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UNBIASED COIN TOSSING WITH A BIASED COIN

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0. Summary. Procedures are exhibited and analyzed for converting a sequence of i.i.d. Bernoulli variables with unknown mean p into a Bernoulli variable with mean $\frac{1}{2}$. The efficiency of several procedures is studied.

1. Introduction. Coins probably have been flipped since the dawn of their invention for the purpose of making impartial decisions. That such decisions are in fact partial is widely accepted. We shall envision an idealized coin which produces i.i.d. Bernoulli variables $X_1, X_2, \cdots$ with unknown parameter $p, 0 < p < 1$. (Such sequences can be closely approximated using radio-active materials.) If $p = \frac{1}{2}$ and $M_1, M_2, \cdots$ is any predetermined sequence of zeros and ones, the sequence $E_1, E_2, \cdots$ defined by $E_\nu \equiv (M_\nu + X_\nu)$ modulo 2 $(\nu \geqq 1)$ is also a sequence of i.i.d. Bernoulli variables with mean $\frac{1}{2}$. If the M sequence represents a "message" in binary format, the E sequence represents an "unbreakable" encoded message. The formula $M_\nu = (E_\nu + X_\nu)$ modulo 2 $(\nu \geqq 1)$ can be used for decoding. Repeated application of any of the procedures to be discussed will produce a Bernoulli sequence with $p = \frac{1}{2}$ from a Bernoulli sequence with $p \in (0, 1)$.

One cannot define a function of the X's which is a Bernoulli variable Z with mean $\frac{1}{2}$ using any (non-randomized) fixed sample size procedure since for any $n \geqq 1$, there are values of $p (0 < p < 1)$ such that $P\{X_i = 0, i = 1, \cdots, n\} > \frac{1}{2}$. An elementary sequential procedure is

Q_1: *Sample $X_1, X_2, \cdots$ sequentially in pairs and stop the first time $2m$ for which $X_{2m} \neq X_{2m-1}$. Set $Z \equiv X_{2m}$.*

This procedure is described by von Neumann (1951) and has been rediscovered by others. The authors are indebted to N. L. Johnson for pointing it out to them, which led to the present investigation. An extension of this procedure to Markov chains has recently been considered by P. Samuelson [1].

Denoting the sample size by N_1, one finds the expected sample size to be

$$EN_1 = p^{-1}q^{-1}, \qquad 0 < p < 1, \qquad q = 1 - p.$$

There are better procedures than Q_1. We shall say that a procedure Q is better than procedure Q' if the corresponding sample sizes satisfy $N \leqq N'$ ($N \leqq N' = \infty$ is permitted) for all sample sequences, with strict inequality for some sequence. In

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Section 2, we investigate a class called "even procedures". Q_1 belongs to this class. The best even procedure Q_2 will require an expected sample size

$$EN_2 = 2\prod_{\nu=1}^{\infty}(1+p^{2^\nu}+q^{2^\nu}), \qquad 0<p<1. \tag{1}$$

In Section 3, we examine a non-even procedure Q_3 which is better than Q_2. We conjecture that Q_3 is "weakly admissible" in the sense that no better procedure exists. In Section 4, we discuss admissibility and show that there is no procedure which is better than (or as good as) all others. In connection with this, we obtain a lower bound for the expected sample size and introduce a fourth (non-even and non-symmetric) procedure Q_4 which has smaller expected sample size than Q_3 for small p.

We tabulate the expected sample size for the range of p one is most likely to encounter, $.4 \leqq p \leqq .6$:

Procedure \ p	.4	.5	.6
Q_1	4.17	4.00	4.17
Q_2	3.57	3.40	3.57
Q_3	3.28	3.10	3.28
Q_4	3.50	3.38	3.55
Lower bound	3.17	3.00	3.17

The margin for improvement in Q_3 in terms of the expected sample size is less than 4% over the entire range of p, $0<p<1$.

2. Even procedures. Let $S_n \equiv \sum_1^n X_\nu$, $n = 1, 2, \cdots$ and $S_0 \equiv 0$. As one samples sequentially from $X_1, X_2, \cdots$ and plots S_n versus $n-S_n$, one generates a "path" in the Euclidean plane. A *stopping set* $\mathscr{S}$ is a set of points (i,j) with i, j nonnegative integers, the *stopping points*. Sampling is stopped as soon as $(n-S_n, S_n)$ is in $\mathscr{S}$. The stopping variable N is defined as the smallest integer $n \geqq 0$ such that $(n-S_n, S_n) \in \mathscr{S}$ if one exists, and $N = \infty$ otherwise. Let $m(i,j)$ denote the number of paths from $(0,0)$ to (i,j) with $N \geqq i+j$. For mathematical convenience we include points (i,j) with $\min(i,j) < 0$ and note that $m(i,j) = 0$ for such points. In order to make all stopping points reachable, we insist that $m(i,j) > 0$ for all $(i,j) \in \mathscr{S}$. With this restriction on $\mathscr{S}$, we call a point (i,j), not in $\mathscr{S}$, a *continuation point* or *inaccessible point* as $m(i,j) > 0$ or $=0$, respectively. We denote the corresponding sets by $\mathscr{C}$ and $\mathscr{I}$. Thus

$$(i,j) \in \mathscr{I} \quad \text{if, and only if,} \quad m(i,j) = 0. \tag{2}$$

