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A NOTE ON THRIFTY STRATEGIES AND MARTINGALES IN A FINITELY ADDITIVE SETTING[1]

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1. Introduction. Let F be a set. Denote by $P(F)$ the set of all finitely additive probability measures defined on all subsets of F. A strategy σ is a sequency $\sigma_0, \sigma_1, \cdots$ where σ_0 is in $P(F)$ and, for all $n > 0$, σ_n is a map from $F \times \cdots \times F$ (n-factors) to $P(F)$. The strategy σ may be viewed as the distribution of a stochastic process $f_1, f_2, \cdots$. That is, σ_0 is the distribution of f_1 and $\sigma_n(f_1, \cdots, f_n)$ is the conditional distribution of f_{n+1} given $(f_1, \cdots, f_n)$. A theory of integration with respect to a strategy σ was developed by Dubins and Savage in [2]. The notation used here is taken mostly from that source.

Let Q_0 be a constant and, for $n > 0$, let Q_n be a real-valued function defined on $F \times \cdots \times F$ (n-factors). Suppose the Q_n are uniformly bounded and, for every $n > 0$ and every n-tuple $(f_1, \cdots, f_n)$,

$$(1) \quad \int Q_n(f_1, \cdots, f_{n-1}, f_n)\, d\sigma_{n-1}(f_1, \cdots, f_{n-1})(f_n) \leqq Q_{n-1}(f_1, \cdots, f_{n-1}).$$

Then the sequence $Q = \{Q_n\}$ is said to be an expectation decreasing semi-martingale with respect to σ ([2], page 29).

If t is a stop rule and $h = (f_1, f_2, \cdots)$, set

$$(2) \qquad Q_t(h) = Q_{t(h)}(f_1, \cdots, f_{t(h)})$$

and

$$(3) \qquad Q(\sigma, t) = \int Q_t\, d\sigma.$$

(The integral is defined in [2] and in Section 2 below.) If $t \leqq t'$, one can show as in [2], page 43, that $Q(\sigma, t) \geqq Q(\sigma, t')$. Define $Q_\infty = \lim_{t\to\infty} Q(\sigma, t)$.

In this note necessary and sufficient conditions are given for a semi-martingale Q to satisfy $Q_\infty = Q_0$. In a countably additive setting a semi-martingale process Q for which $Q_\infty = Q_0$ is a martingale in the sense that equality holds in (1) almost surely. (Example 3.6.2 of [2] shows that this need not be true in our more general setting.) Conditions corresponding to those of this note are easy to find for countably additive processes and are given in [1], page 311. Here conditions are also given for a semi-martingale to be almost a martingale in the sense that $Q_\infty \geqq Q_0 - \epsilon$. As an application, a characterization is given of thrifty strategies for gambling problems, thus solving a problem left open in [2].

2. A slight extension of the Dubins and Savage Integral. Let σ be a strategy and let g be a bounded, real-valued, finitary function defined on

$$H = F \times F \times \cdots.$$

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Then the integral of g with respect to σ is written either $\int g\, d\sigma$ or $E(\sigma, g)$ and is defined in [2], page 17 by induction on the structure of g. That is, $E(\sigma, c) = c$, for c constant, and

$$E(\sigma, g) = \int E(\sigma[f_1], gf_1)\, d\sigma_0(f_1).$$

The definition still makes sense if we no longer insist that g be bounded, but only require that $E(\sigma[f_1], gf_1)$ exist for every f_1 and be uniformly bounded in f_1. A function g with this property will be called conditionally bounded with respect to σ and $I(\sigma)$ will denote the class of such functions for a fixed strategy σ.

THEOREM 1. *For every strategy σ, $I(\sigma)$ is a linear space and $E(\sigma, \cdot)$ is a positive linear functional on $I(\sigma)$.*

PROOF. The proof is straightforward and often uses induction.

3. An identity. Let $Q = \{Q_n\}$ be a sequence where Q_0 is a constant and, for $n > 0$, Q_n is a bounded function from $F \times \cdots \times F$ (n-factors) to the reals. For $h = (f_1, f_2, \cdots)$, define

$$\epsilon_0(h) = \epsilon_0 = Q_0 - \int Q_1(f)\, d\sigma_0(f)$$

and

$$\begin{aligned}(4) \quad \epsilon_n(h) &= \epsilon_n(f_1, \cdots, f_n) \\ &= Q_n(f_1, \cdots, f_n) - \int Q_{n+1}(f_1, \cdots, f_n, f)\, d\sigma_n(f_1, \cdots, f_n)(f).\end{aligned}$$

And, for every stop rule t, set

$$S(Q, \sigma, t)(h) = \textstyle\sum_{n=0}^{t(h)-1} \epsilon_n(h).$$

The extension of Section 2 enables us to prove the next theorem.

