SOME CONVERGENCE THEOREMS FOR INDEPENDENT RANDOM VARIABLES[1]

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1. Introduction. Let a_{nk}, a_n be real numbers and x_n be independent random variables. The convergence of $\sum_{k=1}^{\infty} a_{nk}x_k$ as $n \to \infty$ has been discussed in [4], [5], [7], [14] and [15]. Section 2 of this paper is suggested by Hill's work [7], and is devoted to the convergence of $\sum_{k=1}^{\infty} a_{nk}x_k$ under the condition $\sum_{k=1}^{\infty} a_{nk}^2 = o(\log^{-1} n)$. As an application, we prove the following theorem, relating to a result of Pruitt [14]. *If x_n are identically distributed, $Ex_n = 0$, $Ex_n^2 = 1$, and $\sum_{k=1}^{n} a_{nk}^2 = 1$, then $n^{-\frac{1}{2}}\sum_{k=1}^{n} a_{nk}x_k$ tends to zero* a.e. Section 3 is suggested by Kahane's work [9]. Salem-Zygmund's sample continuity theorem [15] for $\sum_1^{\infty} a_n x_n \cos nt$ is extended from Bernoulli random variables to generalized Gaussian random variables (defined in Section 2). Sections 4 and 5 are devoted to the extensions of Hsu-Robbins' complete convergence theorem [8]; the material in these two sections is closely related to the work of Franck and Hanson [4].

The first counter-example showing that a directed set indexed martingale of bounded variation may diverge pointwise is due to Dieudonné [1]. A simpler counter example is given in Section 6. Section 7 contains some theorems about a.e. unconditional convergence of sums of independent identically distributed random variables, and in Section 8 the following theorem is proved. *If $E \sup_n |x_n| < \infty$, then $\sum_1^{\infty} x_n$ converges* a.e. *implies that $\sum_1^{\infty} Ex_n$ converges.*

2. Extension of Hill's theorems. In this section, we assume that for $n, k = 1, 2, \cdots$, a_{nk} are real numbers and $A_n = \sum_{k=1}^{\infty} a_{nk}^2 < \infty$ for each n.

Lemma 1. *Let x be a random variable, $Ex = 0$ and $|x| \leqq 1$. Then for every real number t,*

$$Ee^{tx} \leqq e^{t^2}. \tag{1}$$

Proof. If $0 < t \leqq 1$, then $E \exp[tx] \leqq 1 + t^2 \leqq \exp[t^2]$. If $t > 1$, then $E \exp[tx] \leqq \exp[t] \leqq \exp[t^2]$. By symmetry, we obtain (1).

In [9], a symmetric random variable x is said to be semi-Gaussian, if there exists $\alpha \geqq 0$ such that for every real number t

$$E \exp[tx] \leqq \exp[\alpha^2 t^2/2]. \tag{2}$$

The minimum of those α satisfying (2) is denoted by $\tau(x)$. Obviously, a $N(0, 1)$ random variable x is semi-Gaussian (with $\tau(x) = 1$), and by Lemma 1, if x is symmetric and bounded by K, x is semi-Gaussian (with $\tau(x) \leqq 2^{\frac{1}{2}}K$). For

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convenience, we introduce the following definition, which is by no means standard.

DEFINITION. A random variable x is said to be generalized Gaussian, if there exists $\alpha \geqq 0$ such that for every real number t, (2) holds. In this case, the minimum of those α satisfying (2) is denoted by $\tau(x)$.

Obviously, if x is generalized Gaussian, so are $-x$ and ax (for $a > 0$) with $\tau(-x) = \tau(x)$ and $\tau(ax) = a\tau(x)$. If x is generalized Gaussian with $\tau(x) \leqq \alpha > 0$, then for $\lambda > 0$, $t > 0$,

$$P[x \geqq \lambda] = P[\lambda x\alpha^{-2} \geqq \lambda^2\alpha^{-2}] \leqq \exp[-\lambda^2/(2\alpha^2)], \tag{3}$$

$$P[|x| \geqq \lambda] \leqq 2 \exp[-\lambda^2/(2\alpha^2)]. \tag{4}$$

The properties of semi-Gaussian random variables has been discussed in [9], pp. 75–77. Some of them can easily be extended to our case. However, the above mentioned properties are enough for our purpose here.

LEMMA 2. *Let x_n be independent, generalized Gaussian with $\tau(x_n) \leqq 2^{\frac{1}{2}}$ and let $T_n = \sum_{k=1}^{\infty} a_{nk}x_k$. Then for each n, the series T_n converges* a.e., *and for every $\epsilon > 0$, $-\infty < t < \infty$,*

$$Ee^{tT_n} \leqq e^{t^2A_n}, \tag{5}$$

$$P[|T_n| > \epsilon] \leqq 2e^{-\epsilon^2/(4A_n)}. \tag{6}$$

PROOF. (6) follows from (5) and (4). To prove (5), let $T_{nm} = \sum_{k=1}^{m} a_{nk}x_k$. By independence,

$$E \exp[tT_{nm}] = \prod_{k=1}^{m} E \exp[ta_{nk}x_k] \leqq \exp[\sum_{k=1}^{m} t^2a_{nk}^2] \leqq \exp[t^2A_n]. \tag{7}$$

Hence T_{nm} is generalized Gaussian with $\tau(T_{nm}) \leqq (2\sum_{k=1}^{m} a_{nk}^2)^{\frac{1}{2}}$, and by (3), for $j \geqq m$,

$$P[|\sum_{k=m}^{j} a_{nk}x_k| > \epsilon] \leqq 2 \exp[-\epsilon^2/(4\sum_{k=m}^{j} a_{nk}^2)]. \tag{8}$$

By hypothesis, $A_n < \infty$ and therefore the left hand side of (8) tends to zero as $j \geqq m \to \infty$. Hence T_n coverges in probability for each n. Since x_n are independent, T_n converges a.e. for each n (see [12], p. 249). By Fatou's lemma and (7), we obtain (5).

