SAMPLING INSPECTION AS A MINIMUM LOSS PROBLEM

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Introduction. In March 1958, in a lecture at Berkeley, Milton Friedman pointed out that statisticians, when asked to recommend a sampling inspection plan for the producer or buyer of a mass product, usually ask him questions which he cannot answer. They first ask: What percentage of defectives would you allow without rejecting the product? If the answer is p_0, the statistician would choose a smaller percentage p_1 and a larger one p_2, and ask questions like: Would you allow a probability of 5 per cent of rejecting the product, if the true percentage is p_1? The answers usually are mere guesses.

However, a competent manager could answer questions like: What is the cost of inspecting a sample of n? What would be your profit or loss if you buy or sell a lot with defective fraction p? What do you do if you reject the product, and what would be your loss in this case? Would it be very expensive to improve the quality of your product? A reasonable production and inspection plan ought to be based solely on these loss functions.

In what follows, we shall leave aside production problems. We shall assume a plant to produce a product of variable quality, the variations being due to accidents we cannot prevent. The only thing the producer can do is to inspect a sample and, if it contains too many defectives, to examine the whole lot and to eliminate the defectives. And the only thing the buyer can do is to inspect a sample and, if it contains too many defectives, to return the product to the producer.

The loss functions will be assumed to be linear functions of the defective fraction p, and the inspection cost to be proportional to the size n of the sample. We shall assume that the same inspection plan is used every day, or in the buyer's case every time he buys a lot, so that in the long run only the average loss counts.

In Sections 1–3 and in Section 4, the minimum loss problem will be discussed from the producer's and from the buyer's point of view separately. In Section 5, the producer's and the buyer's point of view will be combined. It will be shown that the two partners may increase their joint profit by forming a coalition and combining their inspection plans into one.

After having finished an earlier draft of this paper, I learned that S. Moriguti [1] and S. Ura [2] investigated the problem of minimax inspection plans from exactly the same point of view. Moriguti's results are just the same as mine obtained in Section 2 for Case A (p_0n and q_0n large). Ura's results are close to mine obtained in Section 3 for Case B (n large, p_0n not large), but there are slight differences in the numerical values. On the other hand, Ura treated the

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cases $k = 1, 2, 3, 4$ and 5, whereas my calculations stop at $k = 2$. Since my assumptions are a little more general than those of Moriguti and Ura and since their papers are not easily accessible, I shall expose the theory anew.

It would be interesting to extend the theory to sequential tests. A guess concerning the best sequential test will be formulated at the end of Section 4.

1. The producer's problem.

A. The loss function. Suppose a producer takes a sample of his product every day and applies a test. If the lot is accepted, it is sold at a fixed price, but defective units may be sent back by the buyer. The resulting loss is proportional to the defective fraction p, so the loss will be, in the case of acceptance,

$$L' = ap. \tag{1}$$

If the product is rejected, the whole lot will be examined and the defectives will not be sold. Let the cost of inspection of the whole lot be c, and the loss resulting from not selling the defectives bp. Hence, the loss in the case of rejection is

$$L'' = bp + c. \tag{2}$$

On the other hand, if rejected lots are discarded (which is, of course, always the case when inspection is destructive), we have to replace (2) by $L'' = c$, that is, we have to put $b = 0$ in (2).

To these losses, we have to add the cost of inspection of the sample. For the sake of simplicity, we shall suppose that the test is not sequential, so that the cost of inspection is simply fn, where n is the size of the sample. Thus, in the case of acceptance, the loss becomes $L' + fn$. In the case of rejection and 100% inspection, we have already inspected a sample of n at cost fn and we still have to inspect the rest of the lot at cost $c - fn$, so the term fn cancels out and the total loss is simply L''.

For any p, let P and Q be the probabilities of acceptance and rejection by a given test. The expectation value of the loss is

$$L = P(L' + fn) + QL''. \tag{3}$$

This loss function has been considered by Weibull [3] and others. However, as Hamaker [4] has rightly remarked, the sample size is usually small as compared with the size of the lot. Therefore we may, for all practical purposes, replace L'' in (3) by $L'' + fn$, thus obtaining the simpler formula

$$L = PL' + QL'' + fn = L''' + fn, \tag{4}$$

which was also adopted by Anscombe [5] and others.

If rejected lots are discarded, the same formula (4) holds, only we have to put $b = 0$ in (2). The formula thus obtained also holds in the case of destructive testing.

