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ON THE ZEROS OF INFINITELY DIVISIBLE DENSITIES

BY F. W. STEUTEL

The University of Texas[1]

1. Introduction. Making use of a representation theorem for infinitely divisible (inf div) distributions on the nonnegative integers, which is implicit in [3], and its continuous analogue, which is implicit in [5], some properties are proved regarding the zeros of inf div probability density functions (pdf's) on $[0, \infty)$, both in the discrete and in the continuous case.

2. Representation theorems.

THEOREM 1. *A probability distribution* $\{p_n\}$ *on the nonnegative integers, with* $p_0 > 0$, *is* inf div *if ~~and only~~ if*

$$np_n = \sum_{j=0}^{n-1} p_j q_{n-j-1}, \tag{1}$$

where the q's satisfy

$$q_j \geqq 0 \ (j = 0, 1, 2, \cdots); \qquad \sum_{j=1}^{\infty} j^{-1} q_j < \infty. \tag{2}$$

PROOF. From Feller [1] (page 270 seq.) one easily obtains, that $\{p_n\}$ is inf div if and only if its generating function (pgf) $P(z)$ is of the form

$$P(z) = \exp\{-\lambda(1-R(z))\} \qquad (|z| \leqq 1),$$

where $\lambda > 0$ and $R(z)$ is the pgf of some distribution $\{r_n\}$ on the nonnegative integers. Equivalently we have, taking logarithmic derivatives,

$$P'(z) = P(z)Q(z) \qquad (|z| < 1),$$

where $Q(z) = \lambda R'(z)$.

Again equivalently,

$$np_n = \sum_{j=0}^{n-1} p_j q_{n-j},$$

where $q_n = \lambda(n+1)r_{n+1}$, with $\sum_1^{\infty}(n+1)^{-1}q_n = \lambda(1-r_0)$.

In the same way for general distributions on $[0, \infty)$ we have

THEOREM 2. *A distribution function* (df) $F(x)$ *on* $[0, \infty)$ *is* inf div *if and only if it satisfies*

$$\int_0^x u\,dF(u) = \int_0^x F(x-u)\,dP(u), \tag{3}$$

where P is non-decreasing, and

$$\int_1^{\infty} x^{-1}\,dP(x) < \infty. \tag{4}$$

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[1] Now at Twente Institute of Technology, Enschede, Netherlands.

PROOF. According to Feller [2] the Laplace transform $\tilde{F}(\tau)$ of a df on $[0, \infty)$ is inf div if and only if

$$\tilde{F}(\tau) = \exp\{-\int_0^\infty x^{-1}(1-e^{-\tau x})\, dP(x)\},$$

where P is non-decreasing and satisfies (4). Taking logarithmic derivatives and using the convolution theorem yields (3), as we have

$$-d\tilde{F}(\tau)/d\tau = \int_0^\infty xe^{-\tau x}\, dF(x).$$

COROLLARY. *The* pdf $f(x)$ *of a distribution on* $[0, \infty)$ *is* inf div *if and only if*

$$xf(x) = \int_0^x f(x-u)\, dP(u), \tag{5}$$

where P is non-decreasing and satisfies (4).

PROOF. This follows by writing $F(u) = \int_0^u f(t)\, dt$ and changing the order of integration in (3).

3. Zeros of discrete distributions.

LEMMA 1. *If* $\{p_n\}$ *is an* inf div *distribution on the nonnegative integers, with* $p_0 > 0$, *then*

$$\left.\begin{matrix} p_a > 0 \\ q_{b-1} > 0 \end{matrix}\right\} \to p_{a+b} > 0. \tag{6}$$

PROOF. $(a+b)p_{a+b} \geqq p_a q_{b-1} > 0$, hence $p_{a+b} > 0$.

THEOREM 3. *If* $\{p_n\}$ *is an* inf div *distribution on the nonnegative integers, with* $p_0 > 0$, *then*

$$\left.\begin{matrix} p_a > 0 \\ p_b > 0 \end{matrix}\right\} \to p_{a+b} > 0.$$

PROOF. As $bp_b = \sum_{j=0}^{b-1} p_j q_{b-j-1}$, there is a j_0, with $0 \leqq j_0 < b$, such that $p_{j_0} > 0$ and $q_{b-j_0-1} > 0$. It follows by Lemma 1 that $p_{a+b-j_0} > 0$. There are two possibilities.

Case 1. $q_{j_0-1} > 0$ and hence by (6), $p_{a+b} > 0$.

