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Some Properties of Annulus SLE

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Abstract

An annulus SLE_κ trace tends to a single point on the target circle, and the density function of the end point satisfies some differential equation. Some martingales or local martingales are found for annulus SLE_4, SLE_8 and $\mathrm{SLE}_{8/3}$. From the local martingale for annulus SLE_4 we find a candidate of discrete lattice model that may have annulus SLE_4 as its scaling limit. The local martingale for annulus $\mathrm{SLE}_{8/3}$ is similar to those for chordal and radial $\mathrm{SLE}_{8/3}$. But it seems that annulus $\mathrm{SLE}_{8/3}$ does not satisfy the restriction property

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1 Introduction

Schramm-Loewner evolution (SLE) is a family of random growth processes invented by O. Schramm in (12) by connecting Loewner differential equation with a one-dimentional Browinian motion. SLE depend on a single parameter $\kappa \geq 0$, and behaves differently for different value of κ. Schramm conjectured that SLE(2) is the scaling limit of some loop-erased random walks (LERW) and proved his conjuecture with some additional assumptions. He also suggested that SLE(6) and SLE(8) should be the scaling limits of certain discrete lattice models.

After Schramm's paper, there were many papers working on SLE. In the series of papers (4)(5)(6), the locality property of SLE(6) was used to compute the intersection exponent of plane Brwonian motion. In (14), SLE(6) was proved to be the scaling limit of the cite percolation explorer on the triangle lattice. It was proved in (7) that SLE(2) is the scaling limit of the corresponding loop-erased random walk (LERW), and SLE(8) is the scaling limit of some uniform spanning tree (UST) Peano curve. SLE(4) was proved to be the scaling limit of the harmonic exploer in (13). SLE(8/3) satisfies restriction property, and was conjectured in (8) to be the scaling limit of some self avoiding walk (SAW). Chordal SLE(κ, ρ) processes were also invented in (8), and they satisfy one-sided restriction property. For basic properties of SLE, see (11), (3), (16), (15).

The SLE invented by O. Schramm has a chordal and a radial version. They are all defined in simply connected domains. In (17), a new version of SLE, called annulus SLE, was defined in doubly connected domains as follows.

For $p > 0$, let the annulus

$$\mathbb{A}_p = \{z \in \mathbb{C} : e^{-p} < |z| < 1\},$$

and the circle

$$\mathbf{C}_p = \{z \in \mathbb{C} : |z| = e^{-p}\}.$$

Then $\mathbb{A}_p$ is bounded by $\mathbf{C}_p$ and $\mathbf{C}_0$. Let $\xi(t)$, $0 \leq t < p$, be a real valued continuous function. For $z \in \mathbb{A}_p$, solve the annulus Loewner differential equation

$$\partial_t \varphi_t(z) = \varphi_t(z) \mathbf{S}_{p-t}(\varphi_t(z)/\exp(i\xi(t))), \ \ 0 \leq t < p, \ \ \varphi_0(z) = z, \tag{1}$$

where for $r > 0$,

$$\mathbf{S}_r(z) = \lim_{N \to \infty} \sum_{k=-N}^{N} \frac{e^{2kr} + z}{e^{2kr} - z}.$$

For $0 \leq t < p$, let K_t be the set of $z \in \mathbb{A}_p$ such that the solution $\varphi_s(z)$ blows up before or at time t. Then for each $0 \leq t < p$, φ_t maps $\mathbb{A}_p \setminus K_t$ conformally onto $\mathbb{A}_{p-t}$, and maps $\mathbf{C}_p$ onto $\mathbf{C}_{p-t}$. We call K_t and φ_t, respectively, $0 \leq t < p$, the annulus LE hulls and maps, respectively, of modulus p, driven by $\xi(t)$, $0 \leq t < p$. If $(\xi(t)) = \sqrt{\kappa} B(t)$, $0 \leq t < p$, where $\kappa \geq 0$ and $B(t)$ is a standard linear Brownian motion, then K_t and φ_t, $0 \leq t < p$, are called standard annulus SLE$_\kappa$ hulls and maps, respectively, of modulus p. Suppose D is a doubly connected domain with finite modulus p, a is a boundary point and C is a boundary component of D that does not contain a. Then there is f that maps $\mathbb{A}_p$ conformally onto D such that $f(1) = a$ and $f(\mathbf{C}_p) = C$. Let K_t, $0 \leq t < p$, be standard annulus SLE$_\kappa$ hulls. Then $(f(K_t), 0 \leq t < p)$ is called an annulus SLE$_\kappa(D; a \to C)$ chain.

It is known in (17) that annulus SLE_κ is weakly equivalent to radial SLE_κ. so from the existence of radial SLE_κ trace, we know the existence of a standard annulus SLE_κ trace, which is $\beta(t) = \varphi_t^{-1}(\exp(i\xi(t)))$, $0 \le t < p$. Almost surely β is a continuous curve in $\overline{\mathbb{A}_p}$, and for each $t \in [0, p)$, K_t is the hull generated by $\beta((0, t])$, i.e., the complement of the component of $\mathbb{A}_p \setminus \beta((0, t])$ whose boundary contains $\mathbf{C}_p$. It is known that when $\kappa = 2$ or $\kappa = 6$, $\lim_{t\to p} \beta(t)$ exists and lies on $\mathbf{C}_p$ almost surely. In this paper, we prove that this is true for any $\kappa > 0$. And we discuss the density function of the distribution of the limit point. The density function should satisfy some differential equation.

When $\kappa = 2$, $8/3$, 4, 6, or 8, radial and chordal SLE_κ satisfy some special properties. Radial SLE_6 satisfies locality property. Since annulus SLE_6 is (strongly) equivalent to radial SLE_6, so annulus SLE_6 also satisfies the locality property. Annulus SLE_2 is the scaling limit of the corresponding loop-erased random walk. In this paper, we discuss the cases $\kappa = 4$, 8, and $8/3$. We find martingales or local martingales for annulus SLE_κ in each of these cases. From the local martingale for annulus SLE_4, we may construct a harmonic explorer whose scaling limit is annulus SLE_4. The martingales for annulus $\mathrm{SLE}_{8/3}$ are similar to the martingales for radial and chordal $\mathrm{SLE}_{8/3}$, which are used to show that radial and chordal $\mathrm{SLE}_{8/3}$ satisfy the restriction property. However, the martingales for annulus $\mathrm{SLE}_{8/3}$ does not help us to prove that annulus $\mathrm{SLE}_{8/3}$ satisfies the restriction property. On the contrary, it seems that annulus $\mathrm{SLE}_{8/3}$ does not satisfy the restriction property.

2 Annulus Loewner Evolution in the Covering Space

We often lift the annulus Loewner evolution to the covering space. Let e^i denote the map $z \mapsto e^{iz}$. For $p > 0$, let $\mathbb{S}_p = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < p\}$, $\mathbb{R}_p = ip + \mathbb{R}$, and $\mathbf{H}_p(z) = \frac{1}{i}\mathbf{S}_p(e^i(z))$. Then $\mathbb{S}_p = (e^i)^{-1}(\mathbb{A}_p)$ and $\mathbb{R}_p = (e^i)^{-1}(\mathbf{C}_p)$. Solve

$$\partial_t \widetilde{\varphi}_t(z) = \mathbf{H}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t)), \;\; \widetilde{\varphi}_0(z) = z. \tag{2}$$

For $0 \le t < p$, let $\widetilde{K}_t$ be the set of $z \in \mathbb{S}_p$ such that $\widetilde{\varphi}_t(z)$ blows up before or at time t. Then for each $0 \le t < p$, $\widetilde{\varphi}_t$ maps $\mathbb{S}_p \setminus \widetilde{K}_t$ conformally onto $\mathbb{S}_{p-t}$, and maps $\mathbb{R}_p$ onto $\mathbb{R}_{p-t}$. And for any $k \in \mathbb{Z}$, $\widetilde{\varphi}_t(z + 2k\pi) = \widetilde{\varphi}_t(z) + 2k\pi$. We call $\widetilde{K}_t$ and $\widetilde{\varphi}_t$, $0 \le t < p$, the annulus LE hulls and maps, respectively, of modulus p in the covering space, driven by $\xi(t)$, $0 \le t < p$. Then we have $\widetilde{K}_t = (e^i)^{-1}(K_t)$ and $e^i \circ \widetilde{\varphi}_t = \varphi_t \circ e^i$. If $(\xi(t))_{0\le t<p}$ has the law of $(\sqrt{\kappa}B(t))_{0\le t<p}$, then $\widetilde{K}_t$ and $\widetilde{\varphi}_t$, $0 \le t < p$, are called standard annulus SLE_κ hulls and maps, respectively, of modulus p in the covering space.

It is clear that $\mathbf{H}_r$ is an odd function. It is analytic in $\mathbb{C}$ except at the set of simple poles $\{2k\pi + i2mr : k, m \in \mathbb{Z}\}$. And at each pole z_0, the principle part is $\frac{2}{z-z_0}$. For each $z \in \mathbb{C}$, $\mathbf{H}_r(z + 2\pi) = \mathbf{H}_r(z)$, and $\mathbf{H}_r(z + i2r) = \mathbf{H}_r(z) - 2i$.

Let

$$f_r(z) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z).$$

Then f_r is an odd function. It is analytic in $\mathbb{C}$ except at the set of simple poles $\{z \in \mathbb{C} : i\frac{\pi}{r}z = 2k\pi + i2m\pi^2/r, \text{for some } k, m \in \mathbb{Z}\} = \{2m\pi - i2kr : m, k \in \mathbb{Z}\}$. And at each pole z_0, the principle part is $\frac{2}{z-z_0}$. We then compute

$$f_r(z + 2\pi) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z + i2\pi^2/r) = i\frac{\pi}{r}(\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z) - 2i) = f_r(z) + 2\frac{\pi}{r};$$

$$f_r(z+i2r) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z - 2\pi) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z) = f_r(z).$$

Let $g_r(z) = f_r(z) - \mathbf{H}_r(z)$. Then g_r is an odd entire function, and satisfies

$$g_r(z+2\pi) = g_r(z) + 2\pi/r, \quad g_r(z+i2r) = g_r(z) + 2i,$$

for any $z \in \mathbb{C}$. Thus $g_r(z) = z/r$. So we have

$$\mathbf{H}_r(z) = f_r(z) - g_r(z) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z) - \frac{z}{r}. \tag{3}$$

3 Long Term Behaviors of Annulus SLE Trace

In this section we fix $\kappa > 0$ and $p > 0$. Let φ_t and K_t, $0 \le t < p$, be the annulus LE maps and hulls, respectively, of modulus p driven by $\xi(t) = \sqrt{\kappa}B(t)$, $0 \le t < p$. Let $\widetilde{\varphi}_t$ and $\widetilde{K}_t$ be the corresponding annulus LE maps and hulls in the covering space. Let $\beta(t)$ be the corresponding annulus SLE$_\kappa$ trace.

Let $Z_t(z) = \widetilde{\varphi}_t(z) - \xi(t)$. Then we have

$$dZ_t(z) = \mathbf{H}_{p-t}(Z_t(z))dt - \sqrt{\kappa}dB(t).$$

Let $W_t(z) = \frac{\pi}{p-t}Z_t(z)$. Then W_t maps $(\mathbb{S}_p \setminus \widetilde{K}_t, \mathbb{R}_p)$ conformally onto $(\mathbb{S}_\pi, \mathbb{R}_\pi)$. From Ito's formula and equation (3) we have

$$\begin{aligned} dW_t(z) &= \frac{\pi dZ_t(z)}{p-t} + \frac{\pi Z_t(z)}{(p-t)^2} = -\frac{\pi\sqrt{\kappa}dB(t)}{p-t} + \frac{\pi}{p-t}(\mathbf{H}_{p-t}(Z_t(z)) + \frac{Z_t(z)}{p-t})dt \\ &= -\frac{\pi\sqrt{\kappa}dB(t)}{p-t} + \frac{\pi}{p-t}i\frac{\pi}{p-t}\mathbf{H}_{\pi^2/(p-t)}(i\frac{\pi}{p-t}Z_t(z))dt \\ &= -\frac{\pi\sqrt{\kappa}dB(t)}{p-t} + \frac{i\pi^2}{(p-t)^2}\mathbf{H}_{\pi^2/(p-t)}(iW_t(z))dt. \end{aligned}$$

Now we change variables as follows. Let $s = u(t) = \pi^2/(p-t)$. Then $u'(t) = \pi^2/(p-t)^2$. For $\pi^2/p \le s < \infty$, let $\widehat{W}_s(z) = W_{u^{-1}(s)}(z)$. Then there is a standard one dimensional Brownian motion $(B_1(s), s \ge \pi^2/p)$ such that

$$d\widehat{W}_s(z) = \sqrt{\kappa}dB_1(s) + i\mathbf{H}_s(i\widehat{W}_s(z))ds,$$

Let $\widehat{\varphi}_s(z) = \widehat{W}_s(z) - \sqrt{\kappa}B_1(s)$. Then $\partial_s\widehat{\varphi}_s(z) = i\mathbf{H}_s(i\widehat{W}_s(z))$. Let $X_s(z) = \operatorname{Re}\widehat{W}_s(z)$. For $z \in \mathbb{R}_p$, we have $\widehat{W}_s(z), \widehat{\varphi}_s(z) \in \mathbb{R}_\pi$, so $\widehat{W}_s(z) = X_s(z) + i\pi$. Thus for $z \in \mathbb{R}_p$,

$$\partial_s\operatorname{Re}\widehat{\varphi}_s(z) = \operatorname{Re}\partial_s\widehat{\varphi}_s(z) = \operatorname{Re}\left(i\mathbf{H}_s(i(X_s(z)+i\pi))\right) = \lim_{M\to\infty}\sum_{k=-M}^{M}\frac{e^{X_s(z)} - e^{2ks}}{e^{X_s(z)} + e^{2ks}}. \tag{4}$$

Note that $\widehat{W}'_s(z) = \widehat{\varphi}'_s(z)$. So for $z \in \mathbb{R}_p$,

$$\partial_s\widehat{\varphi}'_s(z) = \sum_{k=-\infty}^{\infty}\frac{2e^{X_s(z)}e^{2ks}}{(e^{X_s(z)} + e^{2ks})^2}\widehat{\varphi}'_s(z),$$

which implies that

$$\partial_s \ln |\widehat{\varphi}_s'(z)| = \sum_{k=-\infty}^{+\infty} \frac{2e^{X_s(z)}e^{2ks}}{(e^{X_s(z)}+e^{2ks})^2}. \tag{5}$$

Lemma 3.1. *For every $z \in \mathbb{R}_p$, $X_s(z)$ is not bounded on $[\pi^2/p, \infty)$ almost surely.*

Proof. Suppose the lemma is not true. Then there is $z_0 \in \mathbb{R}_p$ and $a > 0$ such that the probability that $|X_s(z_0)| < a$ for all $s \in [\pi^2/p, \infty)$ is positive. Let X_s denote $X_s(z_0)$. Then we have

$$dX_s = \sqrt{\kappa}dB_1(s) + \left(\lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{X_s}-e^{2ks}}{e^{X_s}+e^{2ks}}\right) ds.$$

Let T_a be the first time that $|X_s| = a$. If such time does not exist, then let $T_a = \infty$. Let $f(x) = \int_{-\infty}^{x} \cosh(s/2)^{-4/\kappa} ds$. Then f maps $\mathbb{R}$ onto $(0, C(\kappa))$ for some $C(\kappa) < \infty$, and $f'(x) = \cosh(x/2)^{-4/k}$. So $f'(x)\frac{e^x-1}{e^x+1} + \frac{\kappa}{2}f''(x) = 0$. Let $U_s = f(X_s)$. Then

$$dU_s = f'(X_s)dX_s + \frac{\kappa}{2}f''(X_s)ds$$

$$= f'(X_s)\sqrt{\kappa}dB_1(s) + f'(X_s)\lim_{M\to\infty}\left(\sum_{k=-M}^{1}\frac{e^{X_s}-e^{2ks}}{e^{X_s}+e^{2ks}} + \sum_{k=1}^{M}\frac{e^{X_s}-e^{2ks}}{e^{X_s}+e^{2ks}}\right) ds$$

$$= f'(X_s)\sqrt{\kappa}dB_1(s) + f'(X_s)\sum_{k=1}^{\infty}\frac{2\sinh(X_s)}{\cosh(2ks)+\cosh(X_s)}ds.$$