Any one of the three sets $\mathscr{S}$, $\mathscr{C}$ and $\mathscr{I}$ determines the other two and the stopping variable N. The condition that N is finite with probability one for $p \in (0, 1)$ is expressed by the identity

$$\sum_{(i,j)\in\mathscr{S}} m(i,j)q^i p^j = 1, \qquad 0<p<1. \tag{3}$$

If this condition is satisfied, an impartial procedure or, briefly, procedure is determined by $\mathscr{S}$ and a Bernoulli variable Z, defined on the paths from (0, 0) to the points of stopping $(N-S_N, S_N)$, which has mean $\frac{1}{2}$ for $0 < p < 1$. If Z exists for a given $\mathscr{S}$ and, for each $(i,j) \in \mathscr{S}$, $k(i,j)$ denotes the number of paths from (0, 0) to (i,j) on which $Z = 1$, we have

$$\sum_{(i,j) \in \mathscr{S}} k(i,j) q^i p^j = \tfrac{1}{2}, \qquad 0 < p < 1. \tag{4}$$

Conversely, if (3) is satisfied and there exist integers $k(i,j)$, $0 \leqq k(i,j) \leqq m(i,j)$, such that (4) holds, we can define Z by subdividing, for each $(i,j) \in \mathscr{S}$, the paths from (0, 0) to (i,j) into two arbitrary subsets of $k(i,j)$ and $m(i,j)-k(i,j)$ paths and assigning the value $Z = 1(Z = 0)$ to each path in the first (second) subset (or vice versa). Two procedures defined on the same stopping set will be considered as equivalent, and our main concern will be to determine stopping sets for which at least one impartial procedure exists and EN is small. We introduce this equivalence as a matter of convenience but do not wish to suggest that the definition of Z is completely unimportant.

If $N < \infty$ with probability one $(0 < p < 1)$ and $m(i,j)$ is even for every $(i,j) \in \mathscr{S}$, we can satisfy (4) by setting

$$k(i,j) = m(i,j)/2 \quad \text{for every} \quad (i,j) \in \mathscr{S}. \tag{5}$$

We call such procedures *even procedures.* Q_1 is an even procedure with stopping set $\mathscr{S}_1 \equiv \{(i,j): i, j = 1, 3, 5, \cdots\}$.

It may be instructive to note that an even procedure is equivalent to a test of Neyman structure. Indeed, a procedure of the type studied in this paper may be viewed as a similar test of size $\frac{1}{2}$ for testing the hypothesis that $X_1, X_2, \cdots$ is a Bernoulli sequence with mean p, $0 < p < 1$. (Alternatives to the hypothesis are not considered.) The very essence of the problem requires that the test be non-randomized, that is, that Z take only the values 0 and 1. For a given stopping rule, the random variable $(N-S_N, S_N)$ is a sufficient statistic. In accordance with the usual definition, Z is a non-randomized test of Neyman structure if the conditional probability of $Z = 1$ given $(N-S_N, N) = (i,j)$ is constant for all $(i,j) \in \mathscr{S}$. Such a test exists (with constant $\frac{1}{2}$) if and only if the numbers $m(i,j)$ are even for all $(i,j) \in \mathscr{S}$.

We turn our attention now toward characterizing even procedures. We shall say that the variable N is "of type k" (whether it stops with certainty or not) if k divides $m(i,j)$ for every $(i,j) \in \mathscr{S} (k = 2, 3, \cdots)$. In the sequel, we denote the binomial coefficient $\binom{i+j}{i}$ by $c(i,j)$. ($c(i,j)$ is the number of ways of reaching (i,j) from (0, 0). $c(i,j) = 0$ if $\min(i,j) < 0$.)

THEOREM 1. *The following are equivalent.*
(a) *N is of type k.*
(b) *k divides $m(i,j)$ for every $(i,j) \in \mathscr{S} \cup \mathscr{I}$.*
(c) *k divides $c(i,j)$ for every $(i,j) \in \mathscr{S}$.*
(d) *k divides $c(i,j)$ for every $(i,j) \in \mathscr{S} \cup \mathscr{I}$.*
(e) *k divides $c(i,j) - m(i,j)$ for every (i,j).*

