THE EXACT DISTRIBUTION OF THE EXTREMAL QUOTIENT

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0. Problem. The only quotients considered up to now are those of two observations taken from different distributions. Instead of these statistics, we consider the quotient Q of the extremes (henceforth called the *extremal quotient*) for $n \geq 2$ independent observations taken from the same distribution. This quotient of the extremes has sometimes been used by climatologists [3]. Since it is obviously not affected by changes of scale, its use may be of interest in cases where the scale plays no role. The sensitivity of the extremal quotient to changes in origin is brought out by consideration of uniform distributions where the extremal quotient for a nonnegative variate has just the opposite qualities of the extremal quotient taken from a nonpositive variate.

The asymptotic distribution of the extremal quotient was given in a previous paper [1]. However, the exact distribution of this statistic has never been studied before.[1]

1. The distribution. Let $f(x)$ and $F(x)$ be the density and cumulative probability function of a variate X where $-\omega_1 \leq X \leq \omega_2$. Let X_n be the largest and X_1 the smallest value in a sample of size n ($n \geq 2$). Then the extremal quotient is $Q = X_n/X_1$. The exact cumulative probability function $H(q)$ will be given in terms of pseudo probability functions

$$\begin{aligned} \Phi_1(q) &= Pr\{\ \ 1 \leq Q \leq q\}, & X &\geq 0, \\ \Phi_2(q) &= Pr\{-1 \leq Q \leq q\}, & X_1 &\leq 0, \quad X_n \geq 0, \\ \Phi_3(q) &= Pr\{\ \ 0 \leq Q \leq q\}, & X &\leq 0. \end{aligned} \tag{1.1}$$

These would be cumulative probability functions of Q if the extremal quotient were restricted to the quadrant indicated by the subscript (see Figure 1). In general the cumulative probability function is

$$H(q) = \begin{cases} \Phi_2(q), & q \leq 0, \\ \Phi_2(0) + \Phi_3(q), & 0 \leq q \leq 1, \\ \Phi_2(0) + \Phi_3(1) + \Phi_1(q), & q \geq 1. \end{cases} \tag{1.2}$$

Integrating the joint density of the extremes,

$$w(x_1, x_2) = n(n-1)f(x_1)[F(x_n) - F(x_1)]^{n-1}f(x_n), \tag{1.3}$$

[1] The authors wish to take this opportunity to thank Mr. J. A. Greenwood, who constructively read a first version of this paper.

