ESTIMATING THE PARAMETERS OF A DIFFERENTIAL PROCESS[1]

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1. Introduction and summary. Let X denote a differential process, i.e., a stochastic process with independent increments for which the distribution of $X(t + h) - X(t)$ depends only on h. The parameter t runs through the interval $[0, +\infty)$, and the usual initial condition $P[X(0) = 0] = 1$ is assumed. Then it is known that the distribution of $X(t)$ is infinitely divisible, i.e., the logarithm of its characteristic function can be written as

$$\log f_{X(t)}(u) = i\gamma tu + t \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - \frac{iux}{1 + x^2}\right) \frac{1 + x^2}{x^2}\, dG(x). \tag{1.1}$$

In this canonical representation, γ is a real constant, and G is a bounded, non-decreasing function, it being permissible always to consider $G(-\infty) = 0$. In the usual probabilistic terminology, the probability law of $X(t)$ is a convolution of a normal law and a (possibly infinite) number of Poisson laws, or a limit of such laws. The function G is called the jump function; its saltus at $x = 0$, $\sigma^2 = G(+0) - G(-0)$, is the variance of the normal component, and its set of points of increase for $x \neq 0$ gives information as to the nature of the Poisson components, viz., the "relative density" of the magnitudes of the discontinuities of the sample function. The purpose of this paper is to derive estimates for this jump function G and for the "trend term", γ. Two estimates of G are obtained, and one estimate is obtained for γ.

In the first method of estimating G, considered in §2, it is assumed that the experimenter can observe a sample function of X at any finite number of values of t that he chooses. Accordingly, for any integer n, let

$$X_{nk} = X\left(\frac{k}{n}\right) - X\left(\frac{k-1}{n}\right).$$

Then the estimate $G^*_{N,n}(u)$ of $G(u)$ is defined as

$$G^*_{N,n}(u) = \frac{1}{N} \sum_{k=1}^{nN} \frac{X^2_{nk}}{1 + X^2_{nk}} I_{[X_{nk} \leqq u]}, \tag{1.2}$$

where

$$I_{[X_{nk} \leqq u]} = \begin{cases} 1 & \text{if } X_{nk} \leqq u \\ 0 & \text{if } X_{nk} > u, \end{cases}$$

and $N = [Tn]$, T being the largest value of t observed. It is proved that this estimate is strongly consistent in the following sense:

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(1.3) $$P\{\lim_{n\to\infty} \lim_{N\to\infty} G^*_{N,n}(u) = G(u) \text{ for all } u \in C(G)\} = 1,$$

where $C(G)$ denotes the set of all values of u at which G is continuous. This estimate is not necessarily an unbiased estimate of $G(u)$ for all $u \in C(G)$.

The second method for estimating G, developed in §3, requires that the experimenter be able to observe all the discontinuities of a sample function of X on a finite interval in addition to being able to observe $X(t)$ at any finite set of values of t. Not only is a consistent estimate obtained for G, but also a consistent estimate is obtained for the variance of the normal component, $\sigma^2 = G(+0) - G(-0)$. Let $\{k_n\}$ denote a sequence of positive integers such that

(1.4) $$\textstyle\sum_n k_n^{-\frac{1}{2}} < \infty.$$

Further, let T_0 be a fixed value of time T, let

(1.5) $$S_n^2 = \sum_{k=1}^{k_n} \{X(kT_0/k_n) - X((k-1)T_0/k_n)\}^2,$$

and

(1.6) D^2 = the sum of the squares of the jumps of the sample function during the time interval $[0, T_0]$.

Finally, let

(1.7) $$J_T(x) = \int_{-\infty}^{x} \frac{b^2}{1+b^2}\, dN_T(b),$$

where, for every Borel set B, $N_T(B)$ denotes the number of discontinuities observed for X during $[0, T]$ whose magnitudes lie in B. The estimate $\hat{G}_{n,T}(u)$ of $G(u)$ is then constructed as follows:

(1.8) $$\hat{G}_{n,T}(u) = \begin{cases} T^{-1}J_T(u) & \text{if } u < 0 \\ T^{-1}J_T(u) + T_0^{-1}(S_n^2 - D^2) & \text{if } u > 0. \end{cases}$$

The estimate $\hat{G}_{n,T}(u)$ is an unbiased estimate of $G(u)$ if $\sigma^2 = 0$ or if $u < 0$, but in any case, $\hat{G}_{n,T}(b) - \hat{G}_{n,T}(a)$ is an unbiased estimate of $G(b) - G(a)$, provided $0 \notin [a, b]$. Also, this estimate is consistent in the following sense:

(1.9) $$\lim_{\substack{T\to\infty \\ n\to\infty}} \hat{G}_{n,T}(u) = G(u)$$

with probability one for every u. In addition, $(1/T_0)\,[S_n^2 - D^2]$ is a consistent estimate of $\sigma^2 = G(+0) - G(-0)$, the variance of the normal component, in the sense that $\lim_{n\to\infty}(1/T_0)\,[S_n^2 - D^2] = \sigma^2$ with probability 1.

In §4 a comparison is made of the two estimates for $G(x)$ obtained in §§2 and 3. It is found that both estimates do agree in a special limiting case. In particular it is proved that

(1.10) $$p \lim_{n\to\infty} G^*_{N,r}(u) = \lim_{n\to\infty} \hat{G}_{n,N}(u) \quad \text{for all} \quad u \in C(\hat{G}_{n,N}).$$

Finally in §5 a consistent and unbiased estimate is derived from the "trend term", γ. This estimate is

$$(1.11) \qquad \hat{\gamma}(t) = \frac{1}{t}\left\{X(t) - \int_{-\infty}^{+\infty} \frac{b^3}{1+b^2}\, dN_t(b)\right\}.$$

It is consistent in the sense that

$$(1.12) \qquad P\{\lim_{t\to\infty} \hat{\gamma}(t) = \gamma\} = 1.$$

Another way of writing this estimate is as follows. Let $J_1, J_2, \cdots, J_n, \cdots$ denote the discontinuities of the sample curve up to time t (not necessarily in order); then

$$\gamma(t) = \frac{1}{t}\left\{X(t) - \sum_n \frac{J_n^3}{1+J_n^2}\right\}$$

2. The first method for estimating G. This method of estimating G is based entirely on necessary conditions for one of the most general central limit theorems. The statement of the theorem is found on page 121 of Gnedenko and Kolmogorov [2], which we restate as follows:

In order that for some suitably chosen constants $A_1, A_2, \cdots, A_n, \cdots$ the distribution functions of sums

$$(2.1) \qquad Y_n = X_{n,1} + X_{n,2} + \cdots + X_{n,k_n} - A_n$$

of independent infinitesimal random variables converge to a limiting distribution function, it is necessary and sufficient that there exist a bounded, non-decreasing function G such that

$$(2.2) \qquad \sum_{k=1}^{k_n} \int_{-\infty}^{u} \frac{x^2}{1+x^2}\, dF_{nk}(x + \alpha_{nk}) \to G(u) \quad \text{as} \quad n \to \infty$$

at all points u at which G is continuous, i.e., at all $u \in C(G)$.

