increasing Lévy processes) only increasing by positive jumps, with Lévy measures $\Pi^+ = \Pi(\cdot \cap (0,\infty))$ and $\Pi^- = \Pi(- \cdot \cap (-\infty, 0))$, respectively. We say that X is a *compound Poisson process*, when $b = \sigma^2 = 0$ and the jump structure is discrete, that is, Π is finite.

The process $\hat{X} := -X$ is called the *dual process*. It has characteristic exponent $\hat{\psi}(\lambda) = \psi(-\lambda)$, that is, its characteristics are $\hat{a} = -a$, $\hat{\sigma}^2 = \sigma^2$ and $\hat{\Pi}(dx) = \Pi(-dx)$.

Resolvents and properties involving hitting times. Denoting the *q-resolvent measure* by

$$U^q(B) = \int_0^\infty e^{-qt} P(X_t \in B)\, dt, \qquad B \in \mathcal{B}(\mathbb{R}),$$

we say, that X is *recurrent* if $U(B) = U^0(B) = \infty$ for every open set $B \ni 0$ in $\mathbb{R}$, *transient* otherwise.

Let $T_B = \inf\{t > 0 : X_t \in B\}$ be the hitting time of $B \in \mathcal{B}(\mathbb{R})$. We define the one-point *q-capacity*

$$c_q := q \int_{-\infty}^{\infty} E\big(\exp(-qT_{\{x\}})\big)\, dx. \tag{1}$$

Loosely speaking, $c_q > 0$ means that X visits every given point with positive probability (this has to be restricted to the half line in the case of a subordinator, of course). For us, the importance of this property lies in the existence of bounded *resolvent densities* u^q.

We say, that 0 *is regular for* B if $T_B = 0$ P-a.s., that is, if X enters B immediately. If $c_q > 0$, then regularity of 0, by which we mean regularity of 0 for $\{0\}$, is equivalent to the existence of continuous resolvent densities u^q. Denote $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$. 0 is called *instantaneous* for X if $T_{\mathbb{R}^*} = 0$ P-a.s., that is, if X leaves the origin immediately; cf. Chapter II of [2] and Bretagnolle [5].

Spaces of sample paths and excursions. It is sometimes convenient to work in the canonical model $(\Omega, \mathcal{A}, (P_x)_{x\in\mathbb{R}}, X)$ where $\Omega = D([0,\infty),\mathbb{R})$ is the set of càdlàg functions ω: $[0,\infty) \to \mathbb{R}$ endowed with the Skorohod topology and the Borel σ-algebra $\mathcal{A} = \mathcal{D}([0,\infty),\mathbb{R})$, X is a Lévy process with characteristic exponent ψ starting from x under P_x, $x \in \mathbb{R}$.

Denote the set of zeros of X by $\mathcal{Z} = \{t \geq 0 : X_t = 0\}$. For all $t \geq 0$ we can look at the previous and next zeros $g_t = \sup(\mathcal{Z} \cap [0,t))$ and $d_t = \inf(\mathcal{Z} \cap [t,\infty))$ and the subpath

$$e_{g_t}(s) = X_{g_t+s}, \quad 0 \leq s < d_t - g_t, \qquad e_{g_t}(s) = \partial, \quad s \geq d_t - g_t$$

where $\partial \notin \mathbb{R}$ is a so-called cemetery. We call e_{g_t} the *excursion of X straddling t*. We also associate trivial excursions $e_t(s) = \partial$, $s \geq 0$, if $0 \leq t \neq g_u$ for all $u \geq 0$. The process $(e_t)_{t\geq 0}$ is called *excursion process in real time*. It has its values in the *excursion space* $(E, \mathcal{E})$ of real-valued càdlàg functions killed when hitting zero, that is, all $\varepsilon \in E$ have the properties that $\varepsilon(s) \neq 0$ for all $s > 0$ and that $\varepsilon(s_0) = \partial$ implies $\varepsilon(s) = \partial$ for all $s > s_0 \geq 0$.

The interesting case is when $\mathcal{Z}$ is not discrete. Then there is a local time process $(\ell_t)_{t\geq 0}$ that increases precisely on the closure $\bar{\mathcal{Z}}$ of $\mathcal{Z}$ and whose inverse $\ell^{-1}(s) = \inf\{t : \ell(t) > s\}$ is a subordinator with closed range $\bar{\mathcal{Z}}$. By standard Itô excursion theory the *excursion process in inverse local time*, $(e_{\ell^{-1}(s-)})_{s\geq 0}$, is a Poisson point process whose intensity measure n^X is called the *excursion measure*. Its marginals at fixed times are the so-called *entrance laws* $n_t^X := n^X(\{\varepsilon \in E : \varepsilon(t) \in \cdot \cap \mathbb{R}\})$ which characterize n^X uniquely; cf. Rogers and Williams [11], Section VI.48.

2.2. *Right inverses of Lévy processes after Evans.* An increasing process K is called a *right inverse* of a Lévy process X if it has càdlàg paths and satisfies $X_{K_x} = x$ for all $x \geq 0$.

In the case where X has no positive jumps (and satisfies $\limsup_{t\to\infty} X_t = \infty$) we can choose K to be the process of first passage times $K_x = A_x^{-1} = T_{[x,\infty)} = T_{\{x\}}$ which is easily seen to be a subordinator by applying the strong Markov property in K_x; cf. [2], Section VII.1.

Dropping the lim sup-condition in the spectrally negative setting, we still get the weaker result of a *partial right inverse K* by which we mean a subordinator killed at an independent exponential ξ satisfying $X_{K_x(\omega)}(\omega) = x$ for all $0 \leq x < \xi(\omega)$ and P-a.e. $\omega \in \Omega$. If we identify the Dirac measures in zero and infinity with exponential distributions of an infinite and zero rate parameter respectively, an infinite killing rate means that there is a.s. no partial right inverse, whereas a zero killing rate means that the partial right inverse is actually a full right inverse.

Let us now provide a brief overview of the results and techniques used in Evans [7].

If right inverses exist, the minimal right inverse is a subordinator. This is the natural choice that is analyzed further. A way to approximate this minimal inverse is by spatial discretization, that is, by defining stopping times

$$T_0 = 0, \qquad T_{k+1}^n = \inf\left\{t \ge T_k^n : X_t = \frac{k+1}{2^n}\right\}, \qquad k \ge 0$$

and processes

$$K_x^n = T_k^n, \qquad \frac{k}{2^n} \le x < \frac{k+1}{2^n}, \quad k \ge 0,$$

for $n \ge 0$. Then, a pathwise argument shows that, if the limit

$$K_x = \inf_{y>x} \sup_{n\ge 0} K_y^n \tag{2}$$

is finite for all $x \ge 0$, then it defines the minimal right inverse. (Note that the infimum only serves for the càdlàg property that might fail otherwise in a countable number of exceptional $x \ge 0$.) Furthermore, if this limit is infinite, no right inverses exist.

THEOREM 1 [7]. *A symmetric Lévy process X possesses a right inverse if and only if it is recurrent and has a nontrivial Gaussian coefficient $\sigma^2 > 0$. More precisely*,

$$E\big(\exp(-qK_x)\big) = \exp\left(-\frac{x}{\sigma^2 u^q(0)}\right),$$

where u^q is the positive and continuous resolvent density, *characterizes the minimal right inverse K which is a subordinator with Laplace exponent $\rho(q) := 1/(\sigma^2 u^q(0))$.*

Let us sketch Evans's proof as this argument will also be useful in the sequel.

The potential theory in Chapter II of [2] allows a restriction to Lévy processes with continuous resolvent densities u^q, $q > 0$, when we have the transform identities

$$E\big(\exp(-qT_{\{x\}})\big) = \frac{u^q(x)}{u^q(0)} \tag{3}$$

for the hitting times of singletons, and right derivatives of u^q in zero

$$\frac{u^q(0) - u^q(x)}{x} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1-\cos\xi}{x^2(q+\psi(\xi/x))}\,d\xi \underset{x\to 0+}{\longrightarrow} \frac{1}{\sigma^2} \in (0,\infty]. \tag{4}$$

Then the Markov property in $T_{\{k2^{-n}\}}$, $k \ge 0$, and the above yield for all $x \ge 0$

$$\begin{aligned} E\left(\exp\left(-q\sup_{n\ge 0} K_x^n\right)\right) &= \lim_{n\to\infty} \big(E\big(\exp(-qT_{\{2^{-n}\}})\big)\big)^{[2^n x]} \\ &= \exp\left(-x\frac{1}{\sigma^2 u^q(0)}\right). \end{aligned} \tag{4'}$$

Now this quantity is positive if and only if the Gaussian coefficient is positive. It tends to one if and only if $u^q(0)$ tends to infinity as q tends to zero. However, this happens if and only if X is recurrent completing the proof.

This theorem says in particular that when $\sigma^2 > 0$, then K has the same law as the linearly time-changed inverse local time of X in zero, whose Laplace exponent κ takes then the form $\kappa(q) = 1/u^q(0)$; cf. Proposition V.4 in [2].

In the beginning of this subsection, we pointed out an identification with classical fluctuation theory in the spectrally negative setting. In a more general setting with positive jumps, the two concepts are different, but show striking structural similarities. The corresponding fluctuation theory result is deeper and not so directly related. First, denoting by $S_t = \sup_{0\le s\le t} X_t$ the supremum process, $M = X - S$ is a strong Markov process. We can associate an inverse local time in zero which is called the ladder time subordinator A^{-1} of X. For an independent exponential random variable τ the local time A_τ in τ and M_τ are independent and one can calculate their distributions. From excursion theory the independence is not very surprising since M_τ only depends on the excursion of M away from zero (excursion of X away from S) started in A_τ.

Suppose now that X is a not necessarily symmetric Lévy process and that there is a minimal right inverse K of X. We introduce $L_t = \inf\{x \ge 0 : K_x > t\}$ and $Z_t = X_t - L_t$. Whereas fluctuation theory studies the bivariate subordinator $(A^{-1}, X \circ A^{-1})$ and the excursions of $M = X - S$, we study here the bivariate subordinator $(K, X \circ K)$ and the excursions of Z. Note that $X \circ K = id$ reduces the study of $(K, X \circ K)$ to K; here id means the identity function on $[0, \infty)$, that is, $(id)(x) = x$. The analogous results to classical fluctuation theory are here

THEOREM 2 [7]. *Suppose, a Lévy process X with characteristic exponent ψ has a minimal right inverse K with Laplace exponent ρ. Let τ be an independent q-exponential random variable.*

(i) *The process Z is a strong Markov process w.r.t. the natural filtration of X.*
(ii) *The state 0 is regular for Z, and L is a corresponding local time.*
(iii) *The random variables L_τ and Z_τ are independent.*
(iv) *L_τ has a $\rho(q)$-exponential distribution.*
(v) *Z_τ has characteristic function*

$$E\big(\exp(i\lambda Z_\tau)\big) = \frac{q(\rho(q) - i\lambda)}{(q + \psi(\lambda))\rho(q)}.$$

(vi) *The family of entrance laws $(n_t^Z)_{t>0}$ (and hence the excursion measure n^Z) is characterized by*

$$\int_0^\infty \int_{\mathbb{R}} \exp(-qt + i\lambda x) n_t^Z(dx)\, dt = \frac{\rho(q) - i\lambda}{q + \psi(\lambda)} - aq \tag{5}$$

where $a \ge 0$ is the drift coefficient of K. Furthermore, we have $a = 0$ if and only if $\{t : Z_t = 0\}$ has zero Lebesgue measure.

