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FREDHOLM DETERMINANT OF A POSITIVE DEFINITE KERNEL OF A SPECIAL TYPE AND ITS APPLICATION[1]

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Let $\rho(x, y)$ be a positive definite symmetric kernel defined over the unit square such that $\rho(x, y) = K(x, y) - \sum_{i=1}^{k} \psi_i(x)\psi_i(y)$, $0 \leqq x$, $y \leqq 1$, where $K(x, y)$ is a bounded symmetric positive definite kernel defined over the unit square, and $\psi_i(x) \in L_2(0, 1)$. Methods of finding Fredholm determinant $D(\lambda)$ of $\rho(x, y)$ in terms of the eigenvalues and the eigenfunctions of $K(x, y)$ are given. A kernel of the type of $\rho(x, y)$ arises as the covariance function of a Gaussian process in the limiting distribution of the modified Cramér-Smirnov test statistic in the k-parameter case which may be described as follows: Let $X_1, \cdots, X_n$ be n independent observations (random variables) from a population with a continuous distribution function $G(x)$. Suppose for every $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k) \in I$, I being an open interval in the k-dimensional Euclidean space R^k, $F(x, \boldsymbol{\theta})$ is a continuous distribution function. Let $\hat{\boldsymbol{\theta}}_n$ be an estimate of $\boldsymbol{\theta}$ obtained from the sample. A test of the hypothesis H: $G(x) = F(x, \boldsymbol{\theta})$ for some unspecified $\boldsymbol{\theta} \in I$ based on the statistic

$$C_n^2 = n \int_{-\infty}^{+\infty} [F_n(x) - F(x, \hat{\boldsymbol{\theta}}_n)]^2 \, dF(x, \hat{\boldsymbol{\theta}}_n) ,$$

is considered and the characteristic function of the asymptotic distribution of C_n^2 is shown to be the Fredholm determinant of a kernel of the type of $\rho(x, y)$ with $K(x, y) = \min(x, y) - xy$ whose eigenvalues and eigenfunctions are known. Results are also used to obtain the limiting distribution of l-sample analogue of C_n^2.

1. Introduction. Let $\rho(x, y)$ be a positive definite symmetric kernel defined over the unit square such that

$$\rho(x, y) = K(x, y) - \sum_{i=1}^{k} \psi_i(x)\psi_i(y) , \qquad 0 \leqq x, y \leqq 1 , \tag{1.1}$$

where $K(x, y)$ is a bounded symmetric positive definite kernel defined over the unit square, with $0 < \lambda_1 < \lambda_2 \cdots$ as its eigenvalues and $f_1(x), f_2(x), \cdots$ corresponding normalized eigenfunctions and $\psi_i(x) \in L_2(0, 1)$, $i = 1, 2, \cdots, k$. In Section 2, methods of obtaining $D(\lambda)$ the F.D. for $\rho(x, y)$ are given by Theorems 2.1 and 2.2. It may be pointed out here that $D(\lambda)$ is also called Fredholm's function (Pogorzelski (1966)) and that $K(x, y)$ can be unbounded satisfying certain conditions, (see Pogorzelski (1966) page 77). Theorem 2.2 gives an expression for $D(\lambda)$ in terms of the eigenvalues and the eigenfunctions of the kernel $K(x, y)$. Thus if $K(x, y)$ is a kernel whose eigenvalues and eigenfunctions are known then Theorem 2.2 gives a method of finding $D(\lambda)$.

A kernel of the type (1.1) arises as the covariance function of a Gaussian

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process involved in the limiting distribution of the modified Cramér–Smirnov test statistic in the k-parameter case which may be described as follows. Let $X_1, \cdots, X_n$ be n independent observations (random variables) from a population with a continuous distribution function $G(x)$. For testing the hypothesis $H_0: G(x) = F(x)$, where $F(x)$ is some specified distribution function, a test based on ω_n^2, defined by (1.2), was proposed by Cramér (1928), Smirnov (1936) and von Mises (1931).

$$(1.2) \qquad \omega_n^2 = n \int_{-\infty}^{+\infty} [F_n(x) - F(x)]^2 \, dF(x) ,$$

where $F_n(x)$ denotes the sample distribution function i.e. $F_n(x) = \nu/n$, ν being the number of X_i $(i = 1, 2, \cdots, n)$ that are less that x. The hypothesis H_0 is rejected for large values of ω_n^2. Following a suggestion of Cramér (1946) Darling (1955) extended the theory of ω_n^2-test to the case when the distribution function $F(x)$ is not completely specified, but depends upon a parameter that must be estimated from the sample. Section 3 of this paper is concerned with the extension of ω_n^2-test to the general case when $F(x)$ depends upon many parameters, say, k of them. Assume that for every $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_k) \in I$, I being an open 'interval' in the k-dimensional Euclidean space R^k, $F(x, \boldsymbol{\theta})$ is a distribution function and $\hat{\boldsymbol{\theta}}_n = (\hat{\theta}_{1n}, \hat{\theta}_{2n}, \cdots, \hat{\theta}_{kn})$ is an estimate of $\boldsymbol{\theta}$ obtained from the sample. Without ambiguity we denote $\hat{\theta}_{in}$ by $\hat{\theta}_i$. Consider a test for the hypothesis $H: G(x) = F(x, \boldsymbol{\theta})$ for some unspecified $\boldsymbol{\theta} \in I$, based on the statistic

$$(1.3) \qquad C_n^2 = n \int_{-\infty}^{+\infty} [F_n(x) - F(x, \hat{\boldsymbol{\theta}}_n)]^2 \, dF(x, \hat{\boldsymbol{\theta}}_n) ,$$

such that H is rejected for sufficiently large values of C_n^2. In Section 3 we obtain the limiting distribution of C_n^2 under the hypothesis H. The characteristic function of the asymptotic distribution of C_n^2 is shown to be the Fredholm determinant of a kernel of the type (1.1) and Theorem 2.2 is applied to find the required characteristic function. Kac, Kiefer and Wolfowitz (1955) considered a test based on C_n^2, when $F(x, \boldsymbol{\theta})$ is a normal distribution $N(x, \mu, \sigma^2)$ when both μ and σ^2 are unknown. We shall obtain the results of Kac, Kiefer and Wolfowitz (1955) as a special case of the theory developed here. The methods used in [6] do not seem to be general enough to be used for an arbitrary $F(x, \boldsymbol{\theta})$. However a suitable combination of techniques of Darling (1955) and Kac, Kiefer and Wolfowitz (1955) is employed here to generalize their results to the present case of continuous $F(x, \boldsymbol{\theta})$ involving k unknown parameters.