Let $v(s) = \int_{\pi^2/p}^{s} f'(X_t)^2 dt$ for $\pi^2/p \le s < T_a$. Let $\widehat{T}_a = v(T_a)$. For $0 \le r < \widehat{T}_a$, let $\widehat{U}_r = U_{v^{-1}(r)}$. Then

$$d\widehat{U}_r = \sqrt{\kappa}dB_2(r) + f'(X_{v^{-1}(r)})^{-1}\sum_{k=1}^{\infty}\frac{2\sinh(X_{v^{-1}(r)})}{\cosh(2ks)+\cosh(X_{v^{-1}(r)})}dr,$$

where $B_2(r)$ is another standard one dimensional Brownian motion. And $\widehat{T}_a$ is a stopping time w.r.t. $B_2(r)$. Let

$$A(r) = f'(X_{v^{-1}(r)})^{-1}\sum_{k=1}^{\infty}\frac{2\sinh(X_{v^{-1}(r)})}{\cosh(2ks)+\cosh(X_{v^{-1}(r)})};$$

$$M(r) = \exp\left(-\int_0^r A(s)\sqrt{\kappa}dB_2(s) - \frac{\kappa}{2}\int_0^r A(s)^2 ds\right).$$

For $0 \le r < \widehat{T}_a$, $|X_{v^{-1}(r)}| < a$, so $|f'(X_{v^{-1}(r)})^{-1}| \le \cosh(a/2)^{4/\kappa}$. And

$$\left|\sum_{k=1}^{\infty}\frac{2\sinh(X_{v^{-1}(r)})}{\cosh(2ks)+\cosh(X_{v^{-1}(r)})}\right| \le \sum_{k=1}^{\infty}\frac{2\sinh(a)}{e^{2ks}/2} = \frac{4\sinh(a)}{e^{2s}-1}.$$

Thus the Nivikov's condition

$$\mathbf{E}\left[\exp\left(\frac{\kappa}{2}\int_0^{\widehat{T}_a} A(s)^2 ds\right)\right] < \infty$$

is satisfied. Let $\mathbf{P}$ denote the original measure for $B_2(r)$. Define $\mathbf{Q}$ on $\widehat{\mathcal{F}}_{\widehat{T}_a}$ such that $d\mathbf{Q}(\omega) = M_{\widehat{T}_a}(\omega)d\mathbf{P}(\omega)$. Then $(\widehat{U}_r, 0 \le r < \widetilde{T}_a)$ is a one dimensional Brownian motion started from 0 and stopped at time $\widehat{T}_a$ w.r.t. the probability law $\mathbf{Q}$. For $0 \le s < T_a$, $|X_s| \le a$, so $|f'(X_s| \ge \cosh(a/2)^{-4/\kappa}$. Thus if $T_a = \infty$, then $\widehat{T}_a = \infty$ too. From the hypothesis of the proof, $\mathbf{P}\{T_a = \infty\} > 0$, so $\mathbf{P}\{\widehat{T}_a = \infty\} > 0$. Since $(\widehat{U}_r, 0 \le r < \widehat{T}_a)$ is a one dimensional Brownian motion w.r.t. $\mathbf{Q}$, so on the event that $\widehat{T}_a = \infty$, $\mathbf{Q}\{\limsup_{r\to\infty} |\widehat{U}_r| < \infty\} = 0$. Thus $\mathbf{Q}\{\limsup_{r\to\infty} |\widehat{U}_r| = \infty\} > 0$. Since $\mathbf{P}$ and $\mathbf{Q}$ are equivalent probability measures, so $\mathbf{P}\{\limsup_{r\to\widehat{T}_a} |\widehat{U}_r| = \infty\} > 0$. Thus $\mathbf{P}\{\limsup_{s\to T_a} |U_s| = \infty\} > 0$. This contradicts the fact that for all $s \in [\pi^2/p, \infty)$, $U_s \in (0, C(\kappa))$ and $C(\kappa) < \infty$. Thus the hypothesis is wrong, and the proof is completed. $\square$

From this lemma and the definition of X_t, we know that for any $z \in \mathbb{R}_p$, $(\operatorname{Re}\widetilde{\varphi}_t(z) - \sqrt{\kappa}B(t))/(p-t)$ is not bounded on $t \in [0,p)$ a.s.. Since for any $k \in \mathbb{Z}$ and $z \in \mathbb{R}_p$, $\widetilde{\varphi}_t(z) - 2k\pi = \widetilde{\varphi}_t(z - 2k\pi)$, so $(\operatorname{Re}\widetilde{\varphi}_t(z) - 2k\pi - \sqrt{\kappa}B(t)))/(p-t) = (\operatorname{Re}\widetilde{\varphi}_t(z-2k\pi) - \sqrt{\kappa}B(t))/(p-t)$ is not bounded on $t \in [0,p)$ a.s., which implies that $X_s(z) - 2ks$ is not bounded on $s \in [\pi^2/p, \infty)$ a.s..

Lemma 3.2. *For every $z \in \mathbb{R}_p$, almost surely $\lim_{s\to\infty} X_s(z)/s$ exists and the limit is an odd integer.*

Proof. Fix $\varepsilon_0 \in (0, 1/2)$ and $z_0 \in \mathbb{R}_p$. Let X_s denote $X_s(z_0)$. There is $b > 0$ such that the probability that $|\sqrt{\kappa}B(t)| \le b + \varepsilon_0 t$ for any $t \ge 0$ is greater than $1 - \varepsilon_0$. Since $\coth(x/2) \to \pm 1$ as $x \in \mathbb{R}$ and $x \to \pm\infty$, so there is $R > 0$ such that when $\pm x \ge R$, $\pm\coth(x/2) \ge 1 - \varepsilon_0$. Let $T = R + b + 1$. If for any $s \ge 0$, $|X_s - 2ks| < T$ for some $k = k(s) \in \mathbb{Z}$, then there is $k_0 \in \mathbb{Z}$ such that $|X_s - 2k_0 s| < T$ for all $s \ge T$. From the argument after Lemma 3.1, the probability of this event is 0. Let s_0 be the first time that $|X_s - 2ks| \ge T$ for all $k \in \mathbb{Z}$. Then s_0 is finite almost surely. There is $k_0 \in \mathbb{Z}$ such that $2k_0 s_0 + T \le X_{s_0} \le 2(k_0+1)s_0 - T$. Let s_1 be the first time after s_0 such that $X_s = 2k_0 s + R$ or $X_s = 2(k_0+1)s - R$. Let $s_1 = \infty$ if such time does not exist. For $s \in [s_0, s_1)$, we have $X_s \in [2k_0 s + R, 2(k_0+1)s - R]$. Note that $(e^x - e^{2ks})/(e^x + e^{2ks}) \to \mp 1$ as $k \to \pm\infty$. So

$$\lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{X_s} - e^{2ks}}{e^{X_s} + e^{2ks}} = 2k_0 + \lim_{M\to\infty} \sum_{k=k_0-M}^{k_0+M} \frac{e^{X_s} - e^{2ks}}{e^{X_s} + e^{2ks}}$$

$$= 2k_0 + \lim_{M\to\infty} \sum_{j=-M}^{M} \frac{e^{X_s - 2k_0 s} - e^{2js}}{e^{X_s - 2k_0 s} + e^{2js}}$$

$$= 2k_0 + \coth(\frac{X_s - 2k_0 s}{2}) + \sum_{j=1}^{\infty} \frac{2\sinh(X_s - 2k_0 s)}{\cosh(2js) + \cosh(X_s - 2k_0 s)}$$

$$\ge 2k_0 + \coth(\frac{X_s - 2k_0 s}{2}) \ge 2k_0 + 1 - \varepsilon_0;$$

and

$$\lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{X_s} - e^{2ks}}{e^{X_s} + e^{2ks}} = 2(k_0+1) + \lim_{M\to\infty} \sum_{j=-M}^{M} \frac{e^{X_s - 2(k_0+1)s} - e^{2js}}{e^{X_s - 2(k_0+1)s} + e^{2js}}$$

$$= 2k_0 + 2 + \coth(\frac{X_s - 2(k_0+1)s}{2}) + \sum_{j=1}^{\infty} \frac{2\sinh(X_s - 2(k_0+1)s)}{\cosh(2js) + \cosh(X_s - 2(k_0+1)s)}$$

$$\le 2k_0 + 2 + \coth(\frac{X_s - 2(k_0+1)s}{2}) \le 2k_0 + 2 + (-1+\varepsilon_0) = 2k_0 + 1 + \varepsilon_0.$$

From equation (4), we have that for $s \in [s_0, s_1)$,

$$(2k_0 + 1 - \varepsilon_0)(s - s_0) \le \mathrm{Re}\,\widehat{\varphi}_s(z_0) - \mathrm{Re}\,\widehat{\varphi}_{s_0}(z_0) \le (2k_0 + 1 + \varepsilon_0)(s - s_0).$$

Note that $X_s = \mathrm{Re}\,\widehat{\varphi}_s(z_0) - \sqrt{\kappa}B_1(s)$, and $(\sqrt{\kappa}B_1(s) - \sqrt{\kappa}B_1(s_0), s \ge s_0)$ has the same distribution as $(\sqrt{\kappa}B(s - s_0), s \ge s_0)$. Let E_b denote the event that $|\sqrt{\kappa}B_1(s) - \sqrt{\kappa}B_1(s_0)| \le b + \varepsilon_0(s - s_0)$ for all $s \ge s_0$. Then $\mathbf{P}(E) > 1 - \varepsilon_0$. And on the event E_b, we have

$$(2k_0 + 1 - \varepsilon_0)(s - s_0) - b - \varepsilon_0(s - s_0) \le X_s - X_{s_0}$$

$$\le (2k_0 + 1 + \varepsilon_0)(s - s_0) + b + \varepsilon_0(s - s_0),$$

from which follows that

$$X_s \le X_{s_0} + (2k_0 + 1 + \varepsilon_0)(s - s_0) + b + \varepsilon_0(s - s_0)$$

$$\le 2(k_0 + 1)s_0 - T + (2k_0 + 1 + \varepsilon_0)(s - s_0) + b + \varepsilon_0(s - s_0)$$

$$= 2(k_0 + 1)s - T + b - (1 - 2\varepsilon_0)(s - s_0) \le 2(k_0 + 1)s - R - 1$$

and

$$X_s \ge X_{s_0} + (2k_0 + 1 - \varepsilon_0)(s - s_0) - b - \varepsilon_0(s - s_0)$$

$$\ge 2k_0 s_0 + T + (2k_0 + 1 - \varepsilon_0)(s - s_0) - b - \varepsilon_0(s - s_0)$$

$$= 2k_0 s + T - b + (1 - 2\varepsilon_0)(s - s_0) \ge 2k_0 s + R + 1.$$

So on the event E_b we have $s_1 = \infty$, which implies that $2k_0 s + R \le X_s \le 2(k_0 + 1)s - R$ for all $s \ge s_0$, and so $\partial_s \mathrm{Re}\,\widehat{\varphi}_s(z_0) \in (2k_0 + 1 - \varepsilon_0, 2k_0 + 1 + \varepsilon_0)$ for all $s \ge s_0$. Thus the event that

$$2k_0 + 1 - \varepsilon_0 \le \liminf_{s\to\infty} \mathrm{Re}\,\widehat{\varphi}_s(z_0)/s \le \limsup_{s\to\infty} \mathrm{Re}\,\widehat{\varphi}_s(z_0)/s \le 2k_0 + 1 + \varepsilon_0$$

has probability greater than $1 - \varepsilon_0$. Since we may choose $\varepsilon_0 > 0$ arbitrarily small, so a.s. $\lim_{s\to\infty} \mathrm{Re}\,\widehat{\varphi}_s(z_0)/s$ exists and the limit is $2k_0 + 1$ for some $k_0 \in \mathbb{Z}$. The proof is now finished by the facts that $X_s(z_0) = \mathrm{Re}\,\widehat{\varphi}_s(z_0) + \sqrt{\kappa}B_1(s)$ and $\lim_{s\to\infty} B_1(s)/s = 0$. $\square$

Let

$$m_- = \sup\{x \in \mathbb{R} : \lim_{s\to\infty} X_s(x + ip)/s \le -1\}$$

and

$$m_+ = \inf\{x \in \mathbb{R} : \lim_{s\to\infty} X_s(x + ip)/s \ge 1\}.$$

Since $X_s(x_1 + ip) < X_s(x_2 + ip)$ if $x_1 < x_2$, so we have $m_- \le m_+$. If the event that $m_- < m_+$ has a positive probability, then there is $a \in \mathbb{R}$ such that the event that $m_- < a < m_+$ has a positive probability. From the definitions, $m_- < a < m_+$ implies that $\lim_{s\to\infty} X_s(a + ip)/s \in (-1, 1)$, which is an event with probability 0 by Lemma 3.2. This contradiction shows that $m_- = m_+$ a.s.. Let $m = m_+$. For any $t \in [0, p)$, $z \in \mathbb{S}_p \setminus \widetilde{K}_t$ and $k \in \mathbb{Z}$, since $\widetilde{\varphi}_t(z + 2k\pi) = \widetilde{\varphi}_t(z) + 2k\pi$, so $Z_t(z + 2k\pi) = Z_t(z) + 2k\pi$, then we have $W_t(z + 2k\pi) = W_t(z) + 2k\pi^2/(p - t)$. Thus $X_s(z + 2k\pi) =$

$X_s(z) + 2ks$ for any $s \in [\pi^2/p, \infty)$, $z \in \mathbb{S}_p \setminus \widetilde{K}_{p-\pi^2/s}$ and $k \in \mathbb{Z}$. If $x \in (m + 2k\pi, m + 2(k+1)\pi)$ for some $k \in \mathbb{Z}$, then $x - 2k\pi > m$ and $x - 2(k+1)\pi < m$. So

$$\lim_{s\to\infty} X_s(x + ip)/s = \lim_{s\to\infty} (X_s(x - 2k\pi + ip) + 2ks)/s \geq 2k + 1$$

and

$$\lim_{s\to\infty} X_s(x + ip)/s = \lim_{s\to\infty} (X_s(x - 2(k+1)\pi + ip) + 2(k+1)s)/s \leq 2k + 1.$$

Therefore $\lim_{s\to\infty} X_s(x + ip)/s = 2k + 1$.

Let $K_p = \cup_{0\leq t<p} K_t$ and $\widetilde{K}_p = \cup_{0\leq t<p} \widetilde{K}_t$. Then $K_p = e^i(\widetilde{K}_p)$, and so $\overline{K_p} = e^i(\overline{\widetilde{K}_p})$.

Lemma 3.3. *$\overline{K_p} \cap \mathbf{C}_p = \{e^{-p+im}\}$ almost surely.*

Proof. We first show that $m + ip \in \overline{\widetilde{K}_p}$. If this is not true, then there is $a, b > 0$ such that the distance between $[m - a + ip, m + a + ip]$ and $\widetilde{K}_t$ is greater than b for all $t \in [0, p)$. From the definition of m, we have $X_s(m \pm a + ip) \to \pm\infty$ as $s \to \infty$. Thus $\operatorname{Re} \widehat{\varphi}_s(m + a + ip) - \operatorname{Re} \widehat{\varphi}_s(m - a + ip) \to \infty$ as $s \to \infty$. So there is $c(s) \in (m - a, m + a)$ such that $\widehat{\varphi}_s'(c(s) + ip) \to \infty$ as $s \to \infty$. Since $\widehat{\varphi}_s$ maps $(\mathbb{S}_p \setminus \widetilde{K}_{p-\pi^2/s}, \mathbb{R}_p)$ conformally onto $(\mathbb{S}_\pi, \mathbb{R}_\pi)$, so by Koebe's $1/4$ theorem, the distance between $c(s) + ip$ and $\widetilde{K}_{p-\pi^2/s}$ tends to 0 as $s \to \infty$. This is a contradiction. Thus $m + ip \in \overline{\widetilde{K}_p}$.