(vii) *If* 0 *is regular for* X, *then* X *has an inverse local time in zero with Laplace exponent* κ. $(n_t^X)_{t>0}$ *is given by*

$$\int_0^\infty \int_{\mathbb{R}} \exp(-qt + i\lambda x) n_t^X(dx)\, dt = \frac{\kappa(q)}{q+\psi(\lambda)}. \tag{6}$$

Evans exploited furthermore the equations (5) and (6) to give a relationship between n_t^Z and n_t^X, in the symmetric setting (when $\rho = \kappa/\sigma^2$ and $a = 0$).

PROOF OF THEOREM 2. Evans's arguments are fine in this more general situation. Let us just say a word on the drift coefficients a of ρ and b of κ. When X is symmetric, that is, X has a positive Gaussian coefficient by Theorem 1, they are multiples of each other and well-known to be trivial due to the latter (see also below).

(vi) As the entrance laws n_t^Z, $t > 0$, satisfy $n_t^Z(\{0\}) = 0$ by definition, but possibly $P(Z_\tau = 0) > 0$, the classical excursion theory argument used by Evans gives here more precisely

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}} e^{-qt} e^{ix\xi} n_t(dx)\, dt &= \int_0^\infty \int_{\mathbb{R}^*} e^{-qt} e^{ix\xi} n_t(dx)\, dt \\ &= \rho(q) \int_{\mathbb{R}^*} e^{ix\xi} \frac{1}{q} P(Z_\tau \in dx)\end{aligned}$$

and the result follows from (v) and, if $a > 0$

$$P(Z_\tau = 0) = \int_0^\infty q e^{-qt} P(Z_t = 0)\, dt = \int_0^\infty q e^{-qt} a v(t)\, dt = \frac{aq}{\rho(q)}$$

where we denoted the potential density of K by v and applied Proposition 1.7 in [3]; cf. also Rogers and Williams [11], Section VI.48.

(vii) We imposed the regularity of 0 for X to have a nice local time process. Note that by the existence of K, the range of X contains $\mathbb{R}^+$. By homogeneity, this implies that $\{t : X_t = 0\}$ has zero Lebesgue measure, so the drift b in κ is zero. □

2.3. *Subordination in the wide sense and right inverses.* Simon [13] considered the following concept. Given a Lévy process X, another Lévy process Y is called subordinate to X in the wide sense if there is a subordinator σ such that $Y = X \circ \sigma$ and σ may be dependent on X, but only locally in the following sense: there is a filtration $\mathcal{F} = (\mathcal{F}_t)_{t\ge 0}$ w.r.t. which X is a Lévy process and σ_s a stopping time for all $s \ge 0$; there is a filtration $(\mathcal{G}_s)_{s\ge 0}$ w.r.t. which (Y, σ) is a Lévy process, $(X_{t\wedge\sigma_s})_{t\ge 0}$ is $\mathcal{G}_s$-measurable and $(X_{\sigma_s+t} - X_{\sigma_s})_{t\ge 0}$ is independent of $\mathcal{G}_s$. In the transient case, Simon characterizes the processes Y that are subordinate to a given process X in terms of the potential measures U and V of X and Y, respectively, and in terms of their excessive functions. Recall that a measurable function $f : \mathbb{R} \to [0, \infty)$ is called excessive w.r.t. X if $E(f(x + X_t))$ increases to $f(x)$ as t decreases to 0, for all $x \in \mathbb{R}$.

THEOREM 3 [13]. *For two transient Lévy processes X and Y are equivalent*:

(a) *Y is subordinate to X in the wide sense.*
(b) *Every excessive function w.r.t. X is excessive w.r.t. Y.*
(c) *There is a (nonnegative) Radon measure μ on $\mathbb{R}$ such that $U = V * \mu$.*

Clearly, $Y = id$ is a transient Lévy process. It is easily seen that the conditions concerning the filtration $\mathcal{F}$ which we choose to be the natural filtration of X are satisfied by the minimal right inverse K and the conditions concerning the filtration $\mathcal{G}$ as well if we let $\mathcal{G}_s$ be generated by $(\sigma_u)_{u\le s}$ and $(X_{t\wedge\sigma_s})_{t\ge 0}$. Some elementary calculations yield

COROLLARY 1. *Let X be a transient Lévy process. The following statements are equivalent*:

(a) *X has a right inverse.*
(b) *Every excessive function w.r.t. X is right-continuous decreasing.*
(c) *There is a (nonnegative) Radon measure μ on $\mathbb{R}$ such that $U = \lambda\!\!\lambda|_{(0,\infty)} * \mu$ where $\lambda\!\!\lambda$ is the Lebesgue measure.*

Let us now see what we can get from Theorem 3 in the recurrent case. Let τ be an independent exponential time with parameter $q > 0$ and let $X^\dagger$ be the process X killed at time τ. Then the theorem applies to $X^\dagger$. As we cannot expect a full inverse, let us kill $Y = id$ as well at some rate $p(q)$ and so analyze the existence of a partial inverse for killed Lévy processes $X^\dagger$. Clearly, as q tends to zero, a partial right inverse of $X^\dagger$ tends to a partial right inverse of X killed at rate $p(0) = \rho(0) \ge 0$ and as right invertibility is a path property, there is a partial right inverse of X if and only if it exists for $X^\dagger$.

COROLLARY 2. *Let X be any Lévy process. Equivalent are*:

(a) *X has a partial right inverse killed at rate $p(0)$.*
(b) *For every excessive function f w.r.t. $X^\dagger$, $e^{-p(q)x} f(x)$ is right-continuous decreasing for sufficiently big $p(q)$.*
(c) *For some (all) $q > 0$ there is a $p(q) > 0$ and a (nonnegative) finite measure μ_q on $\mathbb{R}$ such that $U^q = \frac{1}{p(q)} \operatorname{Exp}(p(q)) * \mu_q$ where $\operatorname{Exp}(p)$ denotes the exponential distribution with parameter (inverse mean) $p > 0$.*
(d) *For some (all) $q > 0$ the function $\lambda \mapsto \frac{p(q)-i\lambda}{q+\psi(\lambda)}$ is the Fourier transform of a (nonnegative) finite measure for $p(q)$ sufficiently big.*

Simon also identifies the measures μ_q. In the notation of Theorem 2 it is given by

$$\mu_q(dx) = a\delta_0(dx) + \int_E \int_0^{\zeta(e)} e^{-qt} \delta_{e(t)}(dx)\, dt\, n^Z(de).$$

Comparing this to Theorem 2(vi), we identify the drift coefficient a of K and an optimal $p(q) = \rho(q)$. This yields the following representation of the Laplace exponent ρ:

$$\rho(q) = \min\left\{p > 0 : \lambda \mapsto \frac{p - i\lambda}{q + \psi(\lambda)} \text{ is Fourier transform of a measure } \mu \geq 0\right\}.$$

Assume that X has a partial right inverse. Then, X visits single points, so that $c_q > 0$, and by Theorem II.16 of [2], u^q is bounded. Furthermore, by Proposition I.12 $x \mapsto u^q(-x)$ is q-excessive w.r.t. X, that is, excessive w.r.t. $X^\dagger$, so by Corollary 2(b), for the optimal value $p(q) = \rho(q)$ we obtain

$$\sup_{x>0} \frac{u^q(0) - u^q(x)}{x} \leq u^q(0)\rho(q) < \infty. \tag{7}$$

If u^q is continuous, the argument of the supremum on the left-hand side is nonnegative, and Evans's argument (4′) yields that the right-hand side is attained in the limit $x = 2^{-n} \downarrow 0$, $n \to \infty$.

These are analytic potential theoretic descriptions of right inverses that are not very explicit. The aim of the sequel is a characterization in terms of the characteristics of X.

3. Existence of right inverses.

3.1. *Statement of the theorems.* Theorem 1 on right inverses for symmetric Lévy processes can be generalized to the nonsymmetric setting in a way considering right inverses K of a Lévy process X and $\hat{K}$ of its dual $\hat{X}$ simultaneously.

Also, the recurrence condition that turns out a necessary condition for the existence of right inverses, unless X is spectrally negative, can be eliminated by the more general notion of partial right inverses. Specifically, any result on right inverses can be reformulated by dropping all recurrence statements and replacing the “right inverses” by “partial right inverses” which is meant to include the possibility of a full right inverse. (The proofs are easily adapted.) This is the natural analogue of killed ladder processes in classical fluctuation theory.

To justify this statement let us show that a transient Lévy process X possesses a full right inverse if and only if it is spectrally negative and $\limsup_{t\to\infty} X_t = \infty$. On the one hand, for spectrally negative processes we choose K to be the inverse supremum process. In the converse direction we can see that positive jumps must not occur. Indeed, if positive jumps occur, then, for some $\varepsilon > 0$, there is positive probability that the first jump J of size $\geq 4\varepsilon$ exceeds the pre-supremum S_J by $\geq 2\varepsilon$ and that X does not return to $(S_J - \varepsilon, S_J + \varepsilon)$ which means that there is positive probability (indeed, probability one by independent trials) that the positive part of the range of X has gaps, so there cannot exist a full right inverse.

Now, the extension of Theorem 1 to the nonsymmetric framework is

THEOREM 4. *A Lévy process X and its dual $\hat{X} = -X$ both have right inverses if and only if X is recurrent with a nontrivial Gaussian coefficient $\sigma^2 > 0$.*

Furthermore, in this case, the Laplace exponents ρ and $\hat{\rho}$ of the minimal right inverses of X and $\hat{X}$ respectively fulfill the identity

$$\rho(q) + \hat{\rho}(q) = \frac{2}{\sigma^2 u^q(0)}. \tag{8}$$

In the symmetric setting ρ and $\hat{\rho}$ both equal (up to a multiplicative constant) the Laplace exponent $\kappa(q) = 1/u^q(0)$ of an inverse local time of X at zero; cf. Proposition V.4(ii) in [2] (see also the discussion before, which motivates the choice of the multiplicative factor). This is no longer true in the nonsymmetric setting, but nevertheless their sum fulfills (8).

The proof of the direct part in Subsection 3.2 uses Evans's argument. The indirect part is a corollary of the next theorem that gives the link with creeping where we say "X can creep upward" if $P(X(T_{[x,\infty)}) = x) > 0$ for some (all) $x > 0$. As has already been mentioned by Evans [7] and is even more apparent in our Theorem 4, the parallels between the existence of right inverses and creeping are striking. Specifically, a Lévy process X has a nontrivial Gaussian component if and only if both X and its dual $\hat{X} = -X$ can creep upward (cf. Theorem VI.19 of [2]). Clearly, upward creeping favors the existence of a right inverse. Also, the converse seems reasonable. However,

THEOREM 5. *Upward creeping is necessary but not sufficient for the existence of* (*partial*) *right inverses.*

The proof of the necessity is given in Subsection 3.3. The insufficiency follows, for example, from Proposition 2 below. We shall now focus on the unbounded variation case, the bounded variation case is postponed to Section 5 which is more elementary.