Kiefer (1959) has studied the limiting distribution of l-sample Cramér–Smirnov test statistic $C_{l,n}^2$ (defined by (4.1)) for testing the hypothesis of goodness of fit. Section 4 indicates the application of Theorem 2.2 in deriving the limiting distribution of modified $C_{l,n}^2$ in the parametric case under consideration.

2. Fredholm determinant for $\rho(x, y)$. For simplicity we find the F.D. associated with $\rho(x, y)$ defined by (1.1) when $k = 3$. The method used is quite general and extension to the case when k is an arbitrary positive integer is clear. Now we introduce the following notation: Let

(2.1) $\rho(x, y) = K(x, y) - \sum_{i=1}^{3} \psi_i(x)\psi_i(y)\,, \qquad 0 \leqq x,\, y \leqq 1,\ \psi_i(x) \in L_2(0, 1)\,,$

where $K(x, y)$, $\{\lambda_i\}$ and $\{f_i(x)\}$ are as defined in Section 1 and $d(\lambda)$ denotes the F.D. associated with $K(x, y)$. Further, let

(2.2) $$\alpha_{ij} = \int_0^1 \psi_i(x) f_j(x)\, dx\,, \qquad i = 1, 2, 3;\ j = 1, 2, \cdots;$$

(2.3) $$c_i(g) = \int_0^1 \psi_i(x) g(x)\, dx\,, \qquad i = 1, 2, 3;$$

(2.4) $$P_{ii}(\lambda) = 1 + \lambda \sum_{j=1}^{\infty} \frac{\alpha_{ij}^2}{1 - \lambda/\lambda_j}\,, \qquad \lambda \neq \lambda_j,\ i = 1, 2, 3;$$

(2.5) $$P_{ik}(\lambda) = \lambda \sum_{j=1}^{\infty} \frac{\alpha_{ij}\alpha_{kj}}{1 - \lambda/\lambda_j}\,, \qquad \lambda \neq \lambda_j,\ i \neq k,\ i, k = 1, 2, 3\,.$$

Set

(2.6) $$\rho_l(x, y) = K(x, y) - \psi_l(x)\psi_l(y)\,, \qquad l = 1, 2, 3;$$

(2.7) $\rho_{lm}(x, y) = K(x, y) - \psi_l(x)\psi_l(y) - \psi_m(x)\psi_m(y)\,, \quad l \neq m,\ l, m = 1, 2, 3\,.$

$\{\lambda_j^{(l)}\}$ and $\{f_j^{(l)}(x)\}$ $l = 1, 2, 3$, denote respectively the eigenvalues (in the order of magnitude) and corresponding normalized eigenfunctions of the kernel $\rho_l(x, y)$. $\{\lambda_j^{(lm)}\}$ and $\{f_j^{(lm)}(x)\}$ are the eigenvalues and the normalized eigenfunctions of $\rho_{lm}(x, y)$. $\alpha_{ij}^{(l)}$, $l = 1, 2, 3$, $(\alpha_{ij}^{(lm)}, l \neq m)$ are defined as in (2.2) with $f_j(x)$ replaced by $f_j^{(l)}(x)(f_j^{(lm)}(x))$. $P_{ik}^{(l)}(\lambda)$ and $P_{ik}^{(l)}(\lambda)(P_{ii}^{(lm)}(\lambda),\ P_{ik}^{(lm)}(\lambda))$ are obtained by replacing α_{ij} by $\alpha_{ij}^{(l)}(\alpha_{ij}^{(lm)})$ and λ_j by $\lambda_j^{(l)}(\lambda_j^{(lm)})$ in the expressions (2.4) and (2.5) that define $P_{ii}(\lambda)$ and $P_{ik}(\lambda)$ respectively. $D_l(\lambda)(D_{lm}(\lambda))$ represents the F.D. for $\rho_l(x, y)(\rho_{lm}(x, y))$.

LEMMA 2.1. *$D_{lm}(\lambda)$, the* F.D. *associated with the kernel $\rho_{lm}(x, y)$ defined by* (2.7) *is given by*

(2.8) $D_{lm}(\lambda) = d(\lambda)P_{ll}(\lambda)P_{mm}^{(l)}(\lambda) = d(\lambda)P_{mm}(\lambda)P_{ll}^{(m)}(\lambda)\,, \quad l \neq m,\ l, m = 1, 2, 3\,.$

PROOF. We prove the first equality in (2.8). Since $\rho(x, y)$ is a positive definite kernel, $\rho_l(x, y)$ being the sum of positive definite kernels is also positive definite. By Theorem 6.2 of Darling (1955) the F.D. of the kernel $\rho_l(x, y)$ is $D_l(\lambda) = d(\lambda)P_{ll}(\lambda)$. The integral equation

(2.9) $$g(x) = \lambda \int_0^1 \rho_{lm}(x, y) g(y)\, dy\,,$$

can be written as

(2.10) $$g(x) = -\lambda c_m(g)\psi_m(x) + \lambda \int_0^1 \rho_l(x, y) g(y)\, dy\,.$$

Using the series expansion for $g(x)$, (Pogorzelski (1966) page 145; Whittaker and Watson (1928) page 228) and arguing as in ([4] page 12) we can prove that only for those values of λ, which are either zeros of $P_{mm}^{(l)}(\lambda)$ or are the eigenvalues of $\rho_l(x, y)$ the equation (2.10) can have a nontrivial solution, i.e., λ is a zero of the entire function $D_l(\lambda)P_{mm}^{(l)}(\lambda)$, with $D_l(0)P_{mm}^{(l)}(0) = 1$. To prove that $D_l(\lambda)P_{mm}^{(l)}(\lambda)$

is the F.D. of $\rho_{lm}(x, y)$ we have to show that for any $\bar{\lambda}$ such that $D_l(\bar{\lambda})P_{mm}^{(l)}(\bar{\lambda}) = 0$, there exists a solution $\bar{g}(x)$ of the integral equation (2.10) such that $\int_0^1 \bar{g}^2(x)\,dx = 1$. In the course of the proof of Theorem 6.2 of Darling (1955) we observe that the zeros of $D_l(\lambda)$ are either simple or double. Hence we have to consider the following three cases:

(i) $\bar{\lambda} \neq \lambda_j^{(l)}$,
(ii) $\bar{\lambda} = \lambda_j^{(l)}$, when $\lambda_j^{(l)}$ is a simple zero of $D_l(\lambda)$, $\alpha_{mj}^{(l)} = 0$,
(iii) $\bar{\lambda} = \lambda_j^{(l)}$, where $\lambda_j^{(l)}$ is a double zero of $D_l(\lambda)$, say,

$$\lambda_j^{(l)} = \lambda_{j+1}^{(l)},\ \alpha_{mj}^{(l)} = \alpha_{m,j+1}^{(l)} = 0\ .$$