Now fix $x_1 < x_2 \in (m, m + 2\pi)$. Then $X_s(x_j + ip)/s \to 1$ as $s \to \infty$ for $j = 1, 2$. So there is s_0 such that $X_s(x_j + ip) \in (s/2, 3s/2)$ for $s \geq s_0$ and $j = 1, 2$. So if $x_0 \in [x_1, x_2]$ and $s \geq s_0$, then $X_s(x_0 + ip) \in (s/2, 3s/2)$, and so

$$\sum_{k=-\infty}^{+\infty} \frac{e^{X_s(x_0+ip)} e^{2ks}}{(e^{X_s(x_0+ip)} + e^{2ks})^2} \leq \sum_{k=-\infty}^{0} e^{2ks - X_s(x_0+ip)} + \sum_{k=1}^{+\infty} e^{X_s(x_0+ip) - 2ks}$$

$$\leq \sum_{k=-\infty}^{0} e^{2ks - s/2} + \sum_{k=1}^{+\infty} e^{3s/2 - 2ks} = \frac{2e^{-s/2}}{1 - e^{-2s}} \leq \frac{2e^{-s/2}}{1 - e^{-2\pi^2/p}}.$$

From equation (5), for all $s \geq s_0$,

$$\partial_s \ln |\widehat{\varphi}_s'(x_0 + ip)| \leq \frac{4e^{-s/2}}{1 - e^{-2\pi^2/p}},$$

which implies that

$$\ln |\widehat{\varphi}_s'(x_0 + ip)| \leq \ln |\widehat{\varphi}_{s_0}'(x_0 + ip)| + \frac{8e^{-s_0/2}}{1 - e^{-2\pi^2/p}}.$$

So there is $M < \infty$ such that $|\widehat{\varphi}_s'(x_0 + ip)| \leq M$ for all $x_0 \in [x_1, x_2]$ and $s \geq s_0$. From Koebe's $1/4$ theorem, we see that $\widetilde{K}_t$ is uniformly bounded away from $[x_1 + ip, x_2 + ip]$ for $t \in [0, p)$. Thus $[x_1 + ip, x_2 + ip] \cap \overline{\widetilde{K}_p} = \emptyset$. Since $x_1 < x_2$ are chosen arbitrarily from $(m, m + 2\pi)$, so $(m + ip, m + 2\pi + ip) \cap \overline{\widetilde{K}_p} = \emptyset$. Thus $\overline{\widetilde{K}_p} \cap [m + ip, m + 2\pi + ip) = \{m + ip\}$. Since $\mathbf{C}_p = e^i([m + ip, m + 2\pi + ip))$, so $\overline{K_p} \cap \mathbf{C}_p = \{e^i(m + ip)\} = \{e^{-p+im}\}$. $\square$

Lemma 3.4. *For every $\varepsilon \in (0, 1)$, there is $C_0 > 0$ depending on ε such that if $q \in (0, \frac{2\pi^2}{\ln(2)}]$, and L_t, $0 \leq t < q$, are standard annulus SLE_κ hulls of modulus q, then the probability that $\cup_{0\leq t<q} L_t \subset \{e^{iz} : |Re\, z| \leq C_0 q\}$ is greater than $1 - \varepsilon$.*

Proof. Let $q_0 = \frac{2\pi^2}{\ln(2)}$. Suppose $q \in (0, q_0]$. Let L_t and ψ_t, $0 \le t < q$, be the annulus LE hulls and maps of modulus q driven by $\sqrt{\kappa}B(t)$, $0 \le t < q$. Let $\widetilde{L}_t$ and $\widetilde{\psi}_t$, $0 \le t < q$, be the corresponding annulus LE hulls and maps in the covering space. There is $b = b(\varepsilon) > 0$ such that the probability that $|\sqrt{\kappa}B(t)| \le b + t/4$ for all $t \ge 0$ is greater than $1 - \varepsilon$. Let $R = \ln(64)$ and $C_0 = (R + b + 1)/\pi$. Let $s_0 = \pi^2/q$. Let $Z_t(z) = \widetilde{\psi}_t(z) - \sqrt{\kappa}B(t)$, $W_t(z) = \pi Z_t(z)/(q - t)$ for $0 \le t < q$. Let $\widehat{W}_s(z) = W_{q-\pi^2/s}(z)$ for $s_0 \le s < \infty$. Then there is another standard one dimensional Brownian motion $B_1(s)$, $s \ge s_0$, such that $\widehat{\psi}_s$ defined by $\widehat{\psi}_s(z) = \widehat{W}_s(z) + \sqrt{\kappa}B_1(s)$ satisfies

$$\partial_s \widehat{\psi}_s(z) = \lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{\widehat{W}_s(z)} + e^{2ks}}{e^{\widehat{W}_s(z)} - e^{2ks}}$$

for $s_0 \le s < \infty$. Let E_ε be the event that $|\sqrt{\kappa}B_1(s) - \sqrt{\kappa}B_1(s_0)| \le b + (s - s_0)/4$ for all $s \ge s_0$. Then $\mathbf{P}(E_\varepsilon) > 1 - \varepsilon$. Fix $z_0 \in \mathbb{S}_q$ with $C_0 q < \operatorname{Re} z_0 < 2\pi - C_0 q$. We claim that in the event E_ε, $\widetilde{\psi}_t(z_0)$ never blows up for $0 \le t < q$. If this claim is justified, then on the event E_ε, $z_0 \notin \widetilde{L}_t$ for any $0 \le t < q$ and $z_0 \in \mathbb{S}_q$ with $C_0 q < \operatorname{Re} z_0 < 2\pi - C_0 q$. So $\cup_{0\le t<q}\widetilde{L}_t$ is disjoint from $\{z \in \mathbb{C} : C_0 q < \operatorname{Re} z < 2\pi - C_0 q\}$. Since $L_t = e^i(\widetilde{L}_t)$, so $\cup_{0\le t<q} L_q$ is disjoint from $\{e^{iz} : C_0 q < \operatorname{Re} z < 2\pi - C_0 q\}$ on the event E_ε. Then we are done.

Assume the event E_ε. Let Z_t denote $Z_t(z_0)$, W_t denote $W_t(z_0)$, $\widehat{W}_s$ denote $\widehat{W}_s(z_0)$, and $\widehat{\psi}_s$ denote $\widehat{\psi}_s(z_0)$. If $\widetilde{\psi}_t(z_0)$ blows up at time $t_* < q$, then $Z_t \to 2k\pi$ for some $k \in \mathbb{Z}$ as $t \to t_*$. Then $\widehat{W}_s - 2ks \to 0$ as $s \to \pi^2/(q - t_*)$. Since $\operatorname{Re} Z_0 = \operatorname{Re} z_0 \in [C_0 q, 2\pi - C_0 q]$, so $\widehat{W}_{s_0} = W_0 \in [C_0\pi, 2s_0 - C_0\pi] \subset (R, 2s_0 - R)$, and so there is a first time $s_1 > s_0$ such that $\operatorname{Re}\widehat{W}_{s_1} \in \{R, 2s_1 - R\}$. Then for $s \in [s_0, s_1]$, we have $\operatorname{Re}\widehat{W}_s \in [R, 2s - R]$. Then

$$\begin{aligned}
|\lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{\widehat{W}_s} + e^{2ks}}{e^{\widehat{W}_s} - e^{2ks}} - 1| &\le \sum_{k=-\infty}^{0} |\frac{e^{\widehat{W}_s} + e^{2ks}}{e^{\widehat{W}_s} - e^{2ks}} - 1| + \sum_{k=1}^{\infty} |\frac{e^{\widehat{W}_s} + e^{2ks}}{e^{\widehat{W}_s} - e^{2ks}} + 1| \\
&\le \sum_{k=-\infty}^{0} \frac{2}{|e^{\widehat{W}_s - 2ks}| - 1} + \sum_{k=1}^{\infty} \frac{2}{|e^{2ks - \widehat{W}_s}| - 1} \le \sum_{k=-\infty}^{0} \frac{4}{e^{R-2ks}} + \sum_{k=1}^{\infty} \frac{4}{e^{2ks-(2s-R)}} \\
&\le \frac{8e^{-R}}{1 - e^{-2s}} \le 16e^{-R} \le \frac{1}{4},
\end{aligned}$$

where we use the fact that $e^{-R} \le \frac{1}{64}$ and $e^{-2s} \le e^{-2s_0} = e^{-2\pi^2/q} \le e^{-2\pi^2/q_0} \le \frac{1}{2}$. Thus

$$\begin{aligned}
|(\widehat{W}_{s_1} - \widehat{W}_{s_0}) - (s_1 - s_0)| &\le |(\widehat{\psi}_{s_1} - \widehat{\psi}_{s_0}) - (s_1 - s_0)| + |\sqrt{\kappa}B_1(s_1) - \sqrt{\kappa}B_1(s_0)| \\
&\le (s_1 - s_0)/4 + b + (s_1 - s_0)/4 = b + (s_1 - s_0)/2.
\end{aligned}$$

Then we have

$$\operatorname{Re}\widehat{W}_{s_1} \ge \operatorname{Re}\widehat{W}_{s_0} + (s_1 - s_0) - b - (s_1 - s_0)/2 \ge C_0\pi + (s_1 - s_0)/2 - b > R$$

and

$$\begin{aligned}
\operatorname{Re}\widehat{W}_{s_1} &\le \operatorname{Re}\widehat{W}_{s_0} + (s_1 - s_0) + b + (s_1 - s_0)/2 \le 2s_0 - C_0\pi + b + 3(s_1 - s_0)/2 \\
&= 2s_1 - (s_1 - s_0)/2 - C_0\pi + b < 2s_1 - R.
\end{aligned}$$

This contradicts that $\mathrm{Re}\,\widehat{W}_{s_1} \in \{R, 2s_1 - R\}$. Thus $\widetilde{\psi}_t(z_0)$ does not blow up for $t \in [0, q)$. Then the claim is justified, and the proof is finished. $\square$

For two nonempty sets $A_1, A_2 \subset \mathbb{A}_p$, we define the angular distance between A_1 and A_2 to be $d_a(A_1, A_2) = \inf\{|\mathrm{Re}\, z_1 - \mathrm{Re}\, z_2| : e^{iz_1} \in A_1, e^{iz_2} \in A_2\}$. For a nonempty set $A \subset \mathbb{A}_p$, we define the angular diameter of A to be $diam_a(A) = \sup\{d_a(z_1, z_2) : z_1, z_2 \in A\}$. If A intersects both A_1 and A_2, then $d_a(A_1, A_2) \le diam_a(A)$. In the above lemma, $\cup_{0 \le t < q} L_t \subset \{e^{iz} : |\mathrm{Re}\, z| \le C_0 q\}$ implies that $diam_a(\cup_{0 \le t < q} L_t) \le 2C_0 q$. Form conformal invariance and comparison principle of extremal distance, we have that for any $d > 0$, there is $h(d) > 0$ such that for any $p > 0$, if for $j = 1, 2$, A_j is a union of connected subsets of $\mathbb{A}_p$, each of which touches both $\mathbf{C}_p$ and $\mathbf{C}_0$, and the extremal distance between A_1 and A_2 in $\mathbb{A}_p$ is greater than $h(d)$, then $d_a(A_1, A_2) > dp$.

Theorem 3.1. $\lim_{t \to p} \beta(t) = e^{-p+im}$ *almost surely.*

Proof. From Lemma 3.3, the distance from e^{-p+im} to K_t tends to 0 as $t \to p$ a.s.. Since K_t is the hull generated by $\beta((0, t])$, so the distance from e^{-p+im} to $\beta((0, t])$ tends to 0 as $t \to p$ a.s.. Suppose the theorem does not hold. Then there is $a, \delta > 0$ such that the event that $\limsup_{t \to p} |e^{-p+im} - \beta(t)| > a$ has probability greater than δ. Let E_1 denote this event. Let $\varepsilon = \delta/4$. Let C_0 depending on ε be as in Lemma 3.4. Let $R = \min\{a, e^{-p}\}$ and $r = \min\{1 - e^{-p}, R \exp(-2\pi h(2C_0 + 1))\}$, where h is the function in the argument before this theorem. Since K_t is generated by $\beta((0, t])$, and $e^{-p+im} \in \overline{K_p}$ a.s., so the distance between e^{-p+im} and $\beta((0, t])$ tends to 0 a.s. as $t \to p$. So there is $t_0 \in (0, p)$ such that the event that the distance between e^{-p+im} and $\beta((0, t_0])$ is less than r has probability greater than $1 - \varepsilon$. Let E_2 denote this event. Let $q_0 = \frac{2\pi^2}{\ln(2)}$, $T = \max\{t_0, p - q_0, -\ln(r + e^{-p})\}$, $p_T = p - T$, and $\xi_T(t) = \xi(T + t) - \xi(T)$ for $0 \le t < p_T$. Let $K_{T,t} = \varphi_T(K_{T+t} \setminus K_T)/e^{i\xi(T)}$ and $\varphi_{T,t}(z) = \varphi_{T+t} \circ \varphi_T^{-1}(\exp(i\xi(T))z)/\exp(i\xi(T))$ for $0 \le t < p_T$. Then one may check that $K_{T,t}$ and $\varphi_{T,t}$, $0 \le t < p_T$, are the annulus LE hulls and maps of modulus p_T driven by ξ_T. Since $\xi_T(t)$ has the same law as $\sqrt{\kappa} B(t)$ and $p_T = p - T \le q_0$, so from Lemma 3.4, the event that $diam_a(\cup_{0 \le t < p_T} K_{T,t}) \le 2C_0 p_T$ has probability greater than $1 - \varepsilon$. Let E_3 denote this event. Since $\mathbf{P}(E_1^c) + \mathbf{P}(E_2^c) + \mathbf{P}(E_3^c) < (1 - \delta) + \varepsilon + \varepsilon < 1$, so $\mathbf{P}(E_1 \cap E_2 \cap E_3) > 0$. This means that the events E_1, E_2 and E_3 can happen at the same time. We will prove that this is a contradiction. Then the theorem is proved.

Assume the event $E_1 \cap E_2 \cap E_3$. Let A_r (A_R, resp.) be the union of connected components of $\{z \in \mathbb{C} : |z - e^{-p+im}| = r\} \cap (\mathbb{A}_p \setminus K_T)$ ($\{z \in \mathbb{C} : |z - e^{-p+im}| = R\} \cap (\mathbb{A}_p \setminus K_T)$, resp.) that touch $\mathbf{C}_p$. From the properties of β in the event E_1 and E_2, we see that A_r and A_R both intersect $K_p \setminus K_T$. Since the distance between e^{-p+im} and K_T is less than r, and $r < R$, so both A_r and A_R are unions of two curves which touch both $\mathbf{C}_p$ and $\mathbf{C}_0 \cup K_T$. Let $B_r = e^{-i\xi(T)} \varphi_T(A_r)$ and $B_R = e^{-i\xi(T)} \varphi_T(A_R)$. Then both B_r and B_R are unions of two curves in $\mathbb{A}_{p_T}$ that touch both $\mathbf{C}_{p_T}$ and $\mathbf{C}_0$.

The extremal distance between A_r and A_R in $\mathbb{A}_p \setminus K_T$ is at least $\ln(R/r)/(2\pi) \ge h(2C_0 + 1)$. Thus the extremal distance between B_r and B_R in $\mathbb{A}_{p_T}$ is at least $h(2C_0 + 1)$. So the angular distance between B_r and B_R is at least $(2C_0 + 1)p_T$. Since A_R and A_r both intersect $K_p \setminus K_T$, so B_R and B_r both intersect $\varphi_T(K_p \setminus K_T)/e^{i\xi(T)} = \cup_{0 \le t < p_T} K_{T,t}$, which implies that $diam_a(\cup_{0 \le t < p_T} K_{T,t}) \ge (2C_0 + 1)p_T$. However, in the event E_3, $diam_a(\cup_{0 \le t < p_T} K_{T,t}) \le 2C_0 p_T$. This contradiction finishes the proof. $\square$

Let's see what can we say about the distribution of $\lim_{t \to p} \beta(t)$. Let $\widetilde{\beta}(t) = \widetilde{\varphi}_t^{-1}(\xi(t))$. Then $\widetilde{\beta}$ is a simple curve in $\mathbb{S}_p$ started from 0, and $\beta(t) = e^i(\widetilde{\beta}(t))$. From Theorem 3.1, $\lim_{t\to p}\widetilde{\beta}(t)$ exists and lies on $\mathbb{R}_p$. We call $\widetilde{\beta}$ an annulus SLE_κ trace in the covering space. Let $m_p + ip$ denote the limit point, where m_p is a real valued random variable.