Theorems 4 and 8 (see Section 5) give complete answers concerning the existence of right inverses when the Gaussian coefficient is positive and when the variation is bounded, respectively. The remaining class of unbounded variation Lévy processes without Gaussian component is more delicate. The boundary between invertible and not invertible processes is not as clear as in those two cases where the behavior was basically dominated by the Gaussian coefficient and the drift coefficient, respectively. Here the jumps (and the convergence generating drift compensator in the Lévy–Khintchine representation) determine the invertibility.

Assume an unbounded variation Lévy process X without Gaussian component. We first characterize the existence of right inverses in an analytic way and then derive some more explicit conditions. Recall that we denote the resolvent densities of X by u^q.

PROPOSITION 1. *An unbounded variation Lévy process has right inverses if and only if u^q is bounded and*

$$\sup_{x>0} \frac{u^q(0) - u^q(x)}{x} < \infty \tag{9}$$

for some (*all*) $q > 0$ ($q \geq 0$ *in the transient case*). *Furthermore*, *in this case*,

$$\rho(q) = \lim_{n\to\infty} \frac{u^q(0) - u^q(2^{-n})}{2^{-n} u^q(0)}$$

is the Laplace exponent of the minimal right inverse.

The only way this supremum can fail to be finite is at $x \downarrow 0$. Recall in particular that $u^0(x) = P(T_x < \infty)u^0(0)$ in the transient case.

Suppose now that the characteristic exponent of the negative jump part "increases much faster" than the characteristic exponent of the positive jump part. Then X has right inverses. Conversely, if the parts do not differ enough, no inverses exist. We give quantitative statements by providing sufficient conditions for inverses (Theorem 6) and showing that they are also necessary (possibly apart from the boundary value) for certain composites of stable processes (Proposition 2).

THEOREM 6. *Let $X = X^+ - X^-$ be a Lévy process split into two independent processes such that X^- is spectrally positive. Let ψ, ψ^+ and ψ^- denote their characteristic exponents so that $\psi(\xi) = \psi^+(\xi) + \psi^-(-\xi)$. Assume for all $\xi \geq \xi_0$*

$$\left|\psi^-(\xi)\right| \geq \xi^\alpha \quad \textit{and} \quad \left|\psi^+(\xi)\right| \leq \xi^{2\alpha-2}/\log^2(\xi) \tag{10}$$

for some $\alpha \in (1, 2]$. Then X possesses a nontrivial partial right inverse K.

In fact, the conditions on the characteristic exponent can be relaxed in the following way. Denoting

$$f(\xi) = |\psi^+(\xi)| \quad \text{and} \quad g(\xi) = |\psi^-(\xi)|,$$

we assume that for some $\varepsilon > 0$ and all $\xi \geq \xi_0$ we have $f(\xi) \leq (1-\varepsilon)g(\xi)$ and that

$$\int^\infty \frac{1}{g(\xi)}\, d\xi < \infty, \qquad \int^\infty \xi \frac{f(\xi)}{g^2(\xi)}\, d\xi < \infty.$$

Then a Lévy process X with characteristic exponent $\psi(\xi) = \psi^+(\xi) + \psi^-(-\xi)$ possesses right inverses.

REMARK 1. Sufficient for the conditions on the characteristic exponent are the following conditions on the Lévy tails at zero:

$$\bar{\Pi}^-(x) \geq x^{-\alpha} \quad \text{and} \quad \bar{\Pi}^+(x) \leq x^{-\beta}, \qquad \beta < 2\alpha - 2,$$

for all $x \le x_0$ as is easily checked:

$$\begin{aligned}
|\psi^-(\xi)| &\ge |\Im(\psi^-(\xi))| = \int_0^\infty (1-\cos(x\xi))\bar{\Pi}^-(x)\,dx \\
&\ge \int_0^{x_0} (1-\cos(x\xi))x^{-\alpha}dx = c\xi^\alpha - \int_{x_0}^\infty (1-\cos(x\xi))x^{-\alpha}\,dx \\
&\ge c_1\xi^\alpha, \\
|\psi^+(\xi)| &\le |\Re(\psi^+(\xi))| + |\Im(\psi^+(\xi)| \\
&= \left|\int_0^\infty \sin(x\xi)\bar{\Pi}^+(x)\,dx\right| + \int_0^\infty (1-\cos(x\xi))\bar{\Pi}^+(x)\,dx \\
&\le \int_0^{\pi/\xi} x\xi x^{-\beta}\,dx + \int_0^{\pi/\xi} y\xi\bar{\Pi}^+(y+\pi/\xi)\,dy \\
&\quad + \int_0^{\pi/\xi} x\xi\bar{\Pi}^+(x)\,dx + \int_{\pi/\xi}^1 2\bar{\nu}(x)\,dx \\
&\le c_2\xi^\beta,
\end{aligned}$$

where we assumed w.l.o.g. that Π^+ is concentrated on $(0, 1)$.

We prove Theorem 6 in Subsection 3.4. As a converse we give

PROPOSITION 2. *Let $X = X^+ - X^-$ be a process, where X^- is spectrally positive stable of index $\alpha \in (1, 2)$ and X^+ is symmetric stable of index $\beta \in (2\alpha - 2, \alpha)$. Then there are no partial right inverses.*

The proof of Proposition 2 is rather technical exploiting the explicit knowledge of the characteristic exponent. It is given as an Appendix.

Theorem 3.4 in Millar [9] shows that the processes considered in Theorem 6 and Proposition 2 can creep upward; cf. also Theorem 3.2 in Blumenthal and Getoor [4] to identify the different kinds of indices used. These results indicate that the existence of right inverses is a much stronger condition than creeping.

3.2. *The case of a positive Gaussian coefficient.* The key to the direct part of the proof of Theorem 4 is the analogue to Evans's formula (4).

LEMMA 1. *Let X have a positive Gaussian coefficient $\sigma^2 \in (0, \infty)$ and call its q-resolvent densities u^q. Then we have*

$$\frac{2u^q(0) - (u^q(x) + u^q(-x))}{x} = \frac{1}{\pi}\int_{-\infty}^\infty \frac{1-\cos(\xi x)}{x}\Re\left(\frac{1}{q+\psi(\xi)}\right)d\xi \underset{x\to 0+}{\longrightarrow} \frac{2}{\sigma^2}.$$

PROOF. X has continuous bounded resolvent densities since $\sigma^2 > 0$. Then we can use Theorem II.19 of Bertoin [2] to obtain

$$\begin{aligned}\frac{2u^q(0) - (u^q(x) + u^q(-x))}{x} &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos(\xi x)}{x}\Re\left(\frac{1}{q+\psi(\xi)}\right)d\xi \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos(\xi)}{x^2}\Re\left(\frac{1}{q+\psi(\xi/x)}\right)d\xi.\end{aligned}$$

Now we can bound using $\Re(\psi) \geq 0$

$$\left|\Re\left(\frac{1}{q+\psi(\lambda)}\right)\right| \leq \frac{1}{q+\sigma^2\lambda^2}.$$

By dominated convergence (recall that $\psi(\lambda) \sim \frac{1}{2}\sigma^2\lambda^2$ as λ tends to infinity), this yields

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1-\cos(\xi)}{x^2}\Re\left(\frac{1}{q+\psi(\xi/x)}\right)d\xi \underset{x\to 0+}{\longrightarrow} \frac{2}{\pi\sigma^2}\int_{-\infty}^{\infty}\frac{1-\cos(\xi)}{\xi^2}\,d\xi.$$

The value of this integral is known to be π, so the limit is $2/\sigma^2 < \infty$. $\square$

The proof of the direct part of Theorem 4 is essentially an adaptation of the proof of Theorem 1:

PROOF OF THE DIRECT PART OF THEOREM 4. (i) Let X be a recurrent Lévy process with nontrivial Gaussian coefficient $\sigma^2 > 0$. We have to show that it has a right inverse. X possesses continuous and bounded resolvent densities u^q. Look at the Laplace transform of the time when first all $j2^{-n}x$, $j = 1, \ldots, 2^n$, have been visited successively, which amounts to adding 2^n independent hitting times of height $2^{-n}x$. Call the time by which this has been accomplished K_x^n. Evans has shown that this quantity increases a.s. to the finite K_x (or infinity, if K_x does not exist). We show that K_x is finite: by the Markov property in $T_{\{j2^{-n}x\}}$, $j = 1, \ldots, 2^n$, and Corollary II.18 in [2]:

$$\begin{aligned}E(\exp(-qK_x)) &= \lim_{n\to\infty} E\big(\exp(-qK_x^n)\big) \\ &= \sup_{n\geq 0}\big(E\big(\exp(-qT_{\{2^{-n}x\}})\big)\big)^{2^n} = \sup_{n\geq 0}\left(\frac{u^q(2^{-n}x)}{u^q(0)}\right)^{2^n},\end{aligned} \tag{11}$$

which by an easy calculation reduces the problem to showing the finiteness of the left derivative $\rho(q)u^q(0)$ of u^q in zero. This is now, once more, a consequence of Corollary II.18 in [2] yielding the domination

$$\frac{u^q(0) - u^q(x)}{x} \leq \frac{2u^q(0) - (u^q(x) + u^q(-x))}{x} \tag{12}$$

and Lemma 1.

The same argument applies to the dual process.

(ii) To show the identity concerning the two Laplace exponents of K and $\hat{K}$ that result from the calculation (11) simply note that the limit as x tends to zero on the right hand side of (12) gives $\rho(q)u^q(0) + \hat{\rho}(q)u^q(0)$ which by Lemma 1 yields the asserted identity. $\square$

3.3. *Creeping.* Theorem 5 states that creeping is necessary but not sufficient for the existence of inverse. We show the former using the Wiener–Hopf factorization in a way which Fourati and Vigon pointed out to us; cf. also [14].

PROOF OF THE NECESSITY PART OF THEOREM 5. The result is established in Section 5 for Lévy processes with bounded variation. Assume therefore that X is an unbounded variation Lévy process which has a partial right inverse K with Laplace exponent ρ. We show that X can creep upward.

Since creeping and the existence of partial right inverses are local properties, we may assume w.l.o.g. that X drifts to $-\infty$ by possibly adding large negative jumps.