THEOREM 2. *If Q_t is in $I(\sigma)$, then $S(Q, \sigma, t)$ is in $I(\sigma)$ and*

$$Q(\sigma, t) = Q_0 - \int S(Q, \sigma, t)\, d\sigma.$$

(Here, Q_t and $Q(\sigma, t)$ are defined by equations (2) and (3).

PROOF. If g is any function on H and $f_1 \varepsilon F$, then gf_1 is that function on H defined by $gf_1(f_2, f_3, \cdots) = g(f_1, f_2, f_3, \cdots)$ ([2], page 14).
Let Qf_1 denote the sequence $\{Q_1 f_1, Q_2 f_1, \cdots\}$. Then

$$S(Q, \sigma, t)f_1 = S(Qf_1, \sigma[f_1], t[f_1]) + \epsilon_0.$$

Arguing by induction on the structure of Q_t, we may assume that for every f_1

$$Qf_1(\sigma[f_1], t[f_1]) = Q_1(f_1) - \int S(Qf_1, \sigma[f_1], t[f_1])\, d\sigma[f_1].$$

Since Q_t is in $I(\sigma)$, $Qf_1(\sigma[f_1], t[f_1])$ is uniformly bounded in f_1. Likewise, Q_1 is bounded. Hence, $S(Q, \sigma, t)$ is conditionally bounded with respect to σ. Moreover,

$$\begin{aligned}Q(\sigma, t) &= \int \{Q_1(f_1) - \int S(Qf_1, \sigma[f_1], t[f_1])\, d\sigma[f_1]\}\, d\sigma_0(f_1) \\ &= \int \{Q_1(f_1) + \epsilon_0\}\, d\sigma_0(f_1) - \int \int \{S(Qf_1, \sigma[f_1], t[f_1]) + \epsilon_0\}\, d\sigma[f_1]\, d\sigma_0(f_1) \\ &= Q_0 - \int S(Q, \sigma, t)\, d\sigma. \quad \square\end{aligned}$$

A special case of Theorem 2 proved useful in [3], Section 1.

4. Application to martingales. In this section the sequence $Q = \{Q_n\}$ is assumed to be an expectation decreasing semi-martingale with respect to the strategy σ. That is, the ϵ_n of the previous section are assumed to be non–negative and the Q_n to be uniformly bounded.

We say σ *totally* ϵ*–conserves* Q *along* $h = (f_1, f_2, \cdots)$ *up to time* n iff $S(Q, \sigma, n)(h) \leqq \epsilon$ (cf. [2], page 48).

According to Theorem 2, $\int S(Q, \sigma, t)\, d\sigma$ is the expected loss or decrease in Q up to time t. The next two theorems give additional information about the connection between $S(Q, \sigma, t)$ and a decrease in Q.

THEOREM 3. *If* $\epsilon > 0$ *and* $Q(\sigma, t) \geqq Q_0 - \epsilon^2$, *then* σ *totally* ϵ*–conserves* Q *up to time* $t(h)$ *with* σ*–probability at least* $1 - \epsilon$.

PROOF. Suppose the conclusion is false. Then $\sigma[S(Q, \sigma, t) > \epsilon] > \epsilon$, whence $\int S(Q, \sigma, t)\, d\sigma > \epsilon^2$. So, by the previous theorem,

$$Q(\sigma, t) < Q_0 - \epsilon^2. \quad \square$$

Now let

$$I = \min \{Q_0, \inf Q_n(f_1, \cdots, f_n)\}$$

$$S = \max \{Q_0, \sup Q_n(f_1, \cdots, f_n)\},$$

where the supremum and infimum are taken over all n and all n-tuples $(f_1, \cdots, f_n)$ of elements of F.

THEOREM 4. *Let* $\epsilon > 0$ *and* $1 \geqq \epsilon' \geqq 0$. *If* σ *totally* ϵ*–conserves* Q *up to time* $t(h)$ *with* σ*–probability at least* $1 - \epsilon'$, *then*

$$Q(\sigma, t) \geqq Q_0 - \epsilon - \epsilon'(S - I).$$

PROOF. By Theorem 2, it is enough to show

$$\int S(Q, \sigma, t)\, d\sigma \leqq \epsilon + \epsilon'(S - I) \tag{5}$$

whenever $\sigma[S(Q, \sigma, t) \leqq \epsilon] \geqq 1 - \epsilon'$.