Suppose the distribution of m_p is absolutely continuous w.r.t. the Lebesgue measure, and the density function $\widetilde{\lambda}(p,x)$ is $C^{1,2}$ continuous. This hypothesis is very likely to be true, but the proof is still missing now. We then have $\int_{\mathbb{R}}\widetilde{\lambda}(p,x)dx = 1$ for any $p > 0$. Since the distribution of $\widetilde{\beta}$ is symmetric w.r.t. the imaginary axis, so is the distribution of $\lim_{t\to p}\widetilde{\beta}(t)$. Thus $\widetilde{\lambda}(p,-x) = \widetilde{\lambda}(p,x)$. Moreover, we expect that when $p \to 0$ the distribution of $(m_p + ip) * \frac{\pi}{p}$ tends to the distribution of the limit point of a strip SLE_κ trace introduced in (18), whose density is $\cosh(x/2)^{-4/\kappa}/C(\kappa)$ for some $C(\kappa) > 0$. If this is true, then the distribution of m_p tends to the point mass at 0 as $p \to 0$.

For $0 \le t < p$, let $\mathcal{F}_t$ be the $\sigma-$algebra generated by $\xi(s)$, $0 \le s \le t$. Fix $T \in [0,p)$. Let $p_T = p - T$. For $0 \le t < p_T$, let $\xi_T(t) = \xi(T+t) - \xi(T)$. Then $\xi_T(t)$ has the same distribution as $\sqrt{\kappa}B(t)$, and is independent of $\mathcal{F}_T$. For $0 \le t < T$, let

$$\widetilde{\varphi}_{T,t}(z) = \widetilde{\varphi}_{T+t} \circ \widetilde{\varphi}_T^{-1}(z + \xi(T)) - \xi(T).$$

Then $\partial_t\widetilde{\varphi}_{T,t}(z) = \mathbf{H}_{p_T - t}(\widetilde{\varphi}_{T,}(z) - \xi_T(t))$, and $\widetilde{\varphi}_{T,0}(z) = z$. Thus $\widetilde{\varphi}_{T,t}(z)$, $0 \le t < p_T$, are annulus LE maps of modulus p_T in the covering space driven by $\xi_T(t)$, $0 \le t < p_T$, and so are independent of $\mathcal{F}_T$. Let

$$\widetilde{\beta}_T(t) = \widetilde{\varphi}_{T,t}^{-1}(\xi_T(t)) = \widetilde{\varphi}_T \circ \widetilde{\varphi}_{T+t}^{-1}(\xi(T_t)) - \xi(T) = \widetilde{\varphi}_T(\widetilde{\beta}(T+t)) - \xi(T), \tag{6}$$

for $0 \le t < p_T$. Then $\widetilde{\beta}_T(t)$, $0 \le t < p_T$, is a standard annulus SLE_κ trace of modulus p_T in the covering space, and is independent of $\mathcal{F}_T$. Thus $\lim_{t\to p_T}\beta_T(t)$ exists and lies on $\mathbb{R}_{p_T}$ a.s.. Let $m_{p_T} + ip_T$ denote the limit point. Then m_{p_T} is independent of $\mathcal{F}_T$, and the density of m_{p_T} w.r.t. the Lebesgue measure is $\widetilde{\lambda}(p_T,\cdot)$. From equation (6), we see $m_{p_T} = \widetilde{\varphi}_T(m_p + ip) - ip_T - \xi(T)$. For $0 \le t < p$, let $\widetilde{\psi}_t(z) = \widetilde{\varphi}_t(z + ip) - i(p-t)$. Then $\widetilde{\psi}_t$ takes real values on $\mathbb{R}$, and $\partial_t\widetilde{\psi}_t(z) = \widehat{\mathbf{H}}_{p-t}(\widetilde{\psi}_t(z) - \xi(t))$, where $\widehat{\mathbf{H}}_r(z) = \mathbf{H}(z + ir) + i$. Let $X_t(z) = \widetilde{\psi}_t(z) - \xi(t)$ for $0 \le t < p_T$. So $m_{p_T} = X_T(m_p)$. From the differential equation for $\widetilde{\psi}_t$, we get

$$dX_t(x) = \widehat{\mathbf{H}}_{p-t}(X_t(x))dt - d\xi(t);$$

and

$$dX_t'(x) = \widehat{\mathbf{H}}_{p-t}'(X_t(x))X_t'(x)dt.$$

Let $a < b \in \mathbb{R}$. Then $\{m_p \in [a,b]\} = \{m_{p_T} \in [X_T(a), X_T(b)]\}$. Since m_{p_T} has density $\widetilde{\lambda}(p_T,\cdot)$ and is independent of $\mathcal{F}_T$, and X_T is $\mathcal{F}_T$ measurable, so

$$\mathbf{E}\left[\mathbf{1}_{\{m_p\in[a,b]\}}|\mathcal{F}_T\right] = \int_{X_T(a)}^{X_T(b)}\widetilde{\lambda}(p-T,x)dx = \int_a^b\widetilde{\lambda}(p-T,X_T(x))X_T'(x)dx.$$

Thus $(\int_a^b\widetilde{\lambda}(p-t,X_t(x))X_t'(x)dx, 0 \le t < p)$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t=0}^p$. Fix $x \in \mathbb{R}$. Choose $a < x < b$ and let $a,b \to x$. Then $(\widetilde{\lambda}(p-t,X_t(x))X_t'(x), 0 \le t < p)$ is a martingale w.r.t. $\{\mathcal{F}_t\}_{t=0}^p$. From Ito's formula, we have

$$-\partial_1\widetilde{\lambda}(r,x) + \widehat{\mathbf{H}}_r'(x)\widetilde{\lambda}(r,x) + \widehat{\mathbf{H}}_r(x)\partial_2\widetilde{\lambda}(r,x) + \frac{\kappa}{2}\partial_2^2\widetilde{\lambda}(r,x) = 0, \tag{7}$$

where ∂_1 and ∂_2 are partial derivatives w.r.t. the first and second variable, respectively.

Let $\widetilde{\Lambda}(p,x) = \int_0^x \widetilde{\lambda}(p,s)ds$ for $p > 0$ and $x \in \mathbb{R}$. Then for any $p > 0$, $\widetilde{\Lambda}(p,\cdot)$ is an odd and increasing function, $\lim_{x\to\pm\infty} \widetilde{\Lambda}(p,x) = \pm\frac{1}{2}$, and $\widetilde{\lambda}(p,x) = \partial_2\widetilde{\Lambda}(p,x)$. Thus for any $r > 0$ and $x \in \mathbb{R}$,

$$\partial_2(-\partial_1\widetilde{\Lambda}(r,x) + \widehat{\mathbf{H}}_r(x)\partial_2\widetilde{\Lambda}(r,x) + \frac{\kappa}{2}\partial_2^2\widetilde{\Lambda}(r,x)) = 0.$$

Since $\widetilde{\Lambda}(r,\cdot)$ is an odd function and $\widehat{\mathbf{H}}_r(0) = 0$, so

$$-\partial_1\widetilde{\Lambda}(r,0) + \widehat{\mathbf{H}}_r(0)\partial_2\widetilde{\Lambda}(r,0) + \frac{\kappa}{2}\partial_2^2\widetilde{\Lambda}(r,0) = 0.$$

Thus for any $r > 0$ and $x \in \mathbb{R}$, we have

$$-\partial_1\widetilde{\Lambda}(r,x) + \widehat{\mathbf{H}}_r(x)\partial_2\widetilde{\Lambda}(r,x) + \frac{\kappa}{2}\partial_2^2\widetilde{\Lambda}(r,x) = 0. \tag{8}$$

And we expect that for any $x \in \mathbb{R}\setminus\{0\}$, $\lim_{r\to 0}\widetilde{\Lambda}(r,x) \to \text{sign}\frac{1}{2}$. On the other hand, if $\widetilde{\Lambda}(r,x)$ satisfies (8), then $\widetilde{\lambda}(r,x) := \partial_2\widetilde{\Lambda}(r,x)$ satisfies (7).

Let $\lambda(r,x) = \sum_{k\in\mathbb{Z}} \widetilde{\lambda}(r, x + 2k\pi)$. Then $\lambda(r,\cdot)$ has a period 2π, and is the density function of the distribution of the argument of $\lim_{t\to r}\beta(t)$, where β is a standard annulus SLE_κ trace of modulus r. So it satisfies $\int_{-\pi}^{\pi}\lambda(r,x)dx = 1$. And $\lambda(r,\cdot)$ is an even function for any $r > 0$. Since $\widehat{\mathbf{H}}_r$ has a period 2π, so $\lambda(r,x)$ also satisfies equation (7). Let $\Lambda(r,x) = \int_0^x \lambda(r,s)ds$. Then $\Lambda(r,x)$ satisfies (8). But $\Lambda(r,x)$ does not satisfies $\lim_{x\to\pm\infty}\Lambda(r,x) = \pm 1$. Instead, we have $\Lambda(r, x + 2\pi) = \Lambda(r,x) + 1$. In the case that $\kappa = 2$, we have some nontrivial solutions to (8). From Lemma 3.1 in (17), we see $-\partial_r\mathbf{H}_r + \mathbf{H}_r\mathbf{H}_r' + \mathbf{H}_r'' = 0$, where the function $\widetilde{\mathbf{S}}_r$ in (17) is the function $\mathbf{H}_r$ here. From the definition of $\widehat{\mathbf{H}}_r$, we may compute that $-\partial_r\widehat{\mathbf{H}}_r + \widehat{\mathbf{H}}_r\widehat{\mathbf{H}}_r' + \widehat{\mathbf{H}}_r'' = 0$. Thus $\Lambda_1(r,x) = \widehat{\mathbf{H}}_r(x)$ and $\Lambda_2(r,x) = r\mathbf{H}_r(x) + x$ satisfy equation (8). So $\lambda_1(r,x) = \widehat{\mathbf{H}}_r'(x)$ and $\lambda_2(r,x) = r\mathbf{H}_r'(x) + 1$ are solutions to (7). In fact, $\lambda_2(r,x)/(2\pi)$ is the distribution of the argument of the end point of a Brownian Excursion in $\mathbb{A}_r$ started from 1 conditioned to hit $\mathbf{C}_r$. From Corollary 3.1 in (17), this is also the distribution of the argument of the limit point of a standard annulus SLE_2 trace of modulus r. So we justified equation (7) in the case $\kappa = 2$.

We may change variables in the following way. For $-\infty < s < 0$, let $\widetilde{\mathrm{P}}(s,y) = \widetilde{\Lambda}(-\frac{\pi^2}{s}, -\frac{\pi}{s}y)$ and $\mathrm{P}(s,y) = \Lambda(-\frac{\pi^2}{s}, -\frac{\pi}{s}y)$. Then for any $s < 0$, $\lim_{y\to\pm\infty}\widetilde{\mathrm{P}}(s,y) = \pm\frac{1}{2}$ and $\mathrm{P}(s, y+2s) = \mathrm{P}(s,y) - 1$. And we expect that $\lim_{s\to-\infty}\widetilde{\mathrm{P}}(s,y) = \int_0^y \cosh(\frac{s}{2})^{-4/\kappa}ds/C(\kappa)$. Let $\mathbf{G}_s(y) = i\mathbf{H}_{-s}(iy - \pi)$ for $s < 0$ and $y \in \mathbb{R}$. From formula (3), we may compute that $\widetilde{\Lambda}(r,x)$ ($\Lambda(r,x)$, resp.) satisfies equation (8) iff $\widetilde{\mathrm{P}}(s,y)$ ($\mathrm{P}(s,y)$, resp.) satisfies

$$-\partial_1\widetilde{\mathrm{P}}(s,y) + \mathbf{G}_s(y)\partial_2\widetilde{\mathrm{P}}(s,y) + \frac{\kappa}{2}\partial_2^2\widetilde{\mathrm{P}}(s,y) = 0. \tag{9}$$

From the equation for $\mathbf{H}_r$ and the definition of $\mathbf{G}_s$, we have $-\partial_s\mathbf{G}_s + \mathbf{G}_s\mathbf{G}_s' + \mathbf{G}_s'' = 0$. Thus $\mathrm{P}_1(s,y) = \mathbf{G}_s(y)$ and $\mathrm{P}_2(s,y) = s\mathbf{G}_s(y) + y$ are solutions to (9). In fact, $\mathrm{P}_1(s,y)$ corresponds to $-\Lambda_2(r,x)/\pi$, and $\mathrm{P}_2(s,y)$ corresponds to $-\pi\Lambda_1(r,x)$.

4 Local Martingales for Annulus SLE$_4$ and SLE$_8$

4.1 Annulus SLE$_4$

Fix $\kappa = 4$. Let K_t and φ_t, $0 \le t < p$, be the annulus LE hulls and maps of modulus p, respectively, driven by $\xi(t) = \sqrt{\kappa}B(t)$. Let $\beta(t)$, $0 \le t < p$, be the trace. For $r > 0$, let $\mathbf{T}_r^{(2)}(z) = \frac{1}{2}\mathbf{S}_r(z^2)$ and $\widetilde{\mathbf{T}}_r^{(2)}(z) = \frac{1}{i}\mathbf{T}_r^{(2)}(e^{iz})$. Solve the differential equations:

$$\partial_t\psi_t(z) = \psi_t(z)\mathbf{T}_{p-t}^{(2)}(\psi_t(z)/e^{i\xi(t)/2}),\ \ \psi_0(z) = z;$$

$$\partial_t\widetilde{\psi}_t(z) = \widetilde{\mathbf{T}}_{p-t}^{(2)}(\widetilde{\psi}_t(z) - \xi(t)/2),\ \ \widetilde{\psi}_0(z) = z.$$

Let P_2 be the square map: $z \mapsto z^2$. Then we have $P_2 \circ \psi_t = \varphi_t \circ P_2$ and $e^i \circ \widetilde{\psi}_t = \psi_t \circ e^i$. Let $L_t := P_2^{-1}(K_t)$ and $\widetilde{L}_t = (e^i)^{-1}(L_t)$. Then ψ_t maps $\mathbb{A}_{p/2} \setminus L_t$ conformally onto $\mathbb{A}_{(p-t)/2}$, and $\widetilde{\psi}_t$ maps $\mathbb{S}_{p/2} \setminus \widetilde{L}_t$ conformally onto $\mathbb{S}_{(p-t)/2}$. Since $K_t = \beta(0,t]$, and β is a simple curve in $\mathbb{A}_p$ with $\beta(0) = 1$, so L_t is the union of two disjoint simple curves opposite to each other, started from 1 and -1, respectively. Let $\alpha_\pm(t)$ denote the curve started from ± 1. Then $\psi_t(\alpha_\pm(t)) = e^i(\pm\xi(t)/2)$.

For each $r > 0$, suppose J_r is the conformal map from $\mathbb{A}_{r/2}$ onto $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < 1\} \setminus [-a_r, a_r]$ for some $a_r > 0$ such that ± 1 is mapped to $\pm\infty$. This J_r is symmetric w.r.t. both x-axis and y-axis, i.e., $J_r(\overline{z}) = \overline{J_r(z)}$, and $J_r(-z) = -J_r(z)$. And $\mathrm{Im}\, J_r$ is the unique bounded harmonic function in $\mathbb{A}_{r/2}$ that satisfies (i) $\mathrm{Im}\, J_r \equiv \pm 1$ on the open arc of $\mathbf{C}_0$ from ± 1 to ∓ 1 in the ccw direction; and (ii) $\mathrm{Im}\, J_r \equiv 0$ on $\mathbf{C}_{r/2}$. Let $\widetilde{J}_r = J_r \circ e^i$.

Lemma 4.1. $-\partial_r\widetilde{J}_r + \widetilde{J}_r'\widetilde{\mathbf{T}}_r^{(2)} + \frac{1}{2}\widetilde{J}_r'' \equiv 0$ *in* $\widetilde{\mathbb{A}}_{r/2}$.