The Wiener–Hopf factorization states

$$U = \hat{V}^* * V$$

where U is the potential measure of X, V and $\hat{V}$ are the potential measures of the ladder height subordinators H and $\hat{H}$, and $\hat{V}^*(dx) = \hat{V}(-dx)$. More precisely, recall that $H = X \circ A^{-1}$ is the height process associated with the ladder time process A^{-1} where A is the local time of X "at the supremum," that is, the local time in zero of $M = S - X$, which is X reflected at its supremum process $S_t = \sup_{0 \le s \le t} X_s$. We refer to [2], Chapter 6. We can associate with $\hat{V}^*$ the potential operator $\hat{\mathcal{V}}^*$ given by

$$\hat{\mathcal{V}}^* f(x) = \int_{(-\infty,0)} f(x+y)\hat{V}^*(dy) = \int_{(0,\infty)} f(x-y)\hat{V}(dy), \qquad f \in \mathcal{C}_0.$$

Let us apply it to the function

$$f(z) = V\big((-z,\infty)\big), \qquad z \le 0,$$

which is continuous (since V is diffuse according to Proposition I.15 in [2], remember X has unbounded variation and is hence regular for $(0,\infty)$, so that the ascending ladder height process is not compound Poisson; cf. after Proposition VI.11). We extend f in any suitable way to obtain a $\mathcal{C}_0$-function on $\mathbb{R}$. For $x \le 0$ we obtain

$$\hat{\mathcal{V}}^* f(x) = \int_{(0,\infty)} f(x-y)\hat{V}(dy) = \hat{V}^* * V\big((-x,\infty)\big) = U\big((-x,\infty)\big).$$

Now the generator $\hat{\mathcal{B}}^*$ of $-\hat{H}$, given by

$$\begin{aligned}\hat{\mathcal{B}}^* g(z) &= -\hat{b}g'(z) + \int_{(-\infty,0)} \big(g(z+y) - g(z)\big)\hat{\nu}^*(dy) \\ &= -\hat{b}g'(z) - \int_{(0,\infty)} \big(g(z) - g(z-y)\big)\hat{\nu}(dy),\end{aligned}$$

is the inverse operator for $-\hat{\mathcal{V}}^*$; cf. after Proposition I.9 in [2] and Berg and Forst [1], Proposition 11.9. Here $\hat{b}$ and $\hat{\nu}$ are the drift coefficient and Lévy measure of $\hat{H}$. We apply this operator to the function

$$g(x) = \hat{\mathcal{V}}^* f(x), \qquad x \in \mathbb{R},$$

to obtain for all $z \le 0$,

$$\begin{aligned} V\big((-z,\infty)\big) = f(z) = -\hat{\mathcal{B}}^* g(z) &= \hat{b} g'(z) + \int_{(0,\infty)} \big(g(z) - g(z-y)\big)\hat{\nu}(dy) \\ &= \hat{b}u(-z) + \int_{(0,\infty)} U\big((-z, y-z]\big)\hat{\nu}(dy). \end{aligned}$$

Now we obtain

$$\begin{aligned} \frac{V([0,x])}{x} &= \frac{f(0) - f(-x)}{x} \\ &= \hat{b}\frac{u(0) - u(x)}{x} + \int_{(0,\infty)} \frac{U((0,x]) - U((y,x+y])}{x}\hat{\nu}(dy). \end{aligned}$$

We can bound the integrand above by the following estimation

$$\begin{aligned} \frac{U((0,x]) - U((y,x+y])}{x} &= \frac{1}{x}\int_0^x \big(u(z) - u(z+y)\big)dz \\ &\le \frac{1}{x}\int_0^x u(0)\big((\rho(0)y)\wedge 1\big)dz = u(0)\big((\rho(0)y)\wedge 1\big) \end{aligned}$$

where the last estimate comes from

$$\begin{aligned} \frac{u(z) - u(z+y)}{u(0)} &= P(T_{\{z\}} < \infty) - P(T_{\{z+y\}} < \infty) \\ &\le P(T_{\{z\}} < \infty) - P(T_{\{z+y\}} < \infty, T_{\{z\}} < \infty) \\ &= P(T_{\{z\}} < \infty)P(T_{\{z+y\}} = \infty | T_{\{z\}} < \infty) \\ &\le P(T_{\{z\}} < \infty)P(T_{\{y\}} = \infty) \le P(T_{\{y\}} = \infty) \le (\rho(0)y)\wedge 1 \end{aligned}$$

by Proposition 1 and the remark thereafter.

Thus we obtain

$$1/b = \sup_{x>0}\frac{V([0,x])}{x} \le \hat{b}\rho(0)u(0) + \int_{(0,\infty)} u(0)((\rho(0)y)\wedge 1)\hat{\nu}(dy) < \infty,$$

that is, H has a positive drift $b > 0$. By Theorem VI.19 in [2] this is equivalent to the upward creeping of X. $\square$

3.4. *The case of a zero Gaussian coefficient.* Let us first give the proof of Proposition 1. It is basically a combination of the results and methods of Simon and Evans as presented in Section 2.

PROOF OF PROPOSITION 1. First assume that an unbounded variation Lévy process has right inverses. Then the discussion of (7) concludes the proof—note that by Bretagnolle [5] and Theorem II.19 in [2] X has continuous resolvent densities.

Conversely, assume (9) and denote the supremum by S. Perform Evans's approximation argument (4′), then

$$E\biggl(\exp\biggl(-q\sup_{n\geq 0}K_x^n\biggr)\biggr)=\inf_{n\geq 0}\biggl(\frac{u^q(2^{-n})}{u^q(0)}\biggr)^{[2^n x]}\geq e^{-xS/u^q(0)}$$

shows that there are right inverses up to height x with positive probability. $\square$

The key to Theorem 6 is the following representation of the resolvent density:

LEMMA 2. *Let X be a Lévy process whose characteristic exponent ψ is such that*

$$\int_0^\infty \biggl|\frac{1}{1+\psi(\xi)}\biggr|\,d\xi<\infty. \tag{13}$$

Then we have for all $x\in\mathbb{R}$

$$u^q(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{q+\psi(-\xi)}\,d\xi.$$

PROOF. Note first that in this setting X has unbounded variation and we have bounded continuous resolvent densities by Bretagnolle [5] and Theorem II.19(i) of [2].

The proof is now complete by the Fourier inversion of the well-known identity

$$\int_{-\infty}^{\infty} e^{i\xi x}u^q(x)\,dx=\frac{1}{q+\psi(\xi)}.$$

This is possible by the integrability of the right hand side, giving the asserted formula Lebesgue-almost everywhere, everywhere by continuity. $\square$

Note that (13) actually only needs to be imposed for $\Im(1/(q+\psi(\xi)))$ and the corresponding condition for the real parts can be expressed by $c_q>0$ which squares with the finiteness of $u^q(0)$; cf. Theorem II.19 in [2]. According to Theorem II.16 in [2], $c_q = 0$ always implies the unboundedness of u^q around zero; but for us $c_q = 0$ is not interesting since it implies that every given point is P-a.s. not attained by X impeding the construction of right inverses, anyway.

PROOF OF THEOREM 6. Note that $\alpha = 2$ has already been treated in Theorem 4. Let therefore $\alpha < 2$ and w.l.o.g. let X have a zero Gaussian coefficient. However, note that the argument still works when the Gaussian coefficient is positive but there are some notational changes in order to keep track of this Gaussian coefficient.

Let us start by observing that X^- has unbounded variation, therefore the spectrally negative process $-X^-$ is regular for $(0, \infty)$ and possesses a nontrivial partial right inverse which coincides with the inverse supremum process. Without loss of generality, we may assume that X does not possess any jumps outside $(-1, 1)$ since we may add them as an independent compound Poisson process later without disturbing the initial behavior that determines the existence of a nontrivial partial right inverse. Denoting the resolvent densities of $-X^-$ by v^q, we take from the preceding lemma

$$0 \le \frac{v^q(0) - v^q(x)}{x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{i\xi x}}{x} \frac{1}{q + \psi^-(\xi)} \, d\xi \tag{14}$$

where we obtain the integrability condition for free, as

$$\left| \frac{1}{1 + \psi^-(\xi)} \right| \le 1 \wedge \xi^{-\alpha}.$$

Recall that the existence of a right inverse squares with the finiteness of the supremum of (14) over $x > 0$, Proposition 1.

As X also has continuous resolvent densities u^q, we can consider the analogous formula for X, where we note that

$$\int_{-\xi_1}^{\xi_1} \left| \frac{1 - e^{i\xi x}}{x} \frac{1}{q + \psi(-\xi)} \right| d\xi \le \int_{-\xi_1}^{\xi_1} \xi \frac{1}{q} \, d\xi.$$

This shows that only $\psi(\xi)$, $\xi \ge \xi_1$, have influence on the integrability and the finiteness of the limit, for an arbitrarily large $\xi_1 \ge \xi_0$ where ξ_0 is such that (10) holds.

Now look at the difference of the expressions for X and $-X^-$:

$$\begin{aligned} &\frac{u^q(0) - u^q(x) - (v^q(0) - v^q(x))}{x} \\ &\qquad = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{i\xi x}}{x} \frac{\psi^+(-\xi)}{(q + \psi(-\xi))(q + \psi^-(\xi))} \, d\xi. \end{aligned}$$

Focusing on $|\xi| \geq \xi_1$ now, we obtain for $\xi_1 \geq \xi_0$ sufficiently large

$$\begin{aligned}
&\frac{1}{2\pi}\int_{(-\infty,-\xi_1]\cup[\xi_1,\infty)}\left|\frac{1-e^{i\xi x}}{x}\frac{\psi^+(-\xi)}{(q+\psi(-\xi))(q+\psi^-(\xi))}\right|d\xi \\
&\quad\leq \frac{1}{\pi}\int_{\xi_1}^{\infty}\left|\frac{x\xi}{x}\frac{\xi^{2\alpha-2}/\log^2(\xi)}{(\xi^\alpha-\xi^{2\alpha-2}/\log^2(\xi))\xi^\alpha}\right|d\xi \\
&\quad\leq \frac{1}{\pi}\int_{\xi_1}^{\infty}\xi\frac{\xi^{2\alpha-2}/\log^2(\xi)}{(\xi^\alpha/2)\xi^\alpha}\,d\xi \\
&\quad= \frac{2}{\pi}\int_{\xi_1}^{\infty}\frac{1}{\xi\log^2(\xi)}\,d\xi,
\end{aligned}$$

a bound independent of x.

Together with the finiteness of the supremum of (14) over $x > 0$ we conclude

$$\sup_{x>0}\frac{u^q(0)-u^q(x)}{x} < \infty,$$

hence X possesses right inverses. $\square$

4. Excursions of X and Z when the Gaussian coefficient is positive.

4.1. *Disjointness of supports of* n^Z *and* $n^{\hat{Z}}(-\cdot)$. Evans started an analysis of the excursions of the strong Markov process $Z = X - L$ where $L_t = \inf\{s \geq 0 : K_s > t\}$. Z corresponds to the process X reflected at its supremum process in classical fluctuation theory; cf. Theorem 2.

The recurrence assumption is not essential for Theorem 4 and similar results. Here, transience may entail an infinite excursion which does not impede the arguments for Theorem 2 nor in the sequel. Evans describes the entrance laws of the excursion measure n^Z of Z rather than the excursion measure itself. Our next result provides a description of the excursion measure n^Z itself in terms of the excursion measure n^X of X.

Evans mentions that the excursion measures of X and Z have the same semigroup but two different entrance laws. In the case of a Brownian motion he explains this by noting the trivial result that all excursions of Z are negative; so, the entrance laws of n^Z are those of n^X confined to the negative half line. Our Theorem 7(ii) generalizes this explanation to the whole class of Lévy processes with a nontrivial Gaussian coefficient: all excursions of Z *start* negative. As the excursions can pass positive after a random initial amount of time, the domains of the entrance laws cannot reflect this.