In case (i) since $\bar{\lambda}$ is not a zero of $D_l(\lambda)$, it is such that $P_{mm}^{(l)}(\bar{\lambda}) = 0$. Noting that $\bar{\lambda}$ is real and $P_{mm}^{(l)\prime}(\lambda) > 0$ for all real λ, $\bar{g}(x)$ given by

$$\bar{g}(x) = \left[-\sum_{j=1}^{\infty} \frac{\alpha_{mj}^{(l)} f_j^{(l)}(x)}{(1-\bar{\lambda}/\lambda_j^{(l)})}\right] \Big/ \left[\sum_{j=1}^{\infty} \frac{\alpha_{mj}^{(l)2}}{(1-\bar{\lambda}/\lambda_j^{(l)})^2}\right]^{\frac{1}{2}}, \tag{2.11}$$

is a solution of (2.10). Thus for any $\bar{\lambda}$ under (*i*) $\bar{g}(x)$ given by (2.11) is a solution of (2.10).

In case (ii) it is necessary that $\alpha_{mj}^{(l)} = 0$, because if $\alpha_{mj}^{(l)} \neq 0$, $\bar{\lambda}$ cannot be a zero of $D_{lm}(\lambda)$. Similarly in case (iii) it is necessary that $\alpha_{mj}^{(l)} = \alpha_{m,j+1}^{(l)} = 0$.

In case (ii) we have two subcases: (a) $\bar{\lambda}$ is such that $D_l(\bar{\lambda}) = 0$ and $P_{mm}^{(l)}(\bar{\lambda}) \neq 0$. This means that $\bar{\lambda}$ is a simple zero of $D_l(\lambda)P_{mm}^{(l)}(\lambda)$ and $f_j^{(l)}(x)$ satisfies (2.10) (Note that $P_{mm}^{(l)}(\lambda) \neq 0$ implies that $c_m(g) = 0$ and $f_j^{(l)}(x)$ satisfies (2.10)). (b) $\bar{\lambda}$ is such that $D_l(\bar{\lambda}) = 0$ and $P_{mm}^{(l)}(\bar{\lambda}) = 0$, i.e. $D_l(\lambda)P_{mm}^{(l)}(\lambda)$ has a double root at $\bar{\lambda} = \lambda_j^{(l)}$. In this case $f_j^{(l)}(x)$ and $\bar{g}(x)$ given by (2.11) are the solutions of (2.10).

Under case (iii) if $\bar{\lambda}$ is such that: (a) $D_l(\bar{\lambda}) = 0$ and $P_{mm}^{(l)}(\bar{\lambda}) \neq 0$, then $\bar{\lambda}$ is a double root of $D_l(\lambda)P_{mm}^{(l)}(\lambda)$ and $f_j^{(l)}(x)$, $f_{j+1}^{(l)}(x)$ satisfy (2.10); (b) $D_l(\bar{\lambda}) = 0$ and $P_{mm}^{(l)}(\bar{\lambda}) = 0$ i.e. $\bar{\lambda}$ is a triple zero of $D_l(\lambda)P_{mm}^{(l)}(\lambda)$, and $f_j^{(l)}(x)$, $f_{j+1}^{(l)}(x)$ and $\bar{g}(x)$ given by (2.11) are solutions of (2.10).

Thus for each zero of $D_l(\lambda)P_{mm}^{(l)}(\lambda)$ we obtain solutions of appropriate multiplicity to (2.10). Hence the first equality in (2.8) follows. The remaining equality in (2.8) follows in the same fashion by writing (2.9) as

$$g(x) = -\lambda c_l(g)\psi_l(x) + \lambda \int_0^1 \rho_m(x, y)g(y)\,dy\ .$$

Application of Lemma 2.1 yields the following theorem.

THEOREM 2.1. *Let* $\rho(x, y)$ *be a positive definite symmetric kernel defined by* (2.1). *Then* $D(\lambda)$ *the* F.D. *associated with* $\rho(x, y)$ *is given by*

$$D(\lambda) = d(\lambda)P_{ii}(\lambda)P_{jj}^{(i)}(\lambda)P_{kk}^{(ij)}(\lambda)\,, \qquad i \neq j \neq k;\ i, j, k = 1, 2, 3\ . \tag{2.12}$$

PROOF. The integral equation

$$g(x) = \lambda \int_0^1 [K(x, y) - \textstyle\sum_{i=1}^{3} \psi_i(x)\psi_i(y)]g(y)\,dy \tag{2.13}$$

can be written as

$$g(x) = -\lambda c_i(g)\psi_i(x) + \lambda \int_0^1 \rho_{jk}(x, y)g(y)\,dy \quad i \neq j \neq k,\ i, j, k = 1, 2, 3\ . \tag{2.14}$$

Observe that $\rho_{jk}(x, y)$ is a positive definite kernel which can have eigenvalues of multiplicity at the most equal to three (see the proof of Lemma 2.1). Using series expansion and proceeding as in [4] an application of Lemma 2.1 yields the result (2.12).

Even if Theorem 2.1 gives a method of obtaining the F.D. associated with $\rho(x, y)$, the method requires the laborious task of finding the eigenvalues and the eigenfunctions of the kernels $K(x, y)$, $\rho_l(x, y)$, and $\rho_{lm}(x, y)$. The following theorem which is a generalization of Theorem 6.2 of Darling (1955) avoids this difficulty by giving an expression for the F.D. of $\rho(x, y)$ in terms of the eigenvalues and the eigenfunctions of the kernel $K(x, y)$.