In this theorem, $F_{nk}(x)$ is the distribution function of $X_{n,k}$, and $\alpha_{nk} = \int_{-\tau}^{\tau} x\, dF_{nk}(x)$ for arbitrary $\tau > 0$. One result of the theorem is that A_n can be selected as $A_n = \alpha_{n1} + \alpha_{n2} + \cdots + \alpha_{nk_n}$. The limiting distribution referred to in the theorem is necessarily infinitely divisible, and the logarithm of its characteristic function is of the form

$$(2.3) \qquad \log f(u) = i\gamma u + \int_{-\infty}^{+\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG(x).$$

The function G in (2.3) is the same function G in (2.2). The couple (γ, G) determine and are determined by this limiting distribution. In case the limiting distribution has finite variance, then it is known that there exists a real constant α and a bounded non-decreasing function $H(x)$ such that

$$(2.4) \qquad \log f(u) = i\alpha u + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux) \frac{1}{x^2}\, dH(x).$$

In this case the limiting distribution determines and is determined by the couple (α, H). It is easily verified that the limiting distribution has finite variance if and only if

$$\int_{-\infty}^{+\infty} (1 + x^2)\, dG(x) < \infty. \tag{2.5}$$

In this case the relation between G and H becomes

$$H(x) = \int_{-\infty}^{x} (1 + x^2)\, dG(x). \tag{2.6}$$

Also

$$\alpha = \gamma + \int_{-\infty}^{+\infty} x\, dG(x). \tag{2.7}$$

In the case of the differential process $X(t)$, let

$$X_{n,k} = X\left(\frac{k}{n}\right) - X\left(\frac{k-1}{n}\right), \qquad k = 1, 2, \cdots, n. \tag{2.8}$$

Letting $n = 1, 2, \cdots$, we see that we have an infinitesimal system of independent random variables for which

$$X_{n1} + X_{n2} + \cdots + X_{n,n} = X(1)$$

for every n, and hence the distribution of $X(1)$ is the limit law of the distributions of this sequence of sums. In this case, $F_{nk}(x)$ is for fixed n the same for all k and is denoted by $F_n(x)$; likewise, $\alpha_n = \alpha_{n,k}$ for $1 \leqq k \leqq n$. Then by (2.2),

$$G_n(u) = n \int_{-\infty}^{u} \frac{x^2}{1 + x^2}\, dF_n(x + \alpha_n) \to G(u) \quad \text{as} \quad n \to \infty \tag{2.9}$$

for all $u \,\varepsilon\, C(G)$. The only deterrent to establishing an estimate for $G(u)$ is the presence of α_n in (2.9). The problem then remains to eliminate the need for α_n; i.e., letting

$$\bar{G}_n(u) = n \int_{-\infty}^{u} \frac{x^2}{1 + x^2}\, dF_n(x), \tag{2.10}$$

the problem is to show that

$$\bar{G}_n(u) \to G(u) \text{ as } n \to \infty$$

for all $u \,\varepsilon\, C(G)$. Accordingly, let

$$G_n^{**}(u) = n \int_{-\infty}^{u} \frac{(x - \alpha_n)^2}{1 + (x - \alpha_n)^2}\, dF_n(x).$$

We now prove

(2.11) LEMMA: *$G_n^{**}(u) \to G(u)$ as $n \to \infty$ for all $u \,\varepsilon\, C(G)$.*

Indeed, for fixed $u \varepsilon C(G)$ and arbitrary $\epsilon > 0$ there exists a $\delta > 0$ and an integer N such that

$$u \pm \delta \varepsilon C(G),$$
$$G(u + \delta) - G(u - \delta) < \epsilon/2,$$
$$| \{G_n(u + \delta) - G_n(u - \delta)\} - \{G(u + \delta) - G(u - \delta)\} | < \epsilon/2,$$

and $| \alpha_n | < \delta$ for all $n > N$. Hence for all $n > N$

$$\left| n_u \int^{u+\alpha_n} \frac{(x - \alpha_n)^2}{1 + (x - \alpha_n)^2} dF_n(x) \right| = | G_n(u + \alpha_n) - G_n(u) |$$
$$\leqq | \{G_n(u + \delta) - G_n(u - \delta)\} - \{G(u + \delta) - G(u - \delta)\} |$$
$$+ | G(u + \delta) - G(u - \delta) | < \epsilon,$$

which proves the lemma.

We now prove:

(2.12) Lemma: *If $u < 0$ and if $u \varepsilon C(G)$, then $\bar{G}_n(u) \to G(u)$ as $n \to \infty$.*

In order to prove this we consider the function

$$f_n(x) = \frac{x^2}{1 + x^2} \frac{1 + (x - \alpha_n)^2}{(x - \alpha_n)^2}.$$

(For all sufficiently large n, $f_n(x)$ is finite for all $x \leqq u$.) Then

$$\bar{G}_n(u) = \int_{-\infty}^{u} f_n(x)\, dG_n^{**}(x),$$

and

$$| \bar{G}(u) - G(u) | \leqq \left| \int_{-\infty}^{u} f_n(x)\, dG_n^{**}(x) - \int_{-\infty}^{u} dG_n^{**}(x) \right|$$
$$+ \left| \int_{-\infty}^{u} dG_n^{**}(x) - G(u) \right|$$
$$\leqq \sup_{x \leqq u} | f_n(x) - 1 | G_n^{**}(u) + | G_n^{**}(u) - G(u) |$$

which converges to zero as $n \to \infty$ because of Lemma (2.11) and the fact that $f_n(x)$ converges uniformly to 1 over any closed set not containing zero.