Let $E^{\pm}$ be the spaces of càdlàg paths $\varepsilon \in E$ such that $\pm\varepsilon(s) > 0$ for all sufficiently small $s > 0$. Set $n^X_{\pm} = 1_{E^{\pm}} n^X$.

THEOREM 7. *Let X be a Lévy process with nontrivial Gaussian component* $\sigma^2 > 0$.

(i) n^X*-a.e. excursion does not oscillate initially, that is,* $n^X = n_+^X + n_-^X$.

(ii) n^Z *and* $n^{\hat{Z}}(-\cdot)$ *are mutually singular. More precisely,*

$$n^Z = \frac{2}{\sigma^2} n_+^X \quad \textit{and} \quad n^{\hat{Z}}(-\cdot) = \frac{2}{\sigma^2} n_-^X(-\cdot).$$

There is nothing special about the coefficient $2/\sigma^2$ of n^X in Theorem 7 since it depends on the choice of a local time of X at zero. Our choice $\kappa(q) = 1/u^q(0)$ was made earlier in accordance with [2]. Actually, also n^Z depends on such a choice, but this one was eased since we have K as a natural inverse local time.

The proof of Theorem 7 is subject of the following subsections.

4.2. *Discreteness of the structure of jumps across zero for excursions.* We analyze the initial and final behavior of excursions of Lévy processes when the Gaussian component does not vanish.

LEMMA 3. *If X is a Lévy process with nontrivial Gaussian component, then excursions do not oscillate initially nor eventually. More precisely,* n^X*-almost every excursion $\varepsilon \in E$ leaves and enters zero continuously, but there are $0 < s(\varepsilon) < t(\varepsilon) < \zeta(\varepsilon)$ where $\zeta(\varepsilon)$ is the lifetime of ε, such that one of the following hold*:

- $\varepsilon(r) < 0$ *for all* $0 < r < s(\varepsilon)$ *and* $\varepsilon(r) < 0$ *for all* $t(\varepsilon) < r < \zeta(\varepsilon)$;
- $\varepsilon(r) < 0$ *for all* $0 < r < s(\varepsilon)$ *and* $\varepsilon(r) > 0$ *for all* $t(\varepsilon) < r < \zeta(\varepsilon)$;
- $\varepsilon(r) > 0$ *for all* $0 < r < s(\varepsilon)$ *and* $\varepsilon(r) < 0$ *for all* $t(\varepsilon) < r < \zeta(\varepsilon)$;
- $\varepsilon(r) > 0$ *for all* $0 < r < s(\varepsilon)$ *and* $\varepsilon(r) > 0$ *for all* $t(\varepsilon) < r < \zeta(\varepsilon)$.

PROOF. Fix a minimum life time $z > 0$ and $\delta > 0$. Since X and $\hat{X} = -X$ can both creep upward, by Theorem VI.19 in [2], we have

$$\lim_{y \downarrow 0} P_y(X_{T_{(-\infty,0]}} = 0) = 1 \quad \text{and} \quad \lim_{y \uparrow 0} P_y(X_{T_{[0,\infty)}} = 0) = 1$$

and so

$$p_\delta = \inf_{0 < y \le \delta} \{P_y(X_{T_{(-\infty,0]}} = 0),\, P_{-y}(X_{T_{[0,\infty)}} = 0)\} \underset{\delta \to 0}{\longrightarrow} 1.$$

Therefore, we can choose $\delta_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} (1 - p_{\delta_n}) < \infty.$$

We introduce the probability measure $Q^z = n^X(\cdot|\zeta > z)$, the law on the excursion space $(E, \mathcal{E})$ of an excursion whose life-time exceeds z. Define $\tau_\delta = \inf\{r > z : -\delta \le \varepsilon(r) \le \delta\}$. Clearly, the law of $\varepsilon(\tau_{\delta_n})$ under Q is concentrated on $[-\delta_n, \delta_n]$ and does not have an atom in zero, so that

$$\sum_{n=1}^{\infty} Q^z\big(\{\varepsilon \in E : \operatorname{sgn}(\varepsilon(r)) = \operatorname{sgn}(\varepsilon(\tau_{\delta_n})) \text{ for all } \tau_{\delta_n} < r < \tau_0\}\big)$$
$$= \sum_{n=1}^{\infty} \int_{[-\delta_n,\delta_n]} P_y\big(\operatorname{sgn}(X_r) = \operatorname{sgn}(y) \text{ for all } 0 < r < T_{\{0\}}\big)$$
$$\times Q^z\big(\varepsilon(\tau_{\delta_n}) \in dy\big) < \infty$$

using the fact that, under Q^z, the canonical process $(\varepsilon(r))_{r\ge z}$ has the semi-group of X killed when hitting zero (at $\tau_0 = \zeta$ and $T_{\{0\}}$, respectively). Now the Borel–Cantelli lemma yields

$$Q^z\big(\text{there is } n \ge 1 \text{ such that } \operatorname{sgn}(\varepsilon(r)) = \operatorname{sgn}(\varepsilon(\tau_{\delta_n})) \text{ for all } \tau_{\delta_n} < r < \tau_0\big) = 1.$$

Allowing life times above $z = 1/k$, $k \ge 1$ integer, we can conclude the property to be n^X-almost sure.

The property at the beginning of the excursion follows by time reversal; cf., for example, Getoor and Sharpe [8], Theorem (4.8), who show that the dual process $\hat{X}$ has as excursion measure the measure which is constructed from n^X by time-reversing excursions. □

This provides the proof of part (i) of Theorem 7. Furthermore, we have:

COROLLARY 3. *For any Lévy process X with nontrivial Gaussian component, n^X-a.e. excursion jumps across the origin at most finitely often in finite time.*

PROOF. Denote the successive jump times across the origin by

$$T_0 = 0, \qquad T_n = \inf\{t > T_{n-1} : \operatorname{sgn}(\varepsilon_t) \ne \operatorname{sgn}(\varepsilon_{t-})\}, \quad n \ge 1.$$

Then $T_1 > 0$ n^X-a.s. by Lemma 3. From T_1 on, ε behaves like a Lévy process started at height $\varepsilon(T_1)$ killed when hitting zero. Assume T_n has a finite limit T. Then the jumps get arbitrarily small since large jumps do not accumulate. Therefore, $\varepsilon(T_n)$ tends to zero, so by quasi-left continuity in T, T is the killing time. This contradicts the fact, that excursions do not oscillate in a neighborhood of their killing time, Lemma 3. □

4.3. *Right inverses for excursions.* The last subsection provided the distinction between excursions that start positive and those that start negative. Let us now focus on those starting positive. Analogous statements apply for those starting negative.

The construction of right inverses after Evans relies on successive first hitting times. For Lévy processes these were analyzed within an infinite horizon. Excursions are defined on finite time intervals, in general. On the one hand, another time reversal argument shall allow us to look at the ends of excursions rather than the beginnings. On the other hand, all calculations have to be exhibited under the semi-group of the Lévy process killed when reaching zero which is still a strong Markov process but the analogous transform representation (3) for the hitting times $T_{\{y\}}$ takes a more complicated form:

LEMMA 4. *Let X be a Lévy process with continuous resolvent densities u^q, $X^\dagger$ its version killed at the first hitting time of a, that is, $X_t^\dagger = X_t$ for $t < T_{\{a\}}$ and $X_t^\dagger = \partial$ for $t \geq T_{\{a\}}$. Then the Laplace transform of a hitting time for $X^\dagger$ is given by*

$$E_x^\dagger\big(\exp(-qT_{\{y\}})\big) = \frac{u^q(y-x)u^q(0) - u^q(a-x)u^q(y-a)}{u^q(0)u^q(0) - u^q(a-y)u^q(y-a)}.$$

PROOF. From the point of view of the potential theory for Markov processes, this result is classical; cf., for example, Lemma 1 in Rogers [10]. The existence of continuous potential densities also allows a direct approach to the central identities

$$u_q^\dagger(x, y) = u^q(y-x) - E_x(e^{-qT_{\{a\}}})u^q(y-a) \quad \text{and} \quad E_x^\dagger(e^{-qT_{\{a\}}}) = \frac{u_q^\dagger(x, a)}{u_q^\dagger(a, a)},$$

the first being based on the Markov property in $T_{\{a\}}$, the second via an approximation of the densities like for Theorem II.19(ii) in [2]. $\square$

Let X now be a Lévy process with continuous resolvent densities $u^q(x)$ and let X have a right inverse K. Then:

PROPOSITION 3. *The probabilities $P(T_{\{a\}} = K_a)$ are positive and tend to one as a tends to zero.*

PROOF. We calculate using the standard approximation of K_a of the proof of Theorem 4 and the transform identity of Lemma 4

$$\begin{aligned} &E_0\big(\exp(-qT_{\{a\}}),\ T_{\{a\}} = K_a\big) \\ &\quad = \lim_{n\to\infty} E_0\big(\exp(-qT_{\{a\}}),\ T_{\{k2^{-n}a\}} < T_{\{(k+1)2^{-n}a\}}, k = 1, \dots, 2^n - 1\big) \end{aligned}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \prod_{k=1}^{2^n-1} E^{\dagger}_{(k-1)2^{-n}a}\big(\exp(-qT_{\{k2^{-n}a\}})\big)\\
&= \lim_{n\to\infty} \prod_{k=1}^{2^n-1} \frac{u^q(2^{-n}a)u^q(0) - u^q((k+1)2^{-n}a)u^q(-k2^{-n}a)}{u^q(0)u^q(0) - u^q(k2^{-n}a)u^q(-k2^{-n}a)}\\
&= \lim_{n\to\infty} \prod_{k=1}^{2^n-1} \frac{1 - \frac{u^q((k+1)2^{-n}a)u^q(-k2^{-n}a)}{u^q(2^{-n}a)u^q(0)}}{1 - \frac{u^q(k2^{-n}a)u^q(-k2^{-n}a)}{u^q(0)u^q(0)}}\bigg(1 - \frac{u^q(0) - u^q(2^{-n}a)}{u^q(0)}\bigg).
\end{aligned}
$$

We proceed by showing two things. First, we show that this quantity is always positive. Second, we show that it tends to 1 as a tends to zero. The analysis of an infinite product is most conveniently carried out by a translation to sums using logarithms. In fact, the above limit of the products is positive if and only if the following sums have a finite limit as n tends to infinity. Note that the above factors are all in $(0, 1]$, so the summands below are nonnegative and we obtain using chiefly $u^q(x) \le u^q(0)$ for all $x \in \mathbb{R}$,

$$
\begin{aligned}
&\sum_{k=1}^{2^n-1}\bigg(\frac{u^q((k+1)2^{-n}a)u^q(-k2^{-n}a)}{u^q(2^{-n}a)u^q(0)} - \frac{u^q(k2^{-n}a)u^q(-k2^{-n}a)}{u^q(0)u^q(0)}\\
&\qquad\qquad + \frac{u^q(0) - u^q(2^{-n}a)}{u^q(0)}\bigg)\\
&\quad= (2^n-1)\frac{u^q(0) - u^q(2^{-n}a)}{u^q(0)}\\
&\qquad + \sum_{k=1}^{2^n-1} \frac{u^q(-k2^{-n}a)}{u^q(0)}\bigg(\frac{u^q((k+1)2^{-n}a)}{u^q(2^{-n}a)} - \frac{u^q(k2^{-n}a)}{u^q(0)}\bigg)\\
&\quad\le \frac{u^q(0) - u^q(2^{-n}a)}{2^{-n}u^q(0)} + \frac{u^q(a)}{u^q(2^{-n}a)} - \frac{u^q(2^{-n}a)}{u^q(0)}\\
&\qquad + \frac{u^q(0) - u^q(2^{-n}a)}{u^q(0)u^q(2^{-n}a)} \sum_{k=2}^{2^n-1} u^q(k2^{-n}a)\\
&\quad\le \rho(q) + \frac{u^q(a)}{u^q(2^{-n}a)} - \frac{u^q(2^{-n}a)}{u^q(0)} + \frac{2^{-n}a\rho(q)u^q(0)}{u^q(0)u^q(2^{-n}a)}(2^n-2)u^q(0)\\
&\quad\xrightarrow[n\to\infty]{} \frac{u^q(a) - u^q(0)}{u^q(0)} + 2a\rho(q)
\end{aligned}
$$

which is finite for all $a > 0$ so that the above products have a positive limit. For the last inequality we also used (7).