Proof. Since $\mathrm{Im}\, \widetilde{J}_r \equiv 0$ on $\mathbb{R}_{r/2}$, by reflection principle, $\widetilde{J}_r$ can be extended analytically across $\mathbb{R}_{r/2}$. And we have $\mathrm{Im}\, \widetilde{J}_r' = \partial_x \mathrm{Im}\, \widetilde{J}_r \equiv 0$ and $\mathrm{Im}\, \widetilde{J}_r'' = \partial_x^2 \mathrm{Im}\, \widetilde{J}_r \equiv 0$ on $\mathbb{R}_{r/2}$. From the equality $\mathrm{Im}\, \widetilde{J}_r(x + ir/2) = 0$, we have $\partial_r \mathrm{Im}\, \widetilde{J}_r + \partial_y \mathrm{Im}\, \widetilde{J}_r/2 \equiv 0$ on $\mathbb{R}_{r/2}$. On $\mathbb{R}_{r/2}$, note that $\mathrm{Im}\, \widetilde{\mathbf{T}}_r^{(2)} \equiv -1/2$, so

$$\begin{aligned}\mathrm{Im}\,(\widetilde{J}_r'\widetilde{\mathbf{T}}_r^{(2)}) &= \mathrm{Re}\, \widetilde{J}_r' \mathrm{Im}\, \widetilde{\mathbf{T}}_r^{(2)} + \mathrm{Im}\, \widetilde{J}_r' \mathrm{Re}\, \widetilde{\mathbf{T}}_r^{(2)} \\ &= -1/2 \mathrm{Re}\, \widetilde{J}_r' = -1/2 \partial_y \mathrm{Im}\, \widetilde{J}_r = \partial_r \mathrm{Im}\, \widetilde{J}_r.\end{aligned}$$

Let $F_r := -\partial_r\widetilde{J}_r + \widetilde{J}_r'\widetilde{\mathbf{T}}_r^{(2)} + \frac{1}{2}\widetilde{J}_r''$. Then $\mathrm{Im}\, F_r \equiv 0$ on $\mathbb{R}_{r/2}$.

For any $k \in \mathbb{Z}$, we see that $\widetilde{J}_r(z)$ is equal to $(-1)^{k+1}\frac{2}{\pi}\ln(z - k\pi)$ plus some analytic function for $z \in \widetilde{\mathbb{A}}_{r/2}$ near $k\pi$. So we may extend $\mathrm{Re}\, \widetilde{J}_r(z)$ harmonically across $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$. Since $\mathrm{Im}\, \widetilde{J}_r$ takes constant value $(-1)^k$ on each interval $(k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$, we have $\mathrm{Re}\, \widetilde{J}_r(\overline{z}) = \mathrm{Re}\, \widetilde{J}_r(z)$. Moreover, the following properties hold: $\partial_r\widetilde{J}_r$ is analytic in a neighborhood of $\mathbb{R}$, $\widetilde{J}_r'$ and $\widetilde{J}_r''$ are analytic in a neighborhood of $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$.

The fact that $\mathrm{Im}\, \widetilde{J}_r$ takes constant value $(-1)^k$ on each $(k\pi, (k+1)\pi)$, $k \in \mathbb{Z}$, implies that $\mathrm{Im}\, \partial_r\widetilde{J}_r$, $\mathrm{Im}\, \widetilde{J}_r'$ and $\mathrm{Im}\, \widetilde{J}_r''$ vanishes on $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$. Since $\mathrm{Im}\, \widetilde{\mathbf{T}}_r^{(2)}$ also vanishes on $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$, so we compute $\mathrm{Im}\, F_r \equiv 0$ on $\mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$.

From $J_r(\overline{z}) = \overline{J_r(z)}$, we find that $\widetilde{J}_r(-\overline{z}) = \overline{\widetilde{J}_r(z)}$. So $\mathrm{Re}\,\widetilde{J}_r(z) = \mathrm{Re}\,\widetilde{J}_r(-\overline{z}) = \mathrm{Re}\,\widetilde{J}_r(-z)$. This means that $\mathrm{Re}\,\widetilde{J}_r$ is an even function, so is $\partial_r \widetilde{J}_r$ and $\widetilde{J}_r''$. And $\widetilde{J}_r'$ is an odd function. Note that $\widetilde{\mathbf{T}}_r^{(2)}$ is an odd function, so F_r is an even function. Since $\widetilde{\mathbf{T}}_r^{(2)}(z)$ is equal to $1/(2z)$ plus some analytic function for z near 0, so the pole of F_r at 0 has order at most 2. However, the coefficient of $1/z^2$ is equal to $2/\pi * 1/2 - 1/2 * 2/\pi = 0$. And 0 is not a simple pole of F_r because F_r is even. So 0 is a removable pole of F_r. Similarly, π is also a removable pole of F_r. Since F_r has period 2π, so every $k\pi$, $k \in \mathbb{Z}$, is a removable pole of F_r. So F_r can be extended analytically across $\mathbb{R}$, and $\mathrm{Im}\,F_r \equiv 0$ on $\mathbb{R}$. Thus $\mathrm{Im}\,F_r \equiv 0$ in $\mathbb{S}_{r/2}$, which implies that $F_r \equiv C$ for some constant $C \in \mathbb{R}$.

Finally, $J_r(-z) = -J_r(z)$ implies that $\widetilde{J}_r(z+\pi) = -\widetilde{J}_r(z)$. Since π is a period of $\widetilde{\mathbf{T}}_r^{(2)}$, we compute $F_r(z+\pi) = -F_r(z)$. So C has to be 0. $\square$

Proposition 4.1. *For any $z \in \mathbb{A}_{p/2}$, $J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$ is a local martingale, from which follows that $\mathrm{Im}\,J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$ is a bounded martingale.*

Proof. Fix $z_0 \in \mathbb{S}_{p/2}$, let $Z_t := \widetilde{\psi}_t(z_0) - \xi(t)/2$, then

$$dZ_t = \widetilde{\mathbf{T}}_{p-t}^{(2)}(Z_t)dt - d\xi(t)/2.$$

Note that $\xi(t)/2 = B(t)$. From Ito's formula and the last lemma, we have

$$d\widetilde{J}_{p-t}(Z_t) = -\partial_r \widetilde{J}_{p-t}(Z_t)dt + \widetilde{J}_{p-t}'(Z_t)dZ_t + \frac{1}{2}\widetilde{J}_{p-t}''(Z_t)dt$$

$$= (-\partial_r \widetilde{J}_{p-t}(Z_t) + \widetilde{J}_{p-t}(Z_t)\widetilde{\mathbf{T}}_{p-t}^{(2)}(Z_t) + \frac{1}{2}\widetilde{J}_{p-t}''(Z_t))dt - \widetilde{J}_{p-t}'(Z_t)d\xi(t)/2 = -\widetilde{J}_{p-t}'(Z_t)d\xi(t)/2.$$

Thus $\widetilde{J}_{p-t}(Z_t)$, $0 \le t < p$, is a local martingale. For any $z \in \mathbb{A}_{p/2}$, there is $z_0 \in \mathbb{S}_{p/2}$ such that $z = e^i(z_0)$. Then

$$J_{p-t}(\psi_t(z)/e^{i\xi(t)/2}) = J_{p-t}(\psi_t(e^i(z_0))/e^{i\xi(t)/2})$$

$$= J_{p-t}(e^i(\widetilde{\psi}_t(z_0) - \xi(t)/2)) = \widetilde{J}_{p-t}(\widetilde{\psi}_t(z_0) - \xi(t)/2.$$

So $J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$, $0 \le t < p$, is a local martingale. Since $|\mathrm{Im}\,J_r(z)| \le 1$ for any $r > 0$ and $z \in \mathbb{A}_{r/2}$, so $\mathrm{Im}\,J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$, $0 \le t < p$, is a bounded martingale. $\square$

Let $h_t(z) = J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$. Then h_t maps $\mathbb{A}_{(p-t)/2} \setminus L_t$ conformally onto $\{z \in \mathbb{C} : |\mathrm{Im}\,z| < 1\} \setminus [-a_{p-t}, a_{p-t}]$ so that $\alpha_\pm(t)$ is mapped to $\pm\infty$. So $\mathrm{Im}\,h_t$ is the unique bounded harmonic function in $\mathbb{A}_{p/2} \setminus L_t$ that vanishes on $\mathbf{C}_{p/2}$, equals to 1 on the arc of $\mathbf{C}_0$ from 1 to -1 in the ccw direction and the north side of $\alpha_+(0,t)$ and $\alpha_-(0,t)$, and equals to -1 on the arc of $\mathbf{C}_0$ from -1 to 1 in the ccw direction and the south side of $\alpha_+(0,t)$ and $\alpha_-(0,t)$.

We have another choice of J_r. Let J_r be the conformal map of $\mathbb{A}_{r/2}$ onto the strip $\{z \in \mathbb{C} : |\mathrm{Im}\,z| < 1\} \setminus [-ib_r, ib_r]$ for some $b_r > 0$ so that ± 1 is mapped to $\pm\infty$. Then $\mathrm{Im}\,J_r$ is the bounded harmonic function in $\mathbb{A}_{r/2}$ determined by the following properties: (i) $\mathrm{Im}\,J_r \equiv \pm 1$ on the open arc of $\mathbf{C}_0$ from ± 1 to ∓ 1 in the ccw direction; and (ii) the normal derivative of $\mathrm{Im}\,J_r$ vanishes on $\mathbf{C}_{r/2}$. Let $\widetilde{J}_r = J_r \circ e^i$. The proposition and lemma in this subsection still hold for J_r and $\widetilde{J}_r$ defined here. The proofs are almost the same. The only difference is at the step when we prove $\mathrm{Im}\,F_r \equiv 0$ on $\mathbb{R}_{r/2}$. Here we have $\mathrm{Re}\,\widetilde{J}_r' = \partial_y \mathrm{Im}\,\widetilde{J}_r$ vanishes on $\mathbb{R}_{r/2}$. Use an argument similar to the proof of the lemma, we can show that $\operatorname{Re} F_r'$ vanishes on $\mathbb{R}_{r/2}$. So $\partial_y \operatorname{Im} F_r$ vanishes on $\mathbb{R}_{r/2}$. Since $\operatorname{Im} F_r$ vanishes on $\mathbb{R}$, $\operatorname{Im} F_r$ has to vanish in $\mathbb{S}_{r/2}$. Use $J_r(-z) = -J_r(z)$, we then conclude that $F_r \equiv 0$. If we let $h_t(z) = J_{p-t}(\psi_t(z)/e^{i\xi(t)/2})$, then $\operatorname{Im} h_t$ is the unique bounded harmonic function in $\mathbb{A}_{p/2} \setminus L_t$ that satisfies the following properties: equals to 1 on the arc of $\mathbf{C}_0$ from 1 to -1 in the ccw direction, and the north side of $\alpha_+(0,t)$ and $\alpha_-(0,t)$, equals to -1 on the arc of $\mathbf{C}_0$ from -1 to 1 in the ccw direction, and the south side of $\alpha_+(0,t)$ and $\alpha_-(0,t)$, and the normal derivatives vanish on $\mathbf{C}_{p/2}$.

Suppose E is a doubly connected domain such that 0 lies in the bounded component of $\mathbb{C} \setminus E$. Fix $v \in \partial_o E$, the outside boundary component of E. Let $\beta(t)$, $0 \le t < p$, be an $\mathrm{SLE}_4(E; v \to \partial_i E)$ trace, where $\partial_i E$ is the inside boundary component of E. So $\beta(t)$ is the image of a standard annulus SLE_4 trace of modulus p under the conformal map from $(\mathbb{A}_p, 1)$ onto (E, v), where p is the modulus of E. Let $D = P_2^{-1}(E)$ and $\{v_+, v_-\} = P_2^{-1}(v)$. $P_2^{-1}(\{\beta(t) : 0 \le t < p\})$ is the union of two disjoint simple curve started from v_+ and v_-, respectively. Let $\alpha_\pm$ denote the curve started from $v_\pm$. Then D is a symmetric ($-D = D$) doubly connected domain, and $\alpha_-(t) = -\alpha_+(t)$ for $0 \le t < p$. Let $D_t = D \setminus \alpha_-([0,t]) \setminus \alpha_+([0,t])$. Let $\gamma_t^\pm$ denote the boundary arc of $\partial_o D_t$ from $\alpha_\pm(t)$ to $\alpha_\mp(t)$ in the ccw direction. Then $\gamma_t^\pm$ contains a boundary arc of $\partial_o D$, one side of $\alpha_+([0,t])$ and one side of $\alpha_-([0,t])$. Let H_t be the bounded harmonic function in D_t which has continuations at $\partial_i D$ and $\gamma_t^\pm$ such that $H_t \equiv 0$ on $\partial_i D$ and $H_t \equiv \pm 1$ on $\gamma_t^\pm$. By the definition of $\mathrm{SLE}_4(E; v \to \partial_i E)$ and conformal invariance of harmonic functions, for any fixed $z_0 \in D$, $H_t(z_0)$, $0 \le t < p$, is a bounded martingale. This H_t corresponds to $\operatorname{Im} h_t$ defined right after Proposition 4.1. We may replace the condition $H_t \equiv 0$ on $\partial_i D$ by $\partial_{\mathbf{n}} H_t \equiv 0$ on $\partial_i D$. Then this H_t corresponds to the $\operatorname{Im} h_t$ defined in the last paragraph. So for any fixed $z_0 \in D$, it is still true that $H_t(z_0)$, $0 \le t < p$, is a bounded martingale.

4.2 Harmonic Explorers for Annulus SLE_4

Let D be a symmetric ($-D = D$) doubly connected subset of hexagonal faces in the planar honeycomb lattice. Two faces of D are considered adjacent if they share an edge. Let $\partial_o D$ and $\partial_i D$ denote the outside and inside component of ∂D, respectively. Suppose v_+ and v_- are vertices that lie on $\partial_o D$, and are opposite to each other, i.e., $v_- = -v_+$. Suppose $\operatorname{Re} v_+ > 0$. Then v_+ and v_- partition the boundary faces of D near $\partial_o D$ into an "upper" boundary component, colored black, and a "lower" boundary component, colored white. All other hexagons in D are uncolored.

Now we construct two curves α_+ and α_- as follows. Let $\alpha_\pm(0) = v_\pm$. Let $\alpha_\pm(1)$ be a neighbor vertex of $\alpha_\pm(0)$ such that $[\alpha_\pm(0), \alpha_\pm(1)]$ is shared by a white hexagon and a black hexagon. At time $n \in \mathbb{N}$, if $\alpha_\pm(n) \notin \partial_i D$, then $\alpha_\pm(n)$ is a vertex shared by a black hexagon, a white hexagon, and an uncolored hexagon, denoted by $f_\pm^n$. Let H_n be the function defined on faces, which takes value 1 on the black faces, -1 on the white faces, 0 on faces that touch $\partial_i D$, and is discrete harmonic at other faces of D. Then $H_n(f_-^n) = -H_n(f_+^n)$. We then color $f_\pm$ black with probability equal to $(1 + H_n(f_\pm^n))/2$ and white with probability equal to $(1 - H_n(f_\pm^n))/2$ such that f_+^n and f_-^n are colored differently. Let $\alpha_\pm(n+1)$ be the unique neighbor vertex of $\alpha_\pm(n)$ such that $[\alpha_\pm(n), \alpha_\pm(n+1)]$ is shared by a white hexagon and a black hexagon. Increase n by 1, and iterate the above process until α_+ and α_- hit $\partial_i D$ at the same time. We always have $\alpha_-(n) = -\alpha_+(n)$, $f_-^n = -f_+^n$, and $H_n(-g) = -H_n(g)$.

From the construction, conditioned on $\alpha_\pm(k)$, $k = 0, 1, \ldots, n$, the expected value of $H_{n+1}(f_\pm^n)$

is equal to $(1+H_n(f^n_\pm))/2-(1-H_n(f^n_\pm))/2=H_n(f^n_\pm)$. And if a face f is colored before time n, then its color will not be changed after time n, so $H_{n+1}(f)=H_n(f)$. Since H_{n+1} and H_n both vanish on the faces near $\partial_i D$, and are discrete harmonic at all other uncolored faces at time $n+1$ and n, resp., so for any face f of D, the conditional value of $H_{n+1}(f)$ w.r.t. $\alpha_\pm(k)$, $k=0,1,\dots,n$ is equal to $H_n(f)$. Thus for any fixed face f_0 of D, $H_n(f_0)$ is a martingale.