For $p = 0$, L'' is larger than L'. For $p = 1$, we may suppose that L' is larger than L'', for it is usually better not to sell a totally defective product than to

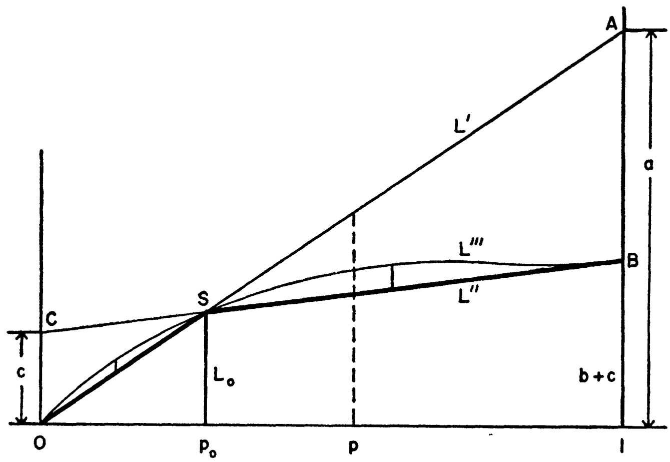

FIG. 1

sell it and to have to take it back with a complaint from the buyer. So the two straight lines (1) and (2) in the (p, L)-plane intersect at a point S with coordinates (p_0, L_0),

$$p_0 = c/(a - b). \tag{5}$$

We shall always put $p + q = 1$ and $p_0 + q_0 = 1$.

If we knew p, the best procedure would be to accept the product for $p < p_0$ and to reject it for $p > p_0$. The loss would then be $L_m = \text{Min}(L', L'')$, or

$$\begin{aligned} L_m &= ap && \text{for } p \leqq p_0, \\ L_m &= bp + c && \text{for } p > p_0. \end{aligned} \tag{6}$$

The function L_m is represented by a heavy line in Fig. 1. We may call L_m the *unavoidable loss.*

The sum $L''' = PL' + QL''$ occurring in (4) is represented, in our diagram, by a curve passing through the point S. For $p = 0$, we may assume that acceptance is certain, and, for $p = 1$, that rejection is certain. So the curve L''' passes through the origin O and the end point B of the line L''.

B. *The minimax loss solution.* If we try to minimize the maximum loss, we only find the following trivial solution: *Reject without taking a sample.* For if we follow this procedure, the loss is L'' and the maximum loss is $b + c$ (point B in the diagram), whereas for all other procedures the maximum loss is larger.

This extremely cautious procedure may be quite reasonable, e.g. in cases where the producer knows that some accident happened in the production process and decides to inspect the whole lot without first taking a sample. In most cases, however, we do not expect p to be large. In those cases, the pessimistic minimax loss procedure is not justified.

C. *Introduction of an a priori distribution.* If we endeavor to formulate our feeling that p is not likely to be large as an exact mathematical hypothesis, we may introduce a process distribution function, $F(p)$, which becomes nearly 1 for unlikely large values of p. We now have to minimize the expectation of the loss,

$$E(L) = \int L \, dF(p) \tag{7}$$

The exact minimum of $E(L)$ can be determined only if the distribution function $F(p)$ is known. The derivative $F'(p)$ is usually called the *process characteristic*. Various functions $F(p)$ or $F'(p)$ have been proposed by several authors. Hamaker [6] gives pictures of five of these functions, and Horsnell [7] lists eight proposals. However, in the discussion to Horsnell's paper, Barnard observes:

"We at Imperial College did some work in trying to find out what sort of process curves do in fact turn up in industry and none we have seen bears the slightest resemblance to those tabulated in Table I".

To this information, Hamaker [6] adds: "Apart from this, industry is constantly changing its products and processes, and by the time we have collected a sufficient number of data for a more detailed analysis some changes may be introduced which completely alter the situation."

Therefore, it seems that the various theoretical solutions of the minimum problem of the loss expectation (7) are of little practical value. We have to admit that we know next to nothing about the actual process function $F(p)$, and we must try to find a practical solution depending only on the loss constants a, b, c, f.