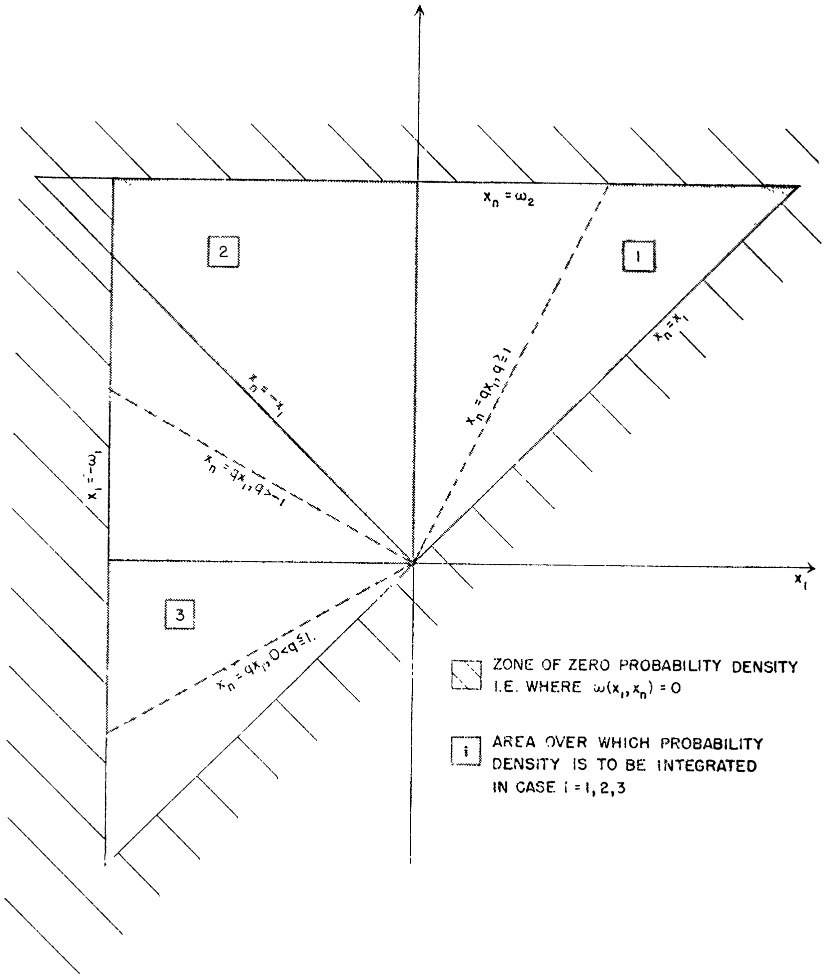

FIG. 1. Zones of integration

over the shaded areas in the proper quadrants of Figure 1, we obtain $\Phi_1(q)$, $\Phi_2(q)$, $\Phi_3(q)$, which, when substituted in (1.2), yield after some simplification

$$
(1.4) \qquad H(q) = \begin{cases} c_1 - n \displaystyle\int_{\omega_2/q}^{0} [F(qx) - F(x)]^{n-1} f(x)\, dx, & q \leq \dfrac{\omega_2}{\omega_1} \leq 0, \\ c_2 - n \displaystyle\int_{-\omega_1}^{0} [F(qx) - F(x)]^{n-1} f(x)\, dx, & \dfrac{\omega_2}{-\omega_1} \leq q \leq 1, \\ c_2 + n \displaystyle\int_{0}^{\omega_2} \left[F(x) - F\left(\dfrac{x}{q}\right)\right]^{n-1} f(x)\, dx, & q \leq 1, \end{cases}
$$

where

$$c_1 = \left[1 - F\left(\frac{\omega_2}{q}\right)\right]^n - [1 - F(0)]^n, \qquad c_2 = 1 - [1 - F(0)]^n, \tag{1.5}$$

and $\omega_1 \geq 0$, $\omega_2 \geq 0$.

It is to be noted that $H(q) = \Phi_1(q)$ or $\Phi_3(q)$ according as the random variable X is always positive or always negative. Thus the pseudo probability functions $\Phi_1(q)$ and $\Phi_3(q)$ *may* be real. Since

$$\Phi_2(0) = 1 - F^n(0) - [1 - F(0)]^n \tag{1.6}$$

can never be unity for finite n, $\Phi_2(q)$ can never be a cumulative probability function. But $\Phi_2(0) \to 1$ as $n \to \infty$ if X can take positive and negative values.

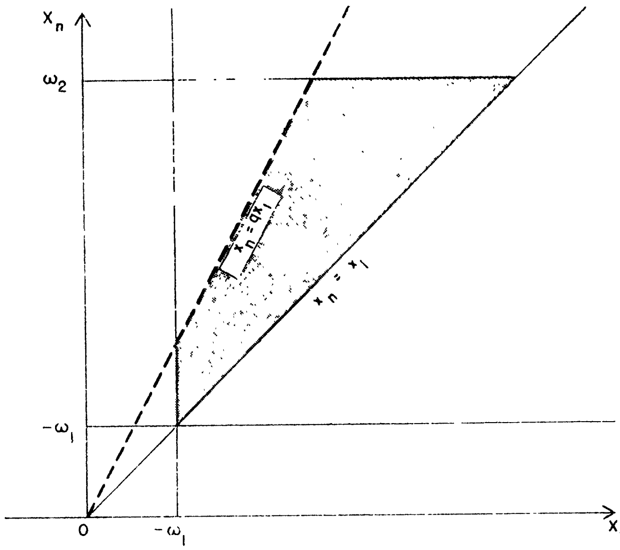

FIG. 2. Area over which probability density $w(x_1, x_n)$ is to be integrated if $\omega_2 > -\omega_1$.