PROOF. Observe that for all (i,j)

$$c(i,j) = c(i-1,j)+c(i,j-1), \tag{6}$$ and

$$m(i,j) = m(i-1,j)I(i-1,j)+m(i,j-1)I(i,j-1), \tag{7}$$

where $I(i,j) \equiv 1$ or 0 as $(i,j) \in \mathscr{C}$ or $\mathscr{S} \cup \mathscr{I}$ respectively. Assume that at least one of (a)–(d) hold. We shall prove (e). Certainly (e) holds for (i,j) such that $i+j \leqq 0$. Let $n \geqq 1$ and assume that (e) holds for all (i,j) for which $i+j < n$. Let $i+j = n$. By the induction assumption,

$$k \quad \text{divides} \quad c(i-1,j)-m(i-1,j). \tag{8}$$

Using (8) alone if $(i-1,j) \in \mathscr{C}$, (8) and any one of (a)–(d) if $(i-1,j) \in \mathscr{S}$, or (8) and (2) if $(i-1,j) \in \mathscr{I}$, we conclude that

$$k \quad \text{divides} \quad c(i-1,j)-m(i-1,j)I(i-1,j). \tag{9}$$

Similarly,

$$k \quad \text{divides} \quad c(i,j-1)-m(i,j-1)I(i,j-1). \tag{10}$$

Finally we complete the induction step by concluding k divides $c(i,j)-m(i,j)$ from (6), (7), (9) and (10).

Conversely we shall derive (a) from (e). With this, it easily follows that (e) implies (a)–(d). Suppose (e) holds but (a) does not. Then there exists an n and some (i,j) with

$$i+j = n, \qquad (i,j) \in \mathscr{S} \quad \text{and} \quad m(i,j) \quad \text{not divisible by} \quad k. \tag{11}$$

Let us assume we have chosen the point (i,j) satisfying (11) with the smallest value of j. We work with the three points (i,j), $(i+1,j-1)$ and $(i+1,j)$. With (6), (7) and (11) we conclude

$$c(i,j) = c(i+1,j)-c(i+1,j-1) \tag{12}$$ and

$$m(i+1,j) = m(i+1,j-1)I(i+1,j-1). \tag{13}$$

From (e), (11) and (12), it follows that k does not divide $c(i+1,j)-c(i+1,j-1)$. In turn, (e) implies

$$k \quad \text{does not divide} \quad m(i+1,j)-m(i+1,j-1). \tag{14}$$

(2), (13) and (14) are compatible only if $I(i+1,j-1) = 0$,

$$(i+1)+(j-1) = n, \qquad (i+1,j-1) \in \mathscr{S}, \tag{15}$$
$$\text{and} \quad m(i+1,j-1) \quad \text{is not divisible by} \quad k.$$

But (15) contradicts the assumption that j is the smallest value satisfying (11).

Letting $k = 2$, it is clear that an even procedure exists if and only if $N < \infty$ with probability one $(0 < p < 1)$ and $c(i,j)$ is even for every $(i,j) \in \mathscr{S} \cup \mathscr{I}$. In particular, there exists an even procedure Q_2 with $\mathscr{S}_2 \cup \mathscr{I}_2 = \{(i,j): c(i,j) \text{ is even}\}$. Of course, no better even procedure exists. We note, in passing, that Theorem 1 directs us to a class of procedures for defining a random variable Z^* which takes k distinct values with the same probability $1/k$.

For the remainder of this section we shall study procedure Q_2. It is helpful to observe Figure 1. The points indicated by the first two symbols correspond to the noncontinuation points $\mathscr{S}_2 \cup \mathscr{I}_2$ of Q_2. Several easily proven facts become apparent.

(i) N_2 is even.

(ii) $m(i,j) = 0$, 1, or 2 as $(i,j) \in \mathscr{I}_2$, $\mathscr{C}_2$, or $\mathscr{S}_2$ respectively.