D. *The minimum regret problem.* The loss L may be split into two parts

$$L = L_m + R. \tag{8}$$

The first term L_m is the unavoidable loss. The second term, the excess of the loss over the unavoidable loss, is called the *regret*. Substituting for L and L_m the expressions (4) and (6), we obtain

$$\begin{aligned} R &= (a - b)(p - p_0)P + fn \qquad && \text{for } p > p_0\,, \\ R &= (a - b)(p_0 - p)Q + fn \qquad && \text{for } p \leqq p_0\,, \end{aligned} \tag{9}$$

$$E(L) = \int L_m \, dF(p) + \int R \, dF(p). \tag{10}$$

The first term on the right is independent of the test procedure. Therefore, in order to minimize $E(L)$, we have to minimize the second term, the expectation of the regret,

$$E(R) = \int R \, dF(p). \tag{11}$$

For any given test, let M be the maximum of the regret $R = R(p)$. The formula (11) implies, of course, that $E(R) \leqq M$. Hence, if we make M small, we are sure that $E(R)$ is always small, no matter what $F(p)$ is. Thus, we are led to the following minimax problem: *To find a test making the maximum of the regret* (9) *as small as possible.*

Let m be the minimum of the maximum regret, M, for all possible tests. It is clear that the minimum exists. Let T_1 be a test for which the maximum regret is just m. The inequality (11) implies for the test T_1 that $E(R) \leqq m$ for all $F(p)$.

On the other hand, let T be a test for which the maximum regret M is larger than m, and let p_1 be a value for which $R(p_1)$ is just M. We now may define a function $F(p)$ which jumps from 0 to 1 for $p = p_1$. For this function $F(p)$ the regret expectation is $E(R) = M > m$.

Hence, if we want to have a test for which the regret expectation is always $\leqq m$, no matter what $F(p)$ is, our only possibility is to take a minimax regret test T_1. All other tests yield, for some $F(p)$ a larger regret expectation, $E(R)$, and hence a larger loss expectation $E(L)$.

Empirically I have found that the minimax regret tests T_1 are also minimum $E(L)$ tests for some suitable process function $F(p)$. The process functions I used were quite reasonable. I assumed, as other authors did, that p may take two values $p_1 < p_0$ and $p_2 > p_0$ with certain probabilities. Processes of this type, which usually produce a satisfactory product but sometimes a bad one, do occur. Thus, we see that the minimax regret tests are quite good also from the minimum $E(L)$ point of view under reasonable assumptions concerning the process function $F(p)$.

The minimax problem will be solved in two cases, (A) and (B). In case (A), p_0n and q_0n are both large. In case (B), the sample size n is still assumed to be large, but np_0 is not. Case (A) was investigated by S. Moriguti [1], case (B) by S. Ura [2]. For sequential tests see Section 4 C.

2. The minimum regret solution in case (A). Let p_0n and q_0n both be large. Let h be the fraction of defectives found in the sample. Let h_0 be a critical fraction near p_0, so that the test

$$\text{accepts, if } h < h_0,$$
$$\text{rejects, if } h > h_0.$$

We are interested only in p-values near h_0 and hence near p_0; for, if p is much larger or much smaller than h_0, rejection or acceptance is nearly certain. For those p-values, the variance of h,

$$\sigma^2 = n^{-1}pq, \tag{12}$$

may be approximated by the constant

$$\sigma_0^2 = n^{-1}p_0q_0. \tag{13}$$

Let Φ be the normal distribution function with mean zero and unit variance. The probability of acceptance may be approximated by

$$P = \Phi((h_0 - p)/\sigma_0). \tag{14}$$

The regret now becomes

$$R_+ = (a - b)(p - p_0)\Phi((h_0 - p)/\sigma_0) + fn \qquad \text{for } p > p_0, \tag{15}$$

$$R_- = (a - b)(p_0 - p)\Phi((p - h_0)/\sigma_0) + fn \qquad \text{for } p \leqq p_0. \tag{16}$$

Putting

$$p - p_0 = \sigma_0 x, \tag{17}$$

$$h_0 - p_0 = \sigma_0 s, \tag{18}$$

we may write, for positive $x = z$,

$$R_+ = R_+(z) = (a - b)\sigma_0 z\Phi(s - z) + fn, \tag{19}$$

and, for negative $x = -z$,

$$R_- = R_-(z) = (a - b)\sigma_0 z\Phi(-s - z) + fn. \tag{20}$$

If s is positive, the function $R_+(z)$ is always larger than $R_-(z)$, hence the maximum of R_+ is larger than the maximum of R_-. Between the two maxima lies the maximum of

$$R_0(z) = (a - b)\sigma_0 z\Phi(-z) + fn. \tag{21}$$

Hence, for $s > 0$, the overall maximum of the regret R is the maximum of R_+, and it is larger than the maximum of R_0. For $s = 0$ the regret is R_0. For $s < 0$, the overall maximum is the maximum of R_-, and it is larger than the maximum of R_0. Hence, in order to minimize the maximum regret, we have to assume $s = 0$, or $h_0 = p_0$. The test now