THEOREM 2.2. *The* F.D. *of a positive definite symmetric kernel* $\rho(x, y) = K(x, y) - \sum_{i=1}^{3} \psi_i(x)\psi_i(y)$ *defined by* (2.1), *is given by*

$$D(\lambda) = d(\lambda)\Delta(\lambda) , \tag{2.15}$$

where $\Delta(\lambda)$ *denotes the determinant of the symmetric matrix*

$$P(\lambda) = (P_{ij}(\lambda)) , \qquad i, j = 1, 2, 3 ; \tag{2.16}$$

and $P_{ii}(\lambda)$, $P_{ik}(\lambda)$ *are defined by* (2.4), (2.5) *respectively.*

PROOF. Write (2.13) as

$$g(x) = -\lambda \sum_{i=1}^{3} c_i(g)\psi_i(x) + \lambda \int_0^1 K(x, y)g(y)\, dy . \tag{2.17}$$

Then, for $\lambda \neq \lambda_j$

$$\begin{aligned} g(x) = -\lambda c_1(g) \sum_{j=1}^{\infty} \frac{\alpha_{1j} f_j(x)}{1 - \lambda/\lambda_j} - \lambda c_2(g) \sum_{j=1}^{\infty} \frac{\alpha_{2j} f_j(x)}{1 - \lambda/\lambda_j} \\ - \lambda c_3(g) \sum_{j=1}^{\infty} \frac{\alpha_{3j} f_j(x)}{1 - \lambda/\lambda_j} . \end{aligned} \tag{2.18}$$

Multiplying (2.18) by $\psi_i(x)$ and integrating we get

$$c_1(g)P_{i1}(\lambda) + c_2(g)P_{i2}(\lambda) + c_3(g)P_{i3}(\lambda) = 0 , \qquad i = 1, 2, 3 ; \tag{2.19}$$

a system of homogeneous equations in $c_i(g)$, which can have a nontrivial solution if and only if $\Delta(\lambda) = 0$. If $\lambda \neq \lambda_j$ all $c_i(g)$ cannot be zero simultaneously, because $c_1(g) = c_2(g) = c_3(g) = 0$, implies that (2.17) is homogeneous which cannot have nontrivial solution unless $\lambda = \lambda_j$ for some j, i.e., $d(\lambda) = 0$. Thus if $\lambda \neq \lambda_j$, then we must have $\Delta(\lambda) = 0$. If $\lambda = \lambda_j$ it is necessary that $\alpha_{ij} = 0$ (see case (ii) in the proof of Lemma 2.1). $P_{ij}(\lambda)$ have simple poles at $\lambda = \lambda_j$. Hence $\Delta(\lambda)$ is finite when $\lambda = \lambda_j$ and we have $d(\lambda_j)\Delta(\lambda_j) = 0$. Therefore, (2.13) has a nontrivial solution only when either λ is such that $\Delta(\lambda) = 0$ or λ is a zero of $d(\lambda)$, i.e., $d(\lambda)\Delta(\lambda) = 0$.

To complete the proof, it remains to show that for every zero of $d(\lambda)\Delta(\lambda)$ there exists a solution of appropriate multiplicity to the integral equation (2.13). For this purpose, it is sufficient to prove that the zeros of $d(\lambda)\Delta(\lambda)$ and those of $D(\lambda)$ in (2.12) are the same. Suppose $\bar{\lambda}$ is a zero of $d(\lambda)$ then obviously $d(\bar{\lambda})\Delta(\bar{\lambda}) = D(\bar{\lambda}) = 0$. Suppose $\bar{\lambda}$ is such that $\Delta(\bar{\lambda}) = 0$ and $d(\bar{\lambda}) \neq 0$. Since $\Delta(\bar{\lambda}) = 0$, there exists a nontrivial solution of (2.19) so that at least one $c_i(g) \neq 0$. Without any loss of generality, let $c_3(g) \neq 0$. Then from the equation

$$g(x) = -\bar{\lambda} c_3(g)\psi_3(x) + \bar{\lambda} \int_0^1 \rho_{12}(x, y)g(y)\,dy\,,$$

we have

$$g(x) = -\bar{\lambda} c_3(g) \sum_{j=1}^{\infty} \frac{\alpha_{3j}^{(12)} f_j^{(12)}(x)}{1 - \bar{\lambda}/\lambda_j^{(12)}}\,, \qquad \bar{\lambda} \neq \lambda_j^{(12)}\,. \tag{2.20}$$

Multiplying both the sides of (2.20) by $\psi_3(x)$ and integrating we get $c_3(g)P_{33}^{(12)}(\bar{\lambda}) = 0$. As $c_3(g) \neq 0$, it follows that $P_{33}^{(12)}(\bar{\lambda}) = 0$ and $\bar{\lambda}$ is a zero of $d(\lambda)P_{11}(\lambda)P_{22}^{(1)}(\lambda)P_{33}^{(12)}(\lambda)$. Thus a zero of $d(\lambda)\Delta(\lambda)$ is a zero of $D(\lambda)$ given by (2.12).

Now we prove that a zero of $D(\lambda)$ given by (2.12) is also a zero of $d(\lambda)\Delta(\lambda)$. Suppose $\bar{\lambda}$ is a zero of $D(\lambda)$ but $d(\bar{\lambda}) \neq 0$. Here we have to consider the following subcases:

(i) $\bar{\lambda} \neq \lambda_j$ and $\bar{\lambda}$ is such that $P_{11}(\bar{\lambda}) = P_{22}(\bar{\lambda}) = P_{33}(\bar{\lambda}) = 0$. By Schwarz's inequality it follows that $P_{ik}(\bar{\lambda}) = 0$ for all $i \neq k$, $i, k = 1, 2, 3$ and hence $\Delta(\bar{\lambda}) = 0$.

(ii) $\bar{\lambda} \neq \lambda_j$, one $P_{ii}(\bar{\lambda})$ is not zero and the remaining are zero, say $P_{11}(\bar{\lambda}) \neq 0$, $P_{22}(\bar{\lambda}) = P_{33}(\bar{\lambda}) = 0$. In this case Schwarz's inequality yields $P_{13}(\bar{\lambda}) = P_{23}(\bar{\lambda}) = P_{12}(\bar{\lambda}) = 0$ and hence $\Delta(\bar{\lambda}) = 0$.

(iii) $\bar{\lambda} \neq \lambda_j$, two of $P_{ii}(\bar{\lambda})$ are not zero and one $P_{ii}(\bar{\lambda})$ is zero, say $P_{11}(\bar{\lambda}) \neq 0$, $P_{22}(\bar{\lambda}) \neq 0$, $P_{33}(\bar{\lambda}) = 0$. Again by Schwarz's inequality we have $P_{13}(\bar{\lambda}) = P_{23}(\bar{\lambda}) = 0$ which means that $\Delta(\bar{\lambda}) = 0$.

(iv) $\bar{\lambda} \neq \lambda_j$, $P_{ii}(\bar{\lambda}) \neq 0$ for all $i = 1, 2, 3$. Since $\bar{\lambda} \neq \lambda_j$, $c_1(g)$, $c_2(g)$, $c_3(g)$ cannot all be zero simultaneously. Suppose $c_3(g) \neq 0$. When $\bar{\lambda} \neq \lambda_j$, $c_1(g)$, $c_2(g)$, $c_3(g)$ satisfy (2.19) with $P_{ii}(\lambda)$ and $P_{ik}(\lambda)$ replaced by $P_{ii}(\bar{\lambda})$ and $P_{ik}(\bar{\lambda})$ respectively. This system has a nontrivial solution only if $\Delta(\bar{\lambda}) = 0$. This completes the proof of the theorem.