Furthermore, the majoration of the sum tends to zero as a tends to zero. Therefore, the limits of the products tend to 1 as a tends to zero. □

PROPOSITION 4. *Let X be a Lévy process with nontrivial Gaussian component. Then n_+^X-a.e. excursion has a nontrivial piece of an inverse.*

PROOF. From the preceding proposition we take that there is a sequence $y_n \downarrow 0$ such that

$$\sum_{n\geq 0} \sup_{0<z\leq y_n} P_0(K_z > T_{\{z\}}) < \infty.$$

In the following argument we shall not look at the initial behavior of excursions that start positive but at the final behavior of excursions that end negative. This amounts to the same by the invariance of the excursion measure under time-reversal; cf. Getoor and Sharpe [8], Theorem 4.8.

Now focus on those excursions that go below the height $-\delta < 0$ and end negative (cf. Lemma 3). We work on the probability space induced by the generic excursion distributed according to the excursion law n^X restricted to excursions passing below $-\delta$ and normalized to be a probability measure P. On this space define variables $\tau_n(\varepsilon) = \inf\{t \geq T_{(-\infty,-\delta)}(\varepsilon) : \varepsilon_t \in (-y_n, 0)\}$, $Z_n(\varepsilon) = -\varepsilon_{\tau_n} \in (0, y_n]$ a.s., (killed) processes $X_t^{(n)}(\varepsilon) = \varepsilon_{\tau_n+t} - \varepsilon_{\tau_n}$, $t \geq 0$, and events $A_n = \{K_{Z_n}^{(n)} > T_{\{Z_n\}}^{(n)}\}$. For every n, $X^{(n)}$ has the same distribution under P as X killed when attaining the random but independent height Z_n under P_0. Clearly,

$$\begin{aligned} P(A_n) = P\big(K_{Z_n}^{(n)} > T_{\{Z_n\}}^{(n)}\big) &= \int_{(0,y_n]} P\big(K_z^{(n)} > T_{\{z\}}^{(n)}\big) P(Z_n \in dz) \\ &\leq \sup_{0<z\leq y_n} P_0(K_z > T_{\{z\}}). \end{aligned}$$

We conclude from the Borel–Cantelli lemma that $P(\limsup_{n\to\infty} A_n) = 0$. This means that a.s. only a finite number of the processes $X^{(n)}$ does not possess an inverse up to height Z_n. In fact, all that we shall make use of, is that, for P-a.e. excursion ε we find an $n \geq 0$ such that the path $X^{(n)}(\varepsilon)$ has an inverse up to $Z_n(\varepsilon) > 0$. Going back to the definition of these quantities, this means that ε, from $\tau_n(\varepsilon)$ to the killing $T_{\{0\}}(\varepsilon)$ rises from $-Z_n(\varepsilon)$ to zero so that there exists an (increasing) "right inverse"

$$k : [-Z_n, 0] \to [\tau_n, T_{\{0\}}] \qquad \text{such that } \varepsilon(k(x)) = x \text{ for all } x \in [-Z_n, 0].$$

The last step is the translation via time reversal and changing signs, still keeping ε fixed. We obtain the negative reversed excursion and its inverse

$$\tilde{\varepsilon}(t) = -\varepsilon(T_{\{0\}} - t) \quad \text{and} \quad \tilde{k}(x) = T_{\{0\}} - k(-x).$$

To be precise, we mention, that this inverse is the maximal and not the minimal inverse but, of course, the existence of a minimal inverse follows trivially now.

Since X and the negative of its time-reversal have the same law, this shows the result for all excursions of X that start positive and exceed δ.

Via any sequence $\delta_n \downarrow 0$ we can include all excursions starting positive completing the proof. □

This is the main tool for the remainder of the proof of Theorem 7.

PROOF OF THEOREM 7(ii). Using $\rho(q) + \hat{\rho}(q) = 2\kappa(q)/\sigma^2$, cf. Theorem 4, and the transform representations of the entrance laws given in Theorem 2, we obtain the corresponding relation of the entrance laws by inversion of the transforms. Just note that

$$\int_0^\infty \int_{\mathbb{R}} \exp(-qt + i\lambda x) n_t^{\hat{Z}}(dx)\, dt = \frac{\hat{\rho}(q) - i\lambda}{q + \hat{\psi}(\lambda)}$$

yields when replacing x by $-x$ and λ by $-\lambda$:

$$\int_0^\infty \int_{\mathbb{R}} \exp(-qt + i\lambda x) n_t^{\hat{Z}}(-dx)\, dt = \frac{\hat{\rho}(q) + i\lambda}{q + \psi(\lambda)}.$$

Since entrance laws characterize the excursion law uniquely, we conclude

$$n^Z + n^{\hat{Z}}(-\cdot) = \frac{2}{\sigma^2} n^X. \tag{15}$$

Applying the preceding proposition, the result is now immediate since excursions of Z cannot be initially invertible. So only initially negative excursions are possible. The dual argument excludes the initially negative excursions from the support of $n^{\hat{Z}}(-\cdot)$. By (15), the proof is complete. □

5. The bounded variation case.

Finally, we analyze the bounded variation case which is quite different from the rest. We both characterize the existence of right inverses and describe the excursion measure of the reflected process. Compared with the last section, the two situations are very far from one another. The behavior of the excursions of Z is not only very different between the cases but also very different when compared to the excursions of X.

5.1. *Right inverses for bounded variation Lévy processes.* The existence characterization gives quite natural conditions. However, we see again that the parallels between the "existence of right inverses" and "upward creeping" fail. Recall first, that a bounded variation process can creep upward if and only if its drift coefficient is positive (cf. [2], Exercise VI.9) with no restriction on the number of positive jumps. Concerning the existence of right inverses, we provide:

THEOREM 8. (i) *A bounded variation recurrent Lévy process has right inverses if and only if it has a positive drift coefficient* $b > 0$ *and a discrete structure of positive jumps*.

(ii) *Suppose* X *is a bounded variation recurrent Lévy process with drift coefficient* $b > 0$ *and Lévy measure* Π. *Provided that* $\Pi((0, \infty)) < \infty$, *the Laplace exponent* ρ *of the minimal right inverse* K *of* X *is given by*

$$b\rho(q) = q + \int_{\mathbb{R}^*} \frac{u^q(0+) - u^q(-y)}{u^q(0+)} \Pi(dy)$$

where u^q *is the bounded* q*-resolvent density of* X *that is continuous apart from zero* (*cf. Bretagnolle* [5]).

The proof of Theorem 8 is based on:

LEMMA 5. *Let* $X_t = t + X_t^+ - X_t^-$ *be a recurrent bounded variation Lévy process decomposed into unit drift, positive and negative jump parts* X^+ *and* X^-. *Let* H *be its ascending ladder height process*; *cf. Chapter VI in* [2]. *Then the following are equivalent*:

(1) X *has a right inverse*.
(2) H *is a compound Poisson subordinator with positive drift added*.
(3) X^+ *is a compound Poisson process*.

PROOF. "(3) ⇒ (2)" By Proposition VI.11(ii) in [2] the supremum process of X increases immediately, and so does H. Since X only has a finite number of positive jumps in any finite interval, so does H since every jump of H corresponds to a jump of X under conservation of their order.

"(2) ⇒ (1)" Define sequences of successive stopping times and processes as follows

$$T_0^- = 0, \qquad S_t^{(0)} = \sup_{0 \le s \le t} X_t, \qquad T_0^+ = \inf\left\{t > 0 : S_t^{(0)} - S_{t-}^{(0)} > 0\right\},$$

$$T_n^- = \inf\left\{t > T_{n-1}^+ : X_t = S_{T_{n-1}^+ -}^{(n-1)}\right\}, \qquad S_t^{(n)} = \sup_{T_n^- \le s \le t} X_s,$$

$$T_n^+ = \inf\left\{t > T_n^- : S_t^{(n)} - S_{t-}^{(n)} > 0\right\}.$$

Then, $T_0^+ > 0$ because it corresponds to the first jump time of H. We conclude $T_n^+ \to \infty$ since X hits the previous maxima again. We define

$$L_t = S_t^{(n)}, \quad T_n^- \le t < T_n^+, \qquad n \ge 0$$

which is nontrivial, as $S^{(0)}$ increases immediately, and without jumps. The right-continuous inverse K of L is a right inverse of X and clearly minimal.

"(1) $\Rightarrow$ (3)" Denote the minimal right inverse by K. Look at the process $Y_t = t - X_t^-$. Assume, X^+ is not compound Poisson, so Y drifts to $-\infty$. However, like X, Y is regular for $(0,\infty)$ and irregular for $(-\infty,0)$, so the set $\mathcal{L}_Y$ of times where Y is at its supremum, is a heavy regenerative set, killed at an independent exponential time with parameter $k \in (0,\infty)$; cf. [2], Exercise VI.6. Denote the first jump time of X^+ of a jump height in (ε,∞) by $\tau_{(\varepsilon,\infty)} \sim \mathrm{Exp}(\bar{\Pi}^+(\varepsilon))$ where Π^+ is the Lévy measure of X^+ and $\mathrm{Exp}(p)$ denotes the exponential distribution with parameter (inverse mean) $p > 0$. Then note that the ladder time process K_Y of Y has a positive drift. Denoting the potential density of K_Y by v, Proposition 1.7 of Bertoin [3] yields

$$P(\xi \in \mathcal{L}_Y) = \frac{v(\xi)}{v(0+)} \underset{\xi \to 0+}{\longrightarrow} 1.$$

This entails that

$$\begin{aligned} p_\varepsilon = P(\tau_{(\varepsilon,\infty)} \in \mathcal{L}_X) &\geq P(\tau_{(\varepsilon,\infty)} \in \mathcal{L}_Y) \\ &= \int_0^\infty P\big(\xi/\bar{\Pi}^+(\varepsilon) \in \mathcal{L}_Y\big) e^{-\xi}\, d\xi \underset{\varepsilon \to 0}{\longrightarrow} 1 \end{aligned}$$

as $\bar{\Pi}^+(\varepsilon) \to \infty$ since X^+ is not compound Poisson.