$$\text{accepts, if } h < p_0,$$

$$\text{rejects, if } h > p_0,$$

and the regret is, for $x = +z$ as well as for $x = -z$,

$$R_0(z) = gz\Phi(-z) + fn, \tag{22}$$

with

$$g = (a - b)\sigma_0. \tag{23}$$

The function $z\Phi(-z)$ is zero for $z = 0$ and again for $z = \infty$. The maximum of the function is

$$C = .170 \qquad (\text{for } z = .752). \tag{24}$$

Hence the maximum regret is

$$R_{\max} = Cg + fn. \tag{25}$$

Substituting g and σ_0 from (23) and (13), we obtain

$$R_{\max} = jn^{-1/2} + fn \tag{26}$$

with

$$j = C(a - b)(p_0 q_0)^{1/2}. \tag{27}$$

We now determine n by minimizing $R_{\max}$. This gives for n the approximation

$$n' = (\tfrac{1}{2}C)^{2/3}(a - b)^{2/3}(p_0 q_0)^{1/3} f^{-2/3} = .193\ ((a - b)/f)^{2/3}(p_0 q_0)^{1/3}. \tag{28}$$

To this approximate value, we have to take the nearest integer n and to make sure that np_0 and nq_0 are really large.

The critical number k of defectives in the sample, i.e. the number which just leads to rejection is the next larger integer to

$$w = np_0. \tag{29}$$

If p_0 is near $\frac{1}{2}$, and $w + \frac{1}{2}$ near an integer, the approximations used here are good, even if n is not very large. However, in many cases p_0 is much less than $\frac{1}{2}$. In these cases, it is no longer admissible to replace σ, as given by (12), by σ_0, as given by (13). For $p > p_0$ the true σ will be larger than σ_0, and for $p < p_0$ less. Hence $R_+(z)$ will become larger and $R_-(z)$ less. To minimize the maximum regret, we have to assume a negative s, which means that the critical value h becomes slightly less than p_0, and the critical number of defectives less than $w + \frac{1}{2}$.

It would be interesting to replace these qualitative considerations by more accurate evaluations, to derive correction terms to the formulas for $R_{\max}$ and n', and to find a more accurate asymptotic formula for the critical integer k. For $p_0 \ll 1$ this has been done by Ura [2]. His asymptotic formula (19) may be written, in our notations, as $k = w + .145$.

3. The minimum regret solution in case (B).

A. *The Poisson approximation.* If np_0 is less than 4, the approximation used in case (A) may no longer be good. Assuming n to be large, we may use the Poisson approximation to the binomial distribution. Putting $u = np$, we obtain for the probability of finding just y defectives in a sample of n

$$P_y = e^{-u}(u^y/y!). \tag{30}$$

If k is the critical value, which just leads to rejection, we have

$$\begin{aligned} P &= P_0 + \cdots + P_{k-1}, \\ Q &= 1 - P. \end{aligned} \tag{31}$$

The regret is

$$\begin{aligned} R_+ &= (a - b)(p - p_0)P + fn \qquad &\text{for } p > p_0, \\ R_- &= (a - b)(p_0 - p)Q + fn \qquad &\text{for } p \leqq p_0. \end{aligned} \tag{32}$$

To get rid of inessential constants we introduce new variables v, w, S instead of p, n, R by putting

(33) $$p/p_0 = v, \qquad p_0 n = w, \qquad (p_0/f)R = S.$$

The Poisson constant is now $u = vw$, and the new regret function S is

(34) $$\begin{aligned} S_+ &= t(v - 1)P + w && \text{for } v > 1, \\ S_- &= t(1 - v)Q + w && \text{for } v \leqq 1, \end{aligned}$$

with

(35) $$t = ((a - b)/f)p_0^2 = (c/f)p_0 .$$

We are now left with only one independent variable t. The procedure for determining v and w and the critical integer k is as follows. We first determine v by maximizing S_+ or S_-, whichever has the largest maximum. The resulting maximum regret $M = S_{\max}$ is, for every choice of k, a function of w. We next determine w so as to minimize M. The resulting minimum m depends only on k. We finally determine k so as to minimize m. The corresponding value of $w = p_0 n$ determines the size n of the sample.

For small t, the best choice of k will be $k = 1$. This means that as soon as one defective is found, the product is rejected. We shall investigate this case in greater detail.

