A COMPARISON OF TESTS ON THE MEAN OF A LOGARITHMICO-NORMAL DISTRIBUTION WITH KNOWN VARIANCE[1, 2]

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1. Summary. Three test procedures are considered for testing an hypothesis on the mean of a logarithmico-normal distribution with known variance. The first is a normal theory test applied to the logarithms of the original data; the second is a normal theory test applied to the original data; and the third is a test based on the Neyman-Pearson Lemma.

The operating characteristics of these tests are developed and some asymptotic properties obtained. It is found that the three procedures give quite different results unless the mean under the null hypothesis is large relative to the standard deviation.

2. Introduction. The studies of the correct transformation to be applied to data in order to more closely fulfill the assumptions underlying a statistical test occupy an important place in the statistical literature. In particular, the use of the logarithmic transformation is widely advocated in cases where the error distribution is known to be logarithmico-normal; or where component effects in the analysis of variance are multiplicative; or where variance heterogeneity is such that the variance is proportional to the square of the mean. The logarithmic transformation would then make the error distribution normal; or cause the effects to be additive; or homogenize the error variance. Thus, a transformation is effected in order to force an observed, and slightly unconventional, model into a well-known and rather well understood model.

The present investigation is concerned with the application of the logarithmic transformation to the problem of testing an hypothesis on the mean of a logarithmico-normal variate with known variance. An experimenter can fail to recognize the need for a transformation and simply proceed to apply normal theory tests to the original data, or he can properly transform the data and then apply a normal theory test to a parameter of the transformed scale. Each of these testing procedures is investigated in detail.

Finally, a third test procedure is developed by using the Neyman-Pearson Lemma for testing simple hypotheses.

A comparison of these tests is then made by means of their operating charac-

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teristics and some asymptotic properties obtained. It is found that the three procedures give quite different results unless the mean under the null hypothesis is large relative to the standard deviation.

3. Statement of the problem. Let y be a normal variate with probability density

$$g(y;\, \mu_y\,,\, \sigma_y^2) = \frac{1}{\sigma_y\sqrt{2\pi}}\, e^{-(y-\mu_y)^2/2\sigma_y^2}. \tag{3.1}$$

Then x, defined through $y = \ln x$, is a logarithmico-normal variate with probability density

$$f(x;\, \mu_y\,,\, \sigma_y^2) = \frac{1}{\sigma_y\sqrt{2\pi}}\, e^{-[\ln x-\mu_y]^2/2\sigma_y^2}\,\frac{1}{x}. \tag{3.2}$$

If the mean and variance of x are designated by μ_x and σ_x^2, respectively, then the following relationships hold [1]:

$$\begin{aligned} \mu_x &= e^{\mu_y+\sigma_y^2/2}, \\ \sigma_x^2 &= e^{2\mu_y+\sigma_y^2}(e^{\sigma_y^2} - 1). \end{aligned} \tag{3.3}$$

Solving (3.3) for μ_y and σ_y^2 gives

$$\begin{aligned} \mu_y &= \ln \frac{\mu_x^2}{\sqrt{\sigma_x^2 + \mu_x^2}}, \\ \sigma_y^2 &= \ln\left[1 + \frac{\sigma_x^2}{\mu_x^2}\right]. \end{aligned} \tag{3.4}$$

If it is assumed that σ_x^2 is known, the problem is how to test the null hypothesis $H_0\colon \mu_x = {}_0\mu_x$, against the simple alternative $H_1 : \mu_x = {}_1\mu_x > {}_0\mu_x$, at a significance level of α, using a sample $O_n : x_1, x_2, \cdots, x_n$, where the x_i are statistically independent.

There is no loss in generality in taking $\sigma_x^2 = 1$, since it is always possible to make a change of variables by dividing the variable, x, by the known standard deviation. Thus, equations (3.4) may be written as

$$\begin{aligned} \mu_y &= \ln \frac{\mu_x^2}{\sqrt{1 + \mu_x^2}}, \\ \sigma_y^2 &= \ln\left[1 + \frac{1}{\mu_x^2}\right], \end{aligned} \tag{3.5}$$

which means (3.1) and (3.2) may be written as

$$\begin{aligned} g(y;\, \mu_y\,,\, \sigma_y^2) &= g(y;\, \mu_x), \\ f(x;\, \mu_y\,,\, \sigma_y^2) &= f(x;\, \mu_x). \end{aligned} \tag{3.6}$$

4. Normal theory test applied to $y = \ln x$. The first test procedure is suggested by the fact that $y = \ln x$ is a normal variate. In fact, under H_0, y is

$N({}_0\mu_y, {}_0\sigma_y^2)$,[4] where ${}_0\mu_y$ and ${}_0\sigma_y^2$ represent the values of (3.5) at $\mu_x = {}_0\mu_x$. Furthermore, since ${}_1\mu_y > {}_0\mu_y$, where ${}_1\mu_y$ is the value of μ_y at $\mu_x = {}_1\mu_x$, it is possible to test H_0': $\mu_y = {}_0\mu_y$ against H_1': $\mu_y = {}_1\mu_y > {}_0\mu_y$ by using the test statistic $[\bar{y} - {}_0\mu_y]\sqrt{n}/{}_0\sigma_y$ with a critical region specified by

$$T_1 = \left\{ (x_1, \cdots, x_n) \,\middle|\, \frac{\dfrac{\sum \ln x_i}{n} - {}_0\mu_y}{{}_0\sigma_y/\sqrt{n}} \geqq z_\alpha \right\}, \tag{4.1}$$

where $\bar{y} = \sum y_i/n$ and z_α is such that

$$\int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \, d\xi = \alpha. \tag{4.2}$$

(In short, the test can be characterized by T_1.) Thus, the testing procedure may be performed by a normal theory test on the transformed scale.