REMARK. It is clear from the method of proof of Theorem 2.1 that the eigenvalues of the kernel defined by (1.1) can be at the most of multiplicity $(k + 1)$.

3. Asymptotic distributions of C_n^2. Now we consider an application of Theorem 2.2 to obtain the characteristic function of the asymptotic distribution of C_n^2 defined by (1.3). Henceforth it is assumed that $F(x, \boldsymbol{\theta})$ is absolutely continuous and $f(x, \boldsymbol{\theta})$ denotes the corresponding density function. We shall use the following basic transformations:

$$u = F(x, \boldsymbol{\theta})\,, \qquad u_j = F(X_j, \boldsymbol{\theta})\,, \qquad i = 1, 2, \cdots, n\,. \tag{3.1}$$

By (3.1) x is defined implicitly as a function of u and $\boldsymbol{\theta}$, except possibly at a denumerable set of values of u, at which x can be defined arbitrarily so as to make the function monotone non-decreasing.

Define

$$\psi_t(x) = \begin{cases} 1 & \text{if } x < t\,, \\ 0 & \text{if } x \geqq t\,; \end{cases} \tag{3.2}$$

then we can write

$$F_n(x) = n^{-1} \textstyle\sum_{j=1}^n \psi_x(X_j) = n^{-1} \sum_{j=1}^n \psi_u(u_j) = G_n(u)\,, \tag{3.3}$$

where $G_n(u) = (1/n)$ (number of u_j's less than u).

3.1. *Case of regular estimates.* First we consider the case of regular estimation, Cramér (1946), when $V(\hat\theta_i) \geqq A_i/n$, $i = 1, 2, \cdots, k$ for some positive A_i. If further the estimates $\hat\theta_i$ are such that $\operatorname{plim}_{n\to\infty} n^{\frac12-\delta}(\hat\theta_i - \theta_i) = 0$ for $\delta > 0$, then we have an extension of Lemma 3.1 of Darling (1955) given below.

LEMMA 3.1. *If*

(i) $\operatorname{plim}_{n\to\infty} n^{\frac12}(\hat\theta_i - \hat\theta_i) = 0\,, \qquad i = 1, 2, \cdots, k;$ *for almost all* x

(ii) $\left|\dfrac{\partial}{\partial\theta_i} F(x, \boldsymbol\theta)\right| \leqq M_i < \infty\,, \qquad i = 1, 2, \cdots, k\,;$

(iii) $\left|\dfrac{\partial^2}{\partial\theta_i^2} F(x, \boldsymbol\theta)\right| < m_i(x)\,, \qquad i = 1, 2, \cdots, k\,;$

(iv) $\left|\dfrac{\partial^2}{\partial\theta_i\partial\theta_j} F(x, \boldsymbol\theta)\right| < m_{ij}(x)\,, \qquad i \neq j,\ i, j = 1, 2, \cdots, k\,;$

(v) $\left|\dfrac{\partial}{\partial\theta_i} f(x, \boldsymbol\theta)\right| < q_i(x)\,, \qquad i = 1, 2, \cdots, k\,;$

where the functions $q_i(x)$ and $m_i^2(x)q_j(x)$ are integrable over $(-\infty, +\infty)$; $m_i(x)$, $m_{ij}(x)$, $q_i(x)$ are square integrable over $(-\infty, +\infty)$, independent of $\boldsymbol\theta$; and the exceptional set does not depend upon $\boldsymbol\theta$; and further if

$$\textstyle\int_{-\infty}^{+\infty} m_i^2(x) f(x, \boldsymbol\theta)\, dx < \infty\,, \qquad \int_{-\infty}^{+\infty} m_{ij}^2(x) f(x, \boldsymbol\theta)\, dx < \infty\,, \qquad i, j = 1, 2, \cdots, k\,;$$

then

$$C_n^2 = C_n^{*2} + \delta_n\,, \tag{3.4}$$

where

$$C_n^{*2} = n \int_{-\infty}^{+\infty}\left[F_n(x) - F(x, \boldsymbol\theta) - \textstyle\sum_{1=i}^k (\hat\theta_i - \theta_i) \dfrac{\partial}{\partial\theta_i} F(x, \boldsymbol\theta)\right]^2 \times f(x, \boldsymbol\theta)\, dx\,, \quad \textit{and} \quad \operatorname{plim}_{n\to\infty} \delta_n = 0\,. \tag{3.5}$$

PROOF. Expanding $F(x, \hat{\boldsymbol\theta})$ and $f(x, \hat{\boldsymbol\theta})$ in Taylor's series about the true value $\boldsymbol\theta$, substitute these in the defining expression (1.3). Use of the assumptions made and the fact that $\sup_x n^{\frac12}|F_n(x) - F(x, \boldsymbol\theta)|$ is bounded in probability, Kolmogorov (1941), yield the required result.

From Lemma 3.1 it is clear that the problem of finding the asymptotic distribution of C_n^2 is equivalent to finding that of C_n^{*2}. If we define

$$g_i(u) = g_i(u, \boldsymbol\theta) = \frac{\partial}{\partial\theta_i} F(x, \boldsymbol\theta)\,, \qquad 0 \leqq u \leqq 1,\ i = 1, 2, \cdots, k\,; \tag{3.6}$$

$$Z_n(u) = n^{\frac12}(G_n(u) - u)\,, \tag{3.7}$$

$$Y_n(u) = Y_n(u, \boldsymbol\theta) = Z_n(u) - \textstyle\sum_{i=1}^k n^{\frac12}(\hat\theta_i - \theta_i) g_i(u)\,, \tag{3.8}$$

then we can write

$$C_n^2 = \int_0^1 Y_n^2(u)\,du + \delta_n\,, \qquad \text{where} \quad \operatorname{plim}_{n\to\infty} \delta_n = 0\,. \tag{3.9}$$

Henceforth we drop the argument $\boldsymbol{\theta}$ and write $Y_n(u) = Y_n(u, \boldsymbol{\theta})$ and $g_i(u) = g_i(u, \boldsymbol{\theta})$ only for the sake of simplicity. The following lemma gives, the limiting form of the stochastic process $Y_n(u)$ defined by (3.8), required to obtain the asymptotic distribution of C_n^2.