Let x_n be independent and $P[x_n = 1] = P[x_n = -1] = \frac{1}{2}$. Then for $t > 0$,

$$E \exp[tx_n] = (e^t + e^{-t})/2 \leqq \exp[t^2/2],$$

since $\exp[t^2/2] = \sum_0^{\infty} t^{2n} 2^{-n}/(n!) \geqq \sum_0^{\infty} t^{2n}/(2n)!$. Hence $\tau(x_n) \leqq 1$ and Lemma 2 implies that $P[|T_n| > \epsilon] \leqq 2 \exp[-\epsilon^2/(2A_n)]$, which was proved by Khintchine [10] and by Salem and Zygmund [15].

THEOREM 1. *Let x_{nk}, $k = 1, 2, \cdots$, be independent, generalized Gaussian with $\sup_k \tau(x_{nk}) \leqq 2^{\frac{1}{2}}$ for each n. Put $T_n = \sum_{k=1}^{\infty} a_{nk}x_{nk}$ and $A_n = \sum_{k=1}^{\infty} a_{nk}^2$. If for every $\alpha > 0$,*

$$\sum_1^{\infty} e^{-\alpha/A_n} < \infty, \tag{9}$$

then the series T_n *converges* a.e. *for each* n, $\lim T_n = 0$ a.e. *and for every* $\epsilon > 0$

$$\sum_1^\infty P[|T_n| > \epsilon] < \infty. \tag{10}$$

PROOF. From Lemma 2, T_n converges a.e. for each n and

$$\sum_1^\infty P[|T_n| > \epsilon] \leqq 2\sum \exp[-\epsilon^2/(4A_n)] < \infty.$$

By Borel-Cantelli's lemma, $\lim T_n = 0$ a.e.

THEOREM 2. *Let* x_{nk}, $k = 1, 2, \cdots$, *be independent and* $x_{nk} - Ex_{nk}$ *be generalized Gaussian with* $\sup_k \tau(x_{nk} - Ex_{nk}) \leqq 2^{\frac{1}{2}}$ *for each* n. *Let* $C_n = \sum_{k=1}^\infty a_{nk}Ex_{nk}$ *converge for each* n *and* $\lim C_n = C$. *If* (9) *holds or if* $A_n = o(\log^{-1} n)$, *then the series* $T_n = \sum_{k=1}^\infty a_{nk}x_{nk}$ *converges* a.e. *for each* n *and* $\lim T_n = C$ a.e.

PROOF. Put $x'_{nk} = x_{nk} - Ex_{nk}$ and $T_n' = \sum_{k=1}^\infty a_{nk}x'_{nk}$. Since $A_n = o(\log^{-1} n)$ implies (9), by Theorem 1, the series T_n' converges a.e. for each n and $\lim T_n' = 0$ a.e. Hence T_n converges a.e. for each n and $\lim T_n = \lim C_n = C$ a.e.

Theorem 2 has been proved by Hill [7] for $P[x_{nk} = 1] = P[x_{nk} = 0] = \frac{1}{2}$. Even in that case, Erdös [3] gives a counter-example to show that the condition $A_n = o(\log^{-1} n)$ in Theorem 2 cannot be replaced by $A_n = O(\log^{-1} n)$.

COROLLARY 1. *Let* x_n *be independent and* $x_n - Ex_n$ *be generalized Gaussian with* $\tau(x_n - Ex_n) \leqq 2^{\frac{1}{2}}$ for $n = 1, 2, \cdots$. *If* $S_n = x_1 + \cdots + x_n$ *and for some* $\alpha > 0$,

$$\lim n^{-\frac{1}{2}} (\log^{-(1+\alpha)/2} n) ES_n = 0,$$

then $\lim n^{-\frac{1}{2}}(\log^{-(1+\alpha)/2} n)S_n = 0$ a.e.

PROOF. For $n = 2, 3, \cdots$, put $a_{nk} = n^{-\frac{1}{2}} \log^{-(1+\alpha)/2} n$ for $1 \leqq k \leqq n$ and $a_{nk} = 0$ for $k > n$. Then $A_n = \sum_{k=1}^n a_{nk}^2 = \log^{-(1+\alpha)} n = o(\log^{-1} n)$. Hence by Theorem 2,

$$\lim_n \sum_{k=1}^\infty a_{nk}x_k = \lim_n (n^{-\frac{1}{2}} \log^{-(1+\alpha)/2} n)S_n = 0, \quad \text{a.e.}$$

This completes the proof.

Let $a_{nk} = (k\sum_{j=1}^n j^{-1})^{-1}$ for $1 \leqq k \leqq n$, $n = 1, 2, \cdots$; and $a_{nk} = 0$ otherwise. For a sequence of real numbers x_n, define

$$t_n = \sum_{j=1}^n j^{-1}x_j / \sum_{j=1}^n j^{-1}. \tag{11}$$

If $\lim t_n = t$, then the sequence x_n is said to be (N, n^{-1}) — summable to value t. It is known ([6], p. 110) that summability (N, n^{-1}) implies summability (C, α) for every $\alpha > 0$. Since the method (N, n^{-1}) is regular ([6], p. 64) and $A_n = \sum_{k=1}^n a_{nk}^2 = O(\log^{-2} n) = o(\log^{-1} n)$, from Theorem 2 we have:

COROLLARY 2. *Let* x_n *be independent and* $x_n - Ex_n$ *be generalized Gaussian with* $\tau(x_n - Ex_n) \leqq 2^{\frac{1}{2}}$ *for* $n = 1, 2, \cdots$ *and* $\lim_n Ex_n = 0$. *Define* t_n *by* (11). *Then* $\lim_n t_n = 0$ a.e. *In particular,* x_n *is summable* (C, α) *to* 0 *for every* $\alpha > 0$.

Corollary 2 has been proved by Hill [7] for $P[x_n = 1] = P[x_n = 0] = \frac{1}{2}$, and it is closely related to a result of Hanson and Koopman [5].

