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INADMISSIBILITY OF THE BEST INVARIANT TEST WHEN THE MOMENT IS INFINITE UNDER ONE OF THE HYPOTHESES

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1. Introduction. Let $(\mathcal{Y}, \mathcal{A}, \lambda_i)(i = 1, 2)$ be probability spaces. For each $i = 1, 2$ and $y \varepsilon \mathcal{Y}$ let $F_i(\cdot, y)$ be a distribution function on the real line R such that $F_i(\cdot, \cdot)$ is $\mathcal{B} \times \mathcal{A}$ measurable where $\mathcal{B}$ is the σ-field of all Borel subsets of the real line R. Assume the distribution of $(X, Y) \varepsilon R \times \mathcal{Y}$ for $\theta \varepsilon R$ and $i = 1, 2$ is given by usual extension of

$$P_{i\theta}((X, Y) \varepsilon C \times D) = \int_D d\lambda_i(y) \int_C F_i(dx - \theta, y)$$

to measurable subsets of $R \times \mathcal{Y}$.

Consider the problem of testing $H_1: i = 1$ versus $H_2: i = 2$. For any level of significance a best invariant test φ_0 is of the form

$$(1.1) \qquad \begin{aligned} \varphi_0(x, y) &= 1 \quad \text{if} \quad \frac{d\lambda_2}{d(\lambda_1 + \lambda_2)}(y) > c \\ &= 0 \quad \text{if} \quad \frac{d\lambda_2}{d(\lambda_1 + \lambda_2)}(y) < c. \end{aligned}$$

We restrict attention to the case that the $F_i(\cdot, y)$ are absolutely continuous with respect to Lebesgue measure for each $y \varepsilon \mathcal{Y}$ and $i = 1, 2$. Denote the density of $F_i(\cdot, y)$ with respect to Lebesgue measure by $f_i(\cdot, y)$.

Lehmann and Stein [1] have shown that if $E_{i0}|X| < \infty$ for $i = 1, 2$ and if

$$(1.2) \qquad \lambda_1\{y: \frac{d\lambda_2}{d(\lambda_1 + \lambda_2)}(y) = c\} = 0$$

then φ_0 is admissible. Condition (1.2) guarantees that φ_0 is the essentially unique best invariant test at some level. Perng [2; Sections 4 and 5] has given examples showing that, with either the moment condition or (1.2) violated, φ_0 may not be admissible. The purpose of this note is to improve Perng's example concerning the moment condition.

Perng has shown that given any $\delta > 0$ one can construct an example in which $E_{i0}|X|^\alpha$ is, for $i = 1, 2$ finite or infinite according as $\alpha < 1 - \delta$ or $\alpha \geqq 1 - \delta$ and for which φ_0 is inadmissible. His example satisfies (1.2). The present example, given in Section 2, also satisfies (1.2) but is such that $E_{10}|X|^\alpha$ is as in Perng's example while $E_{20}|X|^\alpha < \infty$ for all $\alpha > 0$. This suggests the intuitive idea that knowledge of X is useful when the distributions of X under H_1 and H_2 are very different.

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2. The example. Let $\mathcal{Y} = R$ and let λ_i $(i = 1, 2)$ have density g_i with respect to Lebesgue measure where

$$\begin{aligned} g_1(y) = g_2(-y) &= c_1/y^2 && \text{if } y > 1; \\ &= c_2/y^2 && \text{if } y < -1; \\ &= 0 && \text{if } |y| \leqq 1 \end{aligned}$$

with $c_1 + c_2 = 1$.

Let $a > 2$ and $\eta > 0$. For η sufficiently small,

$$[(a - 1 - \eta)/(a - 1 + \eta)]a > 2 \tag{2.1}$$

and

$$[(a - 1 - \eta)/(a - 1 + \eta)](a - 1) > 1. \tag{2.2}$$

Fix $\epsilon > 0$. For a sufficiently close to 2 we have

$$(a - 1)^{1/(1+\epsilon)}(a^{1/(1+\epsilon)} - 1) < 1$$

so that, for η sufficiently small,

$$\begin{aligned} &[(a - 1)(a - 1 - \eta)/(a - 1 + \eta)]^{1/(1+\epsilon)} \\ &\qquad \cdot\{[a(a - 1 + \eta)/(a - 1 - \eta)]^{1/(1+\epsilon)} - 1\} < 1. \end{aligned} \tag{2.3}$$

