ON REGULAR BEST ASYMPTOTICALLY NORMAL ESTIMATES[1]

By Chin Long Chiang

University of California, Berkeley

1. Introduction and summary. This study was initiated in connection with estimating parameters involved in a certain stochastic process of population growth. Because of the nature of distribution functions arising in such studies, the usual methods of estimation result in formulas which are so complex that it is difficult, if not impossible, to obtain explicit solutions for the estimates of the parameters. Investigation of the problem led to an extension of the method of best asymptotically normal estimates developed by Neyman [1]. The estimates derived are termed regular best asymptotically normal estimates (RBAN estimates). This extension can be applied to other problems.

In [1], Neyman considers a whole class of estimates which possess the properties of consistency, of asymptotic normality, and of asymptotic efficiency, and he provides estimates having these asymptotic properties for the case of multinomial distributions. His method is extended in the present paper to a more general case in which random vectors are dealt with. Such an extension was considered by Barankin and Gurland [2], who studied a large class of estimates and showed that if the distributions involved are members of Koopman's family, it is still possible to reach the Cramér-Rao lower bound.

The purposes of the present paper are to discuss a subclass of the estimates considered by Barankin and Gurland and to present simple methods of generating such estimates. The estimates discussed are based on a number of independent random vectors whose distribution functions are not specified. It is proved that under certain regularity conditions, the regular and consistent estimates obtained are asymptotically normal as the number of random vectors tends to infinity. A necessary and sufficient condition for a regular and consistent estimate to have a "minimal" asymptotic covariance matrix is given. An expression is derived for the "minimal" asymptotic covariance matrix. It is also proved that if a function $\mathbf{f}$ satisfies certain conditions, then in order that $\mathbf{f}(\tilde{\boldsymbol{\theta}})$ be an RBAN estimate of $\mathbf{f}(\boldsymbol{\theta})$ at $\mathbf{f}(\boldsymbol{\theta}^0)$, where $\boldsymbol{\theta}^0$ is the true value of the parameter point $\boldsymbol{\theta}$, it is necessary and sufficient that the argument $\tilde{\boldsymbol{\theta}}$ be an RBAN estimate of $\boldsymbol{\theta}$ at $\boldsymbol{\theta}^0$. Methods of generating RBAN estimates are given.

For simplicity of presentation, matrix notation is used throughout this paper. By derivatives of a matrix with respect to a vector (or with respect to a second matrix) is meant the derivatives of the matrix simultaneously with respect to all the components of the vector (or all the elements of the second matrix). The usual rules of differentiation with respect to vectors are used.

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2. Assumptions and definitions. Let

$$Z_\alpha = (Z_{\alpha 1}, Z_{\alpha 2}, \cdots, Z_{\alpha n})', \quad \text{for } \alpha = 1, 2, \cdots, m,$$

be a sequence of independent random vectors taking their values in an n-dimensional Euclidean space R_n. Let

$$\boldsymbol{\theta} = (\theta_1, \cdots, \theta_s)'$$

be a parameter point ranging through a subset $\boldsymbol{\Theta}$ of an s-dimensional Euclidean space. The true value of the parameter, denoted by $\boldsymbol{\theta}^0$, is assumed to be the center of a nondegenerate s-dimensional sphere contained in $\boldsymbol{\Theta}$. For each $\boldsymbol{\theta} \varepsilon \boldsymbol{\Theta}$ and for each α, it is assumed that the vector $\mathbf{Z}_\alpha$ either possesses a probability density or is discrete. The density or frequency function of $\mathbf{Z}_\alpha$ will be denoted by $p_\alpha(\mathbf{z}; \boldsymbol{\theta})$, depending on $\boldsymbol{\theta}$. Let $\boldsymbol{\zeta}_\alpha(\boldsymbol{\theta})$ be the expectation of $\mathbf{Z}_\alpha$ and let $\bar{\boldsymbol{\zeta}}_m(\boldsymbol{\theta})$ be the average of the expectations, i.e.,

$$\bar{\boldsymbol{\zeta}}_m(\boldsymbol{\theta}) = \frac{1}{m} \sum_{\alpha=1}^{m} \boldsymbol{\zeta}_\alpha(\boldsymbol{\theta}).$$

For the sake of simplicity in further formulas, whenever the parameter takes on the true value $\boldsymbol{\theta}^0$, $\boldsymbol{\zeta}_\alpha^0$ will be written for $\boldsymbol{\zeta}_\alpha(\boldsymbol{\theta}^0)$.

The following assumptions will be made throughout the paper.

Assumption 1. The second central moments

$$(1) \qquad \sigma_{Z_{\alpha i} Z_{\alpha j}} = E\{(Z_{\alpha i} - \zeta_{\alpha i}^0)(Z_{\alpha j} - \zeta_{\alpha j}^0) \mid \boldsymbol{\theta} = \boldsymbol{\theta}^0\}, \quad \text{for } \alpha = 1, 2, \cdots, m, \quad i, j = 1, 2, \cdots, n,$$

are finite and the matrix

$$(2) \qquad \boldsymbol{\sigma}_m^* = \left\| \frac{1}{m} \sum_{\alpha=1}^{m} \sigma_{Z_{\alpha i} Z_{\alpha j}} \right\|, \qquad i, j = 1, 2, \cdots, n,$$

tends to a positive definite matrix $\boldsymbol{\sigma}^*$ as m tends to infinity.

Assumption 2. Let

$$\rho = \sqrt{\sum_{i=1}^{n} (Z_{\alpha i} - \zeta_{\alpha i}^0)^2}.$$

Then, for every $\epsilon > 0$,

$$\lim_{m \to \infty} \frac{1}{m} \sum_{\alpha=1}^{m} \int_{|\rho| > \epsilon\sqrt{m}} \rho^2 p_\alpha(\mathbf{z}; \boldsymbol{\theta}^0)\, d\mathbf{z} = 0.$$

Assumption 3. As $m \to \infty$, $\bar{\boldsymbol{\zeta}}_m(\boldsymbol{\theta})$ tends to $\boldsymbol{\zeta}(\boldsymbol{\theta})$ in such a way that $\sqrt{m}\, | \bar{\boldsymbol{\zeta}}_m(\boldsymbol{\theta}) - \boldsymbol{\zeta}(\boldsymbol{\theta})|$ tends to zero. Let $\boldsymbol{\theta}_k = (\theta_1, \theta_2, \cdots, \theta_k)'$, for $1 \leqq k \leqq s$. The function $\boldsymbol{\zeta}(\boldsymbol{\theta})$ has continuous second partial derivatives with respect to $\boldsymbol{\theta}_k$, and the matrix

$$(3) \qquad \mathbf{V}_k(\boldsymbol{\theta}) = \frac{\partial \boldsymbol{\zeta}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_k}$$

has rank k in the neighborhood of the true value $\boldsymbol{\theta}^0 = (\theta_1^0, \theta_2^0, \cdots, \theta_s^0)'$ for every k, $1 \leqq k \leqq s$.

Constant use will be made of the following definitions.

DEFINITION 1. Let $\{\hat{\boldsymbol{\theta}}_m(\mathbf{Z}_1, \cdots, \mathbf{Z}_m)\}$, $m = 1, 2, \cdots$, be a sequence of functions of the observations, taking their values in the s-dimensional space containing $\boldsymbol{\Theta}$. The sequence $\{\hat{\boldsymbol{\theta}}_m\}$ will be said to be a consistent estimate of the parameter point $\boldsymbol{\theta}$ at the true value $\boldsymbol{\theta}^0$ if, as $m \to \infty$, $\hat{\boldsymbol{\theta}}_m$ tends in probability to $\boldsymbol{\theta}^0$. This means that for every $\epsilon > 0$ and $\eta > 0$, there exists a number $m_{\epsilon,\eta}$ such that $m > m_{\epsilon,\eta}$ implies

$$\Pr\{|\hat{\boldsymbol{\theta}}_m - \boldsymbol{\theta}^0| > \epsilon \mid \boldsymbol{\theta} = \boldsymbol{\theta}^0\} < \eta.$$

DEFINITION 2. Let $\mathbf{B}$ be a positive definite matrix such that as $m \to \infty$, the distribution of $\sqrt{m}\,\mathbf{B}^{-1}(\hat{\boldsymbol{\theta}}_m - \boldsymbol{\theta}^0)$ tends to a multivariate normal distribution with a mean zero and a covariance matrix identity; then for $\boldsymbol{\theta}^0$, the estimate $\hat{\boldsymbol{\theta}}_m$ is said to be consistent and asymptotically normal and $m^{-1}\mathbf{B}\mathbf{B}'$ is said to be the asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}_m$.