THEOREM 3. *Let* x_n *be independent, identically distributed. Then* $Ex_1 = 0$ *and* $Ex_1^2 < \infty$, *if and only if for every array* a_{nk} *of real numbers such that* $\lim_n \sum_{k=1}^n a_{nk}^2 = 1$, *we have*

$$\lim_n n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} x_k = 0 \quad \text{a.e.} \tag{12}$$

PROOF. For the "if" part, let $x_k' = x_k$ if $|x_k| \leqq k^{\frac{1}{2}}$, and $x_k' = 0$ if $|x_k| > k^{\frac{1}{2}}$. Let $x_k'' = x_k - x_k'$, $T_n' = n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} x_k'$ and $T_n'' = n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} x_k''$. Then $\lim_n Ex_n' = 0$ and

$$(n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} Ex_k')^2 \leqq n^{-1} \sum_{k=1}^n a_{nk}^2 \cdot \sum_{k=1}^n E^2 x_k' \to 0.$$

Hence $\lim_n n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} Ex_k' = 0$. Since $n^{-\frac{1}{4}} |x_k' - Ex_k'| \leqq 2$ and

$$n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk}(x_k' - Ex_k') = n^{-\frac{1}{4}} \sum_{k=1}^n a_{nk}(x_k' - Ex_k') n^{-\frac{1}{4}},$$

from Theorem 1 and Lemma 1, we have

$$\lim_n n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk}(x_k' - Ex_k') = 0 \quad \text{a.e.}$$

For $\epsilon > 0$, choose $N = N(\epsilon)$ such that $\int_{[|x_1| \geqq N]} x_1^2 \, dP < \epsilon$ and put $x_k^* = x_k$ if $|x_k| \geqq N$ and $x_k^* = 0$ if $|x_k| < N$. Then by the strong law of large numbers,

$$(n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} x_k'')^2 \leqq (1 + o(1)) n^{-1} \sum_1^n (x_k'')^2 \leqq (1 + o(1)) n^{-1} \sum_1^n (x_k^*)^2 + o(1)$$
$$\to E(x_1^*)^2 < \epsilon, \text{ a.e.}$$

Therefore $\lim_n n^{-\frac{1}{2}} \sum_{k=1}^n a_{nk} x_k'' = 0$ a.e. and hence (12) holds.

For the "only if" part, let $a_{nk} = 0$ for $k \neq n$ and $a_{nn} = 1$. Then (12) implies $\lim n^{-\frac{1}{2}} x_n = 0$ a.e. Hence $Ex_1^2 < \infty$. If we put $a_{nk} = n^{-\frac{1}{2}}$ for $1 \leqq k \leqq n$, then (12) implies $\lim (x_1 + \cdots + x_n)/n = 0$ a.e. Hence $Ex_1 = 0$.

The "if" part was conjectured by L. Gleser under the extra condition $\max a_{nk} \to 0$ as $n \to \infty$, where $1 \leqq k \leqq n$. In a personal correspondence, D. Burkholder has given a slightly simpler proof of the "if" part, by first proving (12) under the condition $Ex_1^4 < \infty$ and then truncating x_n at $0 < c < \infty$.

3. Extension of Salem and Zygmund's theorems.

LEMMA 3. *Let a_n and φ_n be real numbers and $Q(t) = \sum_1^m a_n \cos (nt + \varphi_n)$. Then $\|Q'\|_\infty \leqq 2m^2 \|Q\|_\infty$, where $\|Q\|_\infty = \max_t Q(t)$.*

Lemma 3 follows immediately from the definition of Fourier coefficients and is a special case of S. Bernstein's theorem (See [16], II, p. 11): $\|Q'\|_\infty \leqq m \|Q\|_\infty$.

LEMMA 4. *Let x_n and φ_n be random variables. For real numbers a_n, define $Q(t) = \sum_{n=1}^m a_n x_n \cos (nt + \varphi_n)$ and $M(\omega) = \|Q(t)\|_\infty$. Then for $K \geqq 0$,*

$$[M(\omega) \geqq K] \subset \bigcup_{k=1}^{16m^2} [|Q(k/(2m^2))| \geqq K/2]. \tag{13}$$

PROOF. Let $M(\omega) = Q(t(\omega), \omega)$. Then

$$|Q(t') - Q(t(\omega))| \leqq |t' - t(\omega)| \cdot \|Q'\|_\infty \leqq 2m^2 |t' - t(\omega)| \cdot M(\omega).$$

Hence for $|t' - t(\omega)| < (4m^2)^{-1}$,

$$|Q(t')| \geqq M(\omega)(1 - 2m^2 |t' - t(\omega)|) \geqq M(\omega)/2.$$

If $M(\omega) \geqq K$, then there exists an interval $I(\omega)$ of length $\geqq (2m^2)^{-1}$ and $|Q(t, \omega)| \geqq K/2$ for $t \in I(\omega)$. Hence (13) holds.

THEOREM 4. *Let x_n and φ_n be two independent sequences of independent random variables and each x_n be generalized Gaussian with $\tau(x_n) \leqq 2^{\frac{1}{2}}$. Let a_n be real numbers and $\alpha > 0$ such that*

$$\sum_1^\infty a_n^2 \log^{1+\alpha} n < \infty. \tag{14}$$

Then for each t, $\sum_1^\infty a_n x_n \cos(nt + \varphi_n)$ converges a.e. *to a stochastic process $f(t)$. Furthermore there exists another stochastic process $g(t)$ such that for each t, $P[g(t) = f(t)] = 1$ and $g(t)$ is* a.e. *sample continuous.*