Under an alternative $\mu_x > {}_0\mu_x$ the distribution of $\bar{y} = \sum \ln x_i/n$ is

$$N(\mu_y, \sigma_y^2/n),$$

so that the operating characteristic becomes

$$\begin{aligned} \beta_{T_1} &= P\left\{ \bar{y} \leqq z_\alpha \frac{{}_0\sigma_y}{\sqrt{n}} + {}_0\mu_y \right\} \\ &= P\left\{ \frac{\bar{y} - \mu_y}{\sigma_y/\sqrt{n}} \leqq \frac{z_\alpha\, {}_0\sigma_y - (\mu_y - {}_0\mu_y)\sqrt{n}}{\sigma_y} \right\}. \end{aligned} \tag{4.3}$$

By using equations (3.5), one may write this as

$$\beta_{T_1} = \Phi\left\{ \frac{z_\alpha \sqrt{\ln\left(1 + \dfrac{1}{{}_0\mu_x^2}\right)} - \left[\ln \dfrac{\mu_x^2}{\sqrt{1 + \mu_x^2}} - \ln \dfrac{{}_0\mu_x^2}{\sqrt{1 + {}_0\mu_x^2}}\right]\sqrt{n}}{\sqrt{\ln\left(1 + \dfrac{1}{\mu_x^2}\right)}} \right\}, \tag{4.4}$$

which depends upon ${}_0\mu_x$, μ_x, α, and n.

The operating characteristics of the T_1 test were computed for the following four cases:

n	${}_0\mu_x$	α
4	1	.05
4	10	.05
25	1	.05
25	10	.05

The computed values are tabulated in Table I and the corresponding curves are given in Fig. 1, where the following notation is used

$$\delta = \mu_x - {}_0\mu_x. \tag{4.5}$$

[4] The notation $N(\mu, \sigma^2)$ is used to denote a normal distribution with mean μ and variance σ^2.

TABLE I

Probabilities that the T_1-test with sample size n will acept $\mu_x = {}_0\mu_x$ when the mean is at ${}_0\mu_x + \delta$

$\delta/{}_0\mu_x$	$n = 4$		$n = 25$	
	1	10	1	10
0.0	.95	.95	.95	.95
0.2	.88	.89	.53	.74
0.4	.74	.81	.06	.37
0.6	.55	.68	.00	.10
0.8	.33	.54	.00	.01
1.0	.16	.39		
1.2	.05	.26		
1.4	.01	.16		
1.6	.00	.09		
1.8	.00	.03		
2.0	.00	.01		

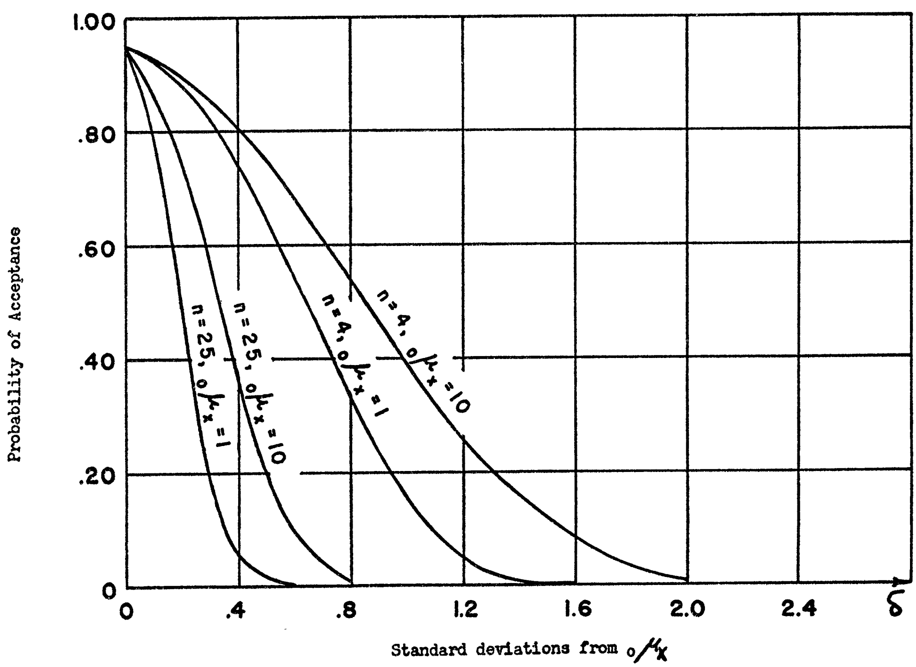

FIG. 1. Operating Characteristics for the T_1 Test

Obviously, the power is not invariant under a translation in ${}_0\mu_x$. For fixed n, the power in discerning a shift of K units (measured in standard deviations) from the null hypothesis decreases as the null hypothesis increases; i.e., the test is a more powerful one when the null hypothesis is small than when it is large. (Since the known variance was assumed to be unity, it might be helpful to rephrase this to read: The region T_1 for testing the mean of a logarithmico-normal variate becomes more powerful against alternatives of the mean greater than the hypothesized one as the ratio of the hypothesized mean to the known standard deviation decreases.)

Further details on the properties of the T_1-test for large ${}_0\mu_x$ are given in Section 7.

5. Normal theory test applied to x. The second procedure to be studied is one which might be applied by the experimenter who, because of either blissful ignorance or wishful thinking, assumes the universe sampled close enough to a normal universe to justify a test based on normal theory.