Case 2. $q_{j_0-1} = 0$. Then, as $p_{j_0} > 0$, there is a j_1, with $0 \leqq j_1 < j_0$, such that $p_{j_1} > 0$ and $q_{j_0-j_1-1} > 0$. It follows that $p_{a+b-j_1} > 0$. Again there are two cases.

Case 2.1. $q_{j_1-1} > 0$ and hence by (6) $p_{a+b} > 0$.

Case 2.2. $q_{j_1-1} = 0$. Then, as $p_{j_1} > 0$, there is a j_2, with $0 \leqq j_2 < j_1$, such that $p_{j_2} > 0$ and $q_{j_1-j_2-1} > 0$. It follows that $p_{a+b-j_2} > 0$.

Proceeding in this way, in a finite number of steps we reach the situation that $p_{a+b-j_m} > 0$, and $q_{j_m-1} > 0$ or $j_m = 0$. Hence $p_{a+b} > 0$.

COROLLARY. *If* $\{p_n\}$ *is an* inf div *distribution on the nonnegative integers,* $p_0 > 0$, *then* $p_1 > 0 \to p_k > 0 \quad (k = 0, 1, 2, \cdots)$.

REMARK. Theorem 3 can also be proved by direct application of the definition of infinite divisibility, without use of Theorem 1.

4. Zeros of densities.

THEOREM 4. *If $f(x)$ is a continuous and* inf div *density on* $(0, \infty)$, *then*

$$f(x_0) = 0 \rightarrow \{f(x) = 0 \quad (x \leqq x_0)\}.$$

PROOF. It is no restriction (this can be achieved by a shift) to assume that for every $\delta > 0$ there is an $x_1 < \delta$ such that $f(x_1) > 0$. We now have to prove that $f(x)$ has no zeros in $(0, \infty)$. Suppose, therefore, that $f(x_1) > 0$ and $f(x_0') = 0$ with $x_0' > x_1$. Then by the continuity of $f(x)$ there is a smallest number x_0 satisfying $x_0 > x_1$ and $f(x_0) = 0$. By (5) we have

$$0 = x_0 f(x_0) = \int_0^{x_0} f(x_0 - u)\, dP(u).$$

As $f(x) > 0$ for all x with $x_1 \leqq x < x_0$, it follows that $\int_{0+}^{x_0 - x_1} dP(u) = 0$, and hence that $\int_{0+} f(x-u)\, dP(u) = 0$ for all $x < x_0 - x_1$. Therefore, by (5), $xf(x) = f(x)P(0)$ for all $x < x_0 - x_1$. It follows from the continuity of f that $f(x) = 0$ for all $x < x_0 - x_1$. As this contradicts our assumption, it follows that x_0 does not exist and that $f(x) \neq 0$ for $x > 0$. This proves the theorem.

COROLLARY. *An* inf div pdf *on* $(0, \infty)$, *which is continuous on* $(0, \infty)$ *and positive on* $(0, \delta)$ *for some* $\delta > 0$, *has no zeros on the positive half-line.*

It does not seem easy to extend the argument of Theorem 4 to pdf's on $(-\infty, \infty)$: if $\phi(t)$ is the characteristic function (ch.f.) of a pdf $f(x)$, having a representation of the form

$$\phi(t) = \exp \int_{-\infty}^{\infty} (e^{itx} - 1)\, d\theta(x),$$

where θ is non-decreasing, then the analogue of (5) becomes (if differentiation is possible)

$$xf(x) = \int_{-\infty}^{\infty} f(x-u) u\, d\theta(u),$$

where however $u\, d\theta(u)$ is not a measure. Theorem 4 provides a generalization of the Corollary to Theorem in [4], if ϕ is the ch.f. of a pdf on $(0, \infty)$ and if ϕ is not integrable.

5. Examples.

Examples of pdf's which are not inf div by the Corollary to Theorem 4 are

1. $f(x) = 6(e^{-x} - 2e^{-2x})^2$ (cf. [6]).
2. $f_\alpha(x) = 1/24 \exp(-x^{\frac{1}{4}})(1 - \alpha \sin x^{\frac{1}{4}})$ for $\alpha = 1$.

$f_0(x)$ is inf div (see [5]). It follows from the closure property of inf div distributions, that f_α cannot be inf div for all α, with $0 \leqq \alpha < 1$, as this would imply that $f_1(x)$ is inf div. The pdf f_α has the same moments for all α (cf. [2], page 224).

From the representation theorems it easily follows that Const. $\{q^n p_n\}$ is inf div if $\{p_n\}$ is inf div. In the same way if $f(x)$ is inf div, then Const. $e^{-\lambda x}f(x)$ is inf div.

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