Now choose $\delta > 0$ arbitrarily small such that $c_\delta = P(\inf_{0 \le s \le \delta} X_s = 0) > 0$. Then we have

$$\begin{aligned} p_\varepsilon c_\delta &= P\Big(S_{\tau_{(\varepsilon,\infty)}} = X_{\tau_{(\varepsilon,\infty)}} = \inf_{0 \le s \le \delta} X_{\tau_{(\varepsilon,\infty)}+s}\Big) \\ &\le P\big(K_{X_{\tau_{(\varepsilon,\infty)}}} > \delta\big) \end{aligned}$$

which in the limit $\varepsilon \to 0$ yields $P(K_{0+} > \delta) \geq c_\delta$ and letting $\delta \to 0$ we get $P(K_{0+} > 0) = 1$ which is absurd since K is a subordinator. $\square$

PROOF OF THEOREM 8(i). We first reduce the discussion to the case of a positive drift. If the drift b is nonpositive and K the minimal inverse, the probability that X enters $(0,\infty)$ immediately is zero, a fortiori $P(K_{0+} = 0) = 0$ which does not agree with the subordinator property of K.

The case of a positive drift has been treated in Lemma 5. $\square$

Part (ii) of Theorem 8 will be derived from Proposition 5 which describes the excursions of Z and which we provide in an own subsection.

5.2. *Excursions of Z in the bounded variation case.* Clearly, the excursions of Z are not as closely related to the excursions of X as in the case of a nontrivial Gaussian component since here X leaves zero continuously whereas Z must be expected to jump from zero. However, we can provide a description in terms of the measures $P_x^\dagger$ of X started in x and killed when first hitting $a = 0$.

PROPOSITION 5. *Let X be a bounded variation Lévy process with positive drift coefficient $b > 0$ and a discrete positive jump structure. Then we have*

$$n^Z(d\varepsilon) = \frac{1}{b} \int_{\mathbb{R}} P_x^\dagger(d\varepsilon)\Pi(dx)$$

where Π is the Lévy measure of X, $P_x^\dagger$ the probability measure on the excursion space E under which X has the law of our given Lévy process started at x and killed when it hits zero.

The factor $1/b$ is not surprising. In fact, admitting the proposition for $b = 1$, it occurs naturally by a linear time change: let X have drift coefficient b and Lévy measure Π, then $Y_t = X_{t/b}$ has unit drift and Lévy measure Π/b. Now

$$\begin{aligned} n^Z(d\varepsilon) = n^{Z(Y)}\big(d\varepsilon(\cdot/b)\big) &= \int_{\mathbb{R}} P_x^\dagger\big(Y \in d\varepsilon(\cdot/b)\big)(\Pi/b)(dx) \\ &= \frac{1}{b} \int_{\mathbb{R}} P_x^\dagger(d\varepsilon)\Pi(dx) \end{aligned}$$

deduces the result for $b \neq 1$.

PROOF OF PROPOSITION 5. Let X be a bounded variation process with a positive drift and a discrete structure of positive jumps. Assume, that excursions started continuously occur with a positive probability. Then, an independent exponential time T has a positive probability to hit a corresponding excursion interval. We reverse the time by introducing the process $X_t^* = X_T - X_{T-t-}$ which has the same distribution as X killed at T. For X^* 0 is irregular for $(-\infty, 0)$, so the descending ladder set is discrete. One of these points (the last one) corresponds to the starting point of the last excursion of X. This leads to a contradiction since X^* always jumps to its new infima.

Look at the compound Poisson case (with unit drift added). We identify the intensity measure n^Z of the Poisson excursion process in local time. It clearly suffices to look at the law of the first excursion. The first excursion, however, starts at the first jump which occurs at an exponential time with parameter $\Pi(\mathbb{R})$. The jump height is independent of the jump time, its distribution is $\Pi(\cdot)/\Pi(\mathbb{R})$. By the Markov property at this jump time, the process $\tilde{X}$ after the jump relative to the position before the jump is a Lévy process started at the jump height. The excursion ends when $\tilde{X}$ first attains zero. Due to the unit drift assumption, the local time coincides with the real time before the first jump. Therefore, passing to local time retains the right frequency factor.

In the general case (bounded variation with discrete positive jump structure and unit drift), let us first focus on the excursions started by a jump in $(-\infty, -1/n) \cup (0, \infty)$. Call the position of the first such jump τ_n. The same reasoning as above yields the corresponding part of the excursion measure, only discounted by the probability

$$p_n = P(Z_{\tau_n -} = 0) = P(X_{\tau_n -} = S_{\tau_n})$$

that X is at its supremum when the first jump of this type occurs. Now, letting n tend to infinity includes all excursions started by a jump, that is, all excursions. Also, a similar argument as for the direction "(3) $\Rightarrow$ (2)" of Lemma 5, shows that p_n tends to 1 so that we get the correct intensity factor. More precisely, denote by $X^{(n)}$ the process X when taking away all jumps in $(-\infty, -1/n) \cup (0, \infty)$, by $T_m \sim \tau_m$ an independent (exponential) time. Then we have, for $m \geq n$,

$$p_m = P\big(X^{(m)}_{\tau_m} = S^{(m)}_{\tau_m}\big) \geq P\big(X^{(n)}_{T_m} = S^{(n)}_{T_m}\big) \underset{m\to\infty}{\longrightarrow} 1. \qquad \square$$

REMARK 2. Having established that excursions start by jumps, there is also a proof via compensation formulas which identifies the excursion measure. We shall give an outline here.

Denote the local time excursion process of Z by $(e_x)_{x\geq 0}$, take any nonnegative bounded continuous functional F on the excursion space. Then

$$\begin{aligned}
&\int_E F(\varepsilon) n^Z(d\varepsilon)\\
&\qquad = E\Bigg(\sum_{0\leq x\leq 1} F(e_x)\Bigg)\\
&\qquad = E\Bigg(\sum_{0\leq t<\infty} 1_{\{L_t\leq 1\}} 1_{\{X_{t-}=L_t\}} 1_{\{\Delta X_t\neq 0\}} F\big(\tilde{X}_s = X_{t+s} - X_{t-}, 0 \leq s \leq \tilde{T}_{\{0\}}\big)\Bigg)\\
&\qquad = E\bigg(\int_{[0,1]}\int_{\mathbb{R}} E_x\big(F(X_s, s \leq T_{\{0\}})\big)\Pi(dx)\, dL(t)\bigg)\\
&\qquad = \frac{1}{b}\int_{\mathbb{R}} E_x\big(F(X_s, s \leq T_{\{0\}})\big)\Pi(dx)
\end{aligned}$$

where we used the compensation formula in excursion theory (cf. Corollary IV.11 in [2]), and the compensation formula for the so-called incursion theory or theory of exit systems (cf. Théorème XX.49 in Dellacherie, Maisonneuve and Meyer [6]).

Proposition 5 entails part (ii) of Theorem 8 as follows:

PROOF OF THEOREM 8(ii). Clearly the drift coefficient of K has to be the reciprocal of the drift coefficient of X, so we obtain a representation

$$\rho(q) = \frac{q}{b} + \frac{1}{b}\int_{(0,\infty)} (1 - e^{-qx})\nu(dx)$$

for some Lévy measure ν/b. This Lévy measure describes the jumps of K, that is, the lengths of the excursions of Z. Therefore

$$\nu(dx) = \int_{\mathbb{R}} P_y(T_{\{0\}} \in dx)\Pi(dy)$$

which yields, by Corollary II.18 in [2],

$$\begin{aligned} b\rho(q) &= q + \int_{\mathbb{R}}\int_{(0,\infty)} (1 - e^{-qx})P_y(T_{\{0\}} \in dx)\Pi(dy) \\ &= q + \int_{\mathbb{R}} \left(1 - \frac{u^q(-y)}{u^q(0+)}\right)\Pi(dy). \end{aligned} \qquad \square$$

5.3. *Further remarks.* We can compare the Lévy measure ν of K to μ, the Lévy measure of the inverse local time A^{-1} at the supremum with Laplace exponent

$$\Phi(q) = \frac{1}{b}q + \int_{(0,\infty]} (1 - e^{-qx})\mu(dx).$$

If we define

$$\lambda(dx) = \frac{1}{b}\int_{\mathbb{R}^-} P_y(T_{\{0\}} = T_{[0,\infty)} \in dx)\Pi(dy)$$

we can express

$$\begin{aligned} \mu(dx) &= \frac{1}{b}\int_{\mathbb{R}^-} P_y(T_{[0,\infty)} \in dx)\Pi(dy) \\ &= \lambda(dx) + \frac{1}{b}\int_{\mathbb{R}^-} P_y(T_{\{0\}} > T_{[0,\infty)} \in dx)\Pi(dy), \\ \nu(dx) &= \frac{1}{b}\int_{\mathbb{R}^*} P_y(T_{\{0\}} \in dx)\Pi(dy) \\ &= \lambda(dx) + \frac{1}{b}\int_{\mathbb{R}^-} P_y(T_{[0,\infty)} < T_{\{0\}} \in dx)\Pi(dy) \\ &\quad + \frac{1}{b}\int_{\mathbb{R}^+} P_y(T_{\{0\}} \in dx)\Pi(dy), \end{aligned}$$

where λ is the only infinite component. In the transient case we interpret these as identities of measures on $(0, \infty]$.

We can derive most of this from Proposition 5. Just note that, denoting the renewal density of the ascending ladder height process H by v, we have

$$\int_{\mathbb{R}^-} P_y(T_{\{0\}} > T_{[0,\infty)})\Pi(dy) = \int_{\mathbb{R}^-} \frac{v(0+) - v(-y)}{v(0+)}\Pi(dy)$$

by [2], Theorem VI.19. The finiteness is obtained as follows. From Lemma 5 we take that the ladder height process is compound Poisson. Therefore there is a right derivative of v in 0. This makes the integrand integrable w.r.t. Π. One obtains the finiteness also by observing that K and A^{-1} coincide up to the first jump of the supremum process.

The argument to establish the existence of a right inverse for Theorem 8 also works in the case of unbounded variation whenever the structure of positive jumps is discrete. However, our converse argument relies crucially on the irregularity of 0 for $(-\infty, 0)$ which X no longer fulfills when the variation is unbounded. We refer to Theorem 6 instead, where we use a different argument to generalize the existence result of Theorem 8.

A comparison of Proposition 5 and Theorem 7 shows the very different behavior of the two cases: whereas in the case of a nontrivial Gaussian component all excursions of Z start continuously from zero, the bounded variation case has no such excursions but initiates every excursion by a jump. The latter behavior can also be contrasted to the behavior of the excursions of X away from zero which all start continuously since a.s. no jump starts from zero, due to the positive drift.