The proof is by induction on the structure of $S(Q, \sigma, t)$. It is easy to check (5) if $S(Q, \sigma, t)$ has structure 0. It remains to prove the inductive step.

Notice that we may assume without loss of generality that $\epsilon_0 \leqq \epsilon$, where ϵ_0 is defined by (4).

Let $A = [S(Q, \sigma, t) > \epsilon]$. Then $\sigma(A) = \int \sigma[f_1](Af_1)\, d\sigma_0(f_1)$, where

$$\begin{aligned} Af_1 &= [S(Q, \sigma, t)f_1 > \epsilon]. \\ &= [S(Qf_1, \sigma[f_1], t[f_1]) > \epsilon - \epsilon_0] \end{aligned}$$

By the inductive assumption,

$$\int S(Qf_1, \sigma[f_1], t[f_1])\, d\sigma[f_1] \leqq (\epsilon - \epsilon_0) + \sigma[f_1](Af_1)(S - I).$$

Hence,

$$\begin{aligned} \int S(Q, \sigma, t)\, d\sigma &= \int \int S(Q, \sigma, t)f_1\, d\sigma[f_1]\, d\sigma_0(f_1) \\ &\leqq \int \{(\epsilon - \epsilon_0) + \sigma[f_1](Af_1)(S - I) + \epsilon_0\}\, d\sigma_0(f_1) \\ &= \epsilon + \sigma(A)(S - I) \\ &\leqq \epsilon + \epsilon'(S - I). \quad \square \end{aligned}$$

Since $Q_\infty = \lim_{t\to\infty} Q(\sigma, t)$, it is possible to use the preceding two theorems to get various conditions that $Q_\infty \geqq Q_0 - \epsilon$. In particular, the following characterization of semi-martingales Q for which $Q_\infty = Q_0$ is immediate.

THEOREM 5. $Q_0 = Q_\infty$ iff, *for every* $\epsilon > 0$ *and every stop rule* t, σ *totally* ϵ*–conserves* Q *up to time* $t(h)$ *with* σ*–probability* 1.

5. Interpretation for thrifty strategies. Let Γ be a gambling house defined on F and let V be the optimal return function as defined in [2], page 41. If f is in F and σ is a strategy available at f in Γ, then the sequence $\{V(f), V(f_1), V(f_2), \cdots\}$ is an expectation decreasing semi-martingale with respect to σ. And $V_\infty = V(\sigma)$ is defined to be $\lim_{t\to\infty} \int V(f_t)\, d\sigma$. Theorem 5 can be specialized to give

THEOREM 5′. $V(\sigma) = V(f)$ iff, *for every* $\epsilon > 0$ *and every stop rule* t, σ *totally* ϵ*–conserves* V *up to time* $t(h)$ *with* σ*–probability* 1.

A strategy σ for which $V(\sigma) = V(f)$ is called thrifty in [2]. The following three finitary sets have been studied in connection with thrifty strategies:

$$A_t = \{h: \epsilon_n(h) = 0 \quad \text{for} \quad n = 0, \cdots, t(h) - 1\}$$

$$B_{t,\epsilon} = \{h: \textstyle\sum_{n=0}^{t(h)-1} \epsilon_n(h) < \epsilon\}$$

$$C_{t,\epsilon} = \{h: \epsilon_n(h) < \epsilon \quad \text{for all} \quad n = 0, \cdots, t(h) - 1\}.$$

Here t is a stop rule, $\epsilon > 0$, and the ϵ_n are as in Section 3. Clearly, $A_t \subset B_{t,\epsilon} \subset C_{t,\epsilon}$. It is shown in [2] that if $\sigma(A_t) = 1$ for all t, then σ is thrifty. It is also shown there that if σ is thrifty, then $\sigma(C_{t,\epsilon}) = 1$ for every t and every $\epsilon > 0$. Both of these results follow from Theorem 5′ which states that σ is thrifty iff $\sigma(B_{t,\epsilon}) = 1$ for every t and every $\epsilon > 0$.

Theorems 3 and 4 could also be used to give conditions for "ϵ–thriftiness."

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