Let $\check{\mathbf{Z}}_m = 1/m \sum_{\alpha=1}^{m} \mathbf{Z}_\alpha$ be the average of the vectors $\mathbf{Z}_\alpha$ and let $\mathbf{S}_m$ be the sample covariance matrix.

DEFINITION 3. An estimate $\hat{\boldsymbol{\theta}}_m$ is said to be regular if

(i) for every $m = 1, 2, \cdots$, the function $\hat{\boldsymbol{\theta}}_m(\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_m)$ is either a function of $\check{\mathbf{Z}}_m$ or a function of $\check{\mathbf{Z}}_m$ and $\mathbf{S}_m$, but it does not depend explicitly either on m or on the individual vectors $\mathbf{Z}_\alpha$; i.e., $\hat{\boldsymbol{\theta}}_m(\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_m) = \hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ or $\hat{\boldsymbol{\theta}}_m(\mathbf{Z}_1, \mathbf{Z}_2, \cdots, \mathbf{Z}_m) = \hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$; and

(ii) $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ has continuous first partial derivatives with respect to $\check{\mathbf{Z}}_m$ when the estimate is a function of $\check{\mathbf{Z}}_m$; or, $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ has continuous first partial derivatives with respect to $\check{\mathbf{Z}}_m$ and $\mathbf{S}_m$ when the estimate is a function of $\check{\mathbf{Z}}_m$ and $\mathbf{S}_m$.

Since the theorems in the following section will be concerned mainly with the derivatives of $\hat{\boldsymbol{\theta}}_m$ with respect to $\check{\mathbf{Z}}_m$, only $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ will be used in Section 3.

3. The main theorems. The purposes of this section are to show that regular and consistent estimates are asymptotically normal, to derive a necessary and sufficient condition for an asymptotically normal estimate to have a "minimal" asymptotic covariance matrix, and to give an expression for the "minimal" covariance matrix.

The following well-known lemmas are stated in appropriate forms.

LEMMA 1. *Let $\mathbf{X}_m$ and $\mathbf{Y}_m$ be s-dimensional random vectors and let $\mathbf{Z}_m$ be an n-dimensional random vector satisfying the relationship*

$$\mathbf{X}_m = \mathbf{D}_m\mathbf{Z}_m + \mathbf{Y}_m,$$

where $\mathbf{D}_m$ is an $s \times n$ random matrix. Suppose that as $m \to \infty$, $\mathbf{D}_m$ tends in probability to a matrix $\mathbf{D}$ with constant elements, $\mathbf{Z}_m$ has a limiting distribution, and $\mathbf{Y}_m$ tends in probability to zero. Then $\mathbf{X}_m$ has the limiting distribution defined by the relation $\mathbf{X} = \mathbf{D}\mathbf{Z}$, where $\mathbf{Z}$ has the limiting distribution of $\mathbf{Z}_m$.

PROOF. $\mathbf{X}_m$ may be written as

$$\mathbf{X}_m = \mathbf{D}\mathbf{Z}_m + (\mathbf{D}_m - \mathbf{D})\mathbf{Z}_m + \mathbf{Y}_m = \mathbf{D}\mathbf{Z}_m + \mathbf{U}_m .$$

According to a theorem by Slutsky [3] (see also Cramér [4], pp. 254, 299; Neyman [1], p. 245), $\mathbf{U}_m$ tends in probability to zero as $m \to \infty$. Furthermore, $\mathbf{D}\mathbf{Z}_m$ has the same limiting distribution as $\mathbf{D}\mathbf{Z}$. The lemma follows from a second application of Slutsky's theorem.

LEMMA 2. *Let Assumption* 2 *be satisfied. As* $m \to \infty$, $\check{\mathbf{Z}}_m$ *tends in probability to* $\boldsymbol{\zeta}^0$, *and the distribution of* $\sqrt{m}(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$ *tends to an n-variate normal distribution with a mean zero and a covariance matrix* $\boldsymbol{\sigma}^*$, *as defined in Assumption* 1.

The lemma is a consequence of the central limit theorem (cf. [5], p. 113) and Lemma 1, since $\sqrt{m}(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0) = \sqrt{m}(\check{\mathbf{Z}}_m - \bar{\boldsymbol{\zeta}}_m) + \sqrt{m}(\boldsymbol{\zeta}_m - \boldsymbol{\zeta}^0)$.

Let

$$\mathbf{A} = \|a_{ki}\|, \qquad k = 1, 2, \cdots, s, \quad i = 1, 2, \cdots, n,$$

be an $s \times n$ matrix of rank $s \leqq n$. Let $\mathbf{A}_k$ stand for the row vector of the matrix $\mathbf{A}$, for $k = 1, 2, \cdots, s$.

THEOREM 1. *Under Assumptions* 1, 2, *and* 3, *the random vector*

$$\sqrt{m}\,\mathbf{X}_m = \sqrt{m}\,\mathbf{A}(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0) \tag{4}$$

has an s-variate asymptotically normal distribution with a mean zero and a covariance matrix $\mathbf{A}\boldsymbol{\sigma}^*\mathbf{A}'$. *Moreover, as* $m \to \infty$, $\mathbf{X}_m$ *tends in probability to zero.*

PROOF. According to Definition 2 of the asymptotic covariance matrix, it is to be shown here that there exists a positive definite matrix, $\mathbf{B}$, say, such that the quantity $\sqrt{m}\,\mathbf{B}\mathbf{A}(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$ has a limiting distribution which is normal with a mean zero and with a covariance matrix identity. Since $\boldsymbol{\sigma}^*$ is a positive definite symmetric matrix, and since $\mathbf{A}$ has rank s, $\mathbf{A}\boldsymbol{\sigma}^*\mathbf{A}'$ is also a positive definite symmetric matrix. Hence, there exists a unique positive definite symmetric matrix, $\mathbf{B}$, such that $\mathbf{B}^2 = \mathbf{A}\boldsymbol{\sigma}^*\mathbf{A}'$, or equivalently,

$$\mathbf{B}^{-1}(\mathbf{A}\boldsymbol{\sigma}^*\mathbf{A}')\mathbf{B}^{-1\prime} = \mathbf{I}_s \qquad (s \times s \text{ identity matrix}).$$

Because of Lemmas 1 and 2, $\sqrt{m}\,\mathbf{B}^{-1}\mathbf{A}(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$ has the same limiting distribution as $\mathbf{B}^{-1}\mathbf{A}\mathbf{Y}$, where $\mathbf{Y}$ is normally distributed with a mean zero and a covariance matrix $\boldsymbol{\sigma}^*$. Consequently, $\sqrt{m}\,\mathbf{B}^{-1}\mathbf{A}(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$ is asymptotically normal with a mean zero and with a covariance matrix $\mathbf{B}^{-1}\mathbf{A}\boldsymbol{\sigma}^*\mathbf{A}'\mathbf{B}^{-1\prime}$, which is an identity matrix. Thus, by Definition 2, the asymptotic covariance matrix of $\sqrt{m}\,\mathbf{A}(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$ is $\mathbf{B}\mathbf{B}' = \mathbf{A}\boldsymbol{\sigma}^*\mathbf{A}'$.

The convergence of $\mathbf{X}_m$ to zero follows immediately from the equation $\mathbf{X}_m = \mathbf{A}(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$ and from the fact that $\check{\mathbf{Z}}_m$ tends in probability to $\boldsymbol{\zeta}^0$ (Lemma 2).

THEOREM 2. *Suppose that* $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ *is a regular and consistent estimate of* $\boldsymbol{\theta}$ *at* $\boldsymbol{\theta}^0$. *Then,*

(i) $\hat{\boldsymbol{\theta}}(\boldsymbol{\zeta}^0) = \boldsymbol{\theta}^0$ *with a probability tending to one, and*

(ii) $\sqrt{m}\,[\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m) - \boldsymbol{\theta}^0]$ *has an s-variate asymptotically normal distribution with a mean zero and a covariance matrix* $\mathbf{A}_0\boldsymbol{\sigma}^*\mathbf{A}_0'$, with

$$\mathbf{A}_0 = \left.\frac{\partial\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)}{\partial\check{\mathbf{Z}}_m}\right|_{\boldsymbol{\zeta}^0}. \tag{5}$$

PROOF. Since $\check{\mathbf{Z}}_m$ tends to $\boldsymbol{\zeta}^0$,

$$\lim_{m\to\infty} \hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m) = \hat{\boldsymbol{\theta}}(\boldsymbol{\zeta}^0),$$

and since $\hat{\boldsymbol{\theta}}$ is consistent, $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ tends also to $\boldsymbol{\theta}^0$; part (i) follows.