Erroneously considering the logarithmico-normal variate x as though it were actually $N(\mu_x, 1)$ leads to the critical region

$$T_2 = \left\{(x_1, \cdots, x_n) \left| \frac{\bar{x} - {}_0\mu_x}{1/\sqrt{n}} \geqq z_\alpha \right. \right\}, \tag{5.1}$$

where $\bar{x} = \sum x_i/n$ and z_α is defined in (4.2).

The calculation of the operating characteristic at any alternative $\mu_x > {}_0\mu_x$ for the T_2-test is an easy matter for the case when $n = 1$. If the mean of x is $\mu_x > {}_0\mu_x$, then $y = \ln x$ is $N(\mu_y, \sigma_y^2)$, where $\mu_y = \ln [\mu_x^2/\sqrt{1 + \mu_x^2}]$ and $\sigma_y^2 = \ln [1 + 1/\mu_x^2]$, and so

$$\begin{aligned}
\beta_{T_2} &= P\{x \leqq z_\alpha + {}_0\mu_x\} = P\{\ln x \leqq \ln [z_\alpha + {}_0\mu_x]\} \\
&= P\left\{\frac{\ln x - \mu_y}{\sigma_y} \leqq \frac{\ln [z_\alpha + {}_0\mu_x] - \mu_y}{\sigma_y}\right\} \\
&= \Phi\left\{\frac{\ln [z_\alpha + {}_0\mu_x] - \mu_y}{\sigma_y}\right\}.
\end{aligned}$$

By using equation (3.5) this becomes

$$\beta_{T_2} = \Phi\left\{\frac{\ln [z_\alpha + {}_0\mu_x] - \ln \dfrac{\mu_x^2}{\sqrt{1 + \mu_x^2}}}{\sqrt{\ln\left(1 + \dfrac{1}{\mu_x^2}\right)}}\right\}, \tag{5.2}$$

which depends only upon ${}_0\mu_x$, μ_x, and α.

The operating characteristic of test T_2 is more difficult to obtain for the case when $n > 1$, because the convolution of n logarithmico-normal variates is needed. Since this could not be obtained in closed form, the particular procedure adopted was to obtain an Edgeworth form of the Gram-Charlier Type A series expansion and then to consider a sufficient number of terms to calculate power correctly to two decimals.

The Edgeworth expansion for the distribution of the variate

$$X = \frac{\xi_n - E(\xi_n)}{\sigma_{\xi_n}}, \tag{5.3}$$

where $\xi_n = x_1 + \cdots + x_n$, with x_i's independent, and where $E(\xi_n)$ and σ_{ξ_n} denote the mean and standard deviation of ξ_n, respectively, is given in Cramér [2, p. 229] as

$$(5.4)\quad F(X) = \Phi(X) - \frac{1}{3!}\gamma_1' \frac{\Phi^{(3)}(X)}{n^{1/2}} + \frac{1}{4!}\gamma_2' \frac{\Phi^{(4)}(X)}{n} + \frac{10}{6!}\gamma_1'^2 \frac{\Phi^{(6)}(X)}{n} + O(n^{-3/2}),$$

where γ_1' and γ_2' are the coefficients of skewness and kurtosis of the x_i variate.

For the logarithmico-normal variate, the skewness and kurtosis become

$$(5.5)\quad \begin{aligned} \gamma_1' &= (\Gamma - 1)^{1/2}(\Gamma + 2) \\ \gamma_2' &= (\Gamma - 1)(\Gamma^3 + 3\Gamma^2 + 6\Gamma + 6), \end{aligned}$$

where

$$\Gamma = e^{\sigma_y^2} = (1 + 1/\mu_x^2).$$

If these results are used, the operating characteristic for T_2 when $n > 1$ at some $\mu_x > {}_0\mu_x$ becomes

$$(5.6)\quad \begin{aligned} \beta_{T_2} &= P\left\{\bar{x} \leqq z_\alpha \frac{1}{\sqrt{n}} + {}_0\mu_x\right\} \\ &= P\left\{\frac{\bar{x} - \mu_x}{1/\sqrt{n}} \leqq \frac{z_\alpha\sqrt{n} + n\,{}_0\mu_x - n\mu_x}{\sqrt{n}}\right\} \\ &= P\{X \leqq z_\alpha - \delta\sqrt{n}\}. \end{aligned}$$

Therefore, for $n > 1$, the operating characteristic for T_2 at a mean $\mu_x > {}_0\mu_x$ may be written as

$$(5.7)\quad \beta_{T_2} = F(z_\alpha - \delta\sqrt{n})$$

where $F(X)$ is given by (5.4) with the coefficients determined by (5.5).

The operating characteristics for the same tests studied in Section 4 have been computed by using the above expansions. The calculated values are given in Table II and the corresponding graphs in Fig. 2.

Now, for $n = 4$, the T_2-test where ${}_0\mu_x$ is equal to 10 standard deviations is more powerful for distinguishing departures less than 1.2 standard deviations than is the T_2-test where ${}_0\mu_x$ is equal to 1 standard deviation. For departures greater than 1.2 standard deviations, the converse is true. Note also that the T_2-test for ${}_0\mu_x = 10$ has an actual α level almost identical with the one for which the test was supposedly constructed; i.e., $\alpha = 0.05$. For ${}_0\mu_x = 1$, however, the true α level is around 0.039 instead of 0.05, as the experimenter had believed. Similar results are true for the case $n = 25$. Thus the T_2 procedure would appear to give a rather satisfactory α level when the value specified by the null hypothesis is large, (or, in general, when the ratio of the value specified by the null hypothesis to the known standard deviation is large.) When the value specified by

TABLE II

Probabilities that the T_2-test with sample size n will accept $\mu_x = {}_0\mu_x$ when the mean is at ${}_0\mu_x + \delta$