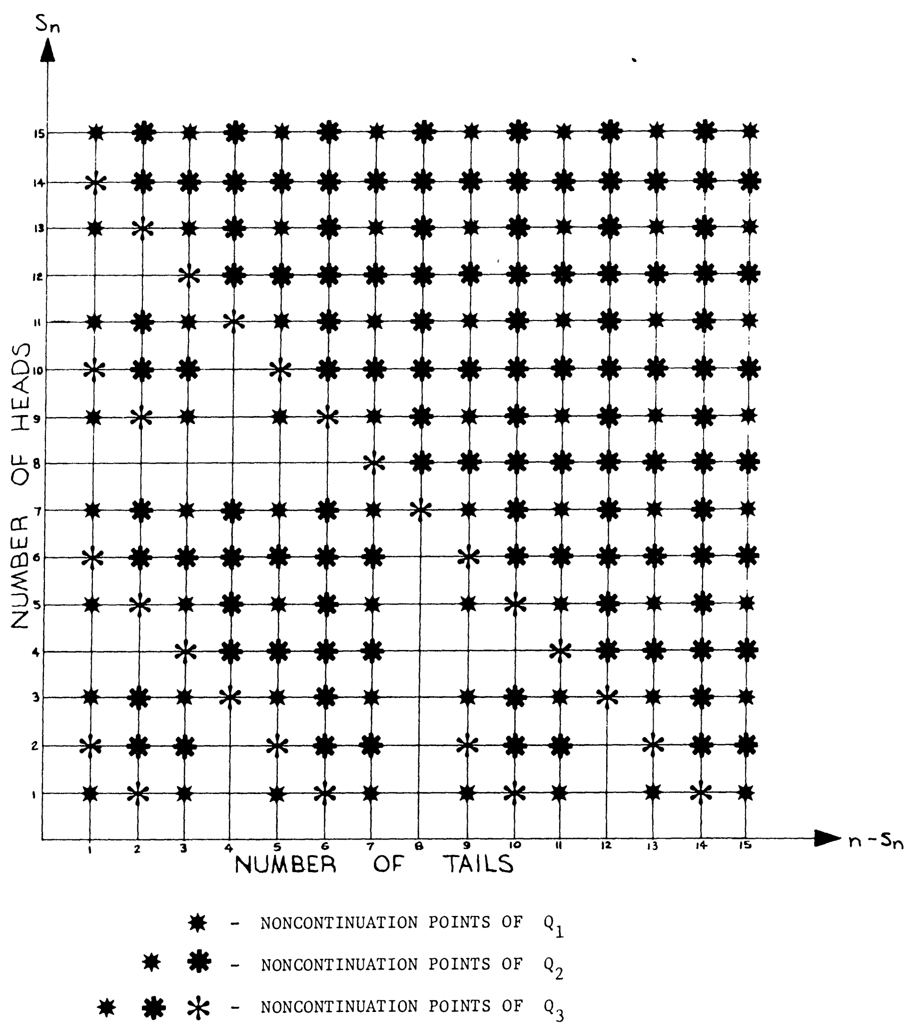

FIG. 1

(iii) For each $v \geqq 0$, the only continuation points (i, j) on the line $i+j = 2^v$ are the two points $(2^v, 0)$ and $(0, 2^v)$.

(iv) For each $v \geqq 1$, if one shifts the origin from $(0, 0)$ to $(2^v, 0)$ or $(0, 2^v)$, the stopping variable which results does not differ from N_2 unless 2^v or more observations are made on X.

(v) A recursive definition for N_2 may be roughly stated as "Stop the first time one reaches a point with an even number (two) of not previously stopped paths to it."

From (ii), it follows that a legitimate definition for Z is given by

$$Z \equiv S_{N_2-1} \text{ modulo } 2. \tag{16}$$

Now let

$$a_n \equiv P\{N_2 > n\}, \qquad n = 0, 1, 2, \cdots; \tag{17}$$

$$A(t) \equiv \textstyle\sum_{r=0}^{\infty} a_r t^r; \qquad A_n(t) \equiv \sum_{r=0}^{n} a_r t^r, \qquad n = 0, 1, 2, \cdots. \tag{18}$$

We have from (iii),

$$a_0 = 1; \quad a_m = p^m + q^m, \qquad m = 2^0, 2^1, 2^2, \cdots; \tag{19}$$

and from (iv),

$$a_{m+n} = a_m a_n, \qquad n = 0, 1, \cdots, m-1; \qquad m = 2^1, 2^2, \cdots. \tag{20}$$

Hence for $m = 2^0, 2^1, \cdots$,

$$\begin{aligned} A_{2m-1}(t) &= A_{m-1}(t) + \textstyle\sum_{n=0}^{m-1} a_m a_n t^{m+n} = A_{m-1}(t)(1 + a_m t^m) \\ &= \cdots = \textstyle\prod_{r=2^0, 2^1, \cdots, m} (1 + a_r t^r). \end{aligned} \tag{21}$$

Letting $m \to \infty$,

$$A(t) = \textstyle\prod_{v=0}^{\infty} \{1 + (pt)^{2^v} + (qt)^{2^v}\}. \tag{22}$$

Setting $t = 1$, we have (in agreement with (1))

$$EN_2 = \textstyle\prod_{v=0}^{\infty} \{1 + p^{2^v} + q^{2^v}\}. \tag{23}$$

Higher moments of N_2 can be obtained from

$$EN_2(N_2 - 1) \cdots (N_2 - k + 1) = kA^{(k-1)}(1). \tag{24}$$

3. A better procedure than Q_2. Under procedure Q_2 there are points $(i, j) \in \mathscr{C}_2$ for which, with certainty, $N = i+j+1$. With Z defined by (16), there is no need to take the Nth observation. (Neither N nor Z depend on it.) This suggests procedure Q_3 given by:

$$\mathscr{S}_3 \cup \mathscr{I}_3 \equiv \{(i, j) : c(i, j) \text{ is even or both } c(i, j+1) \text{ and } c(i+1, j) \text{ are even}\},$$

and $Z \equiv S_{N_3 - \alpha}$ modulo 2, $\alpha = 0$ or 1 as $c(N_3 - S_{N_3}, S_{N_3})$ is odd or even (or [because

N_2 is even] equivalently as N_3 is odd or even). $\mathscr{S}_3 \cup \mathscr{I}_3$ is denoted in Figure 1 by all three symbols. Let

$$b_n \equiv P\{N_3 > n\}, \qquad n = 0, 1, 2, \cdots; \tag{25}$$

$$B(t) \equiv \textstyle\sum_{r=0}^{\infty} b_r t^r; \qquad B_n(t) \equiv \sum_{r=0}^{n} b_r t^r, \qquad n = 0, 1, 2, \cdots. \tag{26}$$

