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AN ITERATED LOGARITHM THEOREM FOR SOME WEIGHTED AVERAGES OF INDEPENDENT RANDOM VARIABLES[1]

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1. Introduction. In [1] Gaposhkin proved that, if $X_1, X_2, \cdots$ is a sequence of uniformly bounded, independent random variables (rv's), each with mean zero and variance one, then, for any $\alpha > 0$

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n} (1-k/n)^{\alpha} X_k}{(2n \log\log n)^{\frac{1}{2}}} = (2\alpha+1)^{-\frac{1}{2}} \text{ a.e.}$$

The purpose of this note is to present the following extension and generalization of Gaposhkin's result:

THEOREM. *Let $X_1, X_2, \cdots$ be a sequence of independent random variables. Suppose that there exists a sequence of positive numbers $c_n = o((\log\log n)^{-\frac{1}{2}})$ and a number $N > 0$ such that, if $n \geqq N$ then*

$$\exp\{(t^2/2n)(1-|t|c_n)\} \leqq E\exp\{tX_m/n^{\frac{1}{2}}\} \leqq \exp\{(t^2/2n)(1+|t|c_n/2)\} \tag{1}$$

for all $m \leqq n$, provided $|t|c_n \leqq 1$.

Let f be a real-valued function which is continuous on [0, 1], and define $S_n = \sum_{k=1}^{n} f(k/n)X_k$. Then

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} S_n \geqq \|f\| \text{ a.e.} \tag{2}$$

where $\|f\| = (\int_0^1 f^2(x)\,dx)^{\frac{1}{2}}$.

Furthermore, any of the following conditions is sufficient to ensure that equality holds in (2):

(i) f is a polynomial:

(ii) f is an absolutely continuous *or* monotone function which can be written as a power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on [0, 1] where $\limsup |a_n|^{1/n} = 1$, $\sum|a_n| < \infty$, $\sum a_n = f(1) = 0$.

(iii) f has a power series representation with radius of convergence greater than 1.

REMARK. Some results of a similar type have been obtained for independent, identically distributed random variables by Strassen [4].

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2. Outline of the proof. The proof of the theorem, for the most part, is similar to the proof of Gaposhkin's theorem. As in most iterated logarithm results, the basic technique used in the proof is the method discovered by Kolmogorov (see [2] or page 260 [3]). In order to apply that technique, the following "exponential bounds" are required:

LEMMA. *Let S be an* rv *and let $\varepsilon > 0$.*

(i) *If $Ee^{tS} \leqq \exp\{(t^2/2)(1+tc/2)\}$ for some $c > 0$ and all $0 \leqq tc \leqq 1$, then $P[S > \varepsilon] < \exp\{(-\varepsilon^2/2)(1-\varepsilon c/2)\}$ if $\varepsilon c \leqq 1$ and $P[S > \varepsilon] < \exp\{-\varepsilon/(4c)\}$ if $\varepsilon c > 1$;*

(ii) *If $\exp\{(t^2/2)(1-tc)\} \leqq Ee^{tS} \leqq \exp\{(t^2/2)(1+tc/2)\}$ for some $c > 0$ and all $0 \leqq tc \leqq 1$, then, for any given $\gamma > 0$, there exist numbers $\varepsilon_0 > 0$ and $\eta_0 > 0$ (both depending on γ) such that $P[S > \varepsilon) > \exp\{(-\varepsilon^2/2)(1+\gamma)\}$ whenever $\varepsilon > \varepsilon_0$ and $\varepsilon c < \eta_0$.*

This lemma can be proved in the same manner as Kolmogorov's exponential bounds are demonstrated. The proof is detailed in [5], but need not be reproduced here.

It should be pointed out, before proceeding further, that $EX_n = 0$ and $EX_n{}^2 = 1$, for every n. To derive these facts it suffices to write the terms of (1) as Taylor series for $t > 0$, cancel the first terms, divide throughout by t (or t^2, respectively) and let $t \downarrow 0$.

There is no loss of generality in assuming that $\|f\| = 1$. Then, using the lemma and the fact that $ES_n{}^2 = \sum_{m=1}^n f^2(m/n) \sim n\|f\| = n$, the inequality (2) can be obtained in a manner similar to that used by Gaposhkin.