LEMMA 3.2. *If*

(i) *we assume the conditions of Lemma* 3.1 (*so that C_n^2 can be written in the form* (3.9)),

(ii) $n^{\frac{1}{2}}(\hat{\theta}_i - \theta_i)$ *is a sum of independently identically distributed random variables*, $i = 1, 2, \cdots, k$,

(iii) *the asymptotic joint distribution of* $(n^{\frac{1}{2}}(\hat{\theta}_1 - \theta_1), n^{\frac{1}{2}}(\hat{\theta}_2 - \theta_2), \cdots, n^{\frac{1}{2}}(\hat{\theta}_k - \theta_k))$ *is normal with mean zero and nonsingular covariance matrix* $\Sigma = (\sigma_{ij})$,

(iv) $\lim_{n\to\infty} E(Z_n(u)n^{\frac{1}{2}}(\hat{\theta}_i - \theta_i)) = h_i(u)$, $0 < u < 1$, $h_i(0) = h_i(1) = 0$, $i = 1, 2, \cdots, k$,

then $Y_n(u)$ converges in distribution to a Gaussian process $Y(u)$ with mean zero and the covariance function $\rho(u, v, \boldsymbol{\theta})$ given by

$$\begin{aligned}\rho(u, v, \boldsymbol{\theta}) = \min(u, v) - uv - \textstyle\sum_{i=1}^k g_i(u)h_i(v) - \sum_{i=1}^k g_i(v)h_i(u) \\ + \textstyle\sum_{i,j=1}^k \sigma_{ij} g_i(u)g_j(v)\,, \qquad 0 \leqq u, v \leqq 1\,.\end{aligned} \tag{3.10}$$

PROOF. The stochastic process $Z_n(u)$ converges in distribution to a Gaussian process with zero and covariance function $K(u, v) = \min(u, v) - uv$, $0 \leqq u$, $v \leqq 1$, see for example Anderson and Darling (1952). Under the assumption (iii) the asymptotic distribution of $\sum_{i=1}^k n^{\frac{1}{2}}(\hat{\theta}_i - \theta_i)g_i(u)$ is normal with mean zero and variance $\sum_{i,j=1}^k \sigma_{ij} g_i(u)g_j(u)$. By multidimensional central limit theorem it follows that $Y_n(u)$ given by (3.8) converges in distribution to a Gaussian process with mean zero. To find the covariance function we have

$$\begin{aligned}\rho_n(u, v, \boldsymbol{\theta}) &= E(Y_n(u)Y_n(v)) \\ &= E(Z_n(u)Z_n(v)) - E(Z_n(u) \textstyle\sum_{i=1}^k n^{\frac{1}{2}}(\hat{\theta}_i - \theta_i)g_i(v)) \\ &\quad - E(Z_n(v) \textstyle\sum_{i=1}^k n^{\frac{1}{2}}(\hat{\theta}_i - \theta_i)g_i(u)) \\ &\quad + E[(\textstyle\sum_{i=1}^k n^{\frac{1}{2}}(\hat{\theta}_i - \theta_i)g_i(u))(\sum_{i=1}^k n^{\frac{1}{2}}(\hat{\theta}_i - \theta_i)g_i(v))]\,.\end{aligned}$$

Under the assumptions (i)–(iv), $\lim_{n\to\infty} \rho_n(u, v, \boldsymbol{\theta}) = \rho(u, v, \boldsymbol{\theta})$, where $\rho(u, v, \boldsymbol{\theta})$ is given by (3.10).

Here we observe that Lemma 3.2 of Darling (1955) is proved under somewhat different conditions from those of the above lemma. In one parameter case Darling assumes that $\hat{\theta}_n$ is "weakly biased," i.e., $\lim_{n\to\infty} nE(\hat{\theta}_n - \theta) = 0$. It may be that $\hat{\theta}_n$ is not weakly biased but at the same time $\lim_{n\to\infty} EZ_n(u)n^{\frac{1}{2}}(\hat{\theta} - \theta) = h(u)$ exists. For example in the case of normal distribution $N(x, \mu, \sigma^2)$ the usual estimate $S^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X})^2$ for σ^2 is not weakly biased but it will be seen later that $h(u)$ defined above exists.

3.2. *Case of efficient estimates.* Now we consider the case when $\hat{\theta}_1, \cdots, \hat{\theta}_k$ are regular, jointly efficient or asymptotically jointly efficient in the sense defined by Cramér ((1946) pages 490–495). (We assume that the covariance matrix Σ of $\hat{\boldsymbol{\theta}}$ is nonsingular. If the rank of Σ is $r < k$ then $\hat{\theta}_1, \cdots, \hat{\theta}_k$ are linearly dependent and as they are unbiased estimates of $\theta_1, \cdots, \theta_k$, we shall have only r parameters to consider). In this case $\Sigma = \Lambda^{-1}$ where $\Lambda = (\lambda_{ij})$ and

$$\lambda_{ij} = E\left(\frac{\partial}{\partial\theta_i} \log f(x, \boldsymbol{\theta}) \cdot \frac{\partial}{\partial\theta_j} \log f(x, \boldsymbol{\theta})\right), \qquad i, j = 1, 2, \cdots, k . \tag{3.11}$$

We also have

$$(\hat{\theta}_i - \theta_i) = \sum_{l=1}^{k}\left[\frac{\sigma_{il}}{n} \sum_{j=1}^{n} \frac{\partial}{\partial\theta_l} \log f(x_j, \boldsymbol{\theta})\right], \qquad i = 1, \cdots, k . \tag{3.12}$$

By Cramér's definition of an efficient estimate, conditions (i) and (v) of Lemma 3.1 are satisfied. We further assume that the remaining conditions hold. In this case $h_{in}(u) = E(Z_n(u)n^{\frac{1}{2}}(\hat{\theta}_i - \theta_i))$ is seen to be independent of n and it is equal to

$$h_{in}(u) = h_i(u) = \sum_{l=1}^{k} \sigma_{il} g_l(u) , \tag{3.13}$$

and since $g_i(1) = g_i(0) = 0$, $h_i(0) = h_i(1) = 0$. Using (3.13) in (3.10) we have the following result.

LEMMA 3.3. *If $(\hat{\theta}_1, \cdots, \hat{\theta}_k)$ are regular, unbiased jointly efficient estimates of $(\theta_1, \cdots, \theta_k)$ then the process $Y_n(u)$ given by* (3.8) *has mean zero and covariance function*

$$\rho(u, v, \boldsymbol{\theta}) = \min(u, v) - uv - \sum_{i,j=1}^{k} \sigma_{ij} g_i(u) g_j(v) , \qquad 0 \leqq u, v \leqq 1 . \tag{3.14}$$

Here we observe that under the conditions of Cramér ((1946) pages 500–504) and those of Lemma 3.2, it can be shown (as in [4]) that if $\hat{\theta}_1, \cdots, \hat{\theta}_k$ are maximum likelihood estimates of $\theta_1, \cdots, \theta_k$ then the process $Y_n(u)$ given by (3.8) converges in distribution to $Y(u)$, a Gaussian process with mean zero and covariance function given by (3.14).