$\delta/{}_0\mu_x$	$n = 4$		$n = 25$	
	1	10	1	10
0.0	.96	.95	.94	.95
0.2	.92	.89	.77	.74
0.4	.85	.80	.39	.36
0.6	.73	.68	.07	.09
0.8	.58	.53	.00	.01
1.0	.41	.37		
1.2	.23	.23		
1.4	.11	.12		
1.6	.04	.06		
1.8	.00	.02		
2.0	.00	.01		

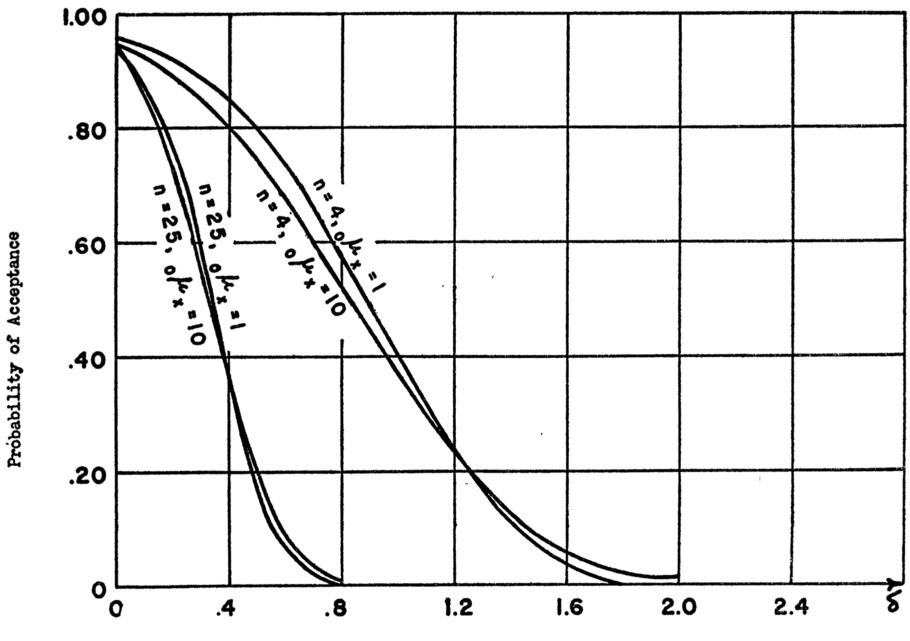

FIG. 2. Operating Characteristics for the T_2 Test

the null hypothesis is small, the experimenter is actually running a smaller risk of rejecting the null hypothesis when true than the risk for which he had constructed the test.

More details on the asymptotic properties of the T_2-test are given in Section 7.

6. Test based on Neyman-Pearson theory. The third test considered is dictated by the Neyman-Pearson Lemma (see [2]) for testing a simple statistical hypothesis. The test may be characterized by the critical region

$$T_3 = \left\{(x_1, \cdots, x_n) \,\middle|\, \frac{\prod f(x_i; \mu_x)}{\prod f(x_i; {}_0\mu_x)} \geqq k\right\}, \tag{6.1}$$

where k is such that

$$\int_{T_3} \cdots \int \prod f(x_i; {}_0\mu_x)\, dx_i = \alpha. \tag{6.2}$$

The inequality in the expression for T_3 can be shown to reduce to

$$\sum (\ln x_i - b/a)^2 \leqq k'$$

where

$$\begin{aligned} a &= \left(\frac{{}_1\sigma_y}{{}_0\sigma_y}\right)^2 - 1, \\ b &= \left(\frac{{}_1\sigma_y}{{}_0\sigma_y}\right)^2 {}_0\mu_y - {}_1\mu_y, \end{aligned} \tag{6.3}$$

and where

$$k' = \left[2\, {}_1\sigma_y^2 \ln k - 2\, {}_1\sigma_y^2 \ln \left(\frac{{}_0\sigma_y}{{}_1\sigma_y}\right)^n + n\, {}_1\mu_y^2 - n \left(\frac{{}_1\sigma_y}{{}_0\sigma_y}\right)^2_0 \mu_y^2\right] \frac{1}{a} + \frac{nb^2}{a^2}.$$

The value of k' must now be found such that (6.2) is satisfied. Under H_0, $z = \ln x - (b/a)$ is $N({}_0\mu_y - b/a, {}_0\sigma_y^2)$. Now the variate

$$\chi'^2 = \frac{\sum z_i^2}{\sigma^2}, \tag{6.4}$$

where z_i are independent and $N(\alpha_i, \sigma^2)$ with α_i not all zero, has a noncentral χ^2 distribution with probability density

$$p(\chi'^2) = \frac{e^{-\frac{1}{2}\chi'^2} e^{-\frac{1}{2}\lambda}}{2^{\frac{1}{2}n}} \sum_{j=0}^{\infty} \frac{(\chi'^2)^{\frac{1}{2}n+j-1}\lambda^j}{2^{2j} j!\,\Gamma(\frac{1}{2}n + j)}, \tag{6.5}$$

where $\lambda = \sum \alpha_i^2/\sigma^2$. Hence the value k' is such that

$$\int_0^{k'/{}_0\sigma_y^2} p(\chi'^2)\, d\chi'^2 = \alpha, \tag{6.6}$$

where the parameter λ in the function $p(\chi'^2)$ is given by

$$\lambda = n({}_0\mu_y - b/a)^2/{}_0\sigma_y^2.$$

If the value of $k'/{}_0\sigma_y^2$ which satisfies (6.6) is denoted by $\chi'^2_{0,\alpha}$, then the critical region (6.1) may be written as

$$T_3 = \left\{(x_1, \cdots, x_n) \,\middle|\, \frac{\sum \{\ln x_i - b/a\}^2}{{}_0\sigma_y^2} \leqq \chi'^2_{0,\alpha}\right\}. \tag{6.7}$$

It is interesting to note that this test, although most powerful for testing H_0 against the simple alternative H_1, is not *uniformly* most powerful against any class of alternatives, since the distribution of the test statistic involves the quantity b/a which depends not only on ${}_0\mu_x$ but also on ${}_1\mu_x$.