Thus, if n is sufficiently large, the extremal quotient may be treated as negative, as was done in [1]. The speed with which the positive part of the distribution of Q shrinks with even fairly small values of n may be seen in Figures 5 and 6. For any initial distribution and sample size an indication of the error committed by using $H(q) = \Phi_2(q)$ when $\omega_1 > 0$, $\omega_2 > 0$ may be found by seeing how close the value of (1.6) is to unity.

If ω_1 is negative, and $\omega_2 > -\omega_1$, then $\omega_2/-\omega_1 \geq Q \geq 1$ (as in Figure 2), and the probability function becomes

$$\begin{aligned} H(q) &= 1 - n \int_{-\omega_1}^{\omega_2/q} \left\{ (n-1) \int_{qx_1}^{\omega_2} [F(x_n) - F(x_1)]^{n-2} f(x_n)\, dx_n \right\} f(x_1)\, dx_1 \\ &= \left[1 - F\left(\frac{\omega_2}{q}\right)\right]^n + n \int_{-\omega_1}^{\omega_2/q} [F(qx) - F(x)]^{n-1} f(x)\, dx. \end{aligned} \tag{1.7}$$

Similarly, if ω_2 is negative, and $-\omega_1 < \omega_2 < 0$, then $\omega_2/-\omega_1 \leq Q \leq 1$ and the probability function is symmetrical to the previous case. These special cases cover, for example, uniform distributions in the intervals $1 \leq x \leq 2$ and $-2 \leq x \leq -1$.

The extremal quotients Q_1 and Q_2 for two variates $X_{(1)}$ and $X_{(2)}$ which are unlimited in both directions and possess mutually symmetrical distributions, have probability functions $H_1(q_1)$ and $H_2(q_2)$ which are linked by

$$H_2(1/q_2) = 1 - H_1(q_1). \tag{1.8}$$

2. Special cases. For a symmetrical limited distribution where $\omega_1 = \omega_2 = \omega$ (say) and $| X | \leq \omega$ we have

$$F(-\omega) = 0, \qquad F(\omega) = 1, \qquad F(0) = \tfrac{1}{2}, \qquad \frac{\omega_2}{-\omega_1} = -1,$$

and (1.5) becomes

$$c_1 = [1 - F(\omega/q)]^n - (\tfrac{1}{2})^n, \qquad c_2 = 1 - (\tfrac{1}{2})^n.$$

For $q = -1, 0, 1$, the probabilities become

$$H(-1) = \tfrac{1}{2} - (\tfrac{1}{2})^n, \qquad H(0) = 1 - (\tfrac{1}{2})^{n-1}, \qquad H(1) = 1 - (\tfrac{1}{2})^n. \tag{2.1}$$

Therefore the probable error about zero is unity, the median of the extremal quotient for a symmetrical distribution converges toward -1, and zero is the practical upper limit for large n.

For symmetrical unlimited distributions, $F(\omega/q)$ vanishes for $q \leq 0$, and the probability consists of *only two* parts, namely

$$H(q) = \begin{cases} 1 - (\tfrac{1}{2})^n - n \displaystyle\int_0^\infty [F(x) - F(qx)]^{n-1} f(x)\,dx, & q \leq 1, \\[2ex] 1 - (\tfrac{1}{2})^n + n \displaystyle\int_0^\infty [F(x) - F(x/q)]^{n-1} f(x)\,dx, & q \geq 1. \end{cases} \tag{2.2}$$

To apply these methods, we consider first four cases of the uniform distribution which give quite unexpected results. By virtue of the scale invariance mentioned in Section 0, we set the length of the interval of variation equal to unity. The first two cases $f(x) = 1$ for $0 \leq x \leq 1$ and $-1 \leq x \leq 0$ obtained from (1.4) are summarized in Table 1. The respective distributions of the extremal quotients for these very closely related distributions have characteristics which are diametrically opposed. The asymptotic values of the medians are mutually reciprocal, and the asymptotic distributions of the reduced quotients are the second and third asymptotic distributions of extreme values [2].