In precisely the same way one can prove

(2.13) Lemma: *If $0 < a < b$ and if $a, b \varepsilon C(G)$, then $\bar{G}_n(b) - \bar{G}_n(a) \to G(b) - G(a)$ as $n \to \infty$.*

With Lemmas (2.12) and (2.13) we can now prove the following:

(2.14) Theorem: *If $u \varepsilon C(G)$, then $\bar{G}_n(u) \to G(u)$.*

Proof: Because of Lemma (2.12) we need only prove this in the case where $u > 0$. From the two inequalities

$$\begin{aligned}
&\frac{1}{1+(\tau+|\alpha_n|)^2}\left\{n\int_{-\tau}^{\tau}x^2\,dF_n(x)-2n\alpha_n^2+n\alpha_n^2(F_n(\tau)-F_n(-\tau))\right\}\\
&\qquad=\frac{1}{1+(\tau+|\alpha_n|)^2}\,n\int_{-\tau}^{\tau}(x-\alpha_n)^2\,dF_n(x)\\
&\qquad\leqq n\int_{-\tau}^{\tau}\frac{(x-\alpha_n)^2}{1+(x-\alpha_n)^2}\,dF_n(x)\leqq n\int_{-\tau}^{\tau}(x-\alpha_n)^2\,dF_n(x)\\
&\qquad\leqq n\int_{-\tau}^{\tau}x^2\,dF_n(x)-n\alpha_n^2
\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}
n\int_{-\tau}^{\tau}x^2\,dF_n(x)&\leqq(1+\tau^2)n\int_{-\tau}^{\tau}\frac{x^2}{1+x^2}\,dF_n(x)\\
&\qquad\leqq(1+\tau^2)n\int_{-\tau}^{\tau}x^2\,dF_n(x)
\end{aligned}\tag{2.16}$$

one obtains

$$\begin{aligned}
n\int_{-\tau}^{\tau}\frac{x^2}{1+x^2}\,dF_n(x)&\leqq n\int_{-\tau}^{\tau}x^2\,dF_n(x)\\
&\leqq\{1+(\tau+|\alpha_n|)^2\}n\int_{-\tau}^{\tau}\frac{(x-\alpha_n)^2}{1+(x-\alpha_n)^2}\,dF_n(x)\\
&\qquad\qquad+\{2-F_n(\tau)+F_n(-\tau)\}n\alpha_n^2\\
&\leqq\{1+(\tau+|\alpha_n|)^2\}n\int_{-\tau}^{\tau}x^2\,dF_n(x)\\
&\qquad\qquad+\{1-(\tau+|\alpha_n|)^2-F_n(\tau)+F_n(-\tau)\}n\alpha_n^2\\
&\leqq(1+\tau^2)(1+(\tau+|\alpha_n|)^2)n\int_{-\tau}^{+\tau}\frac{x^2}{1+x^2}\,dF_n(x)\\
&\qquad+(1+\tau^2)\{1-(\tau+|\alpha_n|)^2-F_n(\tau)+F_n(-\tau)\}n\alpha_n^2.
\end{aligned}\tag{2.17}$$

Now, in the particular situation of this differential process, it is easily seen that the sequence of constants $\{A_n\}$ must necessarily be convergent, and hence $A_n = n\alpha_n$ are bounded. This in turn implies that $\alpha_n \to 0$ as $n \to \infty$ and consequently

$$n\alpha_n^2 \to 0 \quad \text{as} \quad n \to \infty. \tag{2.18}$$

From the inequality (2.17) and for τ always selected such that $\tau, -\tau \in C(G)$, one easily obtains

$$\begin{aligned}
\overline{\lim_n}\, n\int_{-\tau}^{\tau}\frac{x^2}{1+x^2}\,dF_n(x)&\leqq(1+\tau^2)\{G(\tau)-G(-\tau)\}\\
&\leqq(1+\tau^2)^2\,\underline{\lim_n}\, n\int_{-\tau}^{\tau}\frac{x^2}{1+x^2}\,dF_n(x).
\end{aligned}$$

Now let $0 < \tau < u$. Then

$$\underline{\lim_n}\, n\int_{-\infty}^{u}\frac{x^2}{1+x^2}\,dF_n(x)\geqq G(u)-G(\tau)+\frac{1}{1+\tau^2}\{G(\tau)-G(-\tau)\}+G(-\tau),$$

and taking the limit of both sides as $\tau \to 0$, one obtains

$$\underline{\lim}_{n}\, n \int_{-\infty}^{u} \frac{x^2}{1+x^2}\, dF_n(x) \geqq G(u).$$

In a similar fashion,

$$\overline{\lim}_{n}\, n \int_{-\infty}^{u} \frac{x^2}{1+x^2}\, dF_n(x) \leqq G(u),$$

and the theorem is proved.

Theorem (2.14) asserts that

$$\bar{G}_n(u) = \sum_{k=1}^{n} \mathcal{E}\left\{\frac{X_{nk}^2}{1+X_{nk}^2} I_{[X_{nk} \leqq u]}\right\} \to G(u)$$

as $n \to \infty$ for every $u \, \varepsilon \, C(G)$, or

$$\bar{G}_n(u) = n\mathcal{E}\left\{\frac{X_{n1}^2}{1+X_{n1}^2} I_{[X_{n1} \leqq u]}\right\} \to G(u)$$

as $n \to \infty$ for all $u \, \varepsilon \, C(G)$. By the strong law of large numbers,

$$G_{N,n}^*(u) = \frac{1}{N} \sum_{k=1}^{nN} \frac{X_{nk}^2}{1+X_{nk}^2} I_{[X_{nk} \leqq u]} \to \bar{G}_n(u) \tag{2.19}$$

as $N \to \infty$ for every value of u with probability one, for every fixed n. Since $G(u)$ and $G_{N,n}^*(u)$ are nondecreasing in u, one can then write

$$P\{\lim_{n\to\infty} \lim_{N\to\infty} G_{N,n}^*(u) = G(u) \quad \text{for all} \quad u \, \varepsilon \, C(G)\} = 1. \tag{2.20}$$

The estimate to be used is $G_{N,n}^*(u)$ as defined in (2.19). It is strongly consistent in the sense given by (2.20). This estimate, however, is not necessarily an unbiased estimate of $G(u)$ for all values of $u \, \varepsilon \, C(G)$. This is the case when $X(t)$ is a pure Poisson process, i.e.,

$$\log f_{X(t)}(u) = iuat + \lambda t(e^{iub} - 1),$$

where, say, $a > 0$ and $b > 0$. In this case

$$G(x) = \begin{cases} 0 & \text{if} \quad x < b \\ \lambda b^2(1+b^2)^{-1} & \text{if} \quad x > b. \end{cases}$$