Extensive tables of noncentral χ^2 are not yet available, so that it often becomes necessary to use approximations which are discussed, for example, by Patnaik [3] and Abdel-Aty [4]. (Pearson and Hartley [5] promise to include more extensive tables of this nature in their second volume to be published soon.)

When the mean of x is some $\mu_x > {}_0\mu_x$, then the variate $\ln x - (b/a)$ is

$$N(\mu_y - (b/a), \sigma_y^2),$$

where μ_y and σ_y^2 are determined from equation (3.5). Hence, according to (6.4) and (6.5), the quantity

$$\chi'^2 = \sum_1^n \frac{(\ln x_i - b/a)^2}{\sigma_y^2} \tag{6.8}$$

follows a noncentral χ^2-distribution with parameter $\lambda = n(\mu_y - (b/a))^2/\sigma_y^2$ and degrees of freedom equal to n. Using this fact, the operating characteristic of T_3 for some $\mu_x > {}_0\mu_x$ becomes

$$\begin{aligned} \beta_{T_3} &= P\{\chi_0'^2 \geqq \chi_{0,\alpha}'^2\} \\ &= P\left\{\frac{{}_0\sigma_y^2}{{}_1\sigma_y^2} \frac{\sum (\ln x_i - b/a)^2}{{}_0\sigma_y^2} \geqq \frac{{}_0\sigma_y^2}{{}_1\sigma_y^2} \chi_{0,\alpha}'^2\right\} \\ &= P\{\chi'^2 \geqq c\chi_{0,\alpha}'^2\}, \end{aligned} \tag{6.9}$$

where

$$c = \frac{{}_0\sigma_y^2}{{}_1\sigma_y^2} \tag{6.10}$$

and

$$\lambda = \frac{n(\mu_y - b/a)^2}{\sigma_y^2}.$$

The operating characteristics for the following cases were computed:

n	${}_0\mu_x$	${}_1\mu_x$	α
4	1	2	.05
4	1	10	.05
4	10	11	.05
25	1	2	.05

The calculated values are given in Table III and the corresponding curves in Fig. 3.

TABLE III

Probabilities that the T_3-test with sample size n designed to test H_0 : $\mu_x = {}_0\mu_x$ against H_1: $\mu_x = {}_1\mu_x$ will accept H_0 when $\mu_x = {}_0\mu_x + \delta$

$\delta/{}_1\mu_x$	$n = 4$			$n = 25$
	${}_0\mu_x=1$		${}_0\mu_x = 10$	${}_0\mu_x = 1$
	2	10	11	2
0.0	.95	.95	.95	.95
0.1				.79
0.2	.85		.90	.49
0.3				.14
0.4	.64	.67	.80	.02
0.6	.39		.67	.00
0.8	.17		.52	.00
1.0	.06	.08	.36	
1.2	.01		.19	
1.4	.00	.00	.12	
1.6	.00		.05	

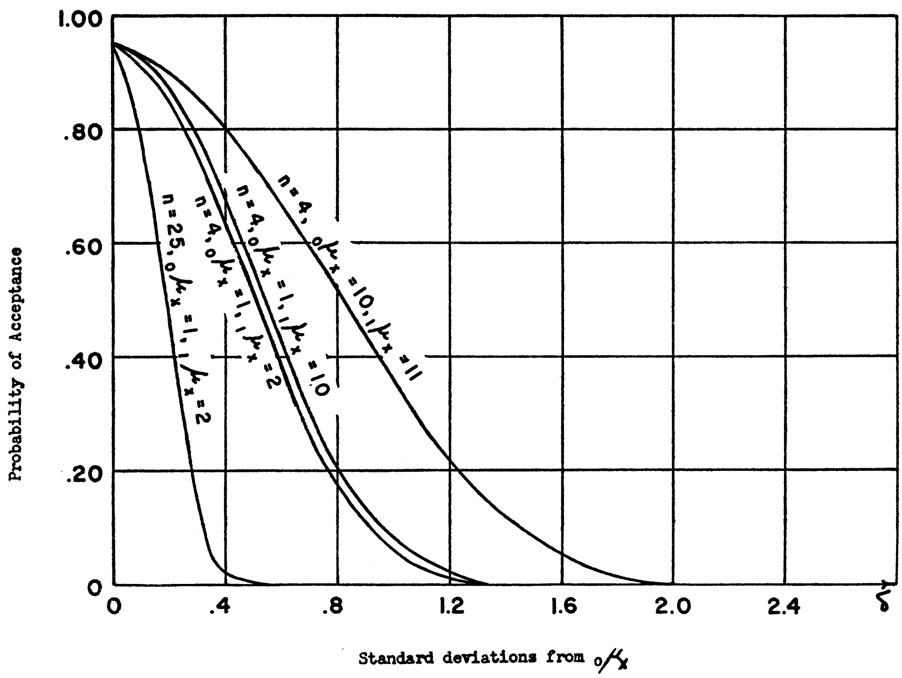

FIG. 3. Operating Characteristics for the T_3 Test

7. Asymptotic properties of the tests for large values of the null hypothesis. If, instead of being a logarithmico-normal variate, x were actually $N(\mu_x, 1)$ then the most powerful test of H_0 against H_1 would be characterized by the critical region

