DENSITY ESTIMATION IN A TOPOLOGICAL GROUP[1]

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1. Introduction. Let $(G, \mathfrak{F}, \mu)$ be a measure space with μ Haar (both left and right) measure on $(G, \mathfrak{F})$ where $\mathfrak{F}$ is the class of Borel sets of the Hausdorff topological group $(G, \mathfrak{I})$. Let X be a $(G, \mathfrak{F})$ valued random variable with distribution given by a measure P which is absolutely continuous with respect to μ and has Radon-Nikodym derivative f with respect to μ. If, for each integer n, $X_1, X_2, \cdots, X_n$ are n independent, identically distributed random variables each having the same distribution as X, Parzen, [3] and Leadbetter, [1], have shown a method for constructing consistent, asymptotically normal estimates for f in the case G is the real line and μ Lebesgue measure. It is the purpose of this note to show that these methods can be extended to the more general case given above.

2. Density estimation. Following the definitions given in [1] a sequence of functions, $\{K_n; n = 1, 2, \cdots\}$, on the group G will be called a δ-sequence with respect to μ if, for each n, K_n is a real-valued, non-negative, symmetric, μ-integrable function on G such that

(1) $\int_G K_n(u)\, d\mu(u) = 1$.

(2) If V is any neighborhood of the identity element $e \in G$, then

(a) $\lim \int_{V^c} K_n(u)\, d\mu(u) = 0$ where V^c denotes the complement of V in G,

(b) $K_n(u) \to 0$ uniformly for almost all $(\mu) u \in V^c$.

Given such a δ-sequence, define for each element $u \in G$ a sequence, $\{f_n(u)\}$, of pointwise density estimators by

$$f_n(u) = (1/n) \textstyle\sum_{j=1}^{n} K_n(uX_j^{-1}).$$

The results of [1] and [3] are then easily modified to fit the present situation, since the properties of the real line used in these results are only those which hold more generally in the topological group case. For example, Theorem 2A of [3] can be restated as follows:

Theorem 1. *If $\{C_n : n = 1, 2, \cdots\}$ is a sequence of constants converging to zero for which $\{C_n K_n^2 : u = 1, 2, \cdots\}$ is a δ-sequence and if f is continuous at $u \in G$, then $(nC_n) \operatorname{Var} f_n(u) \to f(u)$ as $n \to \infty$.*

One result which does not appear in the literature is the following:

Theorem 2. *If the conditions of the above theorem are satisfied and s and t are distinct continuity points of f with $f(s) + f(t) \neq 0$, then $\{f_n(s), f_n(t)\}$ is jointly*

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asymptotically normal and independent in the sense that for any pair of real numbers a and b,

$$P[Z_n(s) \leqq a; \quad Z_n(t) \leqq b] \to (1/2\pi) \int_{-\infty}^{a} \int_{-\infty}^{b} \exp\left[-\tfrac{1}{2}(x^2 + y^2)\right] dy\, dx,$$

where

$$Z_n(s) = [f_n(s) - Ef_n(s)]/[\sigma f_n(s)].$$

PROOF. It is enough (cf. Loève, [2], p. 205) to show that for any pair of real numbers A and B the random variable $AZ_n(s) + BZ_n(t)$ is asymptotically normal. Now

$$\operatorname{Var}[AZ_n(s) + BZ_n(t)] = A^2 + B^2 + 2AB \operatorname{Cov}[f_n(s), f_n(t)]/[\sigma f_n(s)\sigma f_n(t)].$$

This approaches $A^2 + B^2$ as a result of Theorem 2.6 of [1] and Theorem 1 above. By another application of Theorem 1 it is sufficient to show

$$(nC_n)^{\frac{1}{2}} A[f_n(s) - Ef_n(s)] + (nC_n)^{\frac{1}{2}} B[f_n(t) - Ef_n(t)]$$

is asymptotically normal. However by Theorem 2.7 of [1] both members of the above expression are asymptotically normal. Also we can write

$$(nC_n)^{\frac{1}{2}}[f_n(s) - Ef_n(s)] = \sum_{j=1}^{n} U_{nj}$$

$$(nC_n)^{\frac{1}{2}}[f_n(t) - Ef_n(t)] = \sum_{j=1}^{n} V_{nj}$$

as sums of independent, identically distributed random variables, where

$$U_{nj} = (C_n/n)^{\frac{1}{2}}[K_n(sX_j^{-1}) - EK_n(sX_j^{-1})]$$

$$V_{nj} = (C_n/n)^{\frac{1}{2}}[K_n(tX_j^{-1}) - EK_n(tX_j^{-1})].$$

By the central limit theorem (cf. Loève, [2], p. 316) it is sufficient to show that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \sum_{j=1}^{n} P[|AU_{nj} + BV_{nj}| \geqq \epsilon] = 0.$$

But from this same theorem we also get that

$$\lim_{n\to\infty} \sum_{j=1}^{n} P[|U_{nj}| \geqq \epsilon] = 0$$

$$\lim_{n\to\infty} \sum_{j=1}^{n} P[|V_{nj}| \geqq \epsilon] = 0.$$

From these two facts the desired result follows immediately.

REFERENCES

[1] LEADBETTER, M. R. (1963). On the non-parametric estimation of probability densities. Technical Report No. 11, ONR Contract No. NONR-3423(00).

[2] LOÈVE, M. (1955). *Probability Theory*, (1st Ed.). Van Nostrand, New York.

[3] PARZEN, E. (1962). On estimation of a probability density function and mode. *Ann. Math. Statist.* **33** 1065–1076.