If $u \leqq 0$, then for every n, N, it is easily checked that $\mathcal{E}(G_{N,n}^*(u)) = 0$. However, if $0 < x < b$, then for those values of n for which $0 < a/n < x$, one obtains

$$\mathcal{E}(G_{N,n}(x)) = \frac{na^2}{n^2+a^2} > 0, \qquad \text{for all } N,$$

and if $x \geqq b$, then

$$\mathcal{E}(G_{N,n}^*(x)) = ne^{-\lambda/n}\left\{\frac{a^2}{n^2+a^2} + \frac{(a+nb)^2}{n^2+(a+nb)^2}\frac{\lambda}{n}\left(1 + \frac{\lambda}{n}\sum_{i=2}^{\left[\frac{x-a}{b}\right]} \frac{(\lambda/n)^{e-2}}{i!} \cdot \frac{(a+nib)^2}{n^2+(a+nib)^2}\right)\right\}$$

which is clearly greater than $\lambda b^2(1+b^2)^{-1}$.

3. The second method of estimating G. Because of the fact that X is a differential process, once the probability distribution is given for, say, $X(1)$, then it is known for $X(t)$ for every t. Furthermore, for every finite set of values of t, say, $t_1, t_2, \cdots, t_m$, the joint distribution of the random variables $X(t_1), X(t_2), \cdots, X(t_m)$ can be derived from the distribution of $X(1)$. From a probabilistic point of view, two stochastic processes are the same if their corresponding finite dimensional marginal distributions agree. Accordingly, it is found convenient to construct and prove properties of an estimate of G by constructing a process equivalent to X which, because of the equivalence, shall be labeled X.

Using the γ and $G(x)$ in (1.1), a stochastic process in two variables, $Y(b, t)$, is considered, where $b \varepsilon (-\infty, +\infty)$ and $t \varepsilon [0, \infty]$. It is assumed that $Y(b, t)$ is a process with independent "generalized increments" on the (b, t)-half plane over which it is defined, i.e., if $\{A_\mu : \mu \varepsilon M\}$ is a disjoint family of Borel sets on this plane, then the random variables

$$\left\{ \int_{A_\mu} Y(db, dt), \mu \varepsilon M \right\}$$

are independent. The probability distribution of $Y(b, t)$ is assumed to have a characteristic function $f_{Y(b,t)}(u)$ for which

$$\begin{aligned} \log f_{Y(b,t)}(u) &= itu\gamma(G(\infty) - \sigma^2)^{-1} \int_{-\infty}^{b-0} dG(x) \\ &\quad + t \int_{-\infty}^{b-0} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right) \frac{1+x^2}{x^2}\, dG(x), \end{aligned} \tag{3.1}$$

where $\sigma^2 = G(+0) - G(-0)$ is actually the variance of the normal component of $X(1)$. (In case $G(+\infty) = \sigma^2$, then the results stated in this section concerning the estimates of $G(x)$ hold trivially without using $Y(b, t)$.) Then for every t we have

$$\mathrm{X}(t) = \int_{-\infty}^{+\infty} Y(db, t), \tag{3.2}$$

the integral existing with probability one since it converges in distribution. From the process $Y(b, t)$ we construct a sequence of independent stochastic processes, each one in this sequence being a process with independent increments. Let

$$Y_0(t) = Y(+0, t) - Y(-0, t), \tag{3.3}$$

i.e., $Y_0(t)$ is a normal process with mean zero and variance $t\sigma^2$, and let, for fixed positive integer n,

$$Y_{n,k}(t) = \begin{cases} Y\left(\frac{1}{n}, t\right) - Y(+0, t) & \text{if } k = 1 \\ Y\left(\frac{k}{n}, t\right) - Y\left(\frac{k-1}{n}, t\right) & \text{if } k = 0, -1, \pm 2, \pm 3, \cdots. \end{cases} \tag{3.4}$$

Then for every t

$$X(t) = Y_0(t) + \sum_{k=-\infty}^{+\infty} Y_{nk}(t) \tag{3.5}$$

with probability one, since this series of independent random variables does converge in distribution.

The most difficult problem that occurs in this section is to find an estimate for the saltus of $G(x)$ at $x = 0$, i.e., to find an estimate of the variance of the normal component, $Y_0(t)$, of the process. Let $k_1, k_2, \cdots, k_n, \cdots$ denote a sequence of integers such that

$$\sum_n k_n^{-\frac{1}{2}} < \infty. \tag{3.6}$$

Also let

$$\begin{aligned} Y(t) &= X(t) - Y_0(t), \text{ or} \\ Y(t) &= \textstyle\int_{-\infty}^{+\infty} Y(db, t) = \sum_{k=-\infty}^{+\infty} Y_{n.k}(t), \end{aligned} \tag{3.7}$$

and let

$$\begin{aligned} Q_N^2 = \sum_{j=1}^{k_N} \{X(j/k_N) - X((j-1)/k_N)\}^2 \\ - \sum_{j=1}^{k_N} \{Y(j/k_N) - Y((j-1)k_N)\}^2. \end{aligned} \tag{3.8}$$

We can now prove

(3.9) LEMMA: $P\{Q_N^2 \rightarrow \sigma^2 \text{ as } N \rightarrow \infty\} = 1.$

PROOF: From (3.7) and (3.8),

$$Q_N^2 = \sum_{j=1}^{k_N} \{Y_0(j/k_N) - Y_0((j-1)/k_N)\}^2 + 2Z_N, \text{ where}$$

$$\begin{aligned} Z_N = \sum_{j=1}^{k_N} \{(Y_0(j/k_N) - Y_0((j-1)/k_N)) \sum_{k=-\infty}^{+\infty} (Y_{n,k}(j/k_N) \\ - Y_{nk}((j-1)/k_N))\}. \end{aligned} \tag{3.10}$$

Since $Q_N^2 - 2Z_N$ has mean σ^2 and variance $2\sigma^4/k_N$, it converges to σ^2 with probability one. It remains to prove that $Z_N \rightarrow 0$ as $N \rightarrow \infty$ with probability one. It remains to prove that $Z_N \rightarrow 0$ as $N \rightarrow \infty$ with probability one. Toward this end, let, for every fixed k,

$$\begin{aligned} Z_{N,k} = \sum_{j=1}^{k_N} \{Y_0(j/k_N) - Y_0((j - \ )/k_N)\} \\ \times \{Y_{n,k}(j/k_N) - Y_{n,k}((j-1)/k_N)\}. \end{aligned} \tag{3.11}$$