According to Taylor's theorem,

$$\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m) - \hat{\boldsymbol{\theta}}(\boldsymbol{\zeta}^0) = \mathbf{A}_m^*(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$$

with

$$\mathbf{A}_m^* = \left.\frac{\partial\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)}{\partial\check{\mathbf{Z}}_m}\right|_{\boldsymbol{\zeta}^0+\boldsymbol{\delta}_m(\mathbf{Z}_m-\boldsymbol{\zeta}^0)},$$

where $\boldsymbol{\delta}_m$ is an $n \times n$ diagonal matrix having all its diagonal elements between zero and unity. As m tends to infinity, $\boldsymbol{\zeta}^0 + \boldsymbol{\delta}_m(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$ tends in probability to $\boldsymbol{\zeta}^0$. Since the derivatives of $\hat{\boldsymbol{\theta}}$ are assumed to be continuous, $\mathbf{A}_m^*$ tends in probability to $\mathbf{A}_0$. Consequently, $\sqrt{m}\,[\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m) - \boldsymbol{\theta}^0]$ has the same limiting distribution as $\sqrt{m}\,\mathbf{A}_0(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0)$ (Lemma 1). The rest of the proof follows from Theorem 1.

COROLLARY. *Let θ_k be the* kth *element of the vector* $\boldsymbol{\theta}$. *Suppose that $\hat{\theta}_k(\check{\mathbf{Z}}_m)$ is a regular and consistent estimate of θ_k at θ_k^0, then,*

(i) $\hat{\theta}_k(\boldsymbol{\zeta}^0) = \theta_k^0$ *with a probability tending to one, and*

(ii) $\sqrt{m}\,[\hat{\theta}_k(\check{\mathbf{Z}}_m) - \theta_k^0]$ *has an asymptotically normal distribution with a mean zero and a variance* $\mathbf{A}_k\boldsymbol{\sigma}^*\mathbf{A}_k'$, with

$$\mathbf{A}_k = \left.\frac{\partial\hat{\boldsymbol{\theta}}_k(\check{\mathbf{Z}}_m)}{\partial\check{\mathbf{Z}}_m}\right|_{\boldsymbol{\zeta}^0}.$$

The corollary, which is a direct consequence of Theorem 2, may also be verified by considering the following equation:

$$\hat{\theta}_k(\check{\mathbf{Z}}_m) - \hat{\theta}_k(\boldsymbol{\zeta}^0) = \mathbf{A}_k^*(\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0),$$

with

$$\mathbf{A}_k^* = \left.\frac{\partial\hat{\theta}_k(\check{\mathbf{Z}}_m)}{\partial\check{\mathbf{Z}}_m}\right|_{\boldsymbol{\zeta}^0+\boldsymbol{\delta}_m(\check{\mathbf{Z}}_m-\boldsymbol{\zeta}^0)}.$$

Here, the diagonal matrix $\boldsymbol{\delta}_m$ has the same meaning as defined in Theorem 2.

DEFINITION 4. Let $\mathcal{C}^*$ be a class of symmetric positive definite matrices of rank s. A matrix $\mathbf{G} \in \mathcal{C}^*$ is said to be minimal with respect to $\mathcal{C}^*$ if, for every $\mathbf{H} \in \mathcal{C}^*$, the difference $\mathbf{H} - \mathbf{G}$ is positive semidefinite; i.e., for any $1 \times s$ row vector $\mathbf{u}$ and for any $\mathbf{H} \in \mathcal{C}^*$, the quadratic form $\mathbf{u}(\mathbf{H} - \mathbf{G})\mathbf{u}'$ is nonnegative.

Let $\mathcal{C}$ be the class of matrices which are covariance matrices of the limiting distribution of $\sqrt{m}\,[\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m) - \boldsymbol{\theta}^0]$ for some regular and consistent estimate $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$.

Definition 5. A regular and consistent estimate $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ is said to be regular best asymptotically normal (RBAN) if the covariance matrix $\mathbf{A}_0\boldsymbol{\sigma}^*\mathbf{A}_0'$ of the limiting distribution of $\sqrt{m}\,[\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m) - \boldsymbol{\theta}^0]$ is minimal with respect to the class $\mathcal{C}$.

Theorem 3. *Let $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ be a regular and consistent estimate of $\boldsymbol{\theta}^0$. Then,*

(i) *in order for $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ to be* RBAN, *it is sufficient that the matrix*

$$\mathbf{A}_0 = \left.\frac{\partial\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)}{\partial\check{\mathbf{Z}}_m}\right|_{\check{\mathbf{Z}}_m=\boldsymbol{\zeta}^0}$$

satisfy the condition

$$\mathbf{A}_0 = \mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}, \tag{6}$$

where $\mathbf{C}_0 = \mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\mathbf{V}_0$ and

$$\mathbf{V}_0 = \left.\frac{\partial\boldsymbol{\zeta}(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^0};$$

(ii) *the corresponding "minimal" asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ is given by*

$$\boldsymbol{\sigma}_{\hat{\boldsymbol{\theta}}} = m^{-1}\mathbf{C}_0^{-1}; \tag{7}$$

(iii) *condition* (6) *is also necessary if there exists a regular and consistent estimate having the asymptotic covariance matrix $m^{-1}\mathbf{C}_0^{-1}$.*

Proof. If (6) is true, then

$$\begin{aligned}\mathbf{A}_0\boldsymbol{\sigma}^*\mathbf{A}_0' &= (\mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1})\boldsymbol{\sigma}^*(\mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1})' \\ &= \mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\boldsymbol{\sigma}^*\boldsymbol{\sigma}^{*-1}\mathbf{V}_0\mathbf{C}_0^{-1} \\ &= \mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\mathbf{V}_0\mathbf{C}_0^{-1} = \mathbf{C}_0^{-1}.\end{aligned}$$

Thus, the second part of the theorem is an immediate consequence of the first part.

The proof of the first part rests upon the fact that for any regular and consistent estimate $\hat{\boldsymbol{\theta}}$, the equation $\hat{\boldsymbol{\theta}}(\boldsymbol{\zeta}^0) = \boldsymbol{\theta}^0$ holds with a probability tending to one, and this implies that

$$\left.\frac{\partial\hat{\boldsymbol{\theta}}[\boldsymbol{\zeta}(\boldsymbol{\theta})]}{\partial\boldsymbol{\zeta}}\right|_{\boldsymbol{\zeta}^0}\left.\frac{\partial\boldsymbol{\zeta}(\boldsymbol{\theta})}{\partial(\boldsymbol{\theta})}\right|_{\boldsymbol{\theta}^0} = \mathbf{I}_s,$$

which can be rewritten as

$$\mathbf{A}_0\mathbf{V}_0 = \mathbf{I}_s. \tag{8}$$

The asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$, by Theorem 2, has the form,

$$\boldsymbol{\sigma}_{\hat{\boldsymbol{\theta}}} = m^{-1}\mathbf{A}_0\boldsymbol{\sigma}^*\mathbf{A}_0'. \tag{9}$$

To minimize $\mathbf{A}_0\mathfrak{d}^*\mathbf{A}_0'$ subject to condition (8), introduce the Lagrange multiplier

$$\alpha = ||\alpha_{kh}||, \qquad k, h = 1, 2, \cdots, s,$$

differentiate

$$\mathbf{A}_0\mathfrak{d}^*\mathbf{A}_0' - 2\alpha(\mathbf{V}_0'\mathbf{A}_0' - \mathbf{I}_s)$$

with respect to $\mathbf{A}_0$, set the derivative equal to zero to get the equation

$$\mathbf{A}_0\mathfrak{d}^* - \alpha\mathbf{V}_0' = 0, \tag{10}$$

and solve (10) for $\mathbf{A}_0$,

$$\mathbf{A}_0 = \alpha\mathbf{V}_0'\mathfrak{d}^{*-1}. \tag{11}$$

Substituting (11) in (8) gives

$$\alpha\mathbf{V}_0'\mathfrak{d}^{*-1}\mathbf{V}_0 = \mathbf{I}_s,$$

i.e., $\alpha\mathbf{C}_0 = \mathbf{I}_s$, and hence

$$\alpha = \mathbf{C}_0^{-1}. \tag{12}$$

It follows from equations (11) and (12) that

$$\mathbf{A}_0 = \mathbf{C}_0^{-1}\mathbf{V}_0'\mathfrak{d}^{*-1}. \tag{6}$$

It is easy to verify that a regular and consistent estimate satisfying equation (6) will have the property of bestness as defined in Definition 5. Suppose that $\tilde{\mathbf{\theta}}(\check{\mathbf{Z}}_m)$ is any regular and consistent estimate of $\mathbf{\theta}$ at $\mathbf{\theta}^0$, and let $\tilde{\mathbf{A}}_0$ be the corresponding derivative taken at ζ^0; then $\tilde{\mathbf{A}}_0$ also satisfies equation (8). If $\tilde{\mathbf{A}}_0$ does not satisfy condition (6) but $\mathbf{A}_0$ does, then the difference $(\tilde{\mathbf{A}}_0 - \mathbf{A}_0)\mathfrak{d}^*(\tilde{\mathbf{A}}_0 - \mathbf{A}_0)'$ is positive semidefinite. Since