The two examples show that the extremal quotient is very sensitive to operations like translation which, as a rule, have no influence on the distribution.

Consider now a uniform distribution where zero is within the domain of variation of x.

$$F(x) = \omega_1 + x, \qquad f(x) = 1, \qquad -\omega_1 \leq x \leq \omega_2\,, \qquad \omega_1 + \omega_2 = 1. \tag{2.3}$$

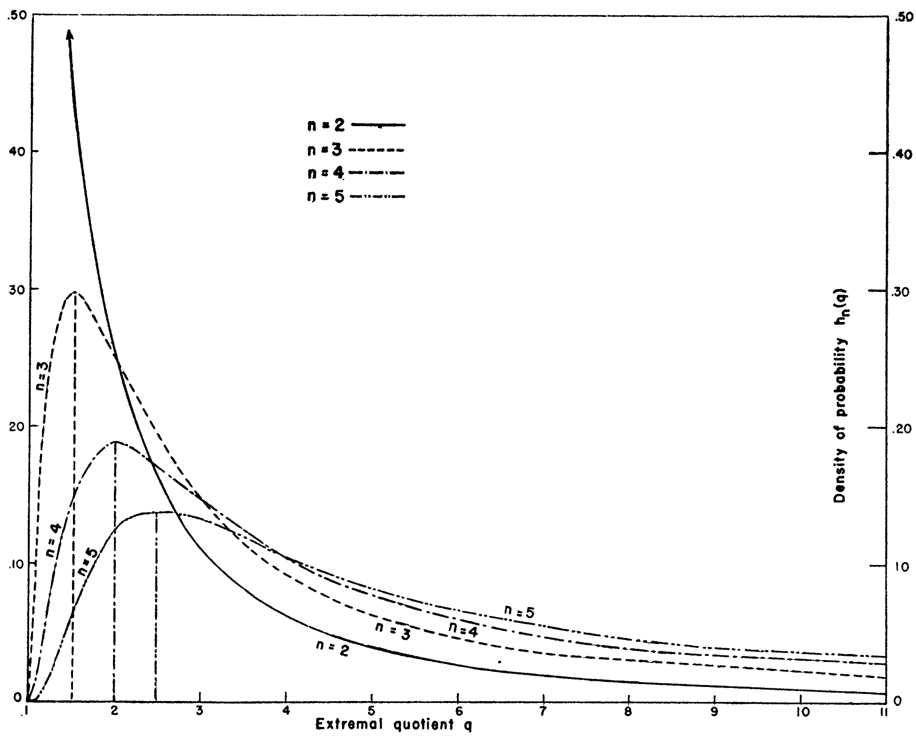

FIG. 3. Densities $h_n(q)$ of the extremal quotient for the uniform distribution $0 \leq x \leq 1$.