The expectation of each summand is zero, since it is the product of two independent random variables of finite expectation and $\mathcal{E} Y_0(t) = 0$. Hence

(3.12) $$\mathcal{E}Z_{N,k} = 0.$$

We may write $Z_{N,k} = \sum_{j=1}^{k_N} U_j V_j$, where $U_j = Y_0(j/k_N) - Y_0((j-1)/k_N)$, and $V_j = Y_{nk}(j/k_N) - Y_{nk}((j-1)/k_N)$. Then one easily obtains Var $(Z_{N,k}) = (1/k_N)\sigma^2 \sum_{j=1}^{k_N} \mathcal{E}V_j^2 = \sigma^2 \mathcal{E}(V_1^2)$. But

$$\mathcal{E}(V_1^2) = \frac{1}{k_N}\left\{ \int_{(k-1)/n}^{k/n} (1+x^2)\, dG(x) + \frac{1}{k_N}\left(\gamma\alpha_{k,n} + \int_{(k-1)/n}^{k/n} x\, dG(x)\right)^2 \right\},$$

where

$$\alpha_{kn} = \frac{G(k/n) - G((k-1)/n)}{G(+\infty) - \sigma^2}.$$

Hence for fixed k

(3.13) $$\operatorname{Var}(Z_{N,k}) = (\sigma^2/k_N)(R + S/k_N),$$

where R and S are both finite and do not depend on N. Hence, for arbitrary $\epsilon > 0$ and because of (3.6), (3.13) and Chebishev's inequality,

(3.14) $$\sum_{N=1}^{\infty} P\{ | Z_{N,k} - 0 | \geq \epsilon\} \leqq \frac{1}{\epsilon^2} \sum_{N=1}^{\infty} \operatorname{Var}(Z_{N,k}) < \infty,$$

which in turn implies (by applying the Borel-Cantelli lemma) that

(3.14a) $$Z_{N,k} \to 0 \quad \text{as} \quad N \to \infty$$

with probability one for every fixed k.

We now show that $Z_N = \sum_{k=-\infty}^{+\infty} Z_{N,k}$ converges with probability one to zero as $N \to \infty$. From (3.1) and (3.4), we have, for $k \neq 0$, $k \neq 1$,

(3.15) $$\log f_{Y_{n,k}(t)}(u) = iut\beta_{n,k} + t \int_{(k-1)/n}^{k/n} (e^{iux} - 1) \frac{1+x^2}{x^2}\, dG(x),$$

where

$$\beta_{nk} = \frac{\gamma(G(k/n) - G((k-1)/n))}{G(\infty) - \sigma^2} + \int_{(k-1)/n}^{k/n} \frac{1}{x}\, dG(x).$$

(3.16) Let $W_{nk}(t) = Y_{nk}(t) - t\beta_{nk}$.

Then since

(3.17) $$\beta = \sum_{k \neq 0,1} | \beta_{nk} | \leqq \frac{\gamma G(\infty)}{G(\infty) - \sigma^2} + G(\infty) < \infty,$$

the series of independent random variables

$$\sum_{k \neq 0,1} W_{nk}(t)$$

converges in law and hence converges with probability one. This implies that $| W_{nk}(t) | \geqq \epsilon$ for only a finite number of values of k with probability one for arbitrary $\epsilon > 0$, in particular for $\epsilon = 1/n$. Also note that if $| W_{nk}(t) | > 0$, then

$| W_{nk}(t) | \geqq (k - 1)/n$. Now let A_i denote the event that i is the largest value of $| k | \geqq 2$ such that $W_{nk}(t) \neq 0$, i.e., $| W_{nk}(t) | \geqq 1/n$, and let A_1 denote the event that $W_{nk}(t) = 0$ for all $| k | \geqq 2$. Because of the fact that $| W_{nk}(t) | \geqq 1/n$ for only a finite number of values of k with probability one, we obtain

$$\sum_{i=1}^{\infty} PA_i = P\left(\bigcup_{i=1}^{\infty} A_i\right) = 1. \tag{3.17a}$$

Now we may write

$$Z_N = S_N + T_N + Z_{N,1} + Z_{N,0}, \tag{3.18}$$

where

$$S_N = \sum_{j=1}^{k_N} \{(Y_0(j/k_N) - Y_0((j-1)/k_N))\sum_{k\neq 0,1} (W_{n,k}(j/k_N) - W_{n,k}((j-1)/k_N))\},$$

$$T_N = \sum_{j=1}^{k_N} \left\{ \frac{1}{k_N} \left(\sum_{k\neq 0,1} \beta_{nk}\right)(Y_0(j/k_N) - Y_0((j-1)/k_N))\right\}.$$

Because of (3.17) and the fact that $\mathcal{E}T_N = 0$,

$$\sum_{N=1}^{\infty} P\{ | T_N | \geqq \epsilon\} \leqq \frac{\beta\sigma^2}{\epsilon^2} \sum_N k_N^{-2} < \infty,$$

and consequently $T_N \to 0$ as $N \to \infty$ with probability one. By (3.14a), $Z_{N,1} + Z_{N,0} \to 0$ as $N \to \infty$ with probability one. Now

$$S_N = \sum_{i=2}^{M} S_N I_{A_i} + S_N I\left(\bigcup_{i=M+1}^{\infty} A_i\right),$$

where, as before, I_S is the indicator of S. By (3.14a), $P\{S_N I_{A_i} \to 0 \text{ as } N \to \infty\} = 1$ for every integer i. Hence $Z_N \to 0$ as $N \to \infty$ over the set $\bigcup_{i=1}^{M} A_i$ for every M except for a set of measure zero, and hence we can conclude by (3.17a) that $P\{Z_N \to 0 \text{ as } N \to \infty\} = 1$. Thus the lemma is proved.