If $n-1<t<n$, and $\alpha_\pm(n-1)$ and $\alpha_\pm(n)$ are defined, let $\alpha_\pm(t)=(n-t)\alpha_\pm(n-1)+(t-(n-1))\alpha_\pm(n)$. Then $\alpha_\pm$ becomes a curve in D. Let $D_t=D\setminus\alpha_+([0,t])\setminus\alpha_-([0,t])$. Note that if the side length of the hexagons is very small compared with the size of D, then for any face f of D, $H_n(f)$ is close to the value of $\widetilde H_n$ at the center of f, where $\widetilde H_n$ is the bounded harmonic function defined on D_n, which has a continuation to $\partial D\setminus\{v_+,v_-\}$ and the two sides of $\alpha_\pm([0,t))$ such that $\widetilde H_n\equiv 0$ on $\partial_i D$, and $\widetilde H_n\equiv\pm1$ on the curve on $\partial_o D_n$ from $\alpha_\pm(n)$ to $\alpha_\mp(n)$ in the ccw direction. From the last section, we may guess that the distribution of $\alpha_\pm$ tends to that of the square root of an annulus $\mathrm{SLE}_4(P_2(D);P_2(v_\pm)\to\partial_i P_2(D))$ trace when the mesh tends to 0. If at each step of the construction of $\alpha_\pm$, we let H_n be the function which is is equal to 1 on the black faces, -1 on the white faces, and is discrete harmonic at all other faces of D including the faces that touch $\partial_i D$, then we get a different pair of curves $\alpha_\pm$. If the mesh is very small compared with the size of D, then for any face f of D, $H_n(f)$ is close to the value of $\widetilde H_n$ at the center of f, where $\widetilde H_n$ is the bounded harmonic function defined on D_n, which has a continuation to $\partial D\setminus\{v_+,v_-\}$ and the two sides of $\alpha_\pm([0,t))$ such that $\partial_{\mathbf n}\widetilde H_n\equiv 0$ on $\partial_i D$, and $\widetilde H_n\equiv\pm1$ on the curve on $\partial_o D_n$ from $\alpha_\pm(n)$ to $\alpha_\mp(n)$ in the ccw direction. So we also expect the law of $\alpha_\pm$ constructed in this way tends to that of the square root of an annulus $\mathrm{SLE}_4(P_2(D);P_2(v_\pm)\to\partial_i P_2(D))$ trace when the mesh tends to 0.

4.3 Annulus SLE_8

Fix $\kappa=8$. Let K_t and φ_t, $0\le t<p$, be the annulus LE hulls and maps, respectively, of modulus p, driven by $\xi(t)=\sqrt{\kappa}B(t)$. For $r>0$, let $\mathbf{T}^{(4)}_r(z)=\frac14\mathbf{S}_r(z^4)$ and $\widetilde{\mathbf{T}}^{(4)}_r(z)=\frac1i\mathbf{T}^{(4)}_r(e^{iz})$. Solve the differential equations:

$$\partial_t\psi_t(z)=\psi_t(z)\mathbf{T}^{(4)}_{p-t}(\psi_t(z)/e^{i\xi(t)/4}),\ \ \psi_0(z)=z;$$

$$\partial_t\widetilde\psi_t(z)=\widetilde{\mathbf{T}}^{(4)}_{p-t}(\widetilde\psi_t(z)-\xi(t)/4),\ \ \widetilde\psi_0(z)=z.$$

Let P_4 be the map: $z\mapsto z^4$. Then we have $P_4\circ\psi_t=\varphi_t\circ P_4$ and $e^i\circ\widetilde\psi_t=\psi_t\circ e^i$. Let $L_t:=P_4^{-1}(K_t)$ and $\widetilde L_t=(e^i)^{-1}(L_t)$. Then ψ_t maps $\mathbb{A}_{p/4}\setminus L_t$ conformally onto $\mathbb{A}_{(p-t)/4}$, and $\widetilde\psi_t$ maps $\mathbb{S}_{p/4}\setminus\widetilde L_t$ conformally onto $\mathbb{S}_{(p-t)/4}$. Let G_r map $\mathbb{A}_{r/4}$ conformally onto $\{z\in\mathbb{C}:|\mathrm{Re}\,z|+|\mathrm{Im}\,z|<1\}\setminus[-a_r,a_r]$ for some $a_r>0$ such that ±1 and $\pm i$ are fixed.

Proposition 4.2. *For any $z\in\mathbb{A}_{r/4}$, $G_{p-t}(\psi_t(z)/e^{i\xi_t/4})$ is a bounded martingale.*

Proof. Let $\widetilde G_r:=G_r\circ e^i$. For any $z\in\mathbb{A}_{p/4}$, there is $w\in\mathbb{S}_{p/4}$ such that $z=e^i(w)$. Then

$$G_{p-t}(\psi_t(z)/e^{i\xi(t)/4})=G_{p-t}(\psi_t(e^{iw})/e^{i\xi(t)/4})=\widetilde G_{p-t}(\widetilde\psi_t(w)-\xi(t)/4).$$

To prove this proposition, it suffices to show that for any $w\in\mathbb{S}_{p/4}$, $\widetilde G_{p-t}(\widetilde\psi_t(w)-\xi(t)/4)$ is a local martingale. Let $Z_t=\widetilde\psi_t(w)-\xi(t)/4$, then

$$dZ_t=\widetilde{\mathbf{T}}^{(4)}_{p-t}(Z_t)dt-dB(t)/\sqrt2.$$

Thus by Ito's formula,

$$d\widetilde{G}_{p-t}(\psi_t(w)-\xi(t)/4) = -\partial_r\widetilde{G}_{p-t}(Z_t)dt + \widetilde{G}'_{p-t}(Z_t)dZ_t + \frac{1}{2}\widetilde{G}''_{p-t}(Z_t)\frac{dt}{2}$$

$$= (-\partial_r\widetilde{G}_{p-t}(Z_t) + \widetilde{G}'_{p-t}(Z_t)\widetilde{\mathbf{T}}^{(4)}_{p-t}(Z_t) + \frac{1}{4}\widetilde{G}''_{p-t}(Z_t))dt - \widetilde{G}'_{p-t}(Z_t)dB(t)/\sqrt{2}.$$

So it suffices to prove the following lemma.

Lemma 4.2. $-\partial_r\widetilde{G}_r + \widetilde{G}'_r\widetilde{\mathbf{T}}^{(4)}_r + \frac{1}{4}\widetilde{G}''_r \equiv 0$ *in* $\mathbb{S}_{r/4}$.

Proof. Let F_r be the left-hand side. Let $Q_r(z) := i(\widetilde{G}_r(z)-1)^2$. Note that $\widetilde{G}_r$ maps $[0,\pi/2]$ and $[-\pi/2,0]$ onto the line segments $[1,i]$ and $[-i,1]$, respectively. Thus $Q_r(z)\to\mathbb{R}$ as $z\in\mathbb{S}_{r/4}$ and $z\to(-\pi/2,\pi/2)$. By reflection principle, Q_r can be extended to an analytic function in a neighborhood of $(-\pi/2,\pi/2)$, and $Q_r(\overline{z}) = \overline{Q_r(z)}$. Since $G_r(\overline{z}) = \overline{G_r(z)}$, so $\widetilde{G}_r(-\overline{z}) = \overline{\widetilde{G}_r(z)}$. It follows that $Q_r(-\overline{z}) = -\overline{Q_r(z)}$. So we have $Q_r(-z) = -Q_r(z)$, and the Taylor expansion of Q_r at 0 is $\sum_{n=0}^{\infty} a_{n,r}z^{2n+1}$. Thus $\widetilde{G}_r(z) = 1+\sum_{n=0}^{\infty} c_{n,r}z^{2n+1/2}$ for z near 0. So $\partial_r\widetilde{G}_r(z) = O(z^{1/2})$ for z near 0, $\widetilde{G}'_r(z) = 1/2c_{1,r}z^{-1/2}+O(z^{3/2})$, and $\widetilde{G}''_r(z) = -1/4c_{1,r}z^{-3/2}+O(z^{1/2})$. Since $\widetilde{\mathbf{T}}^{(4)}_r(z) = 1/(8z)+O(z)$ near 0, so $\widetilde{G}'_r(z)\widetilde{\mathbf{T}}^{(4)}_r(z) = 1/16c_{1,r}z^{-3/2}+O(z^{1/2})$. Then we compute $F_r(z) = O(z^{1/2})$ near 0. Similarly, $F_r(z) = O((z-k\pi/2)^{1/2})$ for z near $k\pi/2$, $k\in\mathbb{Z}$.

For $z\in(k\pi,(k+1/2)\pi)$, $k\in\mathbb{Z}$, $\widetilde{G}_r(z)\in(-1)^k+(1-i)\mathbb{R}$. So $F_r(z)\in(1-i)\mathbb{R}$ for $z\in(k\pi,(k+1/2)\pi)$, $k\in\mathbb{Z}$. Similarly, $F_r(z)\in(1+i)\mathbb{R}$ for $z\in((k-1/2)\pi,k\pi)$, $k\in\mathbb{Z}$. Since $\widetilde{G}_r$ takes real values on $\mathbb{R}_{r/4}$, so F_r also takes real values on $\mathbb{R}_{r/4}$. Let $V_r = \operatorname{Im} F_r$, then $V_r\equiv 0$ on $\mathbb{R}_{r/4}$, and for $k\in\mathbb{Z}$, $\partial_x V+\partial_y V\equiv 0$ on $(k\pi,(k+1/2)\pi)$ and $\partial_x V-\partial_y V\equiv 0$ on $((k-1/2)\pi,k\pi)$, $k\in\mathbb{Z}$. And $V_r(z)\to 0$ as $z\in\mathbb{S}_{r/4}$ and $z\to k\pi/2$, $k\in\mathbb{Z}$. Since $\widetilde{G}_r$ and $\widetilde{\mathbf{T}}^{(4)}_r$ have period 2π, so does F_r. Thus $|V_r|$ attains its maximum in $\overline{\mathbb{S}_{r/4}}$ at some $z_0\in\mathbb{R}\cup\mathbb{R}_{r/4}$. If $z_0\in\mathbb{R}_{r/4}$ or $z_0 = k\pi/2$ for some $k\in\mathbb{Z}$, then $V_r(z_0) = 0$, and so V_r vanishes in $\mathbb{S}_{r/4}$. Otherwise, either $z_0\in(k\pi,(k+1/2)\pi)$ or $z_0\in((k-1/2)\pi,k\pi)$ for some $k\in\mathbb{Z}$. In either cases, we have $\partial_x V_r(z_0) = 0$, so $\partial_y V_r(z_0) = 0$ too. Thus $F'_r(z_0) = 0$. If F_r is not constant in $\mathbb{S}_{r/4}$, then $F_r(z_0) = 1+a_m(z-z_0)^m+O((z-z_0)^{m+1}$ for z near z_0. Then it is impossible that $\operatorname{Im} F_r(z_0)\ge\operatorname{Im} F_r(z)$ for all $z\in\{|z-z_0|<\varepsilon,\operatorname{Im} z\ge\operatorname{Im} z_0\}$ or $\operatorname{Im} F_r(z_0)\le\operatorname{Im} F_r(z)$ for all $z\in\{|z-z_0|<\varepsilon,\operatorname{Im} z\ge\operatorname{Im} z_0\}$. This contradiction shows that F_r has to be constant in $\mathbb{S}_{r/4}$. Since $F_r(z)\to 0$ as $z\to 0$, so this constant is 0. We again conclude that V_r has to vanish in $\mathbb{S}_{r/4}$. $\square$

5 Annulus $\mathrm{SLE}_{8/3}$ and the Restriction Property

In this section, we fix $\kappa = 8/3$ and $\alpha = 5/8$. Let φ_t and K_t, $0\le t<p$, be the annulus LE maps and hulls of modulus p, driven by $\xi(t) = \sqrt{\kappa}B(t)$, $0\le t<p$. Let $\widetilde{\varphi}_t$ and $\widetilde{K}_t$, $0\le t<p$, be the corresponding annulus LE maps and hulls in the covering space. Let $A\ne\emptyset$ be a hull in $\mathbb{A}_p$ w.r.t. $\mathbf{C}_p$ (i.e., $\mathbb{A}_p\setminus A$ is a doubly connected domain whose one boundary component is $\mathbf{C}_p$) such that $1\notin\overline{A}$. So there is $t>0$ such that $K_t\cap A=\emptyset$. Let T_A be the biggest $T\in(0,p]$ such that for $t\in[0,T)$, $K_t\cap A=\emptyset$. Let φ_A be the conformal map from $\mathbb{A}_p\setminus A$ onto $\mathbb{A}_{p_0}$ such that $\varphi_A(1)=1$, where p_0 is equal to the modulus of $\mathbb{A}_p\setminus A$. Let $K'_t = \varphi_A(K_t)$, $0\le t<T_A$. Let $h(t)$ equal p_0 minus the modulus of $\mathbb{A}_{p_0}\setminus K'_t$. Then h is a continuous increasing function with $h(0)=0$. So h maps $[0,T_A)$ onto $[0,S_A)$ for some $S_A\in(0,p_0]$. From Proposition 2.1 in

(17), $L_s = K_{h^{-1}(s)}$, $0 \le s < S_A$, are the annulus LE hulls of modulus p_0, driven by some real continuous function, say $\eta(s)$. Let ψ_s, $0 \le s < S_A$, be the corresponding annulus LE maps. Let $\widetilde{\psi}_s$ and $\widetilde{L}_s$, $0 \le s < S_A$, be the annulus LE maps and hulls, respectively, in the covering space.

Let $f_t = \psi_{h(t)} \circ \varphi_A \circ \varphi_t^{-1}$ and $A_t = \varphi_t(A)$. Then for $0 \le t < T_A$, $e^i(\xi(t)) \notin \overline{A_t}$, and f_t maps $(\mathbb{A}_{p-t} \setminus A_t, \mathbf{C}_{p-t})$ conformally onto $(\mathbb{A}_{p_0-h(t)}, \mathbf{C}_{p_0-h(t)})$. And for any $z_0 \in \mathbf{C}_0 \setminus \overline{A_t}$, if $z \in \mathbb{A}_{p-t} \setminus A_t$ and $z \to z_0$, then $f(z) \to \mathbf{C}_0$. Thus f_t can be extended analytically across $\mathbf{C}_0$ near $e^i(\xi(t))$. A proof similar to those of Lemma 2.1 and 2.2 in (17) shows that $f_t(e^i(\xi(t))) = e^i(\eta(h(t)))$, and $h'(t) = |f_t'(e^i(\xi(t)))|^2$.

Let $\widetilde{\varphi}_A$ be such that $e^i \circ \widetilde{\varphi}_A = \varphi_A \circ e^i$ and $\widetilde{\varphi}_A(0) = 0$. Let $\widetilde{f}_t = \widetilde{\psi}_{h(t)} \circ \widetilde{\varphi}_A \circ \widetilde{\varphi}_t^{-1}$. Then $e^i \circ \widetilde{f}_t = f_t \circ e^i$, and so $e^i \circ \widetilde{f}_t(\xi(t)) = e^i(\eta(h(t)))$. Thus $\widetilde{f}_t(\xi(t)) = \eta(h(t)) + 2k\pi$ for some $k \in \mathbb{Z}$. Now we replace $\eta(s)$ by $\eta(s) + 2k\pi$. Then $\eta(s)$, $0 \le s < S_A$, is still a driving function of L_s, $0 \le s < S_A$. And we have $\widetilde{f}_t(\xi(t)) = \eta(h(t))$. Moreover, we have $h'(t) = \widetilde{f}_t'(\xi(t))^2$.