B. *The case* $k = 1$. For $k = 1$, formulae (31) simplify to

(36) $$\begin{aligned} P &= e^{-u} = e^{-vw}, \\ Q &= 1 - e^{-vw} \end{aligned}$$

The regrets are

(37) $$S_+ = t(v - 1)e^{-vw} + w \qquad (v > 1)$$

(38) $$S_- = t(1 - v)(1 - e^{-vw}) + w. \qquad (v \leqq 1)$$

The maximum of S_+ is found by differentiating with respect to v. The result is

(39) $$v = 1 + w^{-1}.$$

Substituting into (37), we obtain the maximum of S_+,

(40) $$M_+ = tw^{-1}e^{-(w+1)} + w.$$

We shall write this result as

(41) $$M_+ = tf(w) + w,$$

where $f(w)$ is a known function of w. By the same method, we may determine the maximum of S_- as

(42) $$M_- = tg(w) + w,$$

where $g(w)$ is a known function, which may be computed numerically for every value of w.

Plotting M_+ and M_- as functions of w, we get, for different values of t, graphs

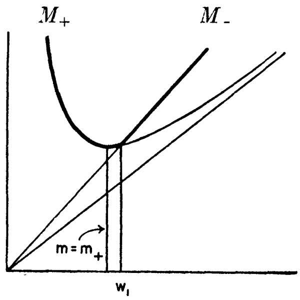

FIG. 2

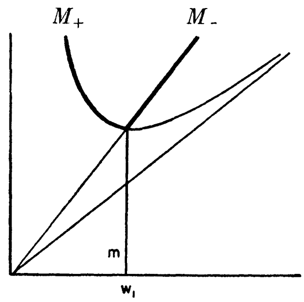

FIG. 3

like those of Fig. 2 or Fig. 3. In every case, we have to consider $M(w) = \text{Max}\ (M_+, M_-)$ as a function of w and to find the minimum of this function $M(w)$.

The function M_+ decreases from ∞ to a minimum m_+ and then increases to ∞. The function M_- is always increasing. The intersection of the curves for M_+ and M_- is given by the equation

$$f(w) = g(w). \tag{43}$$

Since f is a decreasing and g an increasing function, there is only one solution of (43), viz. $w_1 = .868$. This w_1 does not depend on t. We now may distinguish 2 cases:

Case of Fig. 2. If the function M_+ is increasing at w_1, the minimum m of $M(w)$ is equal to the minimum m_+ of M_+.

Case of Fig. 3. If M_+ is not increasing at w_1, the minimum m of $M(w)$ is the common value of M_+ and M_- at w_1.

The condition for M_+ to be increasing at w_1 is

$$tf'(w_1) + 1 > 0. \tag{44}$$

Now $f'(w_1)$ is negative, so condition (44) is satisfied for small t, but not for large t. The limit between the cases of Fig. 2 and Fig. 3 is $t_1 = 2.61$.

For $t \leqq t_1$ the minimum regret m is the minimum m_+ of M_+. Equating the derivative of M_+ to zero, we find

$$t = (w + 1)^{-1} w^2 e^{w+1}, \tag{45}$$

$$m_+ = 2w^{-1}(w + 1)^{-1}. \tag{46}$$

By (45), we may compute t as a function of w and plot the result in the (t, w)-plane. Only the part of the curve between $w = 0$, $t = 0$ and $w_1 = .868$, $t_1 = 2.61$ is needed. For $t \geqq 2.61$, the optimum value of w remains constant $= w_1$, so the next part of the graph for w is a horizontal line $w = w_1$ (see Fig. 4). This line has to be extended to the right until t becomes so large that $k = 2$ would be more favorable than $k = 1$. When does this happen?

C. *The case* $k = 2$. The method for finding the optimum value of w for $k = 2$

is the same as for $k = 1$, the only difference being that the case of Fig. 2 does not occur any more. If we compute the minimum m_+ of the maximum M_+ of the function

$$S_+ = t(v - 1)(1 + vw)e^{-vw} + w, \tag{47}$$

we find that this minimum is larger than m_1 as given by (44) for all values of t below 13. Now a larger minimum means that the case $k = 2$ is less favorable than $k = 1$. So if we want to have $k = 2$, we must assume $t \geqq 13$. For $t = 13$ we are already in the case of Fig. 3, and this holds still more for larger values of t. So we have to compute m from the equation $m = M_+ = M_-$.

Once more, M_+ and M_- are given by formulas like (41) and (42). The functions $f(w)$ and $g(w)$ are more complicated now, but still $f(w)$ is decreasing and $g(w)$ increasing, so that the equation (43) has only one solution, viz. $w_2 = 1.864$.