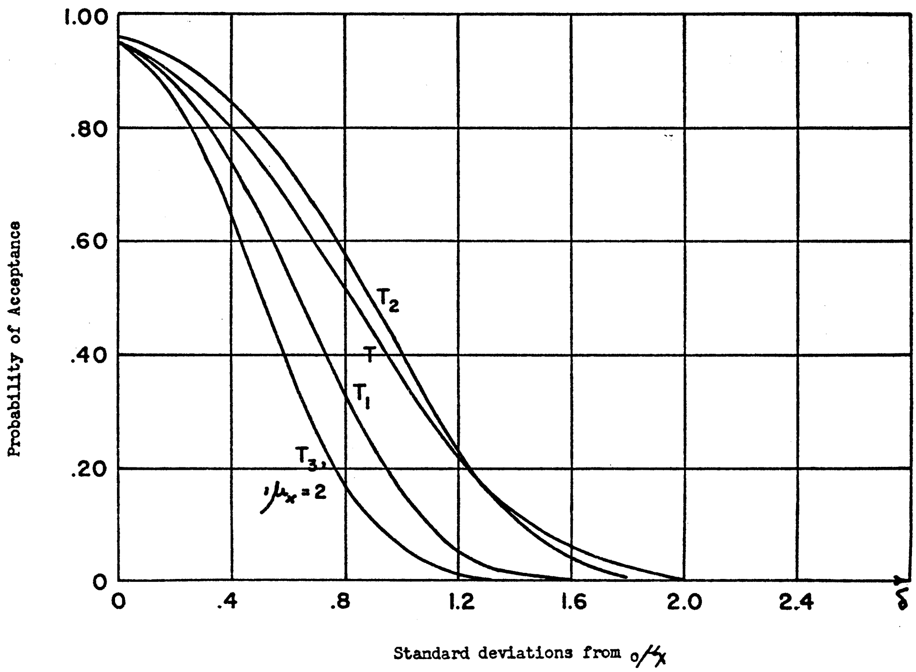

FIG. 4. Comparison of the Operating Characteristics for the T, T_1, T_2, T_3 Tests for $n = 4$, ${}_0\mu_x = 1$

$$T = \left\{ (x_1, \cdots, x_n) \,\middle|\, \frac{\bar{x} - {}_0\mu_x}{1/\sqrt{n}} \geqq z_\alpha \right\} \tag{7.1}$$

and the corresponding operating characteristic would be

$$\beta_T = \Phi(z_\alpha - \delta\sqrt{n}). \tag{7.2}$$

The operating characteristics for the T, T_1, T_2, and T_3 tests for $n = 4$ and ${}_0\mu_x = 1$ are plotted together in Fig. 4, where the scale measures the number of standard deviations away from the hypothesized mean. Similar curves are plotted together in Fig. 5 for the case $n = 4$ and ${}_0\mu_x = 10$. An examination of these curves indicates that the power depends not only on the specific test being used, but also on the specific value of the null hypothesis. In fact, the T_1, T_2, and T_3 operating characteristics given in Fig. 5 cluster closer about the T operating characteristic than do those in Fig. 4. This suggests that, possibly, the approach of all three operating characteristics, as the hypothesized mean is increased, is to the operating characteristic of the T-test. Specific results of this nature will now be proven.

Throughout this entire section ${}_0\mu_x$ will be written simply as μ. Furthermore, an alternative $\mu_x > {}_0\mu_x$ will be written as

$$\mu_x = \mu + \delta, \tag{7.3}$$

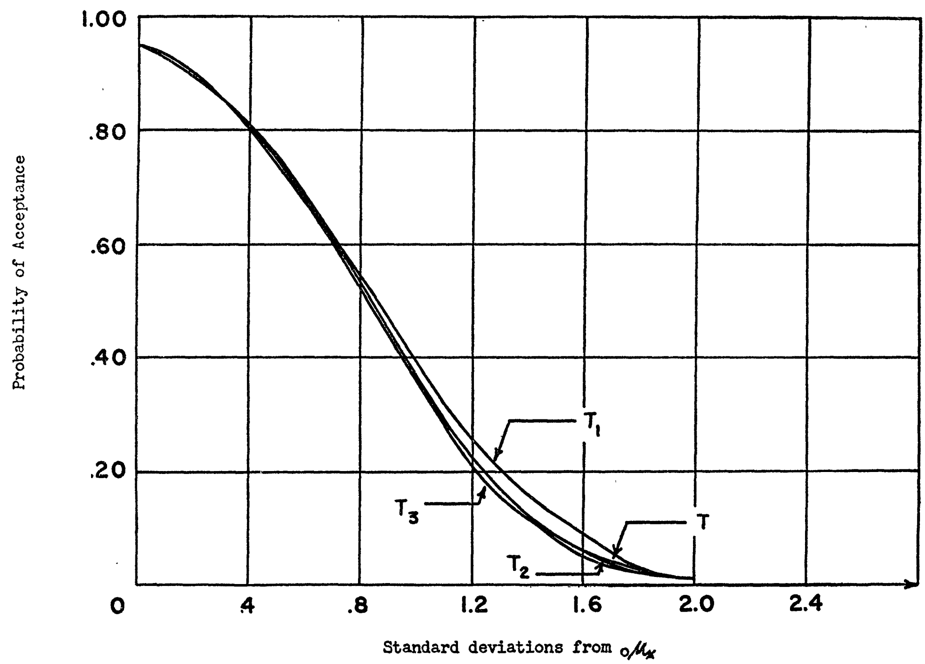

FIG. 5. Comparison of the Operating Characteristics for the T, T_1, T_2, T_3 Tests for $n = 4$, $_0\mu_x = 10$

where δ represents the number of standard deviations from the hypothesized mean.