Unfortunately, Q_N^2 is not an observable random variable. In the definition of Q_N^2 in (3.8), the part that is not observable is

$$L_N^2 = \sum_{j=1}^{k_N} \{Y(j/k_N) - Y((j-1)/k_N)\}^2.$$

We show now that L_N^2 converges with probability one to a bona fide random variable D^2, which is the sum of the squares of the "jumps", i.e.,$L_N^2 \to D^2 = \int_0^1 (Y(dt))^2$. From a practical standpoint D^2 can be considered as "observable" while L_N^2 is observable only if $\sigma^2 = 0$. Thus we prove

(3.19) Lemma: *$L_N^2 \to D^2$ as $N \to \infty$ and D^2 is finite, with probability one.*

Proof: Let us define

$$M_n = k_n^{-\frac{1}{4}}. \tag{3.20}$$

Then from (3.6) and (3.20) we obtain

$$\sum_n M_n^2 < \infty \quad \text{and} \quad \sum_n k_n^{-1} M_n^{-4} < \infty. \tag{3.21}$$

Then for every positive integer n we define two independent differential processes, $U_n(t)$ and $V_n(t)$, for which the logarithms of their characteristic functions are

$$\log f_{V_n(t)}(u) = t\left\{ \int_{-\infty}^{-M_n} + \int_{+M_n}^{+\infty} (e^{iux} - 1) \frac{1+x^2}{x^2} dG(x) \right\},$$

and

$$\log f_{U_n(t)}(u) = iut\gamma_n + t \int_{-M_n}^{+M_n} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{1+x^2}{x^2} dG(x),$$

where

$$\gamma_n = \gamma - \left\{ \int_{-\infty}^{-M_n} + \int_{M_n}^{+\infty} \frac{1}{x} dG(x) \right\}.$$

Clearly one can (equivalently) write $Y(t) = U_n(t) + V_n(t)$ for every n, and $V_n(t)$ has sample functions which are step functions, the jumps of which in absolute value are not less than M_n. Now let

$$\begin{aligned} R_n &= \sum_{j=1}^{k_N} \{U_n(j/k_n) - U_n((j-1)/k_n)\}^2, \\ S_n &= \sum_{j=1}^{k_N} \{U_n(j/k_n) - U_n((j-1)/k_n)\}\{V_n(j/k_n) - V_n((j-1)/k_n)\} \\ T_n &= \sum_{j=1}^{k_N} \{V_n(j/k_n) - V_n(j-1)/k_n)\}^2. \end{aligned} \tag{3.22}$$

Then

$$L_n^2 = R_n + 2S_n + T_n. \tag{3.23}$$

For $\epsilon > 0$, we define

$$G_\epsilon(x) = \begin{cases} G(x) & \text{if} \quad x \leqq -\epsilon \\ G(-\epsilon) & \text{if} \quad |x| < \epsilon \\ G(x) - G(\epsilon) + G(-\epsilon) & \text{if} \quad x \geqq \epsilon. \end{cases}$$

The proof of the lemma will be accomplished by proving that the following three statements are true with probability one:

i) for sufficiently large values of n, T_n is the sum of the squares of the jumps during [0, 1] which are in absolute value $\geqq M_n$.

ii) $R_n \to 0$ as $n \to \infty$, and

iii) $\lim_n T_n$ is finite.

Once we prove i), ii), and iii), the lemma will follow easily. For by i) and (3.20), we have that $T_n \to D^2 = \int_0^1 (Y(dt))^2$ as $n \to \infty$ with probability one. By the

Cauchy inequality, $S_n^2 \leqq R_n T_n$, and thus because of ii) and iii), we have that $S_n \to 0$ as $n \to \infty$ with probability one. Hence the lemma would easily follow. We first prove (ii). Let $g(u)$ denote the characteristic function of

$$U_{jn} = U_n(j/k_n) - U_n(j-1)/k_n).$$

Then

$$\log g(u) = iuk_n^{-1}\left\{\gamma + \int_{M_n \leqq |x|} \frac{1}{x}\, dg(x)\right\} + k_n^{-1} \int_{-M_n}^{M_n} \left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\, dG(x).$$

The first four semi-invariants of U_{jn} are

$$\varkappa_1 = ik_n^{-1}\{\gamma + \int_{|x| \geqq M_n} x^{-1}\, dG(x) + \int_{-M_n}^{M_n} x\, dG(x)\},$$

$$\varkappa_2 = - k_n^{-1} \int_{-M_n}^{M_n}(1+x^2)\, dG(x),$$

$$\varkappa_3 = - ik_n^{-1} \int_{-M_n}^{M_n} x(1+x^2)\, dG(x), \text{ and}$$

$$\varkappa_4 = k_n^{-1} \int_{-M_n}^{M_n} x^2(1+x^2)\, dG(x).$$

Since Var $(U_{jn}^2) = \varkappa_4 + 2\varkappa_2^2 + 4\varkappa_1\varkappa_3 + 4\varkappa_1^2\varkappa_2$, we obtain

$$\text{Var } R_n = \int_{-M_n}^{M_n} x^2(1+x^2)\, dG(x) + (1/k_n)B_n,$$

where the sequence of B_n's is bounded by a finite number, say B. Thus

$$(3.24) \quad \sum_n \text{Var } R_n \leqq \{G(M_1) - G(-M_1)\}(\sum_n M_n^2 + \sum_n M_n^4) + B\sum_n k_n^{-1} < \infty$$

because of (3.21). Consequently (3.24) implies that

$$P\{R_n - \mathcal{E}R_n \to 0 \text{ as } n \to \infty\} = 1.$$

But

$$\mathcal{E}R_n = \int_{-M_n}^{M_n} (1+x^2)\, dG(x) + k_n^{-1}\{\gamma^2 + 2\gamma \int_{|x| \geqq M_n} x^{-1}\, dG(x) + [\int_{|x| \geqq M_n} x^{-1}\, dG(x)]^2 + 2\gamma \int_{-M_n}^{M_n} x\, dG(x) + 2 \int_{|x| \geqq M_n} x^{-1}\, dG(x) \int_{-M_n}^{M_n} x\, dG(x) + [\int_{-M_n}^{M_n} x\, dG(x)]^2\}.$$

By (3.21), $\mathcal{E}R_n \to 0$ as $n \to \infty$.
Hence $P\{R_n \to 0 \text{ as } n \to \infty\} = 1$. Thus ii) is proved. We now prove i). Let A_n denote the event that *at least* 2 jumps of size (in absolute value) $\geqq M_n$ occur in *at least* one of the k_n subintervals of length k_n^{-1}. Then

$$P(A_n) = 1 - \{P[X_n \leqq 1]\}^{k_n},$$

where X_n denotes the number of jumps of size (in absolute value) $\geqq M_n$ during a specific time interval of length k_n^{-1}. Now it is known (e.g., Doob [1], page 423) that X_n has a Poisson distribution with parameter

$$\lambda_n = k_n^{-1} \int_{-\infty}^{\infty} \frac{1+x^2}{x^2}\, dG_{M_n}(x).$$