D. *Comparison of the results for* $k = 1$ *and* 2. The minimum m corresponding to w_1 and $k = 1$ was, according to (41) or (42),

$$m = .1779t + .868, \tag{47}$$

and the minimum corresponding to w_2 and $k = 2$ is given by a similar formula,

$$m = .1227t + 1.864. \tag{48}$$

The linear expressions (47) and (48) become equal for $t_2 = 18.06$.

For $t < t_2$, the m of (47) is less, which means that $k = 1$ is better. For $t > t_2$ the m of (48) is less, which means that $k = 2$ is better. At the point $t_2 = 18.06$ the function w jumps from the constant value $w_1 = .868$ to the constant value $w_2 = 1.864$, and k jumps from 1 to 2. The value 1.864 remains until $k = 3$ becomes better, etc. The behavior of w as a function of t is shown in Fig. 4 (logarithmic scales).

E. *The asymptotic formula for large t.* If t is very large, w is also large and the Poisson distribution may be approximated by a normal distribution. The asymptotic formula for w may be obtained from (28) by putting $q_0 = 1$ and multiplying both sides by p_0. This gives us

$$w = np_0 = .193t, \tag{49}$$

or the dotted line in Fig. 4.

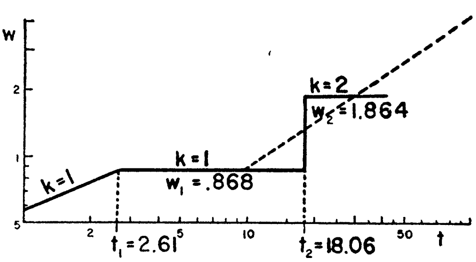

FIG. 4. The (t, w)-diagram

At the end of Section 2, we have seen that, for large w, the critical number of defectives k in a sample of n ought to be chosen slightly less than $w + \frac{1}{2}$. We now see that this is not only true for large values of k and w, but even for the smallest possible values $k = 1$ and $k = 2$. The optimum for w is

$$\text{for } k = 1: w = .868, \quad \text{hence} \quad w + \tfrac{1}{2} = 1.368;$$

$$\text{for } k = 2: w = 1.864, \quad \text{hence} \quad w + \tfrac{1}{2} = 2.364.$$

S. Ura extended the calculations to the cases $k = 3$, 4 and 5. His value $t_2 = 18.3$ is slightly different from mine.

F. *Once more* $k = 1$. For $k = 1$, the Poisson law $p = e^{-u} = e^{-np}$ is a good approximation to the exact formula

$$P = (1 - p)^n \tag{50}$$

even if n is not large, provided p does not exceed .45. Now the largest value of $v = p/p_0$ entering into our calculations was $v_1 = 1 + w_1^{-1} = 2.15$, so, if p_0 does not exceed .2, p will not exceed .43 and the Poisson approximation is justified. Hence *if t lies between* 2 *and* 18, *and p_0 does not exceed* .2, *the results for $k = 1$ obtained by the Poisson approximation may be applied without any modification even to small samples.*

If n, as computed by this method, is less than 4, a direct calculation of the maximum regret M for $n = 1, 2, 3, 4$ is necessary. Also, if p_0 exceeds .2, we have to apply the exact binomial distribution. The exact formulas for the regrets R_+ and R_- are

$$R_+ = (a - b)(p - p_0)q^n + nf,$$

$$R_- = (a - b)(p_0 - p)(1 - q^n) + nf.$$

The maxima M_+ and M_- are readily found by differentiation with respect to p. Very often, the computation of M_- is not necessary. M_- being obviously less than M_+. The maximum of M_+ and M_- is M_n. Thus, we may calculate the sequence $M_1, M_2, \cdots$. As soon as we find an $M_{n+1} \geqq M_n$, we may stop the calculation and take n as the best value. If n turns out to be large, we may replace n by a continuous variable and differentiate with respect to n.

4. The buyer's problem.

A. *The loss functions.* Suppose a buyer gets a product in lots from the producer. If he accepts a lot, he pays a fixed price and uses the product for his own purposes or sells it to others, making a fixed profit minus a loss proportional to the number of defectives. The fixed profit does not enter into our calculations; we are only concerned with the loss due to defective units, viz.

$$L' = ap. \tag{51}$$

If the buyer rejects the lot, he returns it to the producer. In this case, he misses his profit and may have to pay for the transportation, which means that he has a fixed loss

$$L'' = c \tag{52}$$

in the case of rejection.