Thus the null hypothesis $H_0 : \mu_x = {}_0\mu_x$ and the alternative $H_1 : \mu_x = {}_1\mu_x > {}_0\mu_x$ become

$$H_0 : \delta = 0, \tag{7.4}$$
$$H_1 : \delta = \delta_1 ,$$

where $\delta_1 = {}_1\mu_x - {}_0\mu_x$.

The T_1-Test. The behavior of the T_1 test for large μ is summarized in the following theorem:

THEOREM 1. $\lim_{\mu\to\infty} \beta_{T_1} = \beta_T$.

PROOF. Using the notation (7.3), the operating characteristic of the T_1-test may be written as

$$\beta_{T_1} = \Phi\left\{\frac{z_\alpha\sqrt{\ln\left(1+\frac{1}{\mu^2}\right)} - \left[\ln\frac{(\mu+\delta)^2}{\sqrt{1+(\mu+\delta)^2}} - \ln\frac{\mu^2}{\sqrt{1+\mu^2}}\right]\sqrt{n}}{\sqrt{\ln\left(1+\frac{1}{(\mu+\delta)^2}\right)}}\right\} \tag{7.5}$$

$$\equiv \Phi\left\{\frac{z_\alpha A_1 - A_2\sqrt{n}}{A_3}\right\}.$$

Since $\Phi(z)$ is a continuous function of z, it is valid to take the limit sign inside the Φ function.

By using the expansion of the function $\ln(1 + 1/z)$ in the neighborhood of $z = \infty$, it is seen that as μ approaches infinity

$$\frac{A_1}{A_3} = \frac{\frac{1}{\mu}\sqrt{1 + O(\mu^{-2})}}{\frac{1}{\mu + \delta}\sqrt{1 + O(\mu + \delta)^{-2}}} \to 1. \tag{7.6}$$

Note also that

$$\begin{aligned} A_2 &= \ln\left[1 + \frac{\delta}{\mu}\right] + \tfrac{1}{2}\ln\frac{(\mu + \delta)^2}{1 + (\mu + \delta)^2} - \tfrac{1}{2}\ln\frac{\mu^2}{1 + \mu^2} \\ &= \ln\left[1 + \frac{\delta}{\mu}\right] - \tfrac{1}{2}\ln\left[1 + \frac{1}{(\mu + \delta)^2}\right] + \tfrac{1}{2}\ln\left[1 + \frac{1}{\mu^2}\right] \\ &= \frac{\delta}{\mu} + O(\mu^{-2}), \end{aligned}$$

so that

$$\frac{A_2}{A_3} = \frac{\frac{\delta}{\mu} + O(\mu^{-2})}{\frac{1}{\mu + \delta}\sqrt{1 + O(\mu + \delta)^{-2}}} \to \delta. \tag{7.7}$$

Hence, if one uses statements (7.6) and (7.7), the limiting operating characteristic of T_1 becomes

$$\lim_{\mu\to\infty} \beta_{T_1} = \Phi(z - \delta\sqrt{n}),$$

and so

$$\lim_{\mu\to\infty} \beta_{T_1} = \beta_T.$$

Thus, for large values of the hypothesized mean, the T_1-test behaves like the T-test.

The T_2-test. A similar theorem to that proved above will be shown for the T_2-test. The proof involves interchanging the limit and integral signs. As the justification for this, one could use the Lebesgue Dominated Convergence theorem ([2], p. 66], which involves finding an integrable function which bounds the absolute value of the integrand. In cases where the integrand, say $p_n(x)$, is a proper density for all n, Scheffé [7] has shown that a sufficient condition for demonstrating the existence of such a bounding and integrable function is that the limit (as n tends to infinity) of the integrand is also a proper density, say $p(x)$.

It is now possible to proceed to

THEOREM 2. $\lim_{\mu\to\infty} \beta_{T_2} = \beta_T$.

PROOF. The operating characteristic for the T_2 test given in (5.7) may be written as

$$\beta_{T_2} = \int_{\hat{T}_2} \cdots \int \prod_{i=1}^{n} \frac{1}{\sigma_y\sqrt{2\pi}} e^{-[\ln x_i - \mu_y]^2/2\sigma_y^2} \frac{dx_i}{x_i}, \tag{7.8}$$

where $\hat{T}_2$ is the complement set of T_2 and is given by

$$\hat{T}_2 = \left\{(x_1, \cdots, x_n) \,\middle|\, \frac{\bar{x} - {}_0\mu_x}{1/\sqrt{n}} \leqq z_\alpha\right\}. \tag{7.9}$$

Using the notation of (7.3) and letting

$$w_i = x_i - \mu, \tag{7.10}$$

one can write (7.8) as

$$\beta_{T_2} = \int_{\hat{T}_2'} \cdots \int \prod \frac{1}{\sigma_y\sqrt{2\pi}} e^{-[\ln(w_i+\mu)-\mu_y]^2/2\sigma_y^2} \frac{dw_i}{w_i + \mu}, \tag{7.11}$$

where

$$\hat{T}_2' = \{(w_1, \cdots, w_n) \mid \bar{w} \leqq z_\alpha/\sqrt{n}\}. \tag{7.12}$$