$$\begin{aligned}(\tilde{\mathbf{A}}_0 - \mathbf{A}_0)\mathfrak{d}^*(\tilde{\mathbf{A}}_0 - \mathbf{A}_0)' &= (\tilde{\mathbf{A}}_0 - \mathbf{C}_0^{-1}\mathbf{V}_0'\mathfrak{d}^{*-1})\mathfrak{d}^*(\tilde{\mathbf{A}}_0 - \mathbf{C}_0^{-1}\mathbf{V}_0'\mathfrak{d}^{*-1})' \\ &= \tilde{\mathbf{A}}_0\mathfrak{d}^*\tilde{\mathbf{A}}_0' - \mathbf{C}_0^{-1}\mathbf{V}_0'\mathfrak{d}^{*-1}\mathfrak{d}^*\tilde{\mathbf{A}}_0' - \tilde{\mathbf{A}}_0\mathfrak{d}^*\mathfrak{d}^{*-1}\mathbf{V}_0\mathbf{C}_0^{-1} \\ &\quad + \mathbf{C}_0^{-1}\mathbf{V}_0'\mathfrak{d}^{*-1}\mathfrak{d}^*\mathfrak{d}^{*-1}\mathbf{V}_0\mathbf{C}_0^{-1} \\ &= \tilde{\mathbf{A}}_0\mathfrak{d}^*\tilde{\mathbf{A}}_0' - \mathbf{C}_0^{-1}\mathbf{I}_s - \mathbf{I}_s\mathbf{C}_0^{-1} + \mathbf{C}_0^{-1} \\ &= \tilde{\mathbf{A}}_0\mathfrak{d}^*\tilde{\mathbf{A}}_0' - \mathbf{A}_0\mathfrak{d}^*\mathbf{A}_0',\end{aligned}$$

the difference $\tilde{\mathbf{A}}_0\mathfrak{d}^*\tilde{\mathbf{A}}_0' - \mathbf{A}_0\mathfrak{d}^*\mathbf{A}_0'$ is also positive semidefinite. The result follows from Definitions 4 and 5.

To prove part (iii), suppose that there exists a regular and consistent estimate $\hat{\mathbf{\theta}}$ whose derivative taken at ζ^0, $\mathbf{A}_0$, satisfies condition (6). Let $\tilde{\mathbf{\theta}}$ be any other regular and consistent estimate of $\mathbf{\theta}$ at $\mathbf{\theta}^0$ and let $\tilde{\mathbf{A}}_0$ be its derivative taken at ζ^0. In order that the asymptotic covariance matrix $\mathfrak{d}_{\tilde{\theta}} = m^{-1}\tilde{\mathbf{A}}_0\mathfrak{d}^*\tilde{\mathbf{A}}_0'$ of $\tilde{\mathbf{\theta}}$ be minimal, it is obviously necessary that

$$\tilde{\mathbf{A}}_0\mathfrak{d}^*\tilde{\mathbf{A}}_0' = \mathbf{A}_0\mathfrak{d}^*\mathbf{A}_0',$$

which implies that the equation

$$(\tilde{\mathbf{A}}_0 - \mathbf{A}_0)\boldsymbol{\sigma}^*(\tilde{\mathbf{A}}_0 - \mathbf{A}_0)' = \mathbf{0}. \tag{13}$$

Equation (13) holds only if $\tilde{\mathbf{A}}_0 - \mathbf{A}_0 = \mathbf{0}$, i.e.,

$$\tilde{\mathbf{A}}_0 = \mathbf{A}_0 = \mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1},$$

proving part (iii).

It will be shown in the next section that there exist estimates of $\boldsymbol{\theta}$, regular and consistent, whose asymptotic covariance matrix is equal to $m^{-1}\mathbf{C}_0^{-1}$.

REMARK. When a regular and consistent estimate $\hat{\boldsymbol{\theta}}$ is a function of both $\check{\mathbf{Z}}_m$ and $\mathbf{S}_m$, i.e., when $\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$, a sufficient condition imposed on the derivatives of the estimate with respect to $\check{\mathbf{Z}}_m$ and $\mathbf{S}_m$ taken at $\boldsymbol{\zeta}^0$ and $\boldsymbol{\sigma}^*$, i.e., that the estimate have a "minimal" asymptotic covariance matrix, can be deduced by a similar approach. The condition so obtained will be similar to, though not the same as, (6). Both the condition and the corresponding "minimal" asymptotic covariance matrix of the estimate will involve the covariance matrix between $\check{\mathbf{Z}}_m$ and $\mathbf{S}_m$. If the covariance matrix between $\check{\mathbf{Z}}_m$ and $\mathbf{S}_m$ is unknown, it is tedious to obtain estimates having such a "minimal" asymptotic covariance matrix. On the other hand, a condition imposed on the derivative of $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ with respect to $\check{\mathbf{Z}}_m$ taken at $\boldsymbol{\zeta}^0$ and $\boldsymbol{\sigma}^*$ will be the same as (6), and the corresponding asymptotic covariance matrix of the estimate will be equal to $m^{-1}\mathbf{C}_0^{-1}$, *if* the variation of $\mathbf{S}_m$ is neglected. Such negligence is, in a way, not desirable. The essential purpose of this study, however, is not only to deduce a necessary and sufficient condition for an estimate to have "minimal" asymptotic covariance matrix, but also to generate estimates satisfying such a condition or having such a "minimal" asymptotic covariance matrix. Therefore, it seems to be justified to content oneself with condition (6) and with estimates having the asymptotic covariance matrix given by equation (7) (see Section 4).

COROLLARY 1. *Let θ_k be the* kth *element of the vector $\boldsymbol{\theta}$ and $\hat{\theta}_k(\check{\mathbf{Z}}_m)$ be a regular and consistent estimate of θ_k at θ_k^0, for $k = 1, 2, \cdots, s$. Then,*

(i) *in order for $\hat{\theta}_k(\bar{Z}_m)$ to be* RBAN, *for every $k = 1, 2, \cdots, s$, it is sufficient that the matrix $\mathbf{A}_0$ satisfy the condition*

$$\mathbf{A}_0 = \mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}; \tag{6}$$

(ii) *the minimum asymptotic variance of $\hat{\theta}_k(\bar{Z}_m)$ is given by*

$$\delta^2_{\hat{\theta}_k} = m^{-1}\boldsymbol{\varepsilon}_k\mathbf{C}_0^{-1}\boldsymbol{\varepsilon}_k', \tag{14}$$

where $\boldsymbol{\varepsilon}_k = (0, \cdots, 0, 1, 0, \cdots, 0)$ is an $1 \times s$ row vector with the elements zero except the kth *element, which is unity.*

PROOF. In order to prove (i), it is adequate to show that for every $k = 1, 2, \cdots, s$, a sufficient condition for $\hat{\theta}_k(\check{\mathbf{Z}}_m)$ to have minimum asymptotic variance is implied in (6). Since $\hat{\theta}_k$ is regular and consistent, the corollary to

Theorem 2 gives

$$\hat{\theta}_k(\zeta^0) = \theta_k^0 \tag{15}$$

and the asymptotic variance of $\hat{\theta}_k$,

$$\delta_{\hat{\theta}_k}^2 = m^{-1}\mathbf{A}_k\boldsymbol{\delta}^*\mathbf{A}_k',$$

with

$$\mathbf{A}_k = \frac{\partial \hat{\theta}_k}{\partial \check{\mathbf{Z}}_m}\bigg|_{\zeta^0}.$$

By using a similar approach employed in the proof of Theorem 3, minimization of $\mathbf{A}_k\boldsymbol{\delta}^*\mathbf{A}_k'$ leads to the equation

$$\mathbf{A}_k = \boldsymbol{\varepsilon}_k\mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\delta}^{*-1}. \tag{16}$$

Thus, equation (16) is a sufficient condition for $\hat{\theta}_k$ to have the minimum asymptotic variance. Because $\boldsymbol{\varepsilon}_k\mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\delta}^{*-1}$ is the kth row of the matrix $\mathbf{C}_0^{-1}\mathbf{V}_0'\boldsymbol{\delta}^{*-1}$, (16) is the same condition which was imposed on the kth row of matrix $\mathbf{A}_0$. This means that equation (6) implies equation (16), for every $k = 1, 2, \cdots, s$; thus proving part (i). It is of significance to note that whereas condition (6) implies the entire set of s equations (16) for $k = 1, 2, \cdots, s$, the entire set of the s equations (16) also implies equation (6).