We want to prove that $\sum_n PA_n < \infty$. We first note that

$$0 \leqq \lambda_n \leqq \frac{1}{k_n} \frac{1 + M_n^2}{M_n^2} K, \tag{3.24a}$$

where $K = G(+\infty) - G(-\infty)$; for simplicity, let $k = 1$. Now $\sum_n P(A_n) = \sum_n [1 - \{e^{-\lambda_n}(1 + \lambda_n)\}^{k_n}]$ is a convergent series if and only if the infinite product $\Pi_0 = \prod_{n=1}^{\infty} \{e^{-\lambda_n}(1 + \lambda_n)\}^{k_n}$ is convergent (i.e., does not diverge to zero). Note that each term in the product does not exceed 1. Easily, Π_0 converges if and only if $P = \sum_n \log \{e^{-\lambda_n}(1 + \lambda_n)\}^{k_n}$ is absolutely convergent. But $P = \sum_{n=1}^{\infty} \{-k_n\lambda_n + k_n \log (1 + \lambda_n)\}$. But this is equal to

$$\begin{aligned} \sum_n \left\{k_n \lambda_n - k_n \left(\lambda_n - \frac{\theta_n \lambda_n^2}{2}\right)\right\} &\leqq \tfrac{1}{2} \sum_n k_n \lambda_n^2 \\ &\leqq \tfrac{1}{2} \sum_n k_n^{-1} M_n^{-4}(1 + 2M_n^2 + M_n^4) < \infty \end{aligned}$$

by (3.21). Thus $\sum_n P(A_n) < \infty$. Hence by the Borel-Cantelli lemma, the probability that infinitely many of the A_n's occur is zero. Thus i) is proved. In conclusion, we prove iii). To do this, we simply note that, for every n, $V_n(t)$ has only a finite number of jumps with probability one. (The number of these jumps follows a Poisson distribution with parameter λ_n). But also, $U_n(t)$ is a differential process with finite variance and consequently the expectation of the sum of the squares of the jumps is finite. This proves iii). Thus the lemma is proved.

Having obtained lemmas (3.9) and (3.19), we obtain the following theorem without additional proof:

THEOREM: *$Q_N'^2 = \sum_{j=1}^{k_N} \{X(j/k_N) - X((j-1)/k_N)\}^2 - D^2$ converges with probability one to σ^2 as $N \to \infty$.*

Let $N_t(S)$ denote the number of jumps of the sample function during $[0, t)$ of size in S. Then set

$$J_t(x) = \int_{-\infty}^{x} \frac{b^2}{1 + b^2} dN_t(b). \tag{3.25}$$

Now consider

$$J_t(x) - J_t(y) - \int_y^x dG(b) = H_t(x, y) \tag{3.26}$$

If $0 \not\in (y, x]$, $N(y, x]$ is finite with probability one and

$$EN_t(y, x] = \int_{y+}^{x} [(1 + b^2)/b^2] \, dG(b).$$

Then ([1], p. 437), $H_t(x, y)$ is a martingale in both x and $-y$. Since

$$E(| H_t(x, y) |) \leqq \int_{-\infty}^{\infty} dG(b),$$

it follows that the definition of H_t can be extended to $0 \in (y, x]$ and for y converging to $-\infty$. Thus $J_t(x)$ is well-defined and an unbiased estimate of $\int_{-\infty}^{x} dG(b)$ for all x. Hence we obtain the results in Section 1.

4. Comparison of the two estimates of G. The two estimates obtained for G are not necessarily equal since the first estimate obtained can be biased, while the second estimate obtained is unbiased over all intervals which do not contain zero. It does happen that at an intermediate limiting case the two estimates are equal with probability one at all continuity points of the second estimates, and this section is devoted to the proof of this fact. In particular, we prove that

$$(4.1) \qquad p \lim_{n\to\infty} G^*_{N,n}(u) = \lim_{n\to\infty} \hat{G}_{n,N}(u) \quad \text{for all} \quad u \, \varepsilon \, C(\hat{G}_{n,N})$$

This can be expressed as follows

$$(4.2) \quad p_n \lim G^*_{1n}(u) = \begin{cases} \displaystyle\int_{-\infty}^{u} \frac{x^2}{1+x^2}\, dN_1(x) \quad \text{if} \quad u < 0 \\ \displaystyle\int_{-\infty}^{u} \frac{x^2}{1+x^2}\, dN_1(x) + \int_0^1 (dX(t))^2 - \int_{-\infty}^{+\infty} x^2\, dN_1(x) \\ \text{if} \quad x > 0 \end{cases}$$

for every $u \, \varepsilon \, C(\hat{G}_{n,1})$. We shall prove the result in the form expressed in (4.2).

Let $Y_\epsilon(t)$ denote the process formed by the jumps exceeding ϵ in absolute value, Y_0 the normal component and $Y^\epsilon(t)$ the remaining process with the trend term included. As $X_{nk} = X(k/n) - X((k-1)/n)$, we similarly define $Y_{\epsilon nk}$, Y^ϵ_{nk}, and Y_{0nk}. Now

$$(4.3) \qquad nP(|Y_{0nk}| \geqq \alpha) \leqq \frac{1}{\alpha}\sqrt{\frac{2n}{\pi}}\, e^{-n\alpha^2/2}$$

and hence approaches 0. Also

$$(4.4) \qquad nP(|Y^\epsilon_{nk}| \geqq \alpha) \leqq \frac{\operatorname{Var} Y^\epsilon(1)}{(\alpha - \beta/n)^2},$$

$\beta = E(|Y^2_{nk}(1)|)$. Thus if α_n approaches 0 sufficiently slowly as ϵ_n approaches 0, these both become small and hence

$$(4.5) \qquad P\ (\text{for some } k, \ |X_{nk} - Y_{\epsilon nk}| \geqq 2\alpha_n) \to 0.$$

Also if ϵ_n approaches 0 sufficiently slowly, we have already seen that the probability that there are at least two jumps of size larger than ϵ_n in any interval of length $1/n$ approaches 0. Now let $u < 0$ not be the size of a jump. Then for n sufficiently large, there is no jump whose size lies in $(u - 2\alpha_n, u + 2\alpha_n)$. Thus with large probability,

$$(4.6) \qquad G^*_{1n}(u) = \Sigma_1 \frac{X^2_{nk}}{1 + X^2_{nk}},$$

where the sum is over those k's for which there is a jump of value less than μ in $[(k-1)/n, k/n]$, and there is a one-to-one correspondence between such jumps and intervals. Let v_{nk} be the size of the jump corresponding to X_{nk}, i.e., $v_{nk} = X_{nk} - Y_{0nk} - Y_{\epsilon nk}$. Observing that

$$\left|\frac{x^2}{1+x^2} - \frac{y^2}{1+y^2}\right| \leqq \frac{|x-y|}{2},$$

with large probability $G_{1n}^*(u)$ differs from $\int_{-\infty}^u (b^2/(1+b^2))\,dN_1(b)$ by less than $\alpha_n N(-\infty, u)$ and hence the first part of (4.2) is proved. Similarly, the second part of (4.2) holds except possibility for an additive constant. To evaluate this constant we consider $G_{1n}^*(u) - G_{1n}^*(-u)$ where both u and $-u$ are continuity points of N. Then with large probability,

$$G_{1n}^*(u) - G_{1n}^*(-u) = \Sigma_2 \frac{X_{nk}^2}{1+X_{nk}^2}, \tag{4.7}$$

where the sum is over those k's for which there is no jump of absolute value exceeding u in $[(k-1)/n, k/n]$.