PROOF. Put $n(j) = 2^{2^j}$, $Q_j(t) = \sum_{n(j)+1}^{n(j+1)} a_n x_n \cos(nt + \varphi_n)$, $M_j = \|Q_j(t)\|_\infty$, $B_j = \sum_{n(j)+1}^{n(j+1)} a_n^2$, $C_j = \sum_{n(j)+1}^{n(j+1)} a_n^2 \log^{1+\alpha} n$, and $t_k = k/(2n(j+2))$. By Lemma 4,

$$P[M_j \geqq 2j^{-2}] \leqq \sum_{k=1}^{16n(j+2)} P[|Q_j(t_k)| \geqq j^{-2}].$$

For $-\infty < t < \infty$, since x_n is generalized Gaussian with $\tau(x_n) < 2^{\frac{1}{2}}$,

$$\begin{aligned} E \exp[tx_n \cos(ns + \varphi_n)] &= E(E\{\exp[tx_n \cos(ns + \varphi_n)] \mid \varphi_1, \varphi_2 \cdots\}) \\ &\leqq \exp[t^2]. \end{aligned} \tag{15}$$

Hence by independence,

$$E \exp[tQ_j(t_k)] \leqq \exp[t^2 B_j] \leqq \exp[t^2 C_j/(2^j \log 2)^{1+\alpha}]. \tag{16}$$

(15) implies that $x_n \cos(nt + \varphi_n)$ is generalized Gaussian with $\tau(x_n \cos(nt + \varphi_n)) \leqq 2^{\frac{1}{2}}$. By Theorem 1, for each t, $\sum_1^\infty a_n x_n \cos(nt + \varphi_n) = f(t)$ a.e. From (4) and (16),

$$P[|Q_j(t_k)| > j^{-2}] \leqq 2 \exp[-2^{j(1+\alpha)} \log^{1+\alpha} 2/(4C_j j^4)].$$

Therefore

$$\begin{aligned} P[M_j \geqq 2j^{-2}] &\leqq 32n(j+2) \exp[-2^{j(1+\alpha)-2} \log^{1+\alpha} 2/(C_j j^4)] \\ &= \exp[(2^{j+2} + 5) \log 2 - 2^{j(1+\alpha)-2} \log^{1+\alpha} 2/(C_j j^4)] \leqq \exp[-2^{j(1+\alpha/2)}], \end{aligned}$$

if j is large enough. Hence $\sum_1^\infty P[M_j \geqq 2j^{-2}] < \infty$. By Borel-Cantelli's lemma, for almost all ω there exists $j_0(\omega) < \infty$ such that $M_j(\omega) \leqq 2j^{-2}$ if $j \geqq j_0(\omega)$. Hence $\sum_1^\infty M_j < \infty$ a.e. and for almost all ω, $\sum_1^\infty Q_j(t)$ converges uniformly in t to a stochastic process $g(t)$. Since for each fixed ω, $Q_j(t)$ are continuous in t, $g(t)$ is a.e. sample continuous. Since for each t, $\sum_1^\infty Q_j(t) = f(t)$ a.e., $P[f(t) = g(t)] = 1$.

When $P[x_n = 1] = P[x_n = -1] = \frac{1}{2}$ for each n, Theorem 4 is due to Salem and Zygmund [15]. A closely related theorem for semi-Gaussian random variables has been given by Kahane ([9], p. 78).

LEMMA 5. *Let x_j, $j = 1, 2, \cdots, n$, be independent, generalized Gaussian and $\tau(x_j) \leqq (2\tau_j)^{\frac{1}{2}}$. Let N be a stopping variable (or time) relative to $x_1, \cdots, x_n$ and $S_k = x_1 + \cdots + x_k$. Then S_N is generalized Gaussian and $\tau(S_N) \leqq (2\sum_1^n \tau_j)^{\frac{1}{2}}$.*

PROOF. For $-\infty < t < \infty$, put $\varphi_i(t) = E \exp[tx_i]$ and $u_k = \exp[t\sum_1^k x_j]/\prod_1^k \varphi_j(t)$. Then $(u_k, \mathfrak{F}_k, k \leqq n)$ is a martingale ([2], p. 352), where $\mathfrak{F}_k$ is the Borel field generated by $x_1, \cdots, x_k$. Since $1 \leqq N \leqq n$, from [2], p. 303,

$$1 = Eu_1 = Eu_N = E(\exp [tS_N]/\textstyle\prod_1^N \varphi_j(t)).$$

Since $\varphi_j(t) \leqq \exp [t^2\tau_j]$,

$$\begin{aligned} 1 &= \textstyle\sum_{j=1}^n \int_{[N=j]} (\exp [tS_j]/\prod_1^j \varphi_k(t))\, dP \\ &\geqq \textstyle\sum_{j=1}^n \int_{[N=j]} \exp [tS_j - t^2 \sum_1^j \tau_k]\, dP \\ &\geqq \textstyle\sum_{j=1}^n \int_{[N=j]} \exp [tS_j - t^2 \sum_1^n \tau_k]\, dP. \end{aligned}$$

Hence for $-\infty < t < \infty$,

$$\exp [t^2 \textstyle\sum_1^n \tau_k] \geqq E \exp [tS_N]. \tag{17}$$

THEOREM 5. *Let b_n be real numbers, $b_n \geqq 0$, $B_n = b_1 + \cdots + b_n \to \infty$ and $b_1 > 0$. Let x_n be independent, generalized Gaussian, $\tau(x_n) \leqq 2^{\frac{1}{2}}$ and $b_n = o(B_n/\log \log B_n)$. Then*

$$T_n \equiv \textstyle\sum_1^n b_j x_j / B_n \to 0 \quad \text{a.e.} \tag{18}$$

PROOF. For $-\infty < t < \infty$,

$$\textstyle\sum_1^n b_j^2 B_n^{-2} = o[(\log \log B_n)^{-1}] = \epsilon_n / \log \log B_n ,$$

$$E \exp [tT_n] \leqq \exp [t^2 \textstyle\sum_1^n b_j^2/B_n^2] = \exp [\epsilon_n t^2/\log \log B_n],$$

where $\epsilon_n \to 0$. For $\epsilon > 0$, by (4)

$$P[|T_n| > \epsilon] \leqq 2 \exp [-\epsilon^2 \log \log B_n/(4\epsilon_n)].$$

For $\infty > \theta > 1$, let $n(j)$ be the first $m \geqq 1$ such that $\theta^j \leqq B_m < \theta^{j+1}$. It has been shown ([15], p. 247) that $n(j)$ always exists, for j large enough. Then for all large $j \geqq j_0$, say,

$$\begin{aligned} P[|T_{n(j)}| > \epsilon] &\leqq 2 \exp [-\epsilon^2 \log (j \log \theta)/(4\epsilon_{n(j)})] \\ &\leqq 2 \exp [-2 \log (j \log \theta)] = O(j^{-2}). \end{aligned}$$