Let $\widetilde{A} = (e^i)^{-1}(A)$ and $\widetilde{A}_t = (e^i)^{-1}(A_t)$. For any $t \in [0, T_A)$, and $z \in \mathbb{S}_p \setminus \widetilde{A} \setminus \widetilde{K}_t$, we have $\widetilde{f}_t \circ \widetilde{\varphi}_t(z) = \widetilde{\psi}_{h(t)} \circ \widetilde{\varphi}_A(z)$. Taking the derivative w.r.t. t, we compute

$$\partial_t \widetilde{f}_t(\widetilde{\varphi}_t(z)) + \widetilde{f}_t'(\widetilde{\varphi}_t(z))\mathbf{H}_{p-t}(\widetilde{\varphi}_t(z) - \xi(t)) = \widetilde{f}_t'(\xi(t))^2 \mathbf{H}_{p_0-h(t)}(\widetilde{f}_t(\varphi_t(z)) - \widetilde{f}_t(\xi(t))).$$

Since $\widetilde{A}_t = \widetilde{\varphi}_t(A)$ for $0 \le t < T_A$, so for any $t \in [0, T_A)$, and $w \in \mathbb{S}_{p-t} \setminus \widetilde{A}_t$, we have $\widetilde{\varphi}_t^{-1}(w) \in \mathbb{S}_p \setminus \widetilde{A} \setminus \widetilde{K}_t$. Thus

$$\partial_t \widetilde{f}_t(w) = \widetilde{f}_t'(\xi(t))^2 \mathbf{H}_{p_0-h(t)}(\widetilde{f}_t(w) - \widetilde{f}_t(\xi(t))) - \widetilde{f}_t'(w)\mathbf{H}_{p-t}(w - \xi(t)). \tag{10}$$

Recall that

$$\mathbf{H}_r(z) = -i \lim_{M\to\infty} \sum_{k=-M}^{M} \frac{e^{2kr} + e^{iz}}{e^{2kr} - e^{iz}} = \cot(z/2) + \sum_{k=1}^{\infty} -i\left(\frac{e^{2kr} + e^{iz}}{e^{2kr} - e^{iz}} + \frac{e^{-2kr} + e^{iz}}{e^{-2kr} - e^{iz}}\right)$$

$$= \cot(z/2) + \sum_{k=1}^{\infty} \frac{2\sin(z)}{\cosh(2kr) - \cos(z)}.$$

Let

$$S_r = \sum_{k=1}^{\infty} \frac{2}{\cosh(2kr) - 1} = \sum_{k=1}^{\infty} \frac{1}{\cosh^2(kr)}.$$

Then the Laurent sires expansion of $\mathbf{H}_r$ at 0 is $\mathbf{H}_r(z) = 2/z + (S_r - 1/6)z + O(z^2)$.

Apply the following power series expansions:

$$\mathbf{H}_r(z) = 2/z + O(z);$$

$$\widetilde{f}_t'(w) = \widetilde{f}_t'(\xi(t)) + \widetilde{f}_t''(\xi(t))(w - \xi(t)) + O((w - \xi(t))^2);$$

$$\widetilde{f}_t(w) - \widetilde{f}_t(\xi(t)) = \widetilde{f}_t'(\xi(t))(w - \xi(t)) + \frac{\widetilde{f}_t''(\xi(t))}{2}(w - \xi(t))^2 + O((w - \xi(t))^3).$$

After some straightforward computation and letting $w \to \xi(t)$, we get

Lemma 5.1. $\partial_t \widetilde{f}_t(\xi(t)) = -3\widetilde{f}_t''(\xi(t))$.

Now differentiate equation (10) with respect to w. We get

$$\partial_t \widetilde{f}_t'(w) = \widetilde{f}_t'(\xi(t))^2 \widetilde{f}_t'(w)\mathbf{H}'_{p_0-h(t)}(\widetilde{f}_t(w) - \widetilde{f}_t(\xi(t)))$$

$$-\widetilde{f}_t''(w)\mathbf{H}_{p-t}(w-\xi(t)) - \widetilde{f}_t'(w)\mathbf{H}'_{p-t}(w-\xi(t)).$$

Apply the previous power series expansions and the following expansions:

$$\mathbf{H}_r(z) = 2/z + (S_r - 1/6)z + O(z^2);$$

$$\mathbf{H}'_r(z) = -2/z^2 + (S_r - 1/6) + O(z);$$

$$\widetilde{f}_t''(w) = \widetilde{f}_t''(\xi(t)) + \widetilde{f}_t'''(\xi(t))(w-\xi(t)) + O((w-\xi(t))^2);$$

$$\widetilde{f}_t'(w) = \widetilde{f}_t'(\xi(t)) + \widetilde{f}_t''(\xi(t))(w-\xi(t)) + \frac{\widetilde{f}_t'''(\xi(t))}{2}(w-\xi(t))^2 + O((w-\xi(t))^3);$$

$$\widetilde{f}_t(w) - \widetilde{f}_t(\xi(t)) = \widetilde{f}_t'(\xi(t))(w-\xi(t)) + \frac{\widetilde{f}_t''(\xi(t))}{2}(w-\xi(t))^2$$

$$+\frac{\widetilde{f}_t'''(\xi(t))}{6}(w-\xi(t))^3 + O((w-\xi(t))^4).$$

After some long but straightforward computation and letting $w \to \xi(t)$, we get

Lemma 5.2.

$$\frac{\partial_t \widetilde{f}_t'(\xi(t))}{\widetilde{f}_t'(\xi(t))} = \frac{1}{2}\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right)^2 - \frac{4}{3}\frac{\widetilde{f}_t'''(\xi(t))}{\widetilde{f}_t'(\xi(t))}$$

$$+\widetilde{f}_t'(\xi(t))^2(S_{p_0-h(t)} - 1/6) - (S_{p-t} - 1/6).$$

From Ito's formula and the above lemma, we have

$$d\widetilde{f}_t'(\xi(t)) = \partial_t \widetilde{f}_t'(\xi(t))dt + \widetilde{f}_t''(\xi(t))d\xi(t) + \frac{\kappa}{2}\widetilde{f}_t'''(\xi(t))dt$$

$$= \widetilde{f}_t''(\xi(t))d\xi(t) + \widetilde{f}_t'(\xi(t))\left(\frac{1}{2}\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right)^2 + \left(\frac{\kappa}{2} - \frac{4}{3}\right)\frac{\widetilde{f}_t'''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right.$$

$$\left. + \widetilde{f}_t'(\xi(t))^2(S_{p_0-h(t)} - 1/6) - (S_{p-t} - 1/6)\right)dt.$$

Thus

$$d\widetilde{f}_t'(\xi(t))^\alpha = \alpha\widetilde{f}_t'(\xi(t))^{\alpha-1}d\widetilde{f}_t'(\xi(t)) + \alpha(\alpha-1)\widetilde{f}_t'(\xi(t))^{\alpha-2}\frac{\kappa}{2}\widetilde{f}_t''(\xi(t))^2dt$$

$$= \alpha\widetilde{f}_t'(\xi(t))^\alpha\left(\frac{d\widetilde{f}_t'(\xi(t))}{\widetilde{f}_t'(\xi(t))} + (\alpha-1)\frac{\kappa}{2}\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right)^2 dt\right)$$

$$= \alpha\widetilde{f}_t'(\xi(t))^\alpha\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}d\xi(t) + \left(\left(\frac{1}{2} + (\alpha-1)\frac{\kappa}{2}\right)\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}\right)^2\right.\right.$$

$$+\left(\frac{\kappa}{2}-\frac{4}{3}\right)\frac{\widetilde{f}_t'''(\xi(t))}{\widetilde{f}_t'(\xi(t))}+\widetilde{f}_t'(\xi(t))^2\left(S_{p_0-h(t)}-\frac{1}{6}\right)-\left(S_{p-t}-\frac{1}{6}\right)\right)dt\right)$$

$$=\alpha\widetilde{f}_t'(\xi(t))^\alpha\left(\frac{\widetilde{f}_t''(\xi(t))}{\widetilde{f}_t'(\xi(t))}d\xi(t)+\left(h'(t)\left(S_{p_0-h(t)}-\frac{1}{6}\right)-\left(S_{p-t}-\frac{1}{6}\right)\right)dt\right).$$

The last equality uses $\kappa=8/3$, $\alpha=5/8$, and $h'(t)=\widetilde{f}_t'(\xi(t))^2$.

Now we have the following theorem.

Theorem 5.1.

$$M_t=\widetilde{f}_t'(\xi(t))^{5/8}\exp\left(-\frac{5}{8}\int_{p_0-h(t)}^{p-t}\left(S_r-\frac{1}{6}\right)dr\right),$$

$0\le t<T_A$, *is a bounded martingale.*

Proof. From the above computation and Ito's formula, we see tht M_t, $0\le t<T_A$, is a local martingale.

Since f_t maps $\mathbb{A}_{p-t}\setminus A_t$ conformally onto $\mathbb{A}_{p_0-h(t)}$, so by the comparison principle of extremal length, the modulus of $\mathbb{A}_{p_0-h(t)}$ is not bigger than that of $\mathbb{A}_{p-t}$. Thus $p_0-h(t)\le p-t$. Since $S_r>0$ for any $r>0$, so

$$\exp\left(-\frac{5}{8}\int_{p_0-h(t)}^{p-t}\left(S_r-\frac{1}{6}\right)dr\right)\le\exp\left(\frac{5}{48}((p-t)-(p_0-h(t))\right)\le\exp\left(\frac{5p}{48}\right).\tag{11}$$

Let $g_t=f_t^{-1}$, $\widetilde{g}_t=\widetilde{f}_t^{-1}$. Then $g_t\circ e^i=e^i\circ\widetilde{g}_t$. And g_t maps $\mathbb{A}_{p_0-h(t)}$ conformally onto $\mathbb{A}_{p-t}\setminus A_t$. Now $-\ln(g_t(z)/z)$ is a bounded analytic function defined in $\mathbb{A}_{p_0-h(t)}$, $\mathrm{Re}\,(-\ln(g_t(z)/z))\to(p-t)-(p_0-h(t))$ as $z\to\mathbf{C}_{p_0-h(t)}$, and any subsequential limit of $\mathrm{Re}\,(-\ln(g_t(z)/z))$ as $z\to\mathbf{C}_0$ is nonnegative. Thus there are some $C\in\mathbb{R}$ and a positive measure μ_t supported by $\mathbf{C}_0$ of total mass $(p-t)-(p_0-h(t))$ such that for any $z\in\mathbb{A}_{p_0-h(t)}$,

$$-\ln(g_t(z)/z)=\int_{\mathbf{C}_0}\mathbf{S}_{p_0-h(t)}(z/\theta)d\mu_t(\theta)+iC.\tag{12}$$

For any $w\in\mathbb{S}_{p_0-h(t)}$, we have $e^i(w)\in\mathbb{A}_{p_0-h(t)}$, $\ln(g_t(e^i(w)))=i\widetilde{g}_t(w)$, and $\ln(z)=iw$, so

$$-i(\widetilde{g}_t(w)-w)=\int_{\mathbf{C}_0}\mathbf{S}_{p_0-h(t)}(e^i(w)/\theta)d\mu_t(\theta)+iC.$$

If $\widetilde{\mu}_t$ is a measure on $\mathbb{R}$ that satisfies $\mu_t=\widetilde{\mu}_t\circ(e^i)^{-1}$, then for any $w\in\mathbb{S}_{p_0-h(t)}$,

$$\widetilde{g}_t(w)-w=\int_{\mathbb{R}}i\mathbf{S}_{p_0-h(t)}\circ e^i(w-x)d\widetilde{\mu}_t(x)-C=\int_{\mathbb{R}}-\mathbf{H}_{p_0-h(t)}(w-x)d\widetilde{\mu}_t(x)-C.$$

Taking derivative w.r.t. w, we have

$$\widetilde{g}_t'(w)-1=\int_{\mathbb{R}}-\mathbf{H}'_{p_0-h(t)}(w-x)d\widetilde{\mu}_t(x).\tag{13}$$

From equation (3) and the definition of $\mathbf{H}_r$, we have

$$\mathbf{H}_r(z) = i\frac{\pi}{r}\mathbf{H}_{\pi^2/r}(i\frac{\pi}{r}z) - \frac{z}{r} = \frac{\pi}{r}\lim_{M\to\infty}\sum_{k=-M}^{M}\frac{e^{2k\pi^2/r}+e^{-\pi z/r}}{e^{2k\pi^2/r}-e^{-\pi z/r}} - \frac{z}{r}.$$

Thus

$$\mathbf{H}_r'(z) = \frac{\pi^2}{r^2}\sum_{k=-\infty}^{\infty}\frac{-2e^{\pi z/r}e^{2k\pi^2/r}}{(e^{\pi z/r}-e^{2k\pi^2/r})^2} - \frac{1}{r}. \tag{14}$$

So for $z \in \mathbb{R}$, we have $\mathbf{H}_r'(z) < 0$. Apply this to equation (13). We get $\widetilde{g}_t'(\eta(h(t))) > 1$. Thus $\widetilde{f}_t'(\xi(t)) \in (0,1)$. Then from equation (11), we have

$$0 \le M_t \le \exp\left(-\frac{5}{8}\int_{p_0-h(t)}^{p-t}\left(S_r - \frac{1}{6}\right)dr\right) \le \exp\left(\frac{5p}{48}\right).$$

Since M_t, $0 \le t < T_A$, is uniformly bounded, so it is a bounded martingale. $\Box$

Now suppose that A is a smooth hull, i.e., there is a smooth simple closed curve $\gamma : [0,1] \to \mathbb{A}_p \cup \mathbf{C}_0$ with $\gamma((0,1)) \subset \mathbb{A}_p$ and $\gamma(0) \ne \gamma(1) \in \mathbf{C}_0$, and A is bounded by γ and an arc on $\mathbf{C}_0$ between $\gamma(0)$ and $\gamma(1)$.

If $T_A < p$, a proof similar to Lemma 6.3 in (8) shows that $\widetilde{f}_t'(\xi(t)) \to 0$ as $t \to T_A$. Thus $M_t \to 0$ as $t \to T_A$ on the event that $T_A < p$. From now on, we suppose $T_A = p$. Then K_t approaches $\mathbf{C}_p$ as $t \to p$ and is uniformly bounded away from A. Then the modulus of $\mathbb{A}_p \setminus K_t \setminus A$ tends to 0 as $t \to p$. Thus $p_0 - h(t) \to 0$ as $t \to p$. So $S_A = p_0$. Now $A_t = \varphi_t(A)$ is bounded by $\gamma_t = \varphi_t(\gamma)$ and an arc on $\mathbf{C}_0$ between $\gamma_t(0)$ and $\gamma_t(1)$. So A_t is also a smooth hull. Thus f_t and g_t both extend continuously to the boundary of the definition domain. And f_t maps γ_t to an arc on $\mathbf{C}_0$. Let I_t denote this arc. Since $-\ln(g_t(z)/z)$ also extends continuously to $\mathbf{C}_0$, so the measure μ_t in equation (12) satisfies

$$d\mu_t(z) = -\mathrm{Re}\,\ln(g_t(z)/z)/(2\pi)d\mathbf{m}(z) = -\ln|g_t(z)|/(2\pi)d\mathbf{m},$$

where $\mathbf{m}$ is the Lebesgue measure on $\mathbf{C}_0$ (of total mass 2π). Since $\ln|g_t(z)| = 0$ for $z \in \mathbf{C}_0 \setminus I_t$, so μ_t is supported by I_t. Let $\widetilde{\gamma}$ be a continuous curve such that $\gamma = e^i \circ \widetilde{\gamma}$. Let $\widetilde{\gamma}_t = \widetilde{\varphi}_t(\widetilde{\gamma})$ and $\widetilde{I}_t = \widetilde{f}_t(\gamma_t)$. Then $e^i(\widetilde{\gamma}_t) = \gamma_t$, $e^i(\widetilde{I}_t) = I_t$, and $\widetilde{I}_t$ is a real interval. Let $\widetilde{\mu}_t$ be a measure supported by $\widetilde{I}_t$ that satisfies $d\widetilde{\mu}_t(z) = \mathrm{Im}\,\widetilde{g}_t(z)/(2\pi)d\mathbf{m}_{\mathbb{R}}$ for $z \in \widetilde{I}_t$, where $\mathbf{m}_{\mathbb{R}}$ is the Lebesgue measure on $\mathbb{R}$. Since $-\ln|g_t(e^i(z))| = \mathrm{Im}\,\widetilde{g}_t(z)$, so $\mu_t = \widetilde{\mu}_t \circ (e^i)^{-1}$. Thus equation (13) holds for this $\widetilde{\mu}_t$.