3.3. *Characteristic function of the limiting distribution of C_n^2.* The following theorem gives the limiting distribution of C_n^2.

THEOREM 3.1. *If $\hat{\theta}_1, \cdots, \hat{\theta}_k$ are regular unbiased jointly efficient estimates of $\theta_1, \cdots, \theta_k$ and $g_i(u) \in L_2(0, 1)$ where $g_i(u)$ are defined by* (3.6), *then*

$$\lim_{n\to\infty} P(C_n^2 < x) = P\left(\int_0^1 Y^2(u)\, du < x\right) , \tag{3.15}$$

where $Y(u)$ is a Gaussian process with mean zero and covariance function $\rho(u, v, \boldsymbol{\theta})$ given by (3.14).

PROOF. By Lemma 3.3 the process $Y_n(u)$ defined by (3.8) converges in distribution to a Gaussian process with mean zero and covariance function $\rho(u, v, \boldsymbol{\theta})$ given by (3.14). As Σ is a positive definite symmetric matrix there exists a nonsingular matrix $B = (b_{ij})$ such that $\Sigma = B'B$. Define

$$(3.16) \qquad \psi_i(u, \boldsymbol{\theta}) = \psi_i(u) = \sum_{j=1}^{k} b_{ij} g_j(u) ,$$

then (3.14) reduces to

$$(3.17) \qquad \rho(u, v, \boldsymbol{\theta}) = \min(u, v) - uv - \sum_{i=1}^{k} \psi_i(u)\psi_i(v) .$$

By a method similar to that used in ([6] pages 195–197) we can get Kac–Siegert representation [7] for the Gaussian process $Y(u)$ described above and show that the sample functions of $Y(u)$ are continuous with probability one. Hence an application of Donsker's Theorem (1952) gives the required result.

Now we are in a position to apply Theorem 2.2 to find the characteristic function of the limiting distribution under consideration. The characteristic function of the random variable $C^2 = \int_0^1 Y^2(u)\, du$ is given by (see Anderson and Darling (1952))

$$(3.18) \qquad E\{\exp[it \textstyle\int_0^1 Y^2(u)\, du]\} = \prod_{j=1}^{\infty}\left(1 - \frac{2it}{\mu_j}\right)^{-\frac{1}{2}} = [D(2it)]^{-\frac{1}{2}} ,$$

where $\{\mu_j\}$ are the eigenvalues of the kernel $\rho(u, v, \boldsymbol{\theta})$ defined by (3.14) and $D(\mu)$ denotes the F.D. associated with it. We have seen that after making a suitable transformation (3.16), $\rho(u, v, \boldsymbol{\theta})$ can be put in the form (3.17) which is the form of the kernel described in Theorem 2.2 with $K(u, v) = \min(u, v) - uv$, $\lambda_j = \pi^2 j^2$, $f_j(x) = 2^{\frac{1}{2}} \sin \pi j x$, $d(\lambda) = (\sin \lambda^{\frac{1}{2}})/\lambda^{\frac{1}{2}}$. Thus Theorem 2.2 is useful to obtain the characteristic function of C^2.

3.4. *Some properties of C_n^2 test and special cases.* When no unknown parameters are involved, i.e., if $\theta_1, \cdots, \theta_k$ are all known, a test based on ω_n^2 is used for testing the hypothesis $G(x) = F(x, \boldsymbol{\theta})$. The asymptotic distribution of ω_n^2 is the distribution of $\omega^2 = \int_0^1 W^2(u)\, du$, where $W(u)$ is a Gaussian process with zero mean and the covariance function $K(u, v) = \min(u, v) - uv$, $0 \leqq u, v \leqq 1$ and using Kac–Siegert representation for $W(u)$, ω^2 can be written as $\omega^2 = \sum_{j=1}^{\infty} (G_j^2/\pi^2 j^2)$ where $\{G_j\}$ denotes a sequence of independently identically distributed normal variables with mean zero and variance 1. When $\theta_1, \cdots, \theta_k$ are unknown and are estimated from the sample, a test based on C_n^2 can be used. The limiting distribution of C_n^2 is the distribution of the random variable $C^2 = \int_0^1 Y^2(u)\, du$, $Y(u)$ being a Gaussian process with mean zero and covariance function given by (3.14). C^2 can be written as $C^2 = \sum_{j=1}^{\infty} (G_j^2/\mu_j)$, $\{\mu_j\}$ denoting the eigenvalues of $\rho(u, v, \boldsymbol{\theta})$. Noting that $\sum_{i,j}^{k} \sigma_{ij} g_i(u) g_j(v) = \sum_{i=1}^{k} \psi_i(u)\psi_i(v)$ is a positive definite symmetric kernel, it follows by maximum-minimum property of eigenvalues, Riesz and Nagy (1955), that the weights $1/\mu_j$ in C^2 are not greater than the weights $1/\pi^2 j^2$ in ω^2.

In particular if $\psi_i(x) = f_{m_i}(x)/\lambda_{m_i}^{\frac{1}{2}}$ where $m_1, m_2, \cdots, m_k$ are k distinct positive integers then $P_{ik}(\lambda) = 0$ for $i \neq k$ and $P_{ii}(\lambda) = \lambda_{m_i}/(\lambda_{m_i} - \lambda)$ and the number of terms in the infinite product for $D(\lambda)$ is reduced by k and

$$D(\lambda) = \prod_{j \neq m_j}^{\infty} (1 - \lambda/\lambda_j) .$$

This is analogous to reduction in degrees of freedom of usual χ^2 theory.

Case of two unknown parameters. In this case if we write $\sigma_{11} = \sigma_1^2$ $\sigma_{22} = \sigma_2^2$, $\sigma_{12} = \sigma_{21} = r\sigma_1\sigma_2$ and

$$\Sigma = \begin{pmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \sigma_1(1-r^2)^{\frac{1}{2}} & r\sigma_1 \\ 0 & \sigma_2 \end{pmatrix} \begin{pmatrix} \sigma_1(1-r^2)^{\frac{1}{2}} & 0 \\ r\sigma_1 & \sigma_2 \end{pmatrix},$$

$$\psi_1(u) = \sigma_1(1-r^2)^{\frac{1}{2}}g_1(u), \qquad \psi_2(u) = r\sigma_1 g_1(u) + \sigma_2 g_2(u),$$

$$\rho(u, v) = K(u, v) - \psi_1(u)\psi_1(v) - \psi_2(u)\psi_2(v).$$

If $F(x, \boldsymbol{\theta}) = H((x - \theta_1)/\theta_2)$ that is if θ_1 is the location parameter and θ_2 is the scale parameter, then $\rho(u, v, \boldsymbol{\theta})$ involved in the limiting distribution can be shown to be independent of θ_1, θ_2 showing that the test is asymptotically parameter-free in the sense that the limiting distribution does not depend upon the unknown parameters.