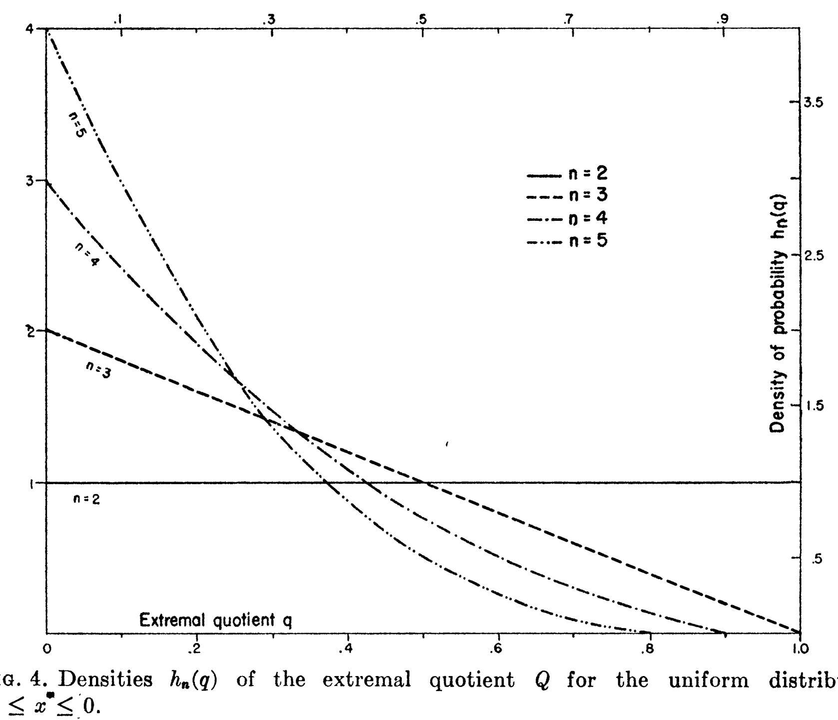

FIG. 4. Densities $h_n(q)$ of the extremal quotient Q for the uniform distribution $-1 \leq x \leq 0$.

TABLE 1

Extremal quotients for two uniform distributions

Limits	$0 \leq x \leq 1$	$-1 \leq x \leq 0$
Cumulative probability $H(q)$	$(1 - 1/q)^{n-1}$	$1 - (1 - q)^{n-1}$
Density $h(q)$	$(n - 1)\ (1 - 1/q)^{n-2}q^{-2}$	$(n - 1)\ (1 - q)^{n-2}$
Range of q	$q \geq 1$	$0 \leq q \leq 1$
Graph	Fig. 3	Fig. 4
Influence of n	density $h(q)$ spreads	density $h(q)$ contracts
Mode	$n/2$	0
Median (for large n)	$(n - 1)/\log 2$	$[\log 2]/(n - 1)$
Mean	does not exist	$1/n$
Moments q^k	do not exist	$1/\binom{n+k-1}{k}$
Reduced variate z	$q/2\tilde{q}$	$q/\tilde{q}$
Asymptotic cumulative probability $H(z)$ for $n \rightarrow \infty$	$e^{-1/z}$	$1 - e^{-z}$