Now $\widetilde{\varphi}_t$ maps $\mathbb{S}_p \setminus \widetilde{K}_t$ conformally onto $\mathbb{S}_{p-t}$. Let Σ_t be the union of $\mathbb{S}_p \setminus \widetilde{K}_t$, its reflection w.r.t. $\mathbb{R}$, and $\mathbb{R} \setminus \overline{\widetilde{K}_t}$. By Schwarz reflection principle, φ_t extends analytically to Σ_t, and maps Σ_t conformally into $\{z \in \mathbb{C} : |\mathrm{Im}\,z| < p-t\}$. For every $z \in A$, the distance from z to the boundary of Σ_t is at least $d_0 = \min\{p, dist(A, K_p)\} > 0$, and the distance from $\varphi_t(z)$ to the boundary of $\{z \in \mathbb{C} : |\mathrm{Im}\,z| < p-t\}$ equals to $p-t$. By Koebe's 1/4 theorem, $|\varphi_t'(z)| \le 4(p-t)/d_0$. Let $H = \max\{\mathrm{Im}\,\widetilde{\gamma}(u) : u \in [0,1]\}$. Since $\widetilde{\gamma}_t = \widetilde{\varphi}_t \circ \widetilde{\gamma}$, so $H_t := \max\{\mathrm{Im}\,\widetilde{\gamma}_t(u) : u \in [0,1]\} \le 4(p-t)H/d_0$. A proof similar as above shows that for any $z \in \widetilde{I}_0$, $|\widetilde{\psi}_{h(t)}(z)| \le 4(p_0 - h(t))/d_1$ for some $d_1 > 0$. Since

$$\widetilde{I}_t = \widetilde{f}_t(\widetilde{\gamma}_t) = \widetilde{\psi}_{h(t)} \circ \widetilde{\varphi}_A(\widetilde{\gamma}) = \widetilde{\psi}_{h(t)}(\widetilde{I}_0), \tag{15}$$

so $|\widetilde{I}_t| \le 4(p-t)/d_1|\widetilde{I}_0|$. Thus $|\mu_t| = |\widetilde{\mu}_t| \le H_t|I_t| \le 16(p-t)(p_0-h(t))H|\widetilde{I}_0|/(d_0d_1)$. Let $C_0 = 16H|\widetilde{I}_0|/(d_0d_1)$, then

$$(p-t)-(p_0-h(t)) = |\mu_t| = |\widetilde{\mu}_t| \le C_0(p-t)(p_0-h(t)). \tag{16}$$

Thus

$$(p_0-h(t))/(p-t) \ge 1 - C_0(p_0-h(t)). \tag{17}$$

Since $\widetilde{\mu}_t$ is supported by $\widetilde{I}_t$, so from equation (13) we have

$$\widetilde{g}_t'(\eta(h(t))) - 1 = \int_{\widetilde{I}_t} -\mathbf{H}'_{p_0-h(t)}(\eta(h(t))-x)d\widetilde{\mu}_t(x).$$

Let $\widetilde{\alpha}(t) = \widetilde{\varphi}_t^{-1}(\xi(t))$. Then $\widetilde{\alpha}(t)$ is a simple curve, and $\alpha(t) = e^i(\widetilde{\alpha}(t)) = \varphi_t^{-1}(e^i(\xi(t)))$ is an annulus $\mathrm{SLE}_{8/3}$ trace. So $K_t = \alpha((0,t])$ for any $t \ge 0$. Thus $\widetilde{K}_t = \cup_{k\in\mathbb{Z}}(\widetilde{\alpha}((0,t]) + 2k\pi)$. Let $\widetilde{\beta}(s) = \widetilde{\varphi}_A(\widetilde{\alpha}(h^{-1}(s)))$ for $0 \le s < p_0$. Since $\widetilde{L}_{h(t)} = \widetilde{\varphi}_A(K_t)$ and $\widetilde{\varphi}_A(z+2k\pi) = \widetilde{\varphi}_A(z)+2k\pi$, so $\widetilde{L}_{h(t)} = \cup_{k\in\mathbb{Z}}(\widetilde{\beta}((0,h(t)]) + 2k\pi)$. Now we compute

$$\widetilde{\psi}_{h(t)}(\widetilde{\beta}(h(t))) = \widetilde{\psi}_{h(t)} \circ \widetilde{\varphi}_A \circ \widetilde{\varphi}_t^{-1}(\xi(t)) = \widetilde{f}_t(\xi(t)) = \eta(h(t)).$$

Thus $\widetilde{\psi}_s$ maps the left and right side of $\beta((0,h(t)))$ to intervals $(b_-(t),\eta(h(t)))$ and $(\eta(h(t)),b_+(t))$, respectively, for some $b_-(t) < \eta(h(t)) < b_+(t)$. Therefore $\widetilde{\psi}_{h(t)}$ maps the $\widetilde{L}_{h(t)}$ to $\cup_{k\in\mathbb{Z}}(b_-(t)+2k\pi, b_+(t)+2k\pi)$. From equation (15), we have $\cup_{k\in\mathbb{Z}}(l(t)+2k\pi, r(t)+2k\pi) \cap \widetilde{I}_t = \emptyset$. So for any $x \in \widetilde{I}_t$ and $k \in \mathbb{Z}$, $|x-(\eta(s)+2k\pi)| \ge \min\{\eta(h(t))-b_-(t), b_+(t)-\eta(h(t))\}$.

As $t \to p$, $\widetilde{\beta}(h(t))$ approaches to a point on $\mathbb{R}_{p_0}$, so the extremal distance between the left side of $\widetilde{\beta}((0,h(t))$ and $\mathbb{R}_{p_0}$ in $\mathbb{S}_{p_0} \setminus \widetilde{L}_{h(t)}$ tends to 0. Since $\widetilde{\varphi}_t$ maps $(\mathbb{S}_{p_0} \setminus \widetilde{L}_{h(t)}, \mathbb{R}_{p_0})$ conformally onto $(\mathbb{S}_{p_0-h(t)}, \mathbb{R}_{p_0-h(t)})$, and the left side of $\widetilde{\beta}((0,h(t))$ is mapped to $(b_-(t),\eta(h(t)))$, so the extremal distance between $(b_-(t),\eta(h(t)))$ and $\mathbb{R}_{p_0-h(t)}$ in $\mathbb{S}_{p_0-h(t)}$ tends to 0 as $t \to p$ by the conformal invariance property of extremal length. Thus $(\eta(h(t))-b_-(t))/(p_0-h(t)) \to +\infty$ as $t \to p$. Similarly, $(b_+(t)-\eta(h(t)))/(p_0-h(t)) \to +\infty$ as $t \to p$.

Suppose $R \ge \ln(2)/\pi$. Then $e^{\pi R} \ge 2$, and so $e^{\pi R}-1 \ge e^{\pi R}/2$. Suppose $r > 0$ and the distance from $x \in \mathbb{R}$ to $\{2k\pi : k \in \mathbb{Z}\}$ is at least rR, then there is $k_0 \in \mathbb{Z}$ such that $2(k_0+1)\pi - rR \ge x \ge 2k_0\pi + rR$. Thus $2rR \le 2\pi$, and so $r \le \pi/R$. From equation (14), we have

$$-\mathbf{H}'_r(x) = \frac{\pi^2}{r^2}\sum_{k=-\infty}^{k_0}\frac{2e^{\pi(x-2k\pi)/r}}{(e^{\pi(x-2k\pi)/r}-1)^2} + \frac{\pi^2}{r^2}\sum_{k=k_0+1}^{+\infty}\frac{2e^{\pi(2k\pi-x)/r}}{(e^{\pi(2k\pi-x)/r}-1)^2} + \frac{1}{r}$$

$$\le \frac{\pi^2}{r^2}\sum_{k=-\infty}^{k_0}\frac{2e^{\pi(2k_0\pi+rR-2k\pi)/r}}{(e^{\pi(2k_0\pi+rR-2k\pi)/r}-1)^2} + \frac{\pi^2}{r^2}\sum_{k=k_0+1}^{+\infty}\frac{2e^{\pi(2k\pi-(2(k_0+1)\pi-rR))/r}}{(e^{\pi(2k\pi-(2(k_0+1)\pi-rR))/r}-1)^2} + \frac{1}{r}$$

$$= 2\frac{\pi^2}{r^2}\sum_{m=0}^{\infty}\frac{2e^{\pi(2m\pi+rR)/r}}{(e^{\pi(2m\pi+rR)/r}-1)^2} + \frac{1}{r} = \frac{\pi^2}{r^2}\sum_{m=0}^{\infty}\frac{4e^{\pi R+2m\pi^2/r}}{(e^{\pi R+2m\pi^2/r}-1)^2} + \frac{1}{r}$$

$$\le \frac{\pi^2}{r^2}\sum_{m=0}^{\infty}\frac{4e^{\pi R+2m\pi^2/r}}{(e^{\pi R+2m\pi^2/r}/2)^2} + \frac{1}{r} = 16\sum_{m=0}^{\infty}e^{-\pi R-2m\pi^2/r} + \frac{1}{r}$$

$$= \frac{\pi^2}{r^2}\frac{16e^{-\pi R}}{1-e^{-2\pi^2/r}} + \frac{1}{r} \le \frac{\pi^2}{r^2}\frac{16e^{-\pi R}}{1-e^{-2\pi^2/(\pi/R)}} + \frac{1}{r} \le 32\frac{\pi^2}{r^2}e^{-\pi R} + \frac{1}{r}.$$

Let $r(t) = p_0 - h(t)$ and $R(t) = \min\{\eta(h(t)) - b_-(t), b_+(t) - \eta(h(t))\}/r(t)$. Then $r(t) \to 0$ and $R(t) \to +\infty$ uniformly in ω as $t \to p$, and for any $x \in \widetilde{I}_t$, the distance from $\eta(h(t)) - x$ and $\{2k\pi : k \in \mathbb{Z}\}$ is at least $r(t)R(t)$. There is $t_0 \in (0,p)$ such that for $t \in [t_0, p)$, $R(t) \ge \ln(2)/\pi$ and $r(t) \le 1/(2C_0)$, where C_0 is as in equation (17). From the above displayed formula, we have $-H'_{r(t)}(x) \le 32\pi^2/r(t)^2 e^{-\pi R(t)} + 1/r$. When $t \in [t_0, p)$, $r(t)/(p-t) \ge 1/2$ by equation (17), and so from the estimation of $-\mathbf{H}'_{r(t)}(x)$ and equation (16), we have

$$\widetilde{g}_t'(\eta(h(t))) - 1 \le |\widetilde{\mu}_t|(32\frac{\pi^2}{r(t)^2}e^{-\pi R(t)} + \frac{1}{r(t)})$$

$$\le C_0(p-t)r(t)(32\frac{\pi^2}{r(t)^2}e^{-\pi R(t)} + \frac{1}{r(t)}) \le 64C_0\pi^2 e^{-\pi R(t)} + C_0(p-t) \to 0,$$

as $t \to p$. Thus $\widetilde{f}_t'(\xi(t)) = 1/\widetilde{g}_t'(\eta(h(t))) \to 1$ as $t \to p$. Recall that the above argument is based on the assumption that $T_A = p$.

Suppose

$$\int_{p_0-h(t)}^{p-t} S_r dr \to 0, \text{ as } t \to p \text{ on the event that } T_A = p. \tag{18}$$

Then $M_t \to 1$ as $t \to T_A$ on the event that $T_A = p$. From the Markov property, we have

$$\widetilde{\varphi}_A'(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_0}^{p}\left(S_r - \frac{1}{6}\right)dr\right) = M_0 = \mathbf{E}\,[\lim_{t\to T_A} M_t] = \mathbf{P}\left(\{T_A = p\}\right).$$

Recall that p_0 is the modulus of $\mathbb{A}_p \setminus A$. Let $K_p = \cup_{0\le t<p} K_t$. Then

$$\mathbf{P}\left(\{K_p \cap A = \emptyset\}\right) = \widetilde{\varphi}_A'(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_0}^{p}\left(S_r - \frac{1}{6}\right)dr\right). \tag{19}$$

If A is not a smooth hull, we may find a sequence of smooth hulls A_n that approaches A. Then $\widetilde{\varphi}_{A_n}'(0) \to \widetilde{\varphi}_A'(0)$ and the modulus of $\mathbb{A}_p \setminus A_n$ tends to the modulus of $\mathbb{A}_p \setminus A$, so equation (19) still holds.

Now suppose B is a hull in $\mathbb{A}_{p_0}$ w.r.t. $\mathbf{C}_{p_0}$. Let $D = A \cup \varphi_A^{-1}(B)$. Then D is a hull in $\mathbb{A}_p$ w.r.t. $\mathbf{C}_p$. Let p_1 be the modulus of $\mathbb{A}_p \setminus D$, which is also the modulus of $\mathbb{A}_{p_0} \setminus B$. Then $\varphi_D = \varphi_B \circ \varphi_A$, so $\widetilde{\varphi}_D = \widetilde{\varphi}_B \circ \widetilde{\varphi}_A$ and $\widetilde{\varphi}_D'(0) = \widetilde{\varphi}_B'(\widetilde{\varphi}_A(0))\widetilde{\varphi}_A'(0) = \widetilde{\varphi}_B'(0)\widetilde{\varphi}_A'(0)$. From the last paragraph,

$$\mathbf{P}\left(K_p \cap D = \emptyset\}\right) = \widetilde{\varphi}_D'(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_1}^{p}\left(S_r - \frac{1}{6}\right)dr\right).$$

Thus

$$\mathbf{P}\left(\{K_p \cap D = \emptyset\}|\{K_p \cap A = \emptyset\}\right) = \widetilde{\varphi}_B'(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_1}^{p_0}\left(S_r - \frac{1}{6}\right)dr\right).$$

If L_s, $0 \le s < p_0$, are standard annulus $\mathrm{SLE}_{8/3}$ hulls of modulus p_0, and $L_{p_0} = \cup_{0\le s<p_0} L_s$, then

$$\mathbf{P}\left(\{L_{p_0} \cap B = \emptyset\}\right) = \widetilde{\varphi}_B'(0)^{5/8} \exp\left(-\frac{5}{8}\int_{p_1}^{p_0}\left(S_r - \frac{1}{6}\right)dr\right).$$

$$= \mathbf{P}\left(\{K_p \cap D = \emptyset\} | \{K_p \cap A = \emptyset\}\right) = \mathbf{P}\left(\{\varphi_A(K_p) \cap B = \emptyset\} | \{K_p \cap A = \emptyset\}\right).$$

Thus conditioned on the event that $K_p \cap A = \emptyset$, $\varphi_A(K_p)$ has the same distribution as L_{p_0}. Then we proved the restriction property of annulus $\mathrm{SLE}_{8/3}$ under the assumption (18).

Unfortunately, the assumption (18) is actually always false. From equation (3) one may compute that S_r is of order $\Theta(1/r^2)$ as $r \to 0$. From (17), $(p-t)-(p_0-h(t)) = |\mu_t|$ is of order $O((p-t)^2)$. In fact, one could prove that it is of order $\Theta((p-t)^2)$. So $\int_{p_0-h(t)}^{p-t} S_r dr$ is uniformly bounded away from 0. Thus it does not tend to 0 as $t \to p$. Therefore we guess that annulus $\mathrm{SLE}_{8/3}$ does not satisfy the restriction property.

Recently, Robert O. Bauer studied in (2) a process defined in a doubly connected domain obtained by conditioning a chordal $\mathrm{SLE}_{8/3}$ in a simply connected domain to avoid an interior contractible compact subset. The process describes a random simple curve connecting two prime ends of a doubly connected domain that lie on the same side, so it is different from the process we study here. That process automatically satisfies the restriction property from the restriction property of chordal $\mathrm{SLE}_{8/3}$. And it satisfies conformal invariance because the set of boundary hulls generates the same σ-algebra as the Hausdorff metric on the space of simple curves.

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