Part (ii) of the corollary can be shown by substituting (16) in the expression $\mathbf{A}_k\boldsymbol{\delta}^*\mathbf{A}_k'$. Simple computation gives $\boldsymbol{\varepsilon}_k\mathbf{C}_0^{-1}\boldsymbol{\varepsilon}_k'$. Equation (14) follows. The right-side member of (14) is identically equal to the kth diagonal element of $m^{-1}\mathbf{C}_0^{-1}$, the asymptotic convariance matrix of $\hat{\boldsymbol{\theta}}$.

The significance of this corollary is that when all of the components of the vector $\boldsymbol{\theta}$ are estimated simultaneously, each of the individual estimates will have the minimum asymptotic variance.

COROLLARY 2. *Let $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ be a regular and consistent estimate of $\boldsymbol{\theta}$ at $\boldsymbol{\theta}^0$. Suppose that the components of the random vector $\check{\mathbf{Z}}_m$ are statistically independent. Then*

(i) *in order for $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ to be* RBAN, *it is sufficient that the matrix $\mathbf{A}_0$ satisfy the condition*

$$\mathbf{A}_0 = \mathbf{G}_0^{-1}\mathbf{V}_0'\mathbf{D}^{*-1},$$

where $\mathbf{D}^*$ is the limit of the diagonal matrix $\mathbf{D}_m^*$ with diagonal elements

$$\frac{1}{m}\sum_{\alpha=1}^{m}\sigma_{Z_{\alpha i}}^2, \qquad \textit{for } i = 1, 2, \cdots, n,$$

and $\mathbf{G}_0 = \mathbf{V}_0'\mathbf{D}^{*-1}\mathbf{V}_0$;

(ii) *the "minimal" asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ is given by*

$$\boldsymbol{\delta}_{\hat{\boldsymbol{\theta}}} = m^{-1}\mathbf{G}_0^{-1}. \tag{17}$$

The corollary is a direct consequence of the substitution of $\sigma_{z_{\alpha i} z_{\alpha i}} = 0$ for $i \neq j; i, j = 1, 2, \cdots, n$ in Theorem 3.

COROLLARY 3. *Let $\hat{\theta}_k(\check{\mathbf{Z}}_m)$ be a regular and consistent estimate of θ_k^0. Suppose that the elements of the random vector $\check{\mathbf{Z}}_m$ are statistically independent. Then,*

(i) *in order for $\hat{\theta}_k(\check{\mathbf{Z}}_m)$ to be* RBAN, *it is sufficient that the vector $\mathbf{A}_k$ satisfy the condition*

$$\mathbf{A}_k = \boldsymbol{\varepsilon}_k \mathbf{G}_0^{-1} \mathbf{V}_0' \mathbf{D}^{*-1},$$

where $\mathbf{D}^*$ and $\mathbf{G}_0$ are defined as in Corollary 2;

(ii) *the minimum asymptotic variance of $\hat{\theta}_k$ is given by*

$$\sigma_{\hat{\theta}_k}^2 = m^{-1} \boldsymbol{\varepsilon}_k \mathbf{G}_0^{-1} \boldsymbol{\varepsilon}_k'.$$

The corollary is a direct consequence of the substitution $\sigma_{z_{\alpha i} z_{\alpha i}} = 0$, for $i \neq j; i, j = 1, 2, \cdots, n$, in Corollary 1.

It is clear that an estimate having the minimal asymptotic covariance matrix in the sense of Definition 5 has a remarkable property of bestness, at least asymptotically. To make this point more apparent, the following general theorem is given.

THEOREM 4. *Let $\hat{\boldsymbol{\theta}}$ be an* RBAN *estimate of $\boldsymbol{\theta}$ for values of $\boldsymbol{\theta}$ in a neighborhood ν of $\boldsymbol{\theta}^0$. Let $\mathbf{f}(\boldsymbol{\theta})$ be a function of $\boldsymbol{\theta}$ with its range in a Euclidean space.*

(i) *If $\mathbf{f}(\boldsymbol{\theta})$ admits continuous partial derivatives $\boldsymbol{\phi}$ in the neighborhood of $\boldsymbol{\theta}^0$, then $\mathbf{f}(\hat{\boldsymbol{\theta}})$ is an* RBAN *estimate of $\mathbf{f}(\boldsymbol{\theta})$ for $\boldsymbol{\theta} \varepsilon \nu$.*

(ii) *If the matrix*

$$\boldsymbol{\phi} = \frac{\partial f}{\partial \boldsymbol{\theta}} \bigg|_{\boldsymbol{\theta}^0}$$

has rank s, then in order for $\mathbf{f}(\tilde{\boldsymbol{\theta}})$ to be an RBAN *estimate of $\mathbf{f}(\boldsymbol{\theta})$, it is necessary and sufficient that the argument $\tilde{\boldsymbol{\theta}}$ be an* RBAN *estimate of $\boldsymbol{\theta}$.*

PROOF. Let $\mathbf{h}(\check{\mathbf{Z}}_m)$ be any other regular and consistent estimate of $\mathbf{f}$, and let

$$\mathbf{H} = \frac{\partial \mathbf{h}(\check{\mathbf{Z}}_m)}{\partial \check{\mathbf{Z}}_m} \bigg|_{\boldsymbol{\zeta}^0}.$$

As in the proof of Theorem 3, we can show that $\mathbf{H}$ must satisfy the relationship

$$\mathbf{H}\mathbf{V}_0 = \boldsymbol{\phi}, \tag{18}$$

where $\mathbf{V}_0$ is the derivative of $\boldsymbol{\zeta}(\boldsymbol{\theta})$ taken at $\boldsymbol{\theta} = \boldsymbol{\theta}^0$, as defined in Theorem 3.

Since $\hat{\boldsymbol{\theta}}$ is RBAN, the limiting distribution of $\sqrt{m}\,\{\mathbf{f}[\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)] - \mathbf{f}(\boldsymbol{\theta}^0)\}$ is normal with a mean zero and a covariance matrix $\boldsymbol{\phi}\mathbf{A}_0\boldsymbol{\sigma}^*\mathbf{A}_0'\boldsymbol{\phi}'$. Similarly, the limiting distribution of $\sqrt{m}\,[\mathbf{h}(\check{\mathbf{Z}}_m) - \mathbf{f}(\boldsymbol{\theta}^0)]$ is normal with a mean zero and a covariance matrix $\mathbf{H}\boldsymbol{\sigma}^*\mathbf{H}'$. To show that $\mathbf{f}(\hat{\boldsymbol{\theta}})$ is best, it is sufficient to show that if $\mathbf{H}$ satisfies (18), then the difference $\mathbf{H}\boldsymbol{\sigma}^*\mathbf{H}' - \boldsymbol{\phi}\mathbf{A}_0\boldsymbol{\sigma}^*\mathbf{A}_0'\boldsymbol{\phi}'$ is positive semidefinite. Let

$$\begin{aligned} \boldsymbol{\pi} &= (\mathbf{H} - \boldsymbol{\phi}\mathbf{A}_0)\boldsymbol{\sigma}^*(\mathbf{H} - \boldsymbol{\phi}\mathbf{A}_0)' \\ &= \mathbf{H}\boldsymbol{\sigma}^*\mathbf{H}' - \boldsymbol{\phi}\mathbf{A}_0\boldsymbol{\sigma}^*\mathbf{H}' - \mathbf{H}\boldsymbol{\sigma}^*\mathbf{A}_0'\boldsymbol{\phi}' + \boldsymbol{\phi}\mathbf{A}_0\boldsymbol{\sigma}^*\mathbf{A}_0'\boldsymbol{\phi}'. \end{aligned}$$

Replacing $\mathbf{A}_0$ by $\mathbf{C}_0^{-1}\mathbf{V}_0'\mathbf{d}^{*-1}$,

$$\boldsymbol{\pi} = \mathbf{H}\mathbf{d}^*\mathbf{H}' - \boldsymbol{\phi}\mathbf{C}_0^{-1}\mathbf{V}_0'\mathbf{H}' - \mathbf{H}\mathbf{V}_0\mathbf{C}_0^{-1}\boldsymbol{\phi}' + \boldsymbol{\phi}\mathbf{C}_0^{-1}\boldsymbol{\phi}',$$

or, owing to (18),

$$\boldsymbol{\pi} = \mathbf{H}\mathbf{d}^*\mathbf{H}' - \boldsymbol{\phi}\mathbf{C}_0^{-1}\boldsymbol{\phi}' = \mathbf{H}\mathbf{d}^*\mathbf{H}' - \boldsymbol{\phi}\mathbf{A}_0\mathbf{d}^*\mathbf{A}_0'\boldsymbol{\phi}'.$$

Since $\boldsymbol{\pi}$ is obviously positive semidefinite, part (i) is proved.