Let Σ_3 denote the complementary sum. Now

$$\frac{1}{1+u^2}\Sigma_2 X_{nk}^2 \leqq G_{1n}^*(u) - G_{1n}^*(-u) \leqq \Sigma_2 X_{nk}^2. \tag{4.8}$$

But

$$\left|\Sigma_3 X_{nk}^2 - \int_{|b|>u} b^2\,dN(b)\right| \leqq 4\alpha_n\left[u + \int_{|b|>u} |b|\,dN(b)\right]$$

with large probability for n sufficiently large. Thus

$$\begin{aligned}\frac{1}{1+u^2}\left(\Sigma X_{nk}^2 - \int_{|b|>u} b^2\,dN(b)\right) - \delta_n &\leqq \hat{G}_{1n}(u) - \hat{G}_{1n}(-u)\\ &\leqq \Sigma X_{nk}^2 - \int_{|b|>u} b^2\,dN(b) + \delta_n\end{aligned}$$

with large probability. However, ΣX_{nk}^2 approaches $\int_0^1 (dX(t))^2$. Hence

$$\lim_{u\to 0} p \lim_{n\to\infty} (G_{1n}^*(u) - G_{1n}^*(-u)) = \int_0^1 (dX(t))^2 - \int_{-\infty}^{\infty} x^2\,dN_1(x), \tag{4.9}$$

which completes the proof of (4.2).

5. Estimating the trend term, γ. As remarked earlier, the probability law of the differential process X is completely determined by a constant γ and a bounded non-decreasing function G as given in (1.1). Two estimates have already been obtained for G; it now remains to find an estimate for γ. In this section we shall derive an unbiased estimate $\hat{\gamma}$ of γ based upon complete observation of the process for $t \in [0, 1]$. Indeed we shall prove that

$$\hat{\gamma} = X(1) - X(0) - \int_{\infty}^{+\infty} \frac{x^3}{1+x^2}\,dN_1(x)$$

is an unbiased estimate of γ. It is trivial to prove that $\hat{\gamma}$ is an unbiased estimate of γ when X is a pure Gaussian process. Hence we shall assume that X is not purely Gaussian in the development that follows.

We can effectively represent $X(t)$ as a sum of three independent differential processes,

$$X(t) = U(t) + V(t) + W(t), \tag{5.1}$$

where

$$\ln f_{U(t)}(u) = -\frac{\sigma^2 u^2 t}{2},$$

$$\ln f_{V(t)}(u) = i\gamma_1 ut + t\int_{-K}^{K}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\,dG(x), \tag{5.2}$$

$$\ln f_{W(t)}(u) = i\gamma_2 ut + t\int_{|x|\geq K}\left(e^{iux} - 1 - \frac{iux}{1+x^2}\right)\frac{1+x^2}{x^2}\,dG(x),$$

where

$$\begin{aligned}\gamma_1 &= \gamma \textstyle\int_{-K}^{K} dG(x)/\int_{-\infty}^{+\infty} dG(x),\\ \gamma_2 &= \gamma \textstyle\int_{|x|\geq K} dG(x)/\int_{-\infty}^{+\infty} dG(x),\end{aligned} \tag{5.3}$$

and $K > 0$ is an arbitrarily selected constant such that both K and $-K$ are points of continuity of G. We first note that

$$-i\frac{d}{du}\ln f_{V(1)}(u) = \left(\gamma_1 + \int_{-K}^{K} x\,dG(x)\right) = E(V(1))$$

exists and is finite. The results of section 3 and the Lebesgue convergence theorem imply that

$$\hat{\gamma}_1 = V(1) - V(0) - \int_{-K}^{K}\frac{x^3}{1+x^2}\,dN_1(x) \tag{5.4}$$

is an unbiased estimate of γ_1. Further, we may write

$$\ln f_{W(1)}(u) = iu\left(\gamma_2 - \int_{|x|\geq K}\frac{1}{x}\,dG(x)\right) + \int_{|x|\geq K}(e^{iux} - 1)\frac{1+x^2}{x^2}\,dG(x). \tag{5.5}$$

Thus $W(1)$ can be expressed as

$$W(1) = \gamma_2 - \int_{|x|\geq K}\frac{1}{x}\,dG(x) + \int_{|x|\geq K} x\,dN_1(x), \tag{5.6}$$

and consequently $W(1) - \int_{|x|\geq K} x\,dN_1(x)$ is a constant. But again by the results of section 3, the random variable $\int_{|x|\geq K}(x/(1+x^2))\,dN_1(x)$ is an unbiased estimate of $\int_{|x|\geq K}(1/x)\,dG(x)$. Hence

$$\hat{\gamma}_2 = W(1) - W(0) - \int_{|x|\geq K}\frac{x^3}{1+x^2}\,dN_1(x) \tag{5.7}$$

is an unbiased estimate of γ_2. Since $\gamma = \gamma_1 + \gamma_2$, then $\hat{\gamma} = X(1) - X(0) - \int_{-\infty}^{+\infty}(x^3/(1+x^2))\,dN_1(x)$ is an unbiased estimate of γ.

Now if we let

$$\hat{\gamma}(t) = t^{-1}\left\{X(t) - X(0) - \int_{-\infty}^{\infty} \frac{x^3}{1+x^2}\, dN_t(x)\right\},$$

not only is $\hat{\gamma}(t)$ an unbiased estimate of γ but is consistent in the sense that

$$P\{\lim_t \gamma(t) = \hat{\gamma}\} = 1.$$

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[2] B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Addison-Wesley Publishing Company, Inc., 1954.