Fix ϵ, $\eta > 0$ and $a > 2$ satisfying (2.1), (2.2) and (2.3). Let

$$\begin{aligned} f_1(x, y) &= \eta^{-1} && \text{if } y > 1,\ y^{1+\epsilon} < x < y^{1+\epsilon} + \eta \\ & && \text{or } y < -1,\ -\eta < x < 0; \\ &= 0 && \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned} f_2(x, y) &= \eta^{-1} && \text{if } y > 1,\ -\eta < x < 0 \\ & && \text{or } y < -1,\ 0 < x < \eta; \\ &= 0 && \text{otherwise.} \end{aligned}$$

Elementary integrations yield $E_{10}|X|^{\alpha} < \infty$ if, and only if, $\alpha < 1 - \epsilon/(1 + \epsilon)$ while $E_{20}|X|^{\alpha} < \infty$ for all $\alpha > 0$. Finally, by (2.2) we can take

$$c_1 < c_2 < c_1[(a - 1)(a - 1 - \eta)/(a - 1 + \eta)]^{1/(1+\epsilon)}. \tag{2.4}$$

Since $c_2 > c_1$, the test of the form (1.1) for level c_1 is given by

$$\begin{aligned} \varphi_0(x, y) &= 1 && \text{if } y \geqq 0 \\ &= 0 && \text{if } y < 0. \end{aligned}$$

Clearly (1.2) is satisfied so long as c is chosen so that $c_1/(c_1 + c_2) < c < c_2/(c_1 + c_2)$. Hence φ_0 is the essentially unique best invariant test at level c_1.

We now define another test φ^* which will be shown to dominate φ_0. Let

$$\begin{aligned}\varphi^*(x, y) &= 1 \qquad \text{if } y < -1, a - 1 \leqq x \leqq |y|^{1+\epsilon};\\ &= 0 \qquad \text{if } y > 1, \max(a - 1, y^{1+\epsilon}) \leqq x \leqq ay^{1+\epsilon};\\ &= \varphi_0(x, y) \qquad \text{otherwise.}\end{aligned}$$

Abbreviate $\varphi_0(X, Y)$ by φ_0 and $\varphi^*(X, Y)$ by φ^* since in the sequel this will cause no confusion. We wish to show that $E_{1\theta}(\varphi^* - \varphi_0) \leqq 0$ and $E_{2\theta}(\varphi^* - \varphi_0) \geqq 0$ with strict inequality in at least one case for some θ.

Clearly for $\theta \leqq a - 1$ we have $E_{2\theta}(\varphi^* - \varphi_0) \geqq 0$ and $E_{1\theta}(\varphi^* - \varphi_0) \leqq 0$ with strict inequality in the former for $a - 1 - \eta < \theta \leqq a - 1$ and in latter for $-\eta < \theta \leqq a - 1$.

Let $\theta > a - 1$. In this case the curves $x = y^{1+\epsilon} + \theta$ and $x = ay^{1+\epsilon}$ intersect with $y \geqq 1$. Thus,

$$\begin{aligned}E_{1\theta}(\varphi^* - \varphi_0) &\leqq c_2\int_{(\theta-\eta)^\omega}^{\infty} y^{-2}\,dy - c_1\int_{[(\theta+\eta)/(a-1)]^\omega}^{\infty} y^{-2}\,dy\\ &= c_2(\theta - \eta)^{-\omega} - c_1[(a - 1)/(\theta + \eta)]^\omega\\ &\leqq (\theta - \eta)^{-\omega}\{c_2 - c_1[(a - 1)(a - 1 - \eta)/(a - 1 + \eta)]^\omega\}\\ &< 0,\end{aligned}$$

where $\omega = (1 + \epsilon)^{-1}$. The last inequality results from (2.4). Now from (2.3) and (2.4) we obtain

$$c_2\{[a(a - 1 + \eta)/(a - 1 - \eta)]^\omega - 1\} < c_1.$$

Thus

$$\begin{aligned}E_{2\theta}(\varphi^* - \varphi_0) &\geqq c_1\int_{(\theta+\eta)^\omega}^{\infty} y^{-2}\,dy - c_2\int_{[(\theta-\eta)/a]^\omega}^{\theta^\omega} y^{-2}\,dy\\ &= c_1[1/(\theta + \eta)]^\omega - c_2\{[a/(\theta - \eta)]^\omega - (1/\theta)^\omega\}\\ &> [1/(\theta + \eta)]^\omega\{c_1 - c_2[[a(\theta + \eta)/(\theta - \eta)]^\omega - 1]\}\\ &\geqq [1/(\theta + \eta)]^\omega\{c_1 - c_2[[a(a - 1 + \eta)/(a - 1 - \eta)]^\omega - 1]\}\\ &> 0.\end{aligned}$$

This completes the proof.

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