Procedure Q_3 has similar counterparts to properties (iii) and (iv) of procedure Q_2 and one finds for $m = 2^1, 2^2, \cdots$ that

$$b_0 = 1; \qquad b_{m-1} = p^{m-1} + q^{m-1}; \qquad b_m = p^m + q^m; \tag{27}$$

$$b_{m+n} = b_m b_n, \qquad n = 0, 1, \cdots, m-2. \tag{28}$$

By direct computation, one can show

$$b_{m+n} = b_m b_n - (pq)^{m-1}, \qquad n = m-1, m = 2^1, 2^2, \cdots. \tag{29}$$

Corresponding to (21) one can obtain for $m = 2^1, 2^2, \cdots$,

$$B_{2m-1}(t) = B_{m-1}(t)(1 + b_m t^m) - (pq)^{m-1} t^{2m-1}.$$

Using the extremes of (21) and the fact that $b_m = a_m$, we find

$$[A_{2m-1}(t)]^{-1} B_{2m-1}(t) = [A_{m-1}(t)]^{-1} B_{m-1}(t) - [A_{2m-1}(t)]^{-1} (pq)^{m-1} t^{2m-1}.$$

Upon summing over m and using the fact that $A_1(t) = B_1(t)$, we conclude

$$[A_{2m-1}(t)]^{-1} B_{2m-1}(t) = 1 - D_m(t), \qquad m = 2^1, 2^2, \cdots,$$

where

$$D_m(t) \equiv \textstyle\sum_{r=2^1,\cdots,m} [A_{2r-1}(t)]^{-1} (pq)^{r-1} t^{2r-1}.$$

Letting $m \to \infty$,

$$B(t) = A(t)(1 - D(t)) \qquad \text{where} \tag{30}$$

$$D(t) \equiv \textstyle\sum_{s=1}^{\infty} \{(pq)^{2^s-1} t^{2^{s+1}-1} / \prod_{v=0}^{s} (1 + (pt)^{2^v} + (qt)^{2^v})\}.$$

In particular, since $B(1) = EN_3$ and $A(1) = EN_2$,

$$EN_3 = \{1 - \textstyle\sum_{s=1}^{\infty} \{(pq)^{2^s-1} / \prod_{v=0}^{s} (1 + p^{2^v} + q^{2^v})\}\} EN_2. \tag{31}$$

4. Problems of existence and admissibility. Let N_5 be the first $n \geqq 2$ for which $\min(n - S_n, S_n) = 1$. There is no procedure that stops sooner than N_5 for the same reason (given in Section 1) that there is no fixed sample size procedure. Hence, for an arbitrary procedure with stopping variable N there is a general lower bound

$$EN \geqq EN_5 = p^{-1} q^{-1} - 1, \qquad 0 < p < 1. \tag{32}$$

We raise the question whether there is a procedure with stopping variable N_5, or, equivalently, whether the inequality in (32) is ever equality. The answer is no as the following theorem shows.

THEOREM 2. *There is no procedure with stopping set*

$$\mathscr{S}_5 = \{(i, j) : i = 1, j \geqq 1 \quad \text{or} \quad j = 1, i \geqq 1\}.$$

PROOF. By equation (4), we must show that there do not exist numbers $a = 0, 1$, or 2,

$$m_k = 0 \quad \text{or} \quad 1, \quad n_k = 0 \quad \text{or} \quad 1, \qquad k = 1, 2, \cdots \tag{33}$$

such that

$$apq + pq \sum_{k=1}^{\infty} (m_k p^k + n_k q^k) = \tfrac{1}{2}, \qquad 0 < p < 1. \tag{34}$$

Clearly we cannot have $m_k = n_k = 0$ for all k nor $m_k = n_k = 1$ for all k. Hence if $a = 2$, then (34) implies $\frac{1}{2} > 2pq$, which is not true for $p = \frac{1}{2}$. The assumption $a = 0$ leads to a similar contradiction. Thus we must have $a = 1$. Since

$$2pq + pq \sum_{k=1}^{\infty} (p^k + q^k) = 1,$$

equation (34) with $a = 1$ is equivalent to

$$\sum_{k=1}^{\infty} \{(2m_k - 1)p^k + (2n_k - 1)q^k\} = 0, \qquad 0 < p < 1. \tag{35}$$

Let $f(p)$ denote the left-hand side of (35). With $k(k-1)\cdots(k-r+1)$ denoted by $k^{(r)}$ ($k^{(0)} = 1$ or 0 as $k >$ or $= 0$), the rth derivative of $f(p)$ is

$$f^{(r)}(p) = \sum_{k=r}^{\infty} k^{(r)}\{(2m_k - 1)p^{k-r} + (-1)^r(2n_k - 1)q^{k-r}\}$$

and

$$f^{(r)}(\tfrac{1}{2}) = \sum_{k=r}^{\infty} k^{(r)} 2^{-k+r}\{(2m_k - 1) + (-1)^r(2n_k - 1)\}.$$