Hence

$$\textstyle\sum_{j_0}^\infty P[|T_{n(j)}| > \epsilon] < \infty. \tag{19}$$

Now let $F_j = [\max_{n(j)<m<n(j+1)} |\sum_{n(j)+1}^m b_i x_i / B_{n(j)}| > \epsilon]$ and let N be the first $n(j) < m < n(j+1)$ such that $|\sum_{n(j)+1}^m b_i x_i| > \epsilon B_{n(j)}$, if there is such one, and $N = n(j+1)$ otherwise. Then N is a stopping variable relative to $x_{n(j)+1}, \cdots, x_{n(j+1)}$. By Lemma 5 and (4),

$$\begin{aligned} P(F_j) &= P[|\textstyle\sum_{n(j)+1}^N b_i x_i / B_{n(j)}| > \epsilon] \\ &\leqq 2 \exp [-\epsilon^2 4^{-1} (\textstyle\sum_{n(j)+1}^{n(j+1)} b_i^2)^{-1} B_{n(j)}^2] \\ &\leqq 2 \exp [-\epsilon^2 4^{-1} (\textstyle\sum_1^{n(j+1)} b_i^2)^{-1} B_{n(j+1)}^2 \theta^{-4}] \\ &\leqq 2 \exp [-\epsilon^2 4^{-1} \theta^{-4} \log \log B_{n(j+1)}/\epsilon_{n(j+1)}] \\ &\leqq 2 \exp [-2 \log \log B_{n(j+1)}], \end{aligned}$$

for all large $j \geqq j_1 (\geqq j_0)$, say. Hence $P(F_j) \leqq O(\exp [-2 \log j]) = O(j^{-2})$.

Therefore

(20) $\sum_{j=j_1}^{\infty} P[\max_{n(j)<m<n(j+1)} |\sum_{n(j)+1}^{m} b_i x_i/B_m| > \epsilon] \leqq \sum_{j_1}^{\infty} P(F_j) < \infty.$

From (19) and (20), by Borel-Cantelli's lemma,

$$P[|\sum_1^n b_i x_i/B_n| > 2\epsilon, \textit{ i. o.}] = 0, \tag{21}$$

and (18) follows immediately.

When $P[x_n = 1] = P[x_n = -1] = \frac{1}{2}$ for each n, Theorem 5 is due to Salem and Zygmund [15]. Even in that case, they give [15] a counter example to show that the condition $b_n = o(B_n/\log\log B_n)$ cannot be replaced by $O(B_n/\log\log B_n)$.

4. Extension of Hsu and Robbins' theorem. Let a_{nk} be real numbers and $A_n = \sum_{k=1}^{\infty} a_{nk}^2$.

THEOREM 6. *Let x_n be independent, identically distributed random variables with $Ex_1 = 0$. Let $a_{nk} = 0$ if $k > n$, and $|a_{nk}| \leqq KA_n$ for some $0 < K < \infty$, $n, k = 1, 2, \cdots$. If for some $0 < \alpha \leqq 1$, $A_n \leqq Kn^{-\alpha}$ and $E|x_1|^{2/\alpha} \leqq K$, then*

$$\sum_1^{\infty} P[|T_n| \geqq \epsilon] < \infty \tag{22}$$

for every $\epsilon > 0$, where $T_n = \sum_{k=1}^{\infty} a_{nk}x_k$.

PROOF. We can assume that $K \geqq 1$ and $A_n > 0$ for each n. For $0 < \beta < \alpha$ and $N = 2, 3, \cdots$, put

$$\begin{aligned} &x_k' = x_k I[x_k \leqq n^{\beta}], \qquad x_k'' = x_k I[x_k \geqq \epsilon n^{\alpha}/(NK^2)], \quad \text{if } a_{nk} \geqq 0;\\ &x_k' = x_k I[x_k \geqq -n^{\beta}], \qquad x_k'' = x_k I[x_k \leqq -\epsilon n^{\alpha}/(NK^2)], \quad \text{if } a_{nk} < 0;\\ &x_k''' = x_k - x_k' - x_k'', \qquad T_n' = \sum_{k=1}^{\infty} a_{nk}x_k',\\ &T_n'' = \sum_{k=1}^{\infty} a_{nk}x_k'', \qquad T_n''' = \sum_{k=1}^{\infty} a_{nk}x_k''', \end{aligned} \tag{23}$$

where $I(A)$ is the indicator function of the set A. If a random variable $x \leqq 1$ a.e., then obviously $E\exp[x] \leqq \exp[Ex + Ex^2]$. Let $0 < t \leqq K^{-1}n^{-\beta}$. Then $ta_{nk}x_k'/A_n \leqq 1$ and $E(a_{nk}x_k') \leqq 0$. Hence

$$\begin{aligned} E\exp[ta_{nk}x_k'/A_n] &\leqq \exp[t^2a_{nk}^2A_n^{-2}E(x_k')^2]\\ &\leqq \exp[t^2a_{nk}^2A_n^{-2}EX_k^2]\\ &\leqq \exp[t^2a_{nk}^2A_n^{-2}K], \end{aligned}$$

since $K \geqq 1$. By independence, $E\exp[tT_n'/A_n] \leqq \exp[t^2K/A_n]$. Hence

$$P[T_n' \geqq \epsilon] = P[tT_n'A_n^{-1} \geqq t\epsilon A_n^{-1}] \leqq \exp[-t(\epsilon - tK)/A_n].$$