Case of normal distribution. If $F(x, \boldsymbol{\theta})$ is a normal distribution function $N(x, \mu, \sigma^2)$ with unknown μ and σ^2, the usual estimates of μ and σ^2 are $\bar{X} = n^{-1}(\sum_{i=1}^n X_i)$ and $S^2 = n^{-1}(\sum_{i=1}^n (X_i - \bar{X})^2)$. Here we observe that S^2 is not weakly biased, however $h_1(u)$ and $h_2(u)$ defined below exist. Let $Y_j = (X_j - \mu)/\sigma$, $j = 1, \cdots, n$, $0 \leqq u \leqq 1$ and

$$\phi(y) = (2\pi)^{-\frac{1}{2}}e^{-y^2/2}, \qquad \Phi(y) = \textstyle\int_{-\infty}^{y} \phi(x)\, dx, \qquad J(u) = \{y : u = \Phi(y)\}.$$

Then

$$\begin{aligned} h_1(u) &= \lim_{n\to\infty} E(Z_n(u)n^{\frac{1}{2}}\bar{Y}) = \lim_{n\to\infty} [nuE\{\bar{Y} \mid Y_1 < J(u)\} - nuE(\bar{Y})] \\ &= -\phi(J(u)) \\ h_2(u) &= \lim_{n\to\infty} E(Z_n(u)n^{\frac{1}{2}}(S^2 - 1)) = \lim_{n\to\infty} E(S^2 \mid Y_1 < J(u)) = -J(u)\phi(J(u)). \end{aligned}$$

In this special case $\sigma_1^2 = 1$, $\sigma_2^2 = 2$, $\sigma_{12} = 0$

$$\psi_1(u) = -\phi(J(u)), \qquad \psi_2(u) = -2^{-\frac{1}{2}}J(u)\phi(J(u))$$

and

$$\rho(u, v, \boldsymbol{\theta}) = \min(u, v) - uv - \phi(J(u))\phi(J(v)) - \tfrac{1}{2}(J(u)J(v))\phi(J(u))\phi(J(v)).$$

This result was derived in [6] by quite a different method.

4. l-sample Cramér–Smirnov test in parametric case. Let X_{ji} $(i = 1, 2, \cdots, n_j;$ $j = 1, 2, \cdots, l)$ be independent random variables with continuous distribution function $G_j(x)$. For every $\boldsymbol{\theta} \in I$ an open interval in R^k let $F(x, \boldsymbol{\theta})$ be an absolutely continuous distribution function. For testing $H_l : G_1(x) = G_2(x) = \cdots = G_l(x) = F(x, \boldsymbol{\theta})$, when the functional form of F is known but $\boldsymbol{\theta}$ is unknown, a natural analogue of C_n^2 defined by (1.3) is given by

$$C_{l,n}^2 = \textstyle\int_{-\infty}^{+\infty} \sum_{j=1}^{l} n_j[F_{n_j}^{(j)}(x) - F(x, \hat{\boldsymbol{\theta}}_N)]^2\, dF(x, \hat{\boldsymbol{\theta}}_N), \tag{4.1}$$

where $N = \sum_{i=1}^{l} n_i$, $n = (n_1, n_2, \cdots, n_l)$, $F_{n_j}^{(j)}(x)$ is the sample distribution function of the jth sample, $\hat{\boldsymbol{\theta}}_N = (\hat{\theta}_{1N}, \cdots, \hat{\theta}_{kN})$ is an estimate of $\boldsymbol{\theta}$ obtained by pooling all the samples.

In this case, under the conditions of Lemma 3.1 and 3.2, it can be shown that if $(\hat{\theta}_{1N}, \cdots, \hat{\theta}_{kN})$ are regular unbiased and efficient estimates for $(\theta_1, \cdots, \theta_k)$ and when $N \to \infty$, $n_j \to \infty$ for each $j = 1, 2, \cdots, l$ such that $\lim_{N\to\infty} n_j/N = a_j$ exists, then the limiting distribution of $C_{l,n}^2$ is the distribution of C_l^2, where

$$C_l^2 = \sum_{j=1}^l \int_0^1 Y_j^2(u)\, du \,, \tag{4.2}$$

$Y_j(u)$ is a Gaussian process with mean zero and covariance function $\rho_j(u, v, \boldsymbol{\theta})$ given by

$$\rho_j(u, v, \boldsymbol{\theta}) = \min(u, v) - uv - a_j \sum_{i,h=1}^k \sigma_{ih} g_i(u) g_h(v) \,. \tag{4.3}$$

$Y_j(u)$ $(j = 1, 2, \cdots, l)$ are mutually independent, $\Sigma = (\sigma_{ij})$ denotes the covariance matrix of the joint limiting distribution of $(N^{\frac{1}{2}}(\hat{\theta}_1 - \theta_1), \cdots, N^{\frac{1}{2}}(\hat{\theta}_k - \theta_k))$ and $g_i(u)$ are defined by (3.6). If $M_j(t)$ denotes the characteristic function of $\int_0^1 Y_j^2(u)\, du$, then the characteristic function $M(t)$ of C_l^2 is

$$M(t) = \prod_{j=1}^l M_j(t) = \prod_{j=1}^l [D_j(2it)]^{-\frac{1}{2}} \,, \tag{4.4}$$

$D_j(\lambda)$ denoting the F.D. associated with $\rho_j(u, v, \boldsymbol{\theta})$ defined by (4.3). $D_j(\lambda)$ can be obtained by means of Theorem 2.2. The characteristic function does depend upon a_j the proportion in which the jth population is sampled.

If $F(x, \boldsymbol{\theta})$ is completely specified and does not involve any unknown parameters the test statistic for testing H_l is defined as

$$\omega_{l,n}^2 = \int_{-\infty}^{+\infty} \sum_{j=1}^l n_j [F_{n_j}^{(j)}(x) - F(x)]^2\, dF(x) \,. \tag{4.5}$$

The limiting distribution of $\omega_{l,n}^2$ given by Kiefer (1959) does not involve a_j.

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