Hence, for $r = 0, 1, 2, \cdots$,

$$f^{(2r)}(\tfrac{1}{2}) = \sum_{k=2r}^{\infty} k^{(2r)} 2^{-k+2r+1} c_k, \tag{36}$$

$$f^{(2r+1)}(\tfrac{1}{2}) = \sum_{k=2r+1}^{\infty} k^{(2r+1)} 2^{-k+2r+2} d_k, \tag{37}$$

where ($f^{(0)}(\frac{1}{2}) \equiv f(\frac{1}{2})$ and)

$$c_k \equiv m_k + n_k - 1, \quad d_k \equiv m_k - n_k, \qquad k = 1, 2, \cdots. \tag{38}$$

Since $f(p)$ is identically zero, it follows that all its derivatives are identically zero and, in particular, that the expressions in (36) and (37) are zero. We first show that $c_k = 0$ for $k = 1, 2, \cdots$. The numbers c_k are restricted to the values $-1, 0, 1$. Since $f^{(2r)}(\frac{1}{2}) = 0$ for $r = 0, 1, \cdots$, it follows that we cannot have $c_k = 1$ for all sufficiently large k, nor $c_k = -1$ for all sufficiently large k. The equation $f^{(0)}(\frac{1}{2}) = 0$ is equivalent to $\sum_{k=1}^{\infty} 2^{-k} c_k = 0$. If not all $c_k = 0$, and m denotes the first integer such that $c_m \neq 0$, then we must have the strict inequality

$$2^{-m} = \left|\sum_{k=m+1}^{\infty} 2^{-k} c_k\right| < \sum_{k=m+1}^{\infty} 2^{-k} = 2^{-m},$$

a contradiction. With all $c_k = 0$, it follows that the sequence $\{d_k\}$ is restricted to the values

$$d_k = -1 \quad \text{or} \quad 1, \qquad k = 1, 2, \cdots. \tag{39}$$

Furthermore, the conditions $f^{(2r+1)}(\frac{1}{2}) = 0$ imply that

$$d_k \quad \text{changes sign infinitely often.} \tag{40}$$

Now consider the equation $f^{(1)}(\frac{1}{2}) = 0$, which is equivalent to

$$\sum_{k=1}^{\infty} 2^{-k} k d_k = 0. \tag{41}$$

Let

$$D_k = \sum_{v=1}^{k} 2^{-v} v\, d_v. \tag{42}$$

We now show that equation (41), subject to conditions (39) and (40), is satisfied (if and) only if the d_k are recursively defined as follows. The value $d_1 = \pm 1$ is arbitrary, and for $k \geqq 2$

$$\begin{aligned} d_k &= \ \ 1 & &\text{if} \quad D_{k-1} < 0 \\ &= \ \ 1 \quad \text{or} \quad -1 & &\text{if} \quad D_{k-1} = 0 \\ &= -1 & &\text{if} \quad D_{k-1} > 0. \end{aligned} \tag{43}$$

First note, by (41) and (42), that

$$|D_k| = \left|\sum_{v=k+1}^{\infty} 2^{-v} v\, d_v\right| < \sum_{v=k+1}^{\infty} 2^{-v} v = 2^{-k}(k+2),$$

with the strict inequality. It is seen by (42) that $2^k D_k$ is an integer. Hence

$$2^k |D_k| \leqq k+1. \tag{44}$$

Clearly d_1 is arbitrary. If $d_1, \cdots, d_{k-1}$ $(k \geqq 2)$ are given, then

$$|D_{k-1} + 2^{-k} k\, d_k| = |D_k| \leqq 2^{-k}(k+1).$$

Equivalently,

$$-(1+d_k)k/2 - \tfrac{1}{2} \leqq 2^{k-1} D_{k-1} \leqq (1-d_k)k/2 + \tfrac{1}{2}$$

or, since $2^{k-1} D_{k-1}$ and $(1 \pm d_k)/2$ are integers,

$$-(1+d_k)k/2 \leqq 2^{k-1} D_{k-1} \leqq (1-d_k)k/2.$$

This immediately implies that d_k must satisfy (43). (It is easy to see that any sequence $\{d_k\}$ defined by (43) satisfies (40) and (41).)

We complete the proof by showing that (43) is incompatible with $f^{(3)}(\frac{1}{2}) = 0$. By (37),

$$96^{-1} f^{(3)}(\tfrac{1}{2}) = A_{1n} + A_{2n} = 0, \tag{45}$$

where

$$A_{1n} \equiv \sum_{k=3}^{n-1} \binom{k}{3} 2^{-k} d_k, \qquad A_{2n} \equiv \sum_{k=n}^{\infty} \binom{k}{3} 2^{-k} d_k, \qquad n = 4, 5, \cdots. \tag{46}$$

Now

$$|A_{2n}| \leqq \sum_{k=n}^{\infty} \binom{k}{3} 2^{-k} = 2^{-n+1}\left\{\binom{n}{3} + \binom{n}{2} + \binom{n}{1} + 1\right\}, \tag{47}$$

and, in particular, $|A_{2,9}| \leqq 65/128$. By direct computation from (43) and (46), $|A_{1,9}| = 79/128$. This contradicts (45) and proves the theorem.