Put $t = n^{-\beta}K^{-1}$. Then for sufficiently large n, $P[T_n' \geqq \epsilon] \leqq \exp[-\epsilon n^{\alpha-\beta}/(2K)]$. Since $\alpha - \beta > 0$,

$$\sum_1^{\infty} P[T_n' \geqq \epsilon] < \infty. \tag{24}$$

Now $P[T_n'' \geqq \epsilon] \leqq nP[|x_1| \geqq \epsilon n^{\alpha}N^{-1}K^{-2}] = nP[(NK^2|x_1|/\epsilon)^{\alpha^{-1}} \geqq n]$. Since $E|x_1|^{2/\alpha} < \infty$ and N, K are fixed constants,

$$\sum_1^\infty P[T_n'' \geqq \epsilon] < \infty. \tag{25}$$

Since $a_{nk}x_k''' \leqq \epsilon/N$, $T_n''' \geqq \epsilon$ implies that there are at least N non-zero x_k''' for $k = 1, 2, \cdots, n$. Hence

$$P[T_n''' \geqq \epsilon] \leqq \binom{n}{N} P^N[|x_1| > n^\beta] \leqq \binom{n}{N} P^N[|x_1|^{\alpha^{-1}} > n^{\beta/\alpha}].$$

By Tchebichev's inequality,

$$P[T_n''' \geqq \epsilon] \leqq \binom{n}{N}(Kn^{-2\beta/\alpha})^N \leqq Mn^{(1-2\beta/\alpha)N},$$

for all large n, where M is a finite constant, depending only on N and K. Choose $\beta = 2\alpha/3$ and $N = 6$. Then for all large n, $P[T_n''' \geqq \epsilon] \leqq Mn^{-2}$. Therefore

$$\sum_1^\infty P[T_n''' \geqq \epsilon] < \infty. \tag{26}$$

From (24), (25) and (26), we have $\sum_1^\infty P[T_n \geqq \epsilon] < \infty$. By symmetry, we obtain $\sum_1^\infty P[T_n \leqq -\epsilon] < \infty$. Hence (22) holds.

When $a_{nk} = n^{-1}$ for $1 \leqq k \leqq n$ and $\alpha = 1$, Theorem 6 is due to Hsu and Robbins [8]. The above demonstration is a modification of Erdös' elegant proof [3] of their theorem. When $1 > \alpha > \frac{1}{2}$, Theorem 6 is implied by a result of Franck and Hanson [4].

Theorem 6 cannot be improved without some modifications, since Erdös [3] has proved that (22) implies $Ex_1 = 0$ and $Ex_1^2 < \infty$ if $a_{nk} = n^{-1}$ for $1 \leqq k \leqq n$ and $\alpha = 1$.

5. Some further extensions. We assumed that x_{nk} are generalized Gaussian in Theorem 1 and that x_n are identically distributed and a_{nk} is a triangular matrix in Theorem 6. Both conditions are essential in their proofs. However, if we are willing to sacrifice some sharpness, further extensions are possible.

Let a_{nk} be real numbers and $A_n = \sum_{k=1}^\infty a_{nk}^2$.

THEOREM 7. *Let x_n be independent, $Ex_n = 0$, $a_{nk} = 0$ if $k > n^\lambda$ for some $1 \leqq \lambda < \infty$ and $|a_{nk}| \leqq KA_n$ for some $K < \infty$. If for some $0 < \alpha \leqq 1$, $A_n \leqq Kn^{-\alpha}$ and $E[|x_n|^{(1+\lambda)/\alpha}(\log + |x_n|)^2] \leqq K$, then*

$$\sum_1^\infty P[|T_n| \geqq \epsilon] < \infty$$

for every $\epsilon > 0$, where $T_n = \sum_{k=1}^\infty a_{nk}x_k$.

PROOF. We can assume that $K \geqq 1$ and $E|x_n|^{(1+\lambda)/\alpha} \leqq K$. For $0 < \beta < \alpha$ and $N = 2, 3, \cdots$, define x_k', x_k'', x_k''', T_n', T_n'' and T_n''' as in (23). Then, as before, we have

$$\sum_1^\infty P[T_n' \geqq \epsilon] < \infty. \tag{27}$$

Put $m = [n^\lambda]$, the integral part of n^λ, and $z_n = (NK^2|x_n|/\epsilon)^{\alpha^{-1}}$. Then for $n > 3$,

$$\begin{aligned} P[T_n'' \geqq \epsilon] &\leqq \sum_{k=1}^m P[(NK^2|x_k|/\epsilon)^{\alpha^{-1}} \geqq n] \\ &\leqq \sum_{k=1}^m P[z_k^{1+\lambda}\log^2 z_k \geqq n^{1+\lambda}\log^2 n] \\ &\leqq \sum_{k=1}^m n^{-(1+\lambda)}\log^{-2} n\cdot E(z_k^{1+\lambda}\log^2 z_k). \end{aligned}$$

By the condition $E[|x_n|^{(1+\lambda)/\alpha}(\log + |x_n|)^2] \leqq K$, there exists a constant $K^* = K^*(N, K, \lambda, \epsilon, \alpha) < \infty$ such that $E(z_n^{1+\lambda} \log^2 z_n) \leqq K^*$. Hence

$$P[T_n'' \geqq \epsilon] \leqq K^* \textstyle\sum_{k=1}^m n^{-(1+\lambda)} \log^{-2} n \leqq K^* n^{-1} \log^{-2} n,$$

$$\textstyle\sum_1^\infty P[T_n'' \geqq \epsilon] < \infty. \tag{28}$$

Now by same reasoning as before, we have

$$P[T_n''' \geqq \epsilon] \leqq \tbinom{m}{N} \textstyle\prod_{k=1}^N P[|x_k| > n^\beta] \leqq \tbinom{m}{N} K^N n^{-N\beta(1+\lambda)/\alpha} = O(n^{N(\lambda - \beta(1+\lambda)/\alpha)}).$$

Choose β so near to α such that $\lambda - \beta(1 + \lambda)/\alpha < 0$, and then choose N so large such that $N(\lambda - \beta(1 + \lambda)/\alpha) \leqq -2$. Then

$$\textstyle\sum_1^\infty P[T_n''' \geqq \epsilon] \leqq \sum_1^\infty O(n^{-2}) < \infty. \tag{29}$$

Therefore $\sum_1^\infty P[T_n \geqq \epsilon] < \infty$. By symmetry, $\sum_1^\infty P[T_n \leqq -\epsilon] < \infty$. This completes the proof.