Proving that equality holds in (2) when f is a polynomial (Condition (i)) also can be accomplished following Gaposhkin, although some extra computations are required to establish Gaposhkin's inequality (24). One would need to know, too, that, for any $j \geqq 0$, $\limsup_{k\to\infty} n_k^{-j}(2n_k \log\log n_k)^{-\frac{1}{2}} \sum_{j=1}^{n_k} k^j X_k \leqq (2j+1)^{-\frac{1}{2}} \leqq 1$ a.e. where n_k is the integral part of c^{2k} for some appropriately chosen $c > 1$; this statement can be proved in the same manner as Kolmogorov's Law of the Iterated Logarithm with the help of our lemma; details appear in [5].

At this juncture, however, Gaposhkin's procedure breaks down in part, necessitating a new approach. We shall now demonstrate that (2) holds with equality under either (ii) or (iii).

Suppose that (ii) holds with f absolutely continuous or that (iii) holds (in which case f is also absolutely continuous and $\sum|a_k| < \infty$). Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that, for any finite number of disjoint intervals in [0, 1], (say (a_n, b_n), $n = 1, 2, \cdots N$),

$$\sum_{n=1}^N |f(b_n)-f(a_n)| < \varepsilon/2 \quad \text{if} \quad \sum_{n=1}^N (b_n - a_n) < \delta.$$

Choose $c > 1$ so close to 1 that

$$c^2 < 2,\ c-1 < \varepsilon/4, \quad (c^2-1)c^{-2} < \delta/2 \quad \text{and} \quad \sum_{j=0}^\infty |a_j|(c^{2j}-1)c^{-2j} < \varepsilon/2. \tag{3}$$

For each $k \geqq 1$, define n_k to be the integral part of c^{2k}; note that $n_{k-1} < n_k$ for all sufficiently large k, say, $k \geqq k_0$, and that $n_k \sim c^{2k}$.

In view of (3), then, $1 - n_{k-1}/n_k < \delta$ for all large k so that if $n_{k-1} < n \leqq n_k$, then $1-(n_{k-1}+1)/n < \delta$.

By absolute continuity, then,

$$\sum_{m=n_{k-1}+1}^{n-1} |f(m/n) - f((m+1)/n)| < \varepsilon/2.$$

A similar, but simpler, argument would establish this inequality under condition (ii), where f is monotone.

Now define $s_n^2 = ES_n^2$ and $t_n^2 = 2 \log\log s_n^2$, for $n \geqq 1$, and, for $k \geqq k_0$,

$$R_k = \max_{n_{k-1} < n \leqq n_k} (s_n t_n)^{-1} |S_n - S_{n_{k-1}}|,$$

$$R_k^{(1)} = \max_{n_{k-1} < n \leqq n_k} (s_n t_n)^{-1} \left|\sum_{m=1}^{n_{k-1}} \{f(m/n) - f(m/n_{k-1})\} X_m\right|, \quad \text{and}$$

$$R_k^{(2)} = \max_{n_{k-1} < n \leqq n_k} (s_n t_n)^{-1} \left|\sum_{m=n_{k-1}+1}^{n} f(m/n) X_m\right|.$$

In addition, we define $S_n^{(k)} = X_{n_{k-1}+1} + \cdots + X_n$, for $k \geqq k_0$ and $n_{k-1} < n \leqq n_k$.

Then, $\sum_{m=n_{k-1}+1}^{n} f(m/n) X_m = \sum_{m=n_{k-1}+1}^{n} \{f(m/n) - f((m+1)/n)\} S_m^{(k)} + f(1) S_n^{(k)}$.

$$R_k^{(2)} \leqq (\varepsilon/2 + |f(1)|) \max_{n_{k-1} < n \leqq n_k} |S_n^{(k)}|/(s_n t_n),$$

and

$$\limsup_{k\to\infty} R_k^{(2)} \leqq |f(1)| + \varepsilon/2$$

in view of Kolmogorov's Theorem.