It is an interesting fact that while no procedure exists for N_5 there is a procedure Q_4 for a slight modification, namely

$$\begin{aligned} N_4 &\equiv N_1 & &\text{if} \quad (N_5 - S_{N_5}, S_{N_5}) = (1, 2), \\ &\equiv N_5 & &\text{otherwise.} \end{aligned}$$

Proof that Q_4 exists. Let

$$\mathscr{S}^* \equiv \{\omega : \omega = (i, 1) \quad \text{or} \quad (1, j);\, i = 2, 4, 6, \cdots;\, j = 4, 6, 8, \cdots\},$$

$$\mathscr{S}^{**} \equiv \{\omega : \omega = (i, 1) \quad \text{or} \quad (1, j);\, i, j = 4, 8, 12, \cdots\},$$

and set

$$g(\omega) \equiv 1 \quad \text{for} \quad \omega \in \mathscr{S}^{**} \qquad \qquad (48)$$
$$\equiv 0 \quad \text{for} \quad \omega \in \mathscr{S}^* - \mathscr{S}^{**}.$$

Consider the condition

$$Z = g(\omega) \quad \text{for every path which continues through} \quad \omega, \qquad \omega \in \mathscr{S}^*. \qquad (49)$$

If we can find a (even) procedure $Q_1{}^*$ for the stopping set $\mathscr{S}_1$ which satisfies the restrictions imposed by (5) with $\mathscr{S} = \mathscr{S}_1$, (48) and (49), then we can define Q_4 by assigning Z according to $Q_1{}^*$ for paths stopping in $\mathscr{S}_1$ and $Z = g(\omega)$ for paths stopping in $\mathscr{S}^*$.

It is sufficient to show that Z can be defined in accordance with (5), (48) and (49) on the paths to points $(i, j) \in \mathscr{S}_1$ with $i, j \geqq 3$. Let (i, j) be any of these points. Then $m(i, j) = m_1 + m_2 + m_3$ where m_1, m_2 and m_3 denote the numbers of paths to (i, j) that pass through the point $(1, 2)$ and through points in $\mathscr{S}^{**}$ and $\mathscr{S}^* - \mathscr{S}^{**}$, respectively. Considering the paths to (i, j) on which $Z = 1$, we must have

$$k(i, j) = \tfrac{1}{2} m(i, j) = k_1 + m_2$$

where k_1 is the number of these paths which pass through $(1, 2)$. Thus we must show that the integer k_1 defined by

$$2k_1 + 2m_2 = m_1 + m_2 + m_3$$

satisfies the inequalities $0 \leqq k_1 \leqq m_1$ or, equivalently, that

$$|m_2 - m_3| \leqq m_1. \qquad (50)$$

Let $m(i, j; i')$ and $m'(i, j; j')$ denote the number of paths to (i, j) which pass through $(i', 1)$ and $(1, j')$, respectively. Then $m'(i, j; j') = m(j, i; j')$ and $m_1 = m'(i, j; 2)$, $m_2 = I_1 + I_2$, $m_3 = I_3 + I_4$, where

$$I_1 \equiv \textstyle\sum_{(i',1) \in \mathscr{S}^{**}} m(i, j; i') = \sum_{v=4,8,\cdots} m(i, j; v)$$

$$I_2 \equiv \textstyle\sum_{(i,j') \in \mathscr{S}^{**}} m'(i, j; j') = \sum_{v=4,8,\cdots} m(j, i; v)$$

$$I_3 \equiv \textstyle\sum_{(i',1) \in \mathscr{S}^* - \mathscr{S}^{**}} m(i, j; i') = \sum_{v=2,6,\cdots} m(i, j; v)$$

$$I_4 \equiv \textstyle\sum_{(1,j') \in \mathscr{S}^* - \mathscr{S}^{**}} m'(i, j; j') = \sum_{v=6,10,\cdots} m(j, i; v).$$

An elementary combinatorial argument shows that

$$m(i, j; v) = 2c((i - v - 1)/2, \quad (j-3)/2) \quad \text{for} \quad (i, j) \in \mathscr{S}_1, \qquad v = 2, 4, \cdots.$$

It follows that

(51) $$m(i,j;2) = m(j,i;2),$$ and

(52) $$m(i,j;v) \quad \text{and} \quad m(j,i;v) \quad \text{are non-increasing in} \quad v.$$

(50) quickly follows from (51) and (52). This completes the proof.

We shall not derive the formula for EN_4 here, but it is not difficult and one finds that $EN_4 < EN_3$ for small values of $p > 0$. This means that Q_3 is not uniformly optimal (i.e., does not minimize EN for $0 < p < 1$) among all procedures.