APPENDIX

Proposition 2 states that $X = X^+ - X^-$ with X^- spectrally positive α-stable and X^+ symmetric β-stable, $0 < 2\alpha - 2 < \beta < \alpha < 2$, has no right inverses.

PROOF OF PROPOSITION 2. We have

$$\Re(\psi^+(\xi)) = |\xi|^\beta, \qquad \Im(\psi^+(\xi)) = 0,$$
$$\Re(\psi^-(\xi)) = |\xi|^\alpha, \qquad \Im(\psi^-(\xi)) = c\,\mathrm{sgn}(\xi)|\xi|^\alpha, \quad c = -\tan(\pi\alpha/2) \in (0,\infty),$$

as we take, for example, from [2], Section VIII.1.

Following the proof of Theorem 6 until the estimations, we obtain for the resolvent densities v^q of $-X^-$ and u^q of X by focusing on the real part of the integrand

$$\begin{aligned}&\frac{u^q(0) - u^q(x) - (v^q(0) - v^q(x))}{x}\\ &\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty}\biggl(\frac{1-\cos(x\xi)}{x}\Re\biggl(\frac{-\psi^+(-\xi)}{(q+\psi^-(\xi))(q+\psi(-\xi))}\biggr)\\ &\qquad\qquad + \frac{\sin(x\xi)}{x}\Im\biggl(\frac{-\psi^+(-\xi)}{(q+\psi^-(\xi))(q+\psi(-\xi))}\biggr)\biggr)d\xi\end{aligned}$$

and we show that this quantity tends to infinity as x tends to zero. More precisely, we show that the sin-term tends to infinity and that the cos-term does not tend to $-\infty$ quicker than the sin-term tends to infinity.

First, we show that the integrand $f(\xi)$ of the sin-term is eventually decreasing when taking away the sin itself. Indeed, when differentiating

$$f(\xi) = c\frac{2q\xi^{\alpha+\beta} + \xi^{\alpha+2\beta} + 2\xi^{2\alpha+\beta}}{(q^2+2q\xi^\alpha+(1+c^2)\xi^{2\alpha})(q^2+2q\xi^\alpha+(1+c^2)\xi^{2\alpha}+2q\xi^\beta+\xi^{2\beta}+2\xi^{\alpha+\beta})}$$

the leading term is negative:

$$f'(\xi) = c\frac{(1+c^2)^2(4\alpha+2\beta-4\alpha-4\alpha)\xi^{6\alpha+\beta}+O(\xi^{5\alpha+2\beta}+\xi^{6\alpha})}{\xi(q^2+2q\xi^\alpha+(1+c^2)\xi^{2\alpha})^2(q^2+2q\xi^\alpha+(1+c^2)\xi^{2\alpha}+2q\xi^\beta+\xi^{2\beta}+2\xi^{\alpha+\beta})^2}.$$

Second, we split the integral into the positive and negative parts of sin, as an alternating sum of eventually decreasing numbers. By restricting to small x we can even assume that the first positive period of sin contains the whole nondecreasing part of f. We may now focus on the first positive and negative parts of sin.

Third, we eliminate the nondecreasing part of f: denote by ξ_0 the barrier after which f is decreasing. Choose $M > 1$ such that

$$\int_{\xi_0}^{M\xi_0} f(\xi)\,d\xi \geq M\xi_0 f(M\xi_0).$$

This is possible since the left-hand side is increasing in M and the right-hand side decreases to zero ($f(\xi) \sim \xi^{\beta-2\alpha}$). Clearly, for all $x < \pi/2M\xi_0$,

$$\int_0^{M\xi_0} \sin(x\xi) f(\xi)\,d\xi \geq \int_{\xi_0}^{M\xi_0} \sin(x\xi) f(\xi)\,d\xi \geq \sin(x\xi_0)M\xi_0 f(M\xi_0),$$

$$-\int_{\pi/x}^{\pi/x+M\xi_0} \sin(x\xi) f(\xi)\,d\xi \leq \sin(Mx\xi_0)M\xi_0 f(\pi/x)$$

and the former dominates the latter if

$$\frac{\sin(x\xi_0)}{\sin(Mx\xi_0)} \geq \frac{f(\pi/x)}{f(M\xi_0)}$$

which is valid below a threshold x_0, say, since the left hand side tends to $1/M$ as x tends to zero, whereas the right hand side tends to zero.

Fourth, the rest of the first positive and negative parts of sin is strong enough to diverge to infinity when x goes to zero:

$$\begin{aligned}\int_0^{2\pi/x} \frac{\sin(x\xi)}{x} f(\xi)\,d\xi &= \int_0^{\pi/x} \frac{\sin(x\xi)}{x}\big(f(\xi) - f(\pi/x+\xi)\big)\,d\xi \\ &\geq \int_{\pi/2x}^{\pi/x} \frac{\sin(x\xi)}{x}\big(f(\xi) - f(\pi/x+\xi)\big)\,d\xi\end{aligned}$$

$$\geq \int_{\pi/2}^{\pi} \frac{\sin(\eta)}{x^2}\, d\eta \big(f(\pi/x) - f(3\pi/2x)\big)$$
$$= -\frac{1}{x^2}\frac{\pi}{2x} f'(c_x\pi/x)$$
$$\geq \varepsilon x^{-2-1-(\beta-2\alpha-1)} = \varepsilon x^{2\alpha-\beta-2} \underset{x\to 0}{\longrightarrow} \infty$$

where the $c_x \in [1, 3/2]$ come from the mean value theorem and do not influence the estimate as they are bounded above and below.

Fifth, look at the cos-term, whose coefficient can be expressed as

$$g(\xi) = \frac{(c^2-1)\xi^{2\alpha+\beta} - \xi^{\alpha+2\beta} - 2q\xi^{\alpha+\beta} - q\xi^{2\beta} - q^2\xi^{\beta}}{(q^2+2q\xi^{\alpha}+(1+c^2)\xi^{2\alpha})(q^2+2q\xi^{\alpha}+(1+c^2)\xi^{2\alpha}+2q\xi^{\beta}+\xi^{2\beta}+2\xi^{\alpha+\beta})}$$

and this is eventually positive if and only if $c^2 > 1$, that is, $\alpha < 3/2$. In these cases, we can therefore neglect the cos-term and this part of the proof is complete.

Let us now take $\alpha \geq 3/2$. Here the argument is more subtle, since, in fact, the cos-terms diverge as well as x tends to zero and so, we need to compare them to the sin-terms

$$\int_0^{\infty} \frac{1-\cos(x\xi)}{x} g(\xi)\, d\xi + \int_0^{\infty} \frac{\sin(x\xi)}{x} f(\xi)\, d\xi$$
$$= \int_0^{\infty} \frac{1-\cos(x\xi)}{x} \left(g(\xi) - \frac{1}{x} f'(\xi)\right) d\xi.$$

It is easily seen that $\xi \in [0, \xi_1]$ do not influence the limit behavior. Furthermore, we have for all $\varepsilon > 0$ and $\tilde{c} := c - \varepsilon > 0$, ξ_1 sufficiently large and $x \leq x_0$ sufficiently small,

$$\int_{\xi_1}^{\infty} \frac{1-\cos(x\xi)}{x} \left(g(\xi) - \frac{1}{x} f'(\xi)\right) d\xi$$
$$\geq \int_{\xi_1}^{\infty} \frac{1-\cos(x\xi)}{x} \left(\gamma_g \xi^{\beta-2\alpha} - \frac{1}{x}\gamma_f \xi^{\beta-2\alpha-1}\right) d\xi$$

where $\gamma_g = (1-\tilde{c}^2)/(1+c^2)^2$ and $\gamma_f = 2\tilde{c}(2\alpha-\beta)/(1+c^2)^2$. Now again, $\xi \in [0, \xi_1]$ do not influence the limit behavior, we can therefore again look at integrals over $[0, \infty)$. Substituting $\eta = x\xi$ yields

$$\int_0^{\infty} \frac{1-\cos(x\xi)}{x} \left(\gamma_g \xi^{\beta-2\alpha} - \frac{1}{x}\gamma_f \xi^{\beta-2\alpha-1}\right) d\xi$$
$$= x^{2\alpha-\beta-2}\gamma_g \bigg(\gamma \int_0^{\infty} (1-\cos(\eta))\eta^{\beta-2\alpha-1}\, d\eta$$
$$- \int_0^{\infty} (1-\cos(\eta))\eta^{\beta-2\alpha}\, d\eta\bigg)$$

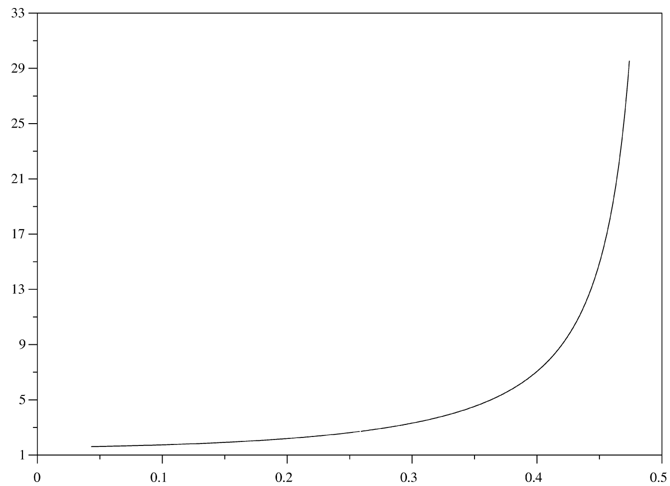

FIG. 1. $a \mapsto \gamma(a, 2-a) \int_0^\infty (1-\cos(\eta))\eta^{a-3}\, d\eta - \int_0^\infty (1-\cos(\eta))\eta^{a-2}\, d\eta$.

and the proof is complete when the term in parentheses is positive. Let us rewrite it as a function of $a = \beta - 2\alpha + 2$ and γ, where $a \in (0, 2-\alpha)$; that is, $\alpha \in [3/2, 2-a)$. As $\gamma(a,\alpha) = 2\tilde{c}(2-a)/(1-\tilde{c}^2)$ ($0 < \tilde{c} = -\tan(\pi\alpha/2) - \varepsilon$) is decreasing in α,

$$\begin{aligned}\gamma(a,\alpha) &\int_0^\infty \big(1-\cos(\eta)\big)\eta^{a-3}\, d\eta - \int_0^\infty \big(1-\cos(\eta)\big)\eta^{a-2}\, d\eta \\ &> \gamma(a, 2-a) \int_0^\infty \big(1-\cos(\eta)\big)\eta^{a-3}\, d\eta - \int_0^\infty \big(1-\cos(\eta)\big)\eta^{a-2}\, d\eta\end{aligned}$$

and the latter is positive by the plot in Figure 1. $\square$

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LABORATOIRE DE PROBABILITÉS
ET MODÈLES ALÉATOIRES
UNIVERSITÉ PIERRE ET MARIE CURIE
175, RUE DU CHEVALERET
F-75013 PARIS
FRANCE
E-MAIL: winkel@proba.jussieu.fr