To prove part (ii), let

$$\tilde{\mathbf{A}}_0 = \frac{\partial\tilde{\boldsymbol{\theta}}}{\partial\check{\mathbf{Z}}_m}\bigg|_{\zeta^0},$$

and consider the matrix $(\boldsymbol{\phi}\tilde{\mathbf{A}}_0 - \boldsymbol{\phi}\mathbf{A}_0)\mathbf{d}^*(\boldsymbol{\phi}\tilde{\mathbf{A}}_0 - \boldsymbol{\phi}\mathbf{A}_0)'$. Using the relations $\mathbf{A}_0\mathbf{d}^* = \mathbf{C}_0^{-1}\mathbf{V}_0'$ and $\mathbf{A}_0\mathbf{V}_0 = \mathbf{I}_s = \tilde{\mathbf{A}}_0\mathbf{V}_0$, simple computation leads to the equation

$$(\boldsymbol{\phi}\tilde{\mathbf{A}}_0 - \boldsymbol{\phi}\mathbf{A}_0)\mathbf{d}^*(\boldsymbol{\phi}\tilde{\mathbf{A}}_0 - \boldsymbol{\phi}\mathbf{A}_0)' = \boldsymbol{\phi}\tilde{\mathbf{A}}_0\mathbf{d}^*\tilde{\mathbf{A}}_0'\boldsymbol{\phi}' - \boldsymbol{\phi}\mathbf{A}_0\mathbf{d}^*\mathbf{A}_0'\boldsymbol{\phi}'.$$

The last difference, then, is positive semidefinite unless $\boldsymbol{\phi}\tilde{\mathbf{A}}_0 = \boldsymbol{\phi}\mathbf{A}_0$. Since $\boldsymbol{\phi}$ has rank s, this implies that $\tilde{\mathbf{A}}_0 = \mathbf{A}_0$, hence the necessity of the condition. The sufficiency follows from part (i).

Corollary. *Let the random vector* $(\hat{\theta}_1(\bar{Z}_m), \cdots, \hat{\theta}_r(\bar{Z}_m))'$ *be a regular and consistent estimate of* $(\theta_1^0, \cdots, \theta_r^0)'$, *for* $r \leqq s$. *Then,*

(i) *in order for the random vector to have "minimal" asymptotic covariance matrix, or for the elements* $\hat{\theta}_k(\bar{Z}_m)$, *for* $k = 1, 2, \cdots, r$, *to have the respective minimum asymptotic variances, it is sufficient that*

$$\begin{bmatrix} \mathbf{A}_1 \\ \vdots \\ \mathbf{A}_r \end{bmatrix} = \boldsymbol{\delta}_r\mathbf{C}_0^{-1}\mathbf{V}_0'\sigma^{*-1},$$

with

$$\boldsymbol{\delta}_r = \begin{bmatrix} 1 & 0 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 1 & \cdot & \cdot & 0 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 1 & 0 & \cdot & \cdot & 0 \end{bmatrix}$$

being an $r \times s$ *matrix;*

(ii) *the minimal asymptotic covariance matrix of the random vector is*

$$m^{-1}\boldsymbol{\delta}_r\mathbf{C}_0^{-1}\boldsymbol{\delta}_r'.$$

The proof of the corollary is obvious.

Random vectors $\mathbf{Z}_\alpha$ considered in this paper are assumed to be independent but not necessarily identical. If identical distribution is assumed, then Assumption 2 is no longer necessary, and Assumptions 1 and 3 may be replaced, respectively, by Assumptions 1′ and 3′.

ASSUMPTION 1′. The second central moments

$$E\{(Z_{\alpha i} - \zeta_i^0)(Z_{\alpha j} - \zeta_j^0) \mid \boldsymbol{\theta} = \boldsymbol{\theta}^0\},$$

for $\alpha = 1, 2, \cdots, m$ and for $i, j = 1, 2, \cdots, n$, are finite.

ASSUMPTION 3′. Let $\boldsymbol{\theta}_k = (\theta_1, \theta_2, \cdots, \theta_k)'$, for $1 \leqq k \leqq s$. The expectation $E(\mathbf{Z}_\alpha) = \boldsymbol{\zeta}(\boldsymbol{\theta})$ has continuous second partial derivatives with respect to $\boldsymbol{\theta}_k$, and the matrix $\mathbf{V}_k(\theta) = \partial\boldsymbol{\zeta}(\boldsymbol{\theta}) / \partial\boldsymbol{\theta}_k$ has rank k in the neighborhood of the true point $\boldsymbol{\theta}^0$, for every k, $1 \leqq k \leqq s$.

In this case the strong law of large numbers can be used and the results will be stronger—the vector $\check{\mathbf{Z}}_m$ tends almost surely to the expectation $\boldsymbol{\zeta}(\boldsymbol{\theta}^0)$, the estimate $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ tends almost surely to the true value $\boldsymbol{\theta}^0$, and all the other probabilistic statements will be stronger.

4. Methods of generating RBAN estimates. In this section methods are given by which RBAN estimates can be obtained. Application of the first method (Theorem 5) requires the knowledge of the matrix $\boldsymbol{\sigma}^*$. In the second method (Theorem 6), such knowledge is not assumed.

THEOREM 5. *Let a quadratic form* $Q(\check{\mathbf{Z}}_m, \boldsymbol{\theta})$ *be defined by*

$$Q(\check{\mathbf{Z}}_m, \boldsymbol{\theta}) = [\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta})]'\boldsymbol{\sigma}^{*-1}(\boldsymbol{\theta})[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta})], \tag{19}$$

where $\boldsymbol{\sigma}^*(\boldsymbol{\theta})$ *is assumed to have continuous second partial derivatives with respect to* $\boldsymbol{\theta}$ *in the neighborhood of the true parameter point* $\boldsymbol{\theta}^0$. *Let Assumptions* 1 *to* 3 *be satisfied. Then,*

(i) *As* $m \to \infty$, *there exists, with a probability tending to one, one and only one function* $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ *which locally minimizes the quadratic form* $Q(\check{\mathbf{Z}}_m, \boldsymbol{\theta})$;

(ii) *The function* $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ *is a consistent estimate of* $\boldsymbol{\theta}^0$;

(iii) $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ *is regular in the sense of Definition* 3;

(iv) $\sqrt{m}\,[\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m) - \boldsymbol{\theta}^0]$ *has an s-variate asymptotically normal distribution; and*

(v) $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ *has the asymptotic covariance matrix* $m^{-1}\mathbf{C}_0^{-1}$, *with* $\mathbf{C}_0 = \mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\mathbf{V}_0$.

Thus $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m)$ is an RBAN estimate.

Because the proofs of Theorems 5 and 6 are analogous, only the proof of Theorem 6 is given.

THEOREM 6. *Let* $\mathbf{S}_m$ *be a consistent estimate of* $\boldsymbol{\sigma}^*$ *at* $\boldsymbol{\theta} = \boldsymbol{\theta}^0$, *and let a quadratic form* $Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta})$ *be defined by*

$$Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta}) = [\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta})]'\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta})]. \tag{20}$$

Let Assumptions 1 *to* 3 *be satisfied. Then,*

(i) *As* $m \to \infty$, *there exists, with a probability tending to one, one and only one function* $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ *which locally minimizes the quadratic form* $Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta})$;

(ii) *The function* $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ *is a consistent estimate of* $\boldsymbol{\theta}^0$;

(iii) $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ *is regular in the sense of Definition* 3;

(iv) $\sqrt{m}\,[\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m) - \boldsymbol{\theta}^0]$ *has an s-variate asymptotically normal distribution; and*

(v) $\hat{\boldsymbol\theta}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ *has the asymptotic covariance matrix* $m^{-1}\mathbf{C}_0^{-1}$, *with* $\mathbf{C}_0 = \mathbf{V}_0'\boldsymbol\sigma^{*-1}\mathbf{V}_0$.
Thus, $\hat{\boldsymbol\theta}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ is an RBAN estimate.

In writing quadratic form (20), it is assumed that for every $m = 2, 3, \cdots$, the matrix $\mathbf{S}_m$ is positive definite with a probability one.