Our last theorem concerns the subject of weak admissibility. Recall that a procedure Q is weakly admissible if there is no better procedure (i.e., no procedure which stops as soon as and sometimes sooner than Q.)

THEOREM 3. *For every procedure Q there is a weakly admissible procedure Q^* which is better than or equivalent to Q.*

We shall need the following lemma.

LEMMA. *Let $R_1, R_2, \cdots$ be a sequence of procedures with stopping variables $M_1, M_2, \cdots$, respectively. If $M_v \to M_\infty$ in law for all $p \in (0, 1)$ as $v \to \infty$ where M_∞ is a stopping variable which is finite with probability one for all $p \in (0, 1)$, then there exists a procedure R_∞ with stopping variable M_∞.*

PROOF OF THEOREM 3. Rather than giving a constructive proof, we shall use Zorn's lemma and some of the terminology commonly used with it. (See, for instance, [2, page 40].) Let P be the class composed of complements of continuation sets $\mathscr{C}' \subset \mathscr{C}$ such that there exists a procedure Q' with continuation set $\mathscr{C}'$. ($\mathscr{C}$ denotes the continuation set of Q.) P is partially ordered by set inclusion. Our lemma verifies that every completely ordered subclass has an upper bound since the number of elements in the subclass is necessarily countable. Zorn's lemma implies that there exists a maximal set in P whose complement we denote by $\mathscr{C}^*$. Consequently, there exists a weakly admissible procedure Q^* which (has continuation set $\mathscr{C}^*$ and) is better than or equivalent to Q.

PROOF OF LEMMA. Let $\omega_1, \omega_2, \cdots$ be an ordering of all points (i, j). By taking a subsequence of $\{R_v\}$ if necessary, we can assume without loss of generality that the classification of the point ω_α as a continuation, stopping or inaccessible point remains the same and $m(\omega_\alpha)$ is constant for procedures $R_\alpha, R_{\alpha+1}, \cdots$, for every α. This is because $P\{M_v = n\}$, when expressed as a linear combination of the *linearly independent* functions $qp^{n-1}, \cdots, q^{n-1}p$, has coefficients which must be *integer* valued. Starting with ω_1, if ω_1 is a stopping point for $R_1, R_2, \cdots$, we let $R_{11}, R_{12}, \cdots$ be a subsequence for which the number of paths stopping at ω_1, and on which $Z = 1$, is a constant, say $k^*(\omega_1)$. If ω_1 is not a stopping point then let $R_{11}, R_{12}, \cdots$ be the original sequence. Proceeding recursively, we let $R_{\alpha 1}, R_{\alpha 2}, \cdots$ equal $R_{\alpha-1,1}, R_{\alpha-1,2}, \cdots$ if ω_α is not a stopping point and let it be some subsequence of $R_{\alpha-1,1}, R_{\alpha-1,2}$ on which the number of paths stopping at ω_α, and on which $Z = 1$ is a constant, say $k^*(\omega_\alpha)$, if ω_α is a stopping point, $\alpha > 1$.

We shall direct our attention to the diagonal sequence $R_{11}, R_{22}, \cdots$. Working with M_∞, we assign $Z = 1$ to $k^*(\omega_\alpha)$ of the paths to ω_α and $Z = 0$ to the remaining $m(\omega_\alpha) - k^*(\omega_\alpha)$ paths to ω_α for each stopping point ω_α. We claim that this defines a procedure R_∞ with stopping variable M_∞. This is because for fixed p, $0 < p < 1$, and fixed $\varepsilon > 0$ there is an α_0 such that for $\alpha \geqq \alpha_0$, the probabilities that $Z = 1$ under R_∞ and $R_{\alpha\alpha}$ differ by less than ε.

REMARKS

(1) Because of Theorem 2, the point $(1, 2)$ cannot be added to the non-continuation set of Q_4. Since the point $(1, 2)$ is a stopping point of Q_3, Theorem 3 guarantees us that at least two non-equivalent weakly admissible procedures exist.

(2) The lemma (used to prove Theorem 3) and Theorem 2 can be used to show that inequality (32) is not "tight" in the sense that $p^{-1}q^{-1} - 1$ is not the greatest lower bound for EN.

(3) We conjecture that there is no uniformly optimal procedure, that is, a procedure which minimizes EN for all $p \in (0, 1)$.

(4) Let $\mathscr{C}$ and $\mathscr{C}' \subset \mathscr{C}$ be two continuation sets. Suppose that the stopping variable N' associated with $\mathscr{C}'$ is finite with probability one $(0 < p < 1)$ and there exists a procedure Q with continuation set $\mathscr{C}$. Then (it is easily shown) there exists a procedure Q' with continuation set $\mathscr{C}'$.

(5) Either Q_3 is weakly admissible or (because of the way procedure Q', referred to in Remark (4), is constructed) there exists a procedure which is equivalent to Q_2 but violates (5). Showing that a procedure is weakly admissible seems to be a difficult task.

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