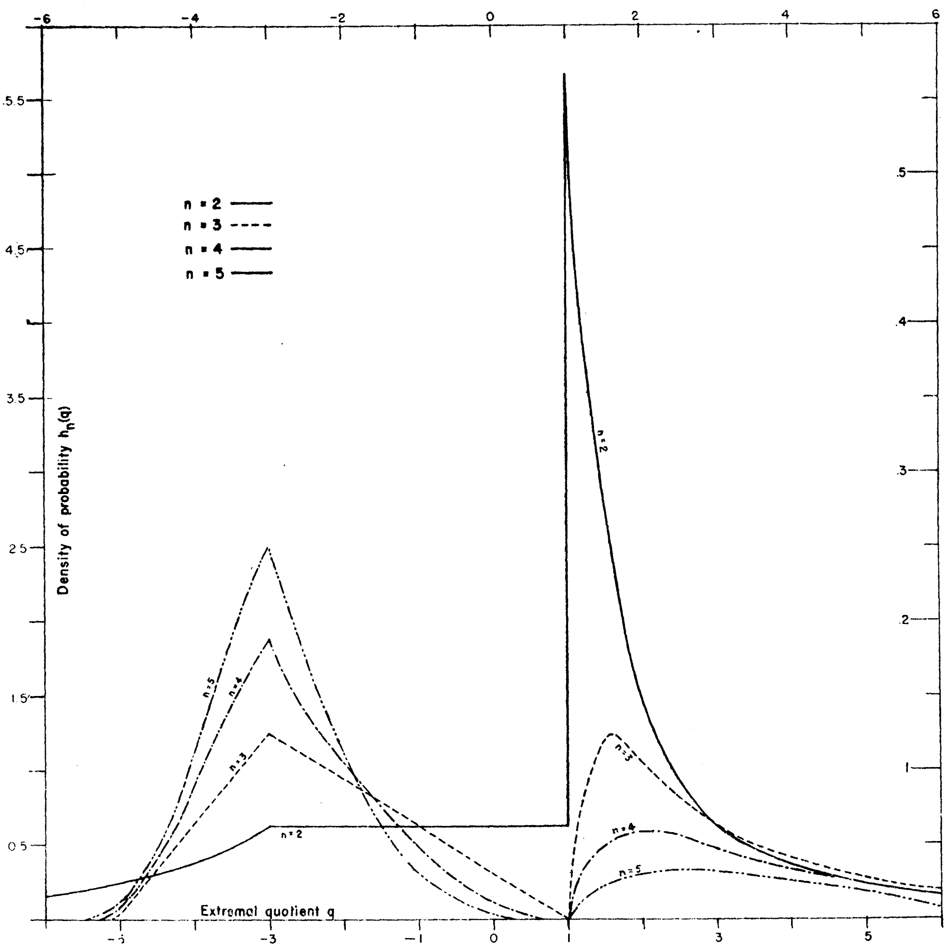

FIG. 5. Densities $h_n(q)$ of the extremal quotient Q for the uniform distribution $-\frac{1}{4} \leq x \leq \frac{3}{4}$.

The general formulas (1.4) and (1.5) lead, after trivial calculations, to the probability of the extremal quotient

$$(2.4)\quad H(q) = \begin{cases} \omega_2^n\,(1 - q^{-1})^{n-1} - 1, & q \leq \dfrac{\omega_2}{-\omega_1}, \\[2ex] 1 - \omega_2^n - (1 - q)^{n-1}\omega_1^n, & \dfrac{\omega_2}{-\omega_1} \leq q \leq 1, \\[2ex] 1 - \omega_2^n\,[1 - (1 - q^{-1})^{n-1}], & q \geq 1, \end{cases}$$

with

$$H\left(\frac{\omega_2}{-\omega_1}\right) = \omega_2 - \omega_2^n, \qquad H(0) = 1 - \omega_2^n - \omega_1^n, \qquad H(1) = 1 - \omega_2^n.$$

The density corresponding to this probability distribution is drawn for $\omega_1 = \frac{1}{4}$, $\omega_2 = \frac{3}{4}$ and $n = 2$ to 5 in Figure 5.

As an example of a symmetrical limited distribution, we put $\omega_1 = \omega_2 = \frac{1}{2}$. These densities are drawn for $n = 2, 3, 4$ in Figure 6. The shapes of these two series are, of course, completely unexpected.

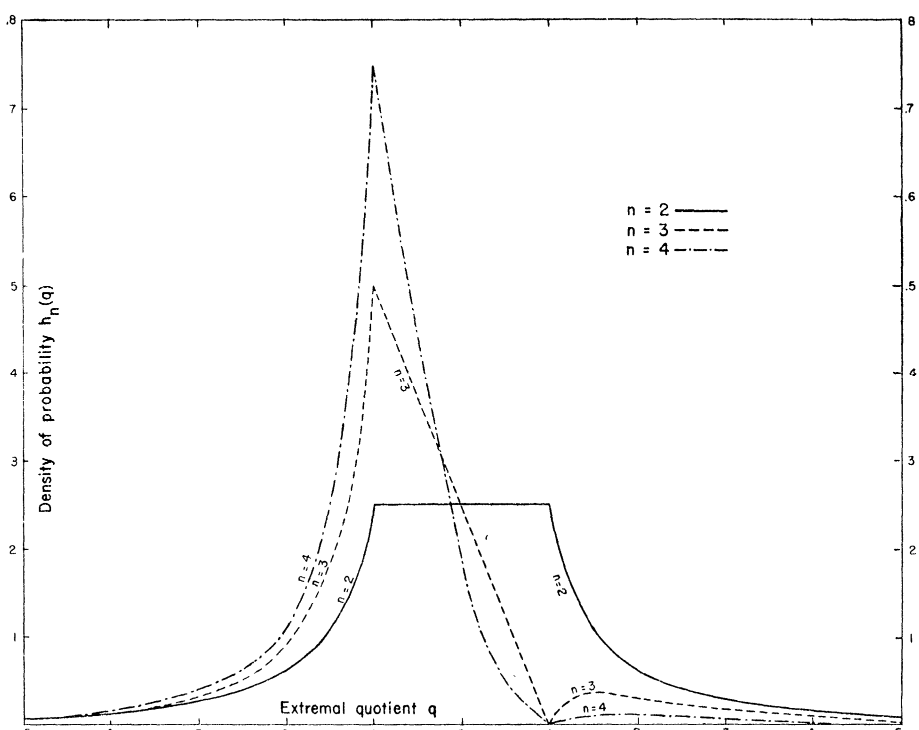

FIG. 6. Densities of the extremal quotient for the uniform distribution $-\frac{1}{2} \leq x \leq \frac{1}{2}$.

