BOOK REVIEW

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WIENER, N., SIEGEL, A., RANKIN, B. AND MARTIN, W. T. *Differential Space, Quantum Systems, and Prediction.* The M.I.T. Press, Cambridge, 1966. x + 176 pp.; $7.50.

Review by H. SALEHI

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For many years it was felt that there is a natural association between the Brownian motion processes and differential space, the prediction theory of stationary stochastic processes and quantum concepts based on the statistical-mechanical differential-space approach. In this book an attempt has been made by the authors to make this association clear. The main part of the book is largely devoted to tracing Wiener's contributions to the Brownian motion process, prediction theory, the factorization problem and his work in the many applications, including the work on quantum theory. We would certainly recommend the book to all persons interested in these subjects. But for those who are not familiar with this area, it would be preferable to read from the well known works by Doob (*Stochastic Processes*) and von Neumann (*Mathematical Foundation of Quantum Mechanics*). What follows below is designed to reveal to the reader the wide scope of this book.

Chapter 1 is an introduction to the remaining chapters. The most important aspect of this chapter is the introduction of probability functional spaces, a generalization of probability spaces. As the authors point out the notion of complete additivity is neither assumed in a probability functional space nor does it follow immediately that the complete additivity can be derived from the postulates of a probability functional space. This lack of complete additivity allows certain computable constructions for random variables, as well as the possibility of certain probabilistic treatment of an individual system in quantum mechanics. The construction method in mathematics is emphasized and examples of the constructional approach are found throughout the book.

Chapter 2 concerns the Brownian motion process and differential space. First a method for constructing the Browian motion process is given. It is then shown that the Brownian motion process, so constructed, is a separable and measurable process. The authors then go on and prove the usual properties of the Brownian motion process such as the continuity and non-bounded variation properties of sample functions almost everywhere. Finally, the relation of the Brownian motion process to noise and information theory, heat conduction in physics, certain phenomena in astromony, etc. are discussed.

In Chapter 3, integration on differential space is taken up. Sections 1 and 2 are about stochastic integrals with respect to the Brownian motion process and may be easily found in many books. An interesting discussion of Fourier-Hermite functions and their relation to non-linear problems is included in this chapter. The rest of Chapter 3 is devoted to discussion of functions of the Brownian motion process and their distributions. In this connection the work of M. Kac on the Brownian motion process and its relation to the Schrödinger equation is briefly discussed.

Chapter 4 is entitled "Matrix factorization and prediction." The factorization of one dimensional case may be described as follows: let $F(\theta)$ be a non-negative (real-valued) integrable function on $(-\pi, \pi]$. Then $F(\theta)$ may be written in the form $F(\theta) = |\Phi(\theta)|^2$ where $\Phi(\theta)$ is an optimal function and square-integrable on $(-\pi, \pi]$ in the form $\Phi(\theta) = \sum_{k=0}^{\infty} a_k e^{ik\theta}$ iff $\int_{-\pi}^{\pi} |\log F(\theta)| \, d\theta < \infty$. In Section 2 a two dimensional regular full rank stationary stochastic process $f_i(n)$, $1 \leqq i \leqq 2$, is considered. Let the 2×2 non-negative hermitian matrix $F(\theta) = [F_{ij}(\theta)]$, $1 \leqq i, j \leqq 2$, be its spectral density. The problem of factorization of F now becomes much harder, because of non-commutativity of matrices. They base their proof on an interesting result essentially due to von Neumann known as the "Alternating projections theorem." The derivation of formula 49 on page 104 is not clear. To get this formula one needs to rearrange the terms of an infinite series which might cause some difficulty. This point, as was clearly mentioned by Wiener and Masani in their paper in *Acta Math.* (1958) p. 108, is due to the fact that the pasts of $f_1(n)$ and $f_2(n)$ may incline at a zero angle. To avoid this difficulty one might approach the problem as was treated by Wiener and Masani in their paper. In Section 5, multiple prediction is considered. Here $f_i(n)$, $1 \leqq i \leqq q$, is a q-dimensional regular full rank stationary stochastic process with the spectral density $F(\theta) = [F_{ij}(\theta)]$, $1 \leqq i, j \leqq q$. For finding the best linear prediction of, say $f(n)$, $n \geqq 0$, in terms of the pasts of $f_1(n), \cdots, f_q(n)$, they use a generalized form of the alternating projections theorem. This generalization does not seem to be suitable as a computational algorithm in the sense that too many limit operations are involved. Here we copy formula 74 on page 113.

$$(1) \quad g_{i,N}(\alpha) = \lim_{M_N \to \infty} [P_N \cdots \lim_{M_2 \to \infty} [P_3 \lim_{M_1 \to \infty} [P_2 P_1]^{M_1}]^{M_2} \cdots]^{M_N} f_i(\alpha).$$

It would have been much better if (1) were replaced by an interesting formula obtained by G. C. Rota in *Bull. Amer. Math. Soc.* (1961) p. 101. Equation (1) using Rota's result may be written as follows:

$$(2) \qquad g_{i,N}(\alpha) = \lim_{n \to \infty} (P_1 P_2 \cdots P_{q-1} P_q)^n (P_q P_{q-1} \cdots P_2 P_1)^n.$$

It is not clear how formula 74 on p. 113 is used to obtain the coefficients involved in formula 75 on p. 113. This is clear, as one can see from pp. 100–102, for the two-dimensional case. Perhaps if (2) were used instead of (1) then even for $q > 2$ the connection would have become clear. The same difficulty that was mentioned for the two-dimensional case in rearranging the terms of an infinite series certainly arises here too. Moreover in this section there is not any indication how one would get the optimal factor $\Phi(\theta)$. In the opinion of the reviewer, good treatments for the factorization of matrix-valued functions are the Wiener and Masani paper mentioned before and the paper by the reviewer in *Trans. Amer. Math. Soc.* (1966), pp. 468–479. In these two papers algorithms are designed for obtaining the optimal factor $\Phi(\theta)$.

In Chapter 5 the authors discuss the "Differential-space theory of quantum systems." The discussion is based on the interesting works of N. Wiener and A. Siegel on quantum mechanics.

Finally there is an appendix "Technique of computing joint probabilities and remarks on the dichotimic algorithm" by R. L. Warnock.

The book is highly readable and with no doubt will give rise to a good deal of interest and research in these fields. The authors have been wise not to overburden the text with all the details in order to make it self-contained, and refer the reader to the appropriate references at the end of the book.