Then, proceeding exactly in accordance with Gaposhkin's handling of $R_k^{(1)}$ with the use of the relations in (3), one discovers that $\limsup R_k^{(1)} \leqq \varepsilon/2$ and, therefore, $\limsup R_k \leqq \varepsilon + |f(1)|$.

It is easily established that $\limsup_{k\to\infty} S_{n_k}/(s_{n_k} t_{n_k}) \leqq 1$ so that it follows, since $s_n^2 \sim n$, that

$$\limsup_{n\to\infty} S_n/(2n \log\log n)^{\frac{1}{2}} \leqq 1 + |f(1)| \text{ a.e.} \tag{4}$$

If (ii) holds, the proof is finished. Under (iii), the radius of convergence of the power series is greater than 1 so that $\sum a_k x^k$ is uniformly convergent on [0, 1].

For each $m \geqq 0$, define $g_m(x) = \sum_{j=0}^{m} a_j x^j$, $h_m(x) = f(x) - g_m(x)$. Since $g_m \to f$ uniformly in x as $m \to \infty$, it is clear that

$$\|g_m\| \to \|f\| = 1 \quad \text{and} \quad \|h_m\| \to 0 \qquad \text{as} \quad m \to \infty.$$

Noting that g_m is a polynomial and h_m is a power series we apply (4) and the fact that equality holds in (2) under condition (i) to find that, for all $m \geqq 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} S_n/(2n \log\log n)^{\frac{1}{2}} \\
&\quad \leqq \limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} \sum_{k=1}^{n} g_m(k/n) X_k \\
&\qquad + \limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} \cdot \sum_{k=1}^{n} h_m(k/n) X_k \\
&\quad \leqq \|g_m\| + \|h_m\| \cdot (1 + |h_m(1)|) \text{ a.e.}
\end{aligned}$$

But $\lim_{m\to\infty} (||g_m|| + ||h_m|| \cdot (1+|h_m(1)|)) = 1$. So,

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}} S_n \leqq 1 \text{ a.e.}$$

3. Consequences and remarks. In view of the complexity of the conditions placed on the random variables in the theorem, it might be useful to indicate some special cases. (1) holds under any of the following circumstances, provided $EX_n = 0$ and $EX_n^2 = 1$:

(a) each X_n has normal $N(0, 1)$ distribution.

(b) for some sequence $0 < a_n = o(n^{\frac{1}{2}}(\log\log n)^{-\frac{1}{2}})$, and all $n \geqq 1$, $|X_n| \leqq a_n$ a.e.

(c) the X_n's are uniformly bounded.

(d) each X_n has Laplace distribution.

(a) is obvious while (c) is a consequence of (b). (b) implies (1) if one uses the argument on page 255 of [3] with $c_n = a_n n^{-\frac{1}{2}}(\log\log n)^{\frac{1}{2}}$. Details of (d) are given in [5].

That Gaposhkin's result is a consequence of our theorem is clear, since $f(x) = (1-x)^\alpha$, $\alpha > 0$ satisfies condition (ii) of the theorem; in fact, we have shown that the Gaposhkin theorem remains valid for a wider class of random variables.

REMARK. It follows from the First Weierstrass Theorem that if f is continuous on [0, 1], then f is approached uniformly be a sequence of polynomials. It is then plausible that (2) should hold with equality for any continuous function, since it is true for polynomials. The author has been unable to establish the conjecture.

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REFERENCES

[1] GAPOSHKIN, V. F. (1965). The Law of the Iterated Logarithm for Cesaro's and Abel's methods of summation. *Theor. Probability Appl.* **10** 411–420.

[2] KOLMOGOROV, A. (1929). Uber des Gesetz des Iterierten Logarithmus. *Math. Ann.* **101** 126–135.

[3] LOÈVE, M. (1963). *Probability Theory*, 3rd ed. Van Nostrand, Princeton.

[4] STRASSEN, V. (1964). An invariance principle for the Law of the Iterated Logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete.* **3** 211–226.

[5] TOMKINS, R. J. (1970). On the Law of the Iterated Logarithm. Ph.D. dissertation, Purdue University.