COROLLARY 3. *Let x_n be independent and $Ex_n = 0$. Assume that there exists $1 \leqq \alpha < \infty$ such that $B(n, \alpha) \equiv \sum_{k>n^\alpha} a_{nk}^2 \leqq K(n \log^2 n)^{-1}$ for some $K < \infty$ and that $A(n, \alpha) = \sum_{k \leqq n^\alpha} a_{nk}^2 \leqq Kn^{-\beta}$ for some $0 < \beta \leqq 1$. If $|a_{nk}| \leqq KA(n, \alpha)$ for $k \leqq n^\alpha$ and $E[|x_n|^{(1+\alpha)/\beta}(\log + |x_n|)^2] \leqq K$, then*

$$\textstyle\sum_1^\infty P[|T_n| \geqq \epsilon] < \infty \tag{30}$$

for every $\epsilon > 0$, where $T_n = \sum_{k=1}^\infty a_{nk}x_k$.

PROOF. Let $b_{nk} = a_{nk}$ if $k \leqq n^\alpha$ and $b_{nk} = 0$ if $k > n^\alpha$. Put $T_n' = \sum_{k=1}^\infty b_{nk}x_k$ and $T_n'' = \sum_{k=1}^\infty (a_{nk} - b_{nk})x_k$. By Theorem 7, $\sum_1^\infty P[|T_n'| \geqq \epsilon] < \infty$. By a theorem of Kolmogorov ([12], p. 236), T_n'' converges a.e. for each n and $E(T_n'')^2 = B(n, \alpha) \leqq K(n \log^2 n)^{-1}$. Hence

$$\textstyle\sum_1^\infty P[|T_n''| \geqq \epsilon] = O(\sum_1^\infty n^{-1} \log^{-2} n) < \infty.$$

Therefore (30) holds.

6. A counter example. Let $(\Omega, \mathfrak{F}, P)$ be a probability space and Λ be a directed set with elements λ. For each $\lambda \in \Lambda$, let $\mathfrak{F}_\lambda \subset \mathfrak{F}$ be a Borel field and x_λ be an integrable, $\mathfrak{F}_\lambda$-measurable random variable. If for every pair $\lambda < \delta$, $\mathfrak{F}_\lambda \subset \mathfrak{F}_\delta$ and $E(x_\delta \mid \mathfrak{F}_\lambda) = x_\lambda$ a.e., then $(x_\lambda, \mathfrak{F}_\lambda, \Lambda)$ is said to be a martingale (see [11]). In [1], Dieudonné constructed a martingale $(x_\lambda, \mathfrak{F}_\lambda, \Lambda)$, which diverges on a set of positive measure and satisfies the conditions: Λ is the family of all finite sets of positive integers, ordered by inclusion, and for each $\lambda \in \Lambda$, $x_\lambda = P(A \mid \mathfrak{F}_\lambda)$. His example shows, among other things, that Doob's martingale convergence theorem ([2], p. 319) cannot be extended to martingales by directed sets without some restrictions on the Borel fields $\mathfrak{F}_\lambda$. However, his example is rather complicated and a simpler counter example can be obtained as follows:

Let Λ be the family of all finite sets λ of positive integers, ordered by inclusion, and let x_n be independent, identically distributed random variables with $P[x_n = 1] = P[x_n = -1] = \frac{1}{2}$. For each $\lambda \in \Lambda$, define

$$S_\lambda = \textstyle\sum_{j \in \lambda} x_j/j, \tag{31}$$

and let $\mathfrak{F}_\lambda$ be the Borel field generated by $(x_j, j \varepsilon \lambda)$. Then by independence, it is easy to verify that $(S_\lambda, \mathfrak{F}_\lambda, \Lambda)$ is a martingale and $ES_\lambda^2 = \sum_{j\varepsilon\lambda} j^{-2} \leqq \sum_1^\infty j^{-2} < \infty$. However, by the following theorem,

$$P[\limsup_\Lambda S_\lambda = \infty] = P[\liminf_\Lambda S_\lambda = -\infty] = 1.$$

THEOREM 8. *Let Λ be the family of all finite sets of positive integers, ordered by inclusion, and let x_n be independent, identically distributed random variables. Define S_λ by* (31) *for $\lambda \varepsilon \Lambda$. If $P[x_1 > 0] > 0$, then*

$$P[\limsup_\Lambda S_\lambda = \infty] = 1. \tag{32}$$

PROOF. Choose $\alpha > 0$ such that $P[x_1 > \alpha] = \beta > 0$. Define $x_n' = \alpha$ if $x_n \geqq \alpha$, and $x_n' = 0$ if $x_n < \alpha$. Then x_n' are independent, identically distributed with $Ex_1' = \alpha\beta > 0$. Let $K = K(\omega) = \{n \mid x_n'(\omega) = \alpha\}$ and $k_n = k_n(\omega) =$ the number of m such that $m \varepsilon K$ and $1 \leqq m \leqq n$. By the strong law of large numbers, there exists a null set A such that for $\omega \not\varepsilon A$, $\lim_n k_n(\omega)/n = \beta$. Let $0 < \epsilon < \beta/4$ and $\omega \not\varepsilon A$. Choose $n_0 = n_0(\omega)$ so large that $|k_n(\omega)/n - \beta| < \epsilon$, if $n \geqq n_0$. For $J \varepsilon \Lambda$, let $J \subset \{1, 2, \cdots, j_0\}$ and choose a positive integer $n_1 = n_1(\omega) \geqq \max(n_0, j_0)$. Put $K_n = K_n(\omega) = \{j \mid j \varepsilon K, n < j \leqq 2n\}$. Then for $n \geqq n_1$,

$$\sum_{j\varepsilon K_n} x_j(\omega)/j \geqq \sum_{j\varepsilon K_n} x_j'(\omega)/j \geqq \alpha(k_{2n} - k_n)/(2n) \geqq \alpha(\beta - 3\epsilon)/2 \geqq \alpha\beta/8.$$