Scheffé's theorem is now used in order to justify bringing the limit sign under the integral sign. Note first that the integrand of (7.11) is a proper density, so that it remains to show that the limit of this density is also a proper density. Also, since the limit of the product is the product of the limits, it is only necessary to consider the behavior of one such factor, namely,

$$f_\mu(w_i;\delta) = \frac{\exp\left[-\left\{\ln(w_i+\mu) - \ln\frac{(\mu+\delta)^2}{\sqrt{1+(\mu+\delta)^2}}\right\}^2 \Big/ 2\ln(1+[\mu+\delta]^{-2}\right]}{\sqrt{2\pi}\ln[1+(\mu+\delta)^{-2}]} \frac{1}{w_i+\mu}, \tag{7.13}$$

which will henceforth be written as

$$f_\mu(w_i;\delta) = \frac{1}{\sqrt{2\pi}\,A_1} e^{-A_2^2/2A_3}. \tag{7.14}$$

Now for a fixed w_i, as $\mu \to \infty$,

$$A_1 = (w_i+\mu)\sqrt{\ln\left(1+\frac{1}{(\mu+\delta)^2}\right)} = \frac{w_i+\mu}{\mu+\delta}\sqrt{1+O(\mu+\delta)^{-2}} \to 1. \tag{7.15}$$

Furthermore,

$$\begin{aligned} A_2 &= \ln\left(\frac{w_i+\mu}{\mu+\delta}\right) + \tfrac{1}{2}\ln\left[1+\frac{1}{(\mu+\delta)^2}\right] \\ &= \ln\left[1+\frac{w_i-\delta}{\mu+\delta}\right] + O(\mu+\delta)^{-2} \\ &= \frac{w_i-\delta}{\mu+\delta} + O(\mu^{-2}) \end{aligned}$$

and

$$A_3 = \ln\left[1 + \frac{1}{(\mu + \delta)^2}\right] = \frac{1}{(\mu + \delta)^2} + O(\mu^{-4}),$$

so that for a fixed w_i

$$\frac{A_2^2}{A_3} \longrightarrow (w_i - \delta)^2. \tag{7.16}$$

Hence, if one makes use of (7.15) and (7.16), it follows that

$$\lim_{\mu\to\infty} f_\mu(w_i\,;\delta) = \frac{1}{\sqrt{2\pi}}\, e^{-(w_i-\delta)^2/2}, \tag{7.17}$$

which is a proper density.

Therefore,

$$\begin{aligned}\lim_{\mu\to\infty} \beta_{T_2} &= \int_{\bar{w}\leqq z_\alpha/\sqrt{n}} \cdots \int \prod \frac{1}{\sqrt{2\mu}}\, e^{-(w_i-\delta)^2/2}\, dw_i \\ &= \int_{-\infty}^{z_\alpha/\sqrt{n}} \frac{\sqrt{n}}{\sqrt{2\pi}}\, e^{-(\xi-\delta)^2/2(1/n)}\, d\xi = \Phi\{z_\alpha - \delta\sqrt{n}\},\end{aligned}$$

and so

$$\lim_{\mu\to\infty} \beta_{T_2} = \beta_T\,.$$

The proof of Theorem 2, above, suggests an interesting property of the logarithmico-normal distribution which is summarized in the following corollary.

Corollary 1. *The standardized logarithmico-normal variate $w = x - \mu_x$ is distributed asymptotically $N(0, 1)$ as $\mu_x \longrightarrow \infty$.*

Proof. The result follows immediately from that part of the proof of Theorem 2 where it was shown that

$$\lim_{\mu\to\infty} f_\mu(w;\delta) = \frac{1}{\sqrt{2\mu}}\, e^{-(w-\delta)^2/2}.$$

(The mean of w may be taken as zero so that $\delta = 0$, which means $\mu = \mu_x$.) The theorem of Scheffé states that this is sufficient to show that

$$\lim_{\mu\to\infty} \int_S f_\mu(w;0)\, dw = \int_S \frac{1}{\sqrt{2\pi}}\, e^{-w^2/2}\, dw$$

for all Borel sets S in R.

Another property of the logarithmico-normal distribution follows readily from this corollary.

Corollary 2. *The standardized logarithmico-normal variate $w = x - \mu_x$ is distributed asymptotically $N(0, 1)$ as $\sigma_y^2 \longrightarrow 0$.*

PROOF. Corollary 1 showed that $x - \mu_x$ is asymptotically normal as $\mu_x \to \infty$. According to (3.5),

$$\sigma_y^2 = \ln\left[1 + \frac{1}{\mu_x^2}\right],$$

which means $\mu_x \to \infty$ if and only if $\sigma_y^2 \to 0$. Hence, $x - \mu_x$ is asymptotically $N(0, 1)$ as $\sigma_y^2 \to 0$.

The result of Corollary 2 was also obtained by Yuan [8] who considered the normal variate $y = (1/c)\ln[(x - a)/b]$ and showed that

$$\lim_{c \to 0} y = \frac{x - m}{\sigma}.$$

This, according to Yuan, would imply x is asymptotically normal as c approaches zero. The quantity c corresponds to σ_y.

The T_3-test. One would expect that a similar result to Theorems 1 and 2 would be true for the T_3-test. Since the logarithmico-normal distribution approaches the normal distribution as $\mu_x \to \infty$, one would conjecture that for large values of μ the most powerful tests based on the two distributions could be interchanged with a guarantee of similar calculated risks.

The corresponding theorem to those given above reads:

THEOREM 3. $\lim_{\mu \to \infty} \beta_{T_3} = \beta_T$.

The details of this proof, which are not included here, may be found in Severo [9]. The theorem is proved as a special case of more general results which are summarized in two theorems. The first is concerned with the uniqueness of the most powerful critical region for testing a simple hypothesis as a parameter of the distribution is allowed to pass to its limit. This uniqueness is demonstrated up to a set of measure less than an $\epsilon > 0$. The second theorem then justifies the convergence of the power function to the power function of the limiting critical region.

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