To both losses, (51) and (52), the cost of inspection fn must be added. Hence the expected loss is

$$L = PL' + QL'' + fn = apP + cQ + fn. \tag{53}$$

Comparing (51) and (52) with (1) and (2), we see that the minimum loss problem is the same as before, only with $b = 0$.

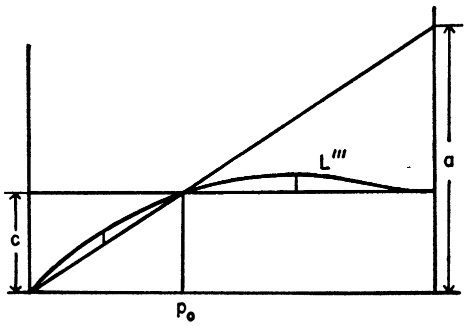

FIG. 5

B. *The cost per non-defective unit.* The expressions (51) and (52) represent the buyer's loss for every lot he has ordered. However, what really matters is his cost per non-defective unit. We may assume that defective units are worthless to him, and that every year he needs a certain number of non-defective units. Now what is the average price he has to pay for these?

Following Hamaker [4], I shall use the following notation:

$f_r = \int Q \, d\, F(p) =$ average fraction of lots rejected,

$f_{da} = \int p \, P \, d\, F(p) =$ average fraction defective accepted,

$f_{ea} = \int (1 - p)P \, d\, F(p) =$ average fraction effective accepted,

$N =$ number of units in a lot.

All integrals are from 0 to 1.

The average number of non-defectives accepted per lot is

$$Nf_{ea} = N(1 - f_r - f_{da}). \tag{54}$$

Let the price to be paid for an accepted lot be a, and the cost of a rejected lot c. The average cost of a lot is

$$(1 - f_r)a + f_r c + fn = a - f_r(a - c) + fn. \tag{55}$$

Hence the average cost of a non-defective unit is

$$K = [a - f_r(a - c) + fn]/N(1 - f_r - f_{da}). \tag{56}$$

We may assume f_r and f_{da} to be small as compared with 1, and fn small as compared with a. Neglecting powers and products of small terms, we obtain

$$(57)\quad NK = a(1 - f_r((a - c)/a) + (fn/a) + f_r + f_{da}) = a + f_{da}a + f_rc + fn.$$

In order to minimize K for given N, we have to minimize the expression

$$(58)\qquad f_{da}a + f_rc + fn = \int (apP + cQ + fn)\, dF(p).$$

Now this is just the expectation of the loss L of equation (53)

$$(59)\qquad E(L) = \int L\, dF(p).$$

Hence, no matter whether we start with the loss per lot or with the loss per non-defective unit, the buyer's minimum loss problem is just the same as the producer's, only with $b = 0$. Hence, the minimum regret solutions, obtained in Section 2 and Section 3 for non-sequential tests, may be applied without any modification to the buyer's problem.

The method of approximation used in passing from the fractional expression (56) to the linear expression (57) is due to Hamaker [4]. The same method can be applied whenever factors such as $1 - f_{da}$ or $1 - f_r - f_{da}$ occur in the denominator of a cost formula. As an example, Horsnell's cost formula for non-destructive inspection([7], Formula (1)) may be mentioned.

C. *Sequential tests.* For sequential tests there is an essential difference between the producer's and the buyer's loss function. If the buyer finds many defectives, his best action is to stop sampling and to return the product to the producer. But if the producer finds many defectives, and if inspection is not destructive, he may go on sampling without increasing his loss, because he has to inspect the whole lot and to eliminate the defectives anyhow.

Therefore, a producer's sequential test will have an acceptance line but no rejection line. As long as we do not cross the acceptance line, sampling goes on until the whole lot is inspected. The sampling plan is defined by an increasing sequence of numbers $n_0 < n_1 < n_2 \cdots$. The lot is accepted when a sample of n_0 contains no defectives, or when a sample of n_1 contains just 1 defective, etc. Anscombe [8] investigated sequential sampling plans of this type, in which the numbers n_k form an arithmetical sequence

$$(60)\qquad n_k = n_0 + kd.$$

Quite generally, let P_k be the probability of acceptance with k defectives in a sample of n_k. The loss in this case would be

$$(61)\qquad L_k = ap + fn_k,$$

and, in the case of 100% inspection,

$$(62)\qquad L'' = bp + c.$$

For given p, the loss expectation is

$$(63) \qquad L = \sum P_k L_k + QL'' = apP + \sum fn_k P_k + (bp + c)Q.$$

The problem is, once more, to minimize the loss expectation

$$(64) \qquad E(L) = \int L\, dF(p)$$

or, if $F(p)$ is not known, to minimize the maximum of the regret

$$(65) \qquad R = L - L_m .$$

For the case of an arithmetical sequence (60), good approximations to P, Q and $\sum n_k P_k$ are obtainable from Wald's theory [10]. I do not know whether an arithmetical sequence is most economical.