Hence

$$\sum_{n_1 < j\varepsilon K} x_j(\omega)/j = \sum_{m=0}^\infty \sum_{j\varepsilon K_{2^m n_1}} x_j(\omega)/j = \infty.$$

Therefore $\sup_{\lambda > J} S_\lambda(\omega) = \infty$ for every $J \varepsilon \Lambda$ and hence $\limsup_\Lambda S_\lambda(\omega) = \infty$.

7. Unconditional convergence. Theorem 8 implies that the net S_λ defined as in (31) by independent, identically distributed random variables x_n can never converge a.e., except for the trivial case $P[x_n = 0] = 1$. On the other hand, if x_n are independent identically distributed with mean zero, and if either x_1 is symmetric or $E(|x_1| \log^+ |x_1|) < \infty$, then by a theorem of Marcinkiewicz and Zygmund [13], $\sum_1^\infty x_j/j$ converges a.e. Naturally, one can ask: under what conditions, for every sequence $\{J_n\} \subset \Lambda$ satisfying $J_1 \subset J_2 \subset \cdots$, do we have the a.e. convergence for the sequence S_{J_n} as $n \to \infty$?

THEOREM 9. *Let x_n be independent, identically distributed, $Ex_1 = 0$ and $\{J_n\}$ be a monotone increasing sequence of finite sets of positive integers. If either* (i) *x_1 is symmetric, or* (ii) *$E(|x_1| \log^+ |x_1|) < \infty$, then S_{J_n} converges* a.e. *as $n \to \infty$, where S_{J_n} is defined by* (31). *If moreover $\lim_n J_n = \{1, 2, \cdots\}$, then*

$$\lim_n S_{J_n} = \sum_1^\infty x_j/j \quad \text{a.e.} \tag{33}$$

PROOF. For $0 < c < \infty$, define $x_n' = x_n$ if $|x_n| \leqq cn$ and $x_n' = 0$ if $|x_n| > cn$. Since x_n are independent, identically distributed and $E|x_1| < \infty$, we have

$$\sum_1^\infty P[|x_n| \geqq cn] < \infty, \qquad \sum_1^\infty \sigma^2(x_n')/n^2 < \infty. \tag{34}$$

Under condition (ii),

$$\sum_1^\infty \int_{[x_j>cj]} x_j\, dP/j \leqq \sum_{j=1}^\infty \sum_{k=j}^\infty c(k+1)P[(ck+c \geqq x_1 > ck]/j$$
$$\leqq \sum_{k=1}^\infty c(k+1)(1+\log k)P[ck+c \geqq x_1 > ck] < \infty.$$

Similarly,

$$\sum_1^\infty \int_{[x_j<-cj]} -x_j\, dP/j < \infty.$$

Hence (ii) implies

$$(35)\qquad \sum_{j\varepsilon J_n} |E(x_j')/j| \leqq \sum_1^\infty |E(x_j')/j| \leqq \sum_1^\infty \int_{[|x_j|>cj]} |x_j|\, dP/j < \infty.$$

Under condition (i), $Ex_n' = 0$ for every n. Therefore $\sum_{j\varepsilon J_n} E(x_j')/j$ converges as $n \to \infty$. By Kolmogorov's three series theorem ([12], p. 237), from (34), we have the a.e. convergence of S_{J_n} as $n \to \infty$. If moreover, $\lim_n J_n = \{1, 2, \cdots\}$, then (33) follows from ([2], pp. 118–119).

8. A convergence theorem for sums of means. The most difficult part of the proofs for Kolmogorov's three series theorem lies in demonstrating the convergence of the sums of means of the truncated random variables in the necessary part. The two known methods for proving this fact are that of symmetrization (see [12], p. 237) and characteristic functions (see [2], p. 111). The following theorem furnishes us another proof.

THEOREM 10. *Let x_n be independent, $S_n = x_1 + \cdots + x_n$ and $E \sup_{n\geqq1} |x_n| < \infty$. If S_n converges* a.e., *then $\sum_1^\infty Ex_n$ converges.*

PROOF. Put $z = \sup_n |x_n|$, $m_n = Ex_n$ and $S_\infty = \lim_n S_n$. For $0 < c < \infty$, define t = first $n \geqq 1$ such that $|S_n| \geqq c$, if there is such one, and $t = \infty$ otherwise. Then $P[t = \infty] > 0$ if c is large enough. For $k = 1, 2, \cdots$, put $s \equiv t(k) = \min(t, k)$. Then $\lim_k S_s = S_t$ a.e. Since $|S_s| \leqq z + c$, by the Lebesgue dominated convergence theorem $\lim_k ES_s = ES_t$. It is easy to see that (or by [2], p. 303) $ES_s = E\sum_1^s m_j$. Hence $ES_{t(k)} = ES_s = \sum_1^k m_j P[t \geqq j]$ and $m_k = (ES_{t(k)} - ES_{t(k-1)})/P[t \geqq k]$. Since $P[t \geqq k]$ monotonically decreases to $P[t = \infty] > 0$ and $ES_{t(k)}$ converges as $k \to \infty$, $\sum_1^\infty m_n$ converges by summation by parts.

By oral communication, S. M. Samuels has given another proof of Theorem 10.

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