We now show how the methods have to be altered to cover the case where a symmetrical unlimited distribution cannot be regarded as a single function, as for instance in the so-called first Laplacean distribution, where the formulas for the two symmetrical branches differ,

$$(2.5)\qquad \begin{aligned} f(x) &= f_1(x) = \tfrac{1}{2}e^{x}, & F(x) &= F_1(x) = \tfrac{1}{2}e^{x}, & x &\le 0,\\ f(x) &= f_2(x) = \tfrac{1}{2}e^{-x}, & F(x) &= F_2(x) = 1 - \tfrac{1}{2}e^{-x}, & x &\ge 0. \end{aligned}$$

TABLE 2

Extremal quotients for $n = 2$

Name	Initial density $f(x)$	Condition on x	Density $h(q)$	Condition on q
Laplace	$\frac{1}{2}e^{x}$ $\frac{1}{2}e^{-x}$	$x \le 0$ $x \ge 0$	$1/[2(1-q)^2]$ $1/[2(1+q)^2]$	$q \le 0$ $q \ge 0$
Exponential	e^{-x} e^{x}	$x \ge 0$ $x \le 0$	$2/(1+q)^2$ $2/(1+q)^2$	$q \ge 1$ $0 \le q \le 1$
Gamma	$e^{-x}x^{k-1}/\Gamma(k)$	$x \ge 0$	$\dfrac{2\Gamma(2k)q^{k-1}}{\Gamma^2(k)\ (1+q)^{2k}}$	$q \ge 1$
Normal	$e^{-x^2/2}/\sqrt{2\pi}$	$-\infty < x < \infty$	$\dfrac{1}{\pi(1+q^2)}$	$-\infty < q < \infty$
Cauchy	$1/[\pi(1+x^2)]$	$-\infty < x < \infty$	$\dfrac{-\log q^2}{\pi^2(1-q^2)}$	$-\infty < q < \infty$

In formula (1.4), which is valid even if the functional form varies under the integral sign, we have to use f_2 and F_2 for positive values of x, and f_1 and F_1 for negative values of x. Accordingly we have

$$(2.6)\qquad H(q) = \begin{cases} 1 - (\tfrac{1}{2})^n - n\displaystyle\int_{-\infty}^{0} [F_2(qx) - F_1(x)]^{n-1} f_1(x)\,dx, & q \le 0,\\ 1 - (\tfrac{1}{2})^n - n\displaystyle\int_{-\infty}^{0} [F_1(qx) - F_1(x)]^{n-1} f_1(x)\,dx, & 0 < q < 1,\\ 1 - (\tfrac{1}{2})^n + n\displaystyle\int_{0}^{\infty} \left[F_2(x) - F_2\left(\frac{x}{q}\right)\right]^{n-1} f_2(x)\,dx, & q \ge 1. \end{cases}$$

It is easily seen that the middle term holds for $q \ge 0$. Thus the probability function and density consist of only two branches which join at $q = 0$.

The degenerate case, $n = 2$, is shown for different initial distributions in Table 2.

If, as in the case of the Cauchy distribution, the initial distribution possesses no moments and does not vanish at $x = 0$, the density of Q becomes infinite at $q = 0$ for $n = 2$.

On the whole, the theory leads to surprisingly complicated results even for the simplest distributions as long as the sample size is small.

REFERENCES

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[2] E. J. GUMBEL, "The return period of flood flows," *Annals of Math. Stat.*, Vol. 12 (1941), pp. 163–190.
[3] V. A. CONRAD, *Methods in Climatology*, Harvard University Press, 1946.