PROOF. Differentiation of the quadratic form with respect to $\boldsymbol\theta$ leads to the equation

$$\mathbf{W}(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol\theta) = \mathbf{V}'(\boldsymbol\theta)\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol\zeta(\boldsymbol\theta)] = 0, \tag{21}$$

with $\mathbf{V}(\boldsymbol\theta) = \partial\boldsymbol\zeta(\boldsymbol\theta) / \partial\boldsymbol\theta$. Clearly, the derivative of Q with respect to $\boldsymbol\theta$ is equal to $-2\mathbf{W}$. In order to prove part (i) of Theorem 6, we have to show that as $m \to \infty$, equation (21) has, with a probability tending to one, a root, $\hat{\boldsymbol\theta}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$, in the neighborhood of the true parameter point $\boldsymbol\theta^0$, however small is the neighborhood, and that at the point $\hat{\boldsymbol\theta}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$, the quadratic form attains a minimum.

Equation (21) is satisfied for $\check{\mathbf{Z}}_m = \boldsymbol\zeta^0$, $\mathbf{S}_m = \boldsymbol\sigma^*$, and $\boldsymbol\theta = \boldsymbol\theta^0$, since $\boldsymbol\zeta^0$ is written for $\boldsymbol\zeta(\boldsymbol\theta^0)$. Clearly the function $\mathbf{W}(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol\theta)$ possesses continuous first partial derivatives with respect to $\check{\mathbf{Z}}_m$ and $\mathbf{S}_m$. Assumption 3 on the differentiability of the function $\boldsymbol\zeta(\boldsymbol\theta)$ implies that $\mathbf{W}(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol\theta)$ possesses also continuous first partial derivatives with respect to $\boldsymbol\theta$. The derivatives taken at the point $(\boldsymbol\zeta^0, \boldsymbol\sigma^*, \boldsymbol\theta^0)$ are

$$\begin{aligned}\frac{\partial\mathbf{W}}{\partial\boldsymbol\theta}\bigg|_{\boldsymbol\zeta^0,\boldsymbol\delta^*,\boldsymbol\theta^0} &= \frac{\partial\mathbf{V}'(\boldsymbol\theta)}{\partial\boldsymbol\theta}\,\mathbf{S}_m^{-1}\,[\check{\mathbf{Z}}_m - \boldsymbol\zeta(\boldsymbol\theta)]\bigg|_{\boldsymbol\zeta^0,\boldsymbol\delta^*,\boldsymbol\theta^0} - \mathbf{V}'(\boldsymbol\theta)\mathbf{S}_m^{-1}\mathbf{V}(\boldsymbol\theta)\bigg|_{\boldsymbol\zeta^0,\boldsymbol\delta^*,\boldsymbol\theta^0}\\ &= -\,\mathbf{V}_0'\boldsymbol\sigma^{*-1}\mathbf{V}_0,\end{aligned}$$

where $\mathbf{V}_0 = \mathbf{V}(\boldsymbol\theta^0)$ is of rank s and $\boldsymbol\sigma^*$ is positive definite; hence, $\mathbf{V}_0'\boldsymbol\sigma^{*-1}\mathbf{V}_0$ is positive definite. It follows from the implicit function theorem ([6], p. 117) that (a) there exists a region R containing $(\boldsymbol\zeta^0, \boldsymbol\sigma^*)$ and a rectangular parallelepiped (θ^*, θ^{**}) containing $\boldsymbol\theta^0$ such that for every point, $(\mathbf{Z}, \mathbf{S})$, say, inside the region R, equation (21) holds for one and only one point $\boldsymbol\theta = \hat{\boldsymbol\theta}(\mathbf{Z}, \mathbf{S})$ inside the parallelepiped (θ^*, θ^{**}); (b) the Jacobian $|\partial\mathbf{W} / \partial\boldsymbol\theta|$ taken inside (θ^*, θ^{**}) will have a constant sign; (c) the function $\hat{\boldsymbol\theta}(\mathbf{Z}, \mathbf{S})$ is a continuous function and possesses continuous partial derivatives with respect to $\mathbf{Z}$ and $\mathbf{S}$; and (d) substitutions of $\mathbf{Z} = \boldsymbol\zeta^0$ and $\mathbf{S} = \boldsymbol\sigma^*$ into $\hat{\boldsymbol\theta}(\mathbf{Z}, \mathbf{S})$ lead to the equation $\hat{\boldsymbol\theta}(\boldsymbol\zeta^0, \boldsymbol\sigma^*) = \boldsymbol\theta^0$, the true parameter point.

Because of the convergence of $\check{\mathbf{Z}}_m$ to $\boldsymbol\zeta^0$ and $\mathbf{S}_m$ to $\boldsymbol\sigma^*$, as $m \to \infty$, with a probability tending to one, the point $(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ will be inside the region R, however small is R, and thus the equation (21) will hold for one and only one point $\hat{\boldsymbol\theta}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ inside (θ^*, θ^{**}).

It may be convenient to point out here that (c) and (d) imply the consistency of the estimate $\hat{\boldsymbol\theta}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$.

To show that the quadratic form $Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol\theta)$ attains a minimum at the point $\hat{\boldsymbol\theta}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$, we let $\boldsymbol\theta_k = (\theta_1, \theta_2, \cdots, \theta_k)'$, for $1 \leqq k \leqq s$, and let $\mathbf{V}_k(\boldsymbol\theta) = \partial\boldsymbol\zeta(\boldsymbol\theta) / \partial\boldsymbol\theta_k$, which is of rank k. For $k = s$, $\boldsymbol\theta_k = \boldsymbol\theta$ and $\mathbf{V}_k(\boldsymbol\theta) = \mathbf{V}(\boldsymbol\theta)$. The second partial derivatives of $Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}_k$ are

$$\frac{\partial^2 Q}{\partial \boldsymbol{\theta}_k^2} = -2\,\frac{\partial \mathbf{V}_k'(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}_k}\,\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta})] + 2\mathbf{V}_k'(\boldsymbol{\theta})\mathbf{S}_m^{-1}\mathbf{V}_k(\boldsymbol{\theta}). \tag{22}$$

The first term on the right-hand side of (22) tends in probability to zero, and the second term is positive definite in the neighborhood of $\boldsymbol{\theta}^0$, for every $1 \leqq k \leqq s$. Because of the convergence of $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, S_m)$ to $\boldsymbol{\theta}^0$, this means that the matrix $\mathbf{V}'(\boldsymbol{\theta})\mathbf{S}_m^{-1}\mathbf{V}(\boldsymbol{\theta})$ is positive definite for $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ and that all of the principal minors of the matrix are positive. It follows that $Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta})$ is a minimum at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ ([7], pp. 51–52).

Regularity of the estimate $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ is implied in (c).

Since $\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$ is a regular and consistent estimate of $\boldsymbol{\theta}^0$, it follows from Theorem 2 that $\sqrt{m}\,[\hat{\boldsymbol{\theta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m) - \boldsymbol{\theta}^0]$ is asymptotically normal, proving part (iv).

We now write

$$\begin{aligned}\mathbf{W}(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta}^0) - \mathbf{W}(\check{\mathbf{Z}}_m, \mathbf{S}_m, \hat{\boldsymbol{\theta}}) &= \left\{\frac{\partial \mathbf{W}}{\partial \boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\right\}(\boldsymbol{\theta}^0 - \hat{\boldsymbol{\theta}}) \\ &= \left\{\left(\frac{\partial \mathbf{V}'}{\partial \boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\right)\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta}^*)] - \mathbf{V}'(\boldsymbol{\theta}^*)\mathbf{S}_m^{-1}\mathbf{V}(\boldsymbol{\theta}^*)\right\}(\boldsymbol{\theta}^0 - \hat{\boldsymbol{\theta}}),\end{aligned}$$

where $\boldsymbol{\theta}^* = \boldsymbol{\theta}^0 + \boldsymbol{\delta}_m(\boldsymbol{\theta}^0 - \hat{\boldsymbol{\theta}})$, with $\boldsymbol{\delta}_m$ being an $s \times s$ diagonal matrix having all diagonal elements between zero and unity. Transposing the derivative of $\mathbf{W}$ to the other side of the equality sign gives

$$\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0 = \left\{\mathbf{V}'(\boldsymbol{\theta}^*)\mathbf{S}_m^{-1}\mathbf{V}(\boldsymbol{\theta}^*) - \left(\frac{\partial \mathbf{V}'}{\partial \boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\right)\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta}^*)]\right\}^{-1}\mathbf{V}_0'\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0],$$