In the case of a buyer who is entitled to return rejected lots, a sequential test may have an acceptance line and a rejection line. The loss function is

$$(66) \qquad L = apP + cQ + fE(n).$$

The lot size will be assumed to be large as compared with $E(n)$, so that in all probability a decision is reached long before the lot is exhausted. If the acceptance and rejection lines are straight and parallel, we have a Wald probability ratio test. I am inclined to believe that the minimum regret tests corresponding to the loss function (66) are just probability ratio tests, but I cannot prove it. It would be very interesting to calculate the parameters of these tests as functions of a, c and f.

If the lot size is not very large as compared with $E(n)$, we have to use, in working out a probability ratio test, the hypergeometric distribution instead of the binomial one; see [9].

5. Application of game theory. Until now, we have considered the producer's and the buyer's problem separately. However, we may combine the two points of view and regard the whole transaction as a two-person non-zero sum game.

A. *The rules of the game.* The producer may inspect a sample and may decide to examine the whole lot and to eliminate the defectives, or he may send the product as it is.

The buyer may inspect a sample and may return the product as soon as he finds at least one defective, or he may accept the product and use it or sell it to others.

The producer pays a constant production cost and gets a fixed price if the product is accepted. If his own inspection does not accept it, he is still able to make the fixed price minus $bp + c$. If the buyer returns the lot, the producer gets the fixed price minus $bp + c + c'$. In any case, he has to pay fn for the inspection of a sample of n.

The buyer pays the fixed price if he accepts the sample. By using or selling it himself, he makes a fixed profit minus ap. If he returns the product, he has, to pay c'' for cost of transportation. Besides, he has to pay fn_1 for the inspection of a sample of n_1.

B. *The three players and their strategies.* In order to apply the theory of games, we may introduce "Nature" as a third player, whose profit or loss is such that the sum of the three profits is zero. Nature may influence the defective fraction p by causing accidents in the production process. The producer can do nothing against this except repairing the damage, so that next day the odds for a new accident are exactly what they were before.

So a "move" of Nature means a value of p, valid one day. The difference between Nature and the other players is that Nature is not interested in increasing her profit. Nature's only "strategy" is to produce random numbers p according to a distribution function $F(p)$.

A strategy of the producer or buyer means an inspection plan. The buyer's best strategy has been investigated already in Section 4. If the buyer knows by previous experience the distribution $G(p)$ of the p-values the producer sends him, he may find an inspection plan which minimizes the expectation value of his loss

$$E(L) = \int L(p)\, dG(q). \tag{67}$$

If the function $G(p)$ is not known, the buyer may adopt the minimax regret strategy.

The producer's strategy has to be considered anew, because the rules of the game are more complicated than those adopted in Section 1. The formulas for the producer's loss if his own inspection rejects the product are the same as in Section 1. On the other hand, if the product is sent to the buyer, the producer's profit or loss depends on the defective fraction p and on the buyer's strategy. Even if we assume the buyer's strategy to be known, the producer's loss expectation will be a rather complicated expression. The assumption made in Section 1 that the loss expectation is proportional to p, may be a useful approximation.

C. *The possibility of coalitions.* Until now, we have investigated the strategy of the three players separately. Next we have to consider the possibility of two players forming a coalition with the aim of making their joint profit as large as possible.

Of course, a coalition of one of the thinking players with Nature with the aim of ruining the other one makes no sense, but the producer and the buyer may well agree upon a combined sampling plan which would maximize the sum of their profits.

By combining the two sampling inspections into one, the two partners can avoid the additional losses c' and c'' which arise when the product is sent and returned to the producer. This means: The inspection has to be made only at the producer's, e.g. by a neutral agent. If n is the total size of the sample, the combined loss of the producer and the buyer is

$$L_1 = ap + fn, \quad \text{if accepted,}$$

and

$$L_2 = bp + c, \quad \text{if rejected.}$$

Now this combined loss function is exactly the same as the producer's loss function adopted in Section 1. Hence we may apply the theory developed in Sections 1–3 to find the best strategy of the coalition.

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