since $\mathbf{W}(\check{\mathbf{Z}}_m, \mathbf{S}_m, \hat{\boldsymbol{\theta}}) = 0$ and $\mathbf{W}(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta}^0) = \mathbf{V}_0'\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0]$. By Lemma 1, $\sqrt{m}\,(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^0)$ has the same limiting distribution as

$$\sqrt{m}\left\{\mathbf{V}'(\boldsymbol{\theta}^*)\mathbf{S}_m^{-1}\mathbf{V}(\boldsymbol{\theta}^*) - \left(\frac{\partial \mathbf{V}'}{\partial \boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\right)\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta}^*)]\right\}^{-1}\mathbf{V}_0'\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0],$$

or as

$$\sqrt{m}\,\{\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\mathbf{V}_0\}^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}^0], \tag{23}$$

since $\check{\mathbf{Z}}_m \to \boldsymbol{\zeta}^0$, $\mathbf{S}_m \to \boldsymbol{\sigma}^*$, $\boldsymbol{\theta}^* \to \boldsymbol{\theta}^0$, and

$$\left\{\mathbf{V}'(\boldsymbol{\theta}^*)\mathbf{S}_m^{-1}\mathbf{V}(\boldsymbol{\theta}^*) - \left(\frac{\partial \mathbf{V}'}{\partial \boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^*}\right)\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta}^*)]\right\} \to \mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\mathbf{V}_0.$$

According to Theorem 1, the quantity (23) is asymptotically normal with a mean zero and with an asymptotic covariance matrix

$$[\{\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\mathbf{V}_0\}^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}]\boldsymbol{\sigma}^*[\{\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\mathbf{V}_0\}^{-1}\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}]' = \{\mathbf{V}_0'\boldsymbol{\sigma}^{*-1}\mathbf{V}_0\}^{-1},$$

proving the theorem.

In the practical application of the preceding methods there may be difficulties in solving such equations as (21) for $\hat{\boldsymbol{\theta}}$, since the function $\boldsymbol{\zeta}$ might be a complicated function of the parameter $\boldsymbol{\theta}$. In such cases, devices suggested by Neyman [1] may be used.

In the light of Theorem 4, an estimate of a function $\mathbf{f}(\boldsymbol{\theta})$, say $\hat{\mathbf{f}}$, may first be found and then $\hat{\mathbf{f}}$ may be used to obtain the estimate of the parameter $\boldsymbol{\theta}$, provided that the function $\mathbf{f}$ satisfies the regularity conditions assumed in Theorem 4. One possible such function is $\mathbf{f}(\boldsymbol{\theta}) = \boldsymbol{\zeta}(\boldsymbol{\theta})$. Assumption 3 on the function $\boldsymbol{\zeta}$ implies that the regularity conditions in Theorem 4 are satisfied. Thus the quadratic form (20) may be minimized with respect to $\boldsymbol{\zeta}$ to obtain the RBAN estimate $\hat{\boldsymbol{\zeta}}$ and then the equation $\hat{\boldsymbol{\zeta}} = \mathbf{f}(\hat{\boldsymbol{\theta}})$ may be solved for $\hat{\boldsymbol{\theta}}$. In doing so, however, it should be remembered that $\boldsymbol{\zeta}$ is a vector of n components, and $\boldsymbol{\theta}$ is a vector of $s \leqq n$ components. To ensure a unique solution of $\hat{\boldsymbol{\theta}}$ from the estimate $\hat{\boldsymbol{\zeta}}$, the function $\boldsymbol{\zeta}$ must be subject to $n - s = r$ restrictions before estimation takes place. Let the restrictions be represented by the equation

$$\mathbf{F}^0 \equiv \mathbf{F}(\boldsymbol{\zeta}) = 0, \tag{24}$$

where $\mathbf{F}^0$ is an $r \times 1$ column vector. Equation (24) is deduced by eliminating the parameter $\boldsymbol{\theta}$ from the equation $\boldsymbol{\zeta} = \mathbf{f}(\boldsymbol{\theta})$, with $\mathbf{f}$ denoting the function of the parameter $\boldsymbol{\theta}$. The estimate obtained by this procedure is identically equal to the one found by directly applying Theorem 6.

A second modification of the methods is suggested for the purpose of deriving an explicit formula for the estimate $\hat{\boldsymbol{\zeta}}$. Under Assumption 3 on the differentiability of the function $\boldsymbol{\zeta}$, the function $\mathbf{F}(\boldsymbol{\zeta})$ has continuous partial derivatives with respect to $\boldsymbol{\zeta}$ and the matrix of the derivatives has rank r. Using Taylor's theorem, we write the reduced form of the restrictions (24),

$$\mathbf{F}^*(\boldsymbol{\zeta}, \check{\mathbf{Z}}_m) \equiv \mathbf{F} + \mathbf{T}(\boldsymbol{\zeta} - \check{\mathbf{Z}}_m) = 0, \tag{25}$$

where $\mathbf{F} = \mathbf{F}(\check{\mathbf{Z}}_m)$ and

$$\mathbf{T} = \left.\frac{\partial \mathbf{F}(\boldsymbol{\zeta})}{\partial \boldsymbol{\zeta}}\right|_{\hat{\mathbf{Z}}_m}.$$

The idea is to minimize the quadratic form (20) with respect to $\boldsymbol{\zeta}$, subject to the reduced form, (25), instead of the original restrictions (24). The resulting estimate is also RBAN. This is shown in the following lemma proved in [1], p. 257, in the case of multinomial random variables.

LEMMA 4. *Let Q denote the quadratic form $Q(\check{\mathbf{Z}}_m, \boldsymbol{\theta})$ or $Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta})$. If the minimization of the quadratic form Q under restriction* (24) *leads to an* RBAN *estimate of the expectation $\boldsymbol{\zeta}^0$, then the minimization of the same quadratic form under the reduced restriction* (25) *will also lead to an* RBAN *estimate of $\boldsymbol{\zeta}^0$.*

An explicit formula for the estimate of $\boldsymbol{\zeta}^0$ is given in the following:

LEMMA 5. *The function $\hat{\boldsymbol{\zeta}}(\check{\mathbf{Z}}_m, \mathbf{S}_m)$, which minimizes the quadratic form*

$$Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \boldsymbol{\theta}) = [\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta})]'\mathbf{S}_m^{-1}[\check{\mathbf{Z}}_m - \boldsymbol{\zeta}(\boldsymbol{\theta})] \tag{20}$$

subject to the reduced restrictions (25), *is given by*

$$\hat{\zeta}(\check{\mathbf{Z}}_m, \mathbf{S}_m) = \check{\mathbf{Z}}_m - \mathbf{S}_m\mathbf{T}'\mathbf{P}^{-1}\mathbf{F}, \tag{26}$$

with $\mathbf{P} = \mathbf{T}\mathbf{S}_m\mathbf{T}'$. *The corresponding quadratic form is given by*

$$Q(\check{\mathbf{Z}}_m, \mathbf{S}_m, \hat{\zeta}) = \mathbf{F}'\mathbf{P}^{-1}\mathbf{F}. \tag{27}$$

The first part of the lemma can be proved easily by using the Lagrange method as outlined in the proof of Theorem 3. A direct computation gives the second part of the lemma.

In obtaining RBAN estimates in a practical problem, the essential part of the work is deducing the side restrictions (24). Once the side restrictions are deduced RBAN estimates can be obtained by a straightforward computation. A detailed description of the procedure is given in [8].

An application of the methods has been made to a stochastic process of flour beetles, and the work is being prepared for publication elsewhere.

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REFERENCES

[1] J. Neyman, "Contribution to the theory of the χ^2 test," *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability*, University of California Press, 1949, pp. 239–273.

[2] E. W. Barankin and J. Gurland, "On asymptotically Normal, Efficient Estimators: I," *University of California Publications in Statistics*, Vol. 1, pp. 89–130.

[3] E. Slutsky, "Über stochastische Asymptoten und Grenzwerte," *Metron.*, Vol. 5 (1925).

[4] H. Cramér, *Mathematical Methods of Statistics*, Princeton University Press, 1946.

[5] H. Cramér, *Random Variables and Probability Distributions*, Cambridge Tracts in Mathematics, No. 36, Cambridge, 1937.

[6] R. Courant, *Differential and Integral Calculus*, Vol. II. (English translation by E. J. McShane) Interscience Publishers, Inc., N. Y., 1949.

[7] H. Hancock, *Lectures on the Theory of Maxima and Minima of Functions of Several Variables*, University of Cincinnati Bulletin No. 13, 1903.

[8] C. L. Chiang, "On the design of mass medical surveys," *Human Biology*, Vol. 23 (1951), pp. 242–271.