

A note on the scaling limits of contour functions of Galton-Watson trees

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Abstract

Recently, Abraham and Delmas constructed the distributions of super-critical Lévy trees truncated at a fixed height by connecting super-critical Lévy trees to (sub)critical Lévy trees via a martingale transformation. A similar relationship also holds for discrete Galton-Watson trees. In this work, using the existing works on the convergence of contour functions of (sub)critical trees, we prove that the contour functions of truncated super-critical Galton-Watson trees converge weakly to the distributions constructed by Abraham and Delmas.

Keywords: Galton-Watson trees; Branching processes; Lévy trees; contour functions; scaling limit.

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1 Introduction

In this note, we are interested in studying the scaling limits of contour functions of Galton-Watson trees. Since the (sub)critical case has been extensively studied, see e.g. [6] and [11], we mainly focus on the super-critical case.

In [6], it was shown that the scaling limits of contour functions of (sub)critical Galton-Watson trees are the height processes which encode the Lévy trees; see also [11] and references therein for some recent developments. In the super-critical case, however, it is not so convenient to use contour functions to characterize either Galton-Watson trees or Lévy trees which may have infinite mass. To this end, Abraham and Delmas in [1] showed that the distributions of super-critical continuous state branching processes ('CSBPs' in short) stopped at a fixed time are absolutely continuous w.r.t. (sub)critical CSBPs, via a martingale transformation. Since the Lévy trees code the genealogy of CSBPs, they further defined the distributions of the super-critical Lévy tree truncated at a fixed height by a similar change of probability.

In this work, we shall show that such distributions defined in [1] arise as the weak limits of scaled contour functions of super-critical Galton-Watson trees cut at a given level. Our main result is Theorem 3.3. A key to this result is the observation shown in Lemma 2.1 which could be regarded as a discrete counterpart of martingale transformation for Lévy trees constructed in [1]. Then by a collection of related results on convergence of Galton-Watson processes to CSBPs, we obtain our main result.

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Let us mention some related works here. In [16], the authors studied the local time processes of contour functions of binary Galton-Watson trees in continuous time. Duquesne and Winkel in [7] and [8] also constructed super-critical Lévy trees as increasing limits of Galton-Watson trees. In their works, the trees are viewed as metric spaces and the convergence holds in the sense of the Gromov-Hausdorff distance; see Remark 3.5 below for a discussion of the relation between the present work and [8].

The paper is organized as follows. In Section 2, we recall some basic definitions and facts on trees and branching processes. In Section 3, we present our main result and its proof.

We shall assume that all random variables in the paper are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$.

2 Trees and branching processes

2.1 Discrete trees and Galton-Watson trees

We present the framework developed in [17] for trees; see also [13] for more notation and terminology. Introduce the set of labels

$$\mathbb{U} = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n,$$

where $\mathbb{N}^* = \{1, 2, \dots\}$ and by convention $(\mathbb{N}^*)^0 = \{\emptyset\}$.

An element of \mathbb{U} is thus a sequence $w = (w^1, \dots, w^n)$ of elements of \mathbb{N}^* , and we set $|w| = n$, so that $|w|$ represents the generation of w or the height of w . If $w = (w^1, \dots, w^m)$ and $v = (v^1, \dots, v^n)$ belong to \mathbb{U} , we write $wv = (w^1, \dots, w^m, v^1, \dots, v^n)$ for the concatenation of w and v . In particular $w\emptyset = \emptyset w = w$.

A (finite or infinite) rooted ordered tree \mathbf{t} is a subset of \mathbb{U} such that

1. $\emptyset \in \mathbf{t}$.
2. $(w^1, \dots, w^n) \in \mathbf{t} \setminus \{\emptyset\} \implies (w^1, \dots, w^{n-1}) \in \mathbf{t}$.
3. For every $w \in \mathbf{t}$, there exists a finite integer $k_w \mathbf{t} \geq 0$ such that if $k_w \mathbf{t} \geq 1$, then $wj \in \mathbf{t}$ for any $1 \leq j \leq k_w \mathbf{t}$ ($k_w \mathbf{t}$ is the number of children of $w \in \mathbf{t}$).

Then \emptyset is called the root of tree \mathbf{t} . Let \mathbf{T}^∞ denote the set of all such trees \mathbf{t} . For each $u \in \mathbb{U}$, define $\mathbf{T}_u = \{\mathbf{t} \in \mathbf{T}^\infty : u \in \mathbf{t}\}$. We endow \mathbf{T}^∞ with the σ -algebra $\sigma(\mathbf{T}_u, u \in \mathbb{U})$; see [17] for details. Given a tree \mathbf{t} , we call an element in the set $\mathbf{t} \subset \mathbb{U}$ a node of \mathbf{t} . Denote the height of a tree \mathbf{t} by $|\mathbf{t}| := \max\{|\nu| : \nu \in \mathbf{t}\}$. For $h = 0, 1, 2, \dots$, define $r_h \mathbf{t} = \{\nu \in \mathbf{t} : |\nu| \leq h\}$, which is a finite tree. Denote by $\#\mathbf{t}$ the number of nodes of \mathbf{t} . Let

$$\mathbf{T} := \{\mathbf{t} \in \mathbf{T}^\infty : \#\mathbf{t} < \infty\}$$

be the set of all finite trees.

We say that $w \in \mathbf{t}$ is a leaf of \mathbf{t} if $k_w \mathbf{t} = 0$ and set

$$\mathcal{L}(\mathbf{t}) := \{w \in \mathbf{t} : k_w \mathbf{t} = 0\}.$$

So $\mathcal{L}(\mathbf{t})$ denotes the set of leaves of \mathbf{t} and $\#\mathcal{L}(\mathbf{t})$ is the number of leaves of \mathbf{t} .

To code the finite trees, we introduce the so-called *contour functions*; see [13] for details. Suppose that the tree $\mathbf{t} \in \mathbf{T}$ is embedded in the half-plane in such a way that edges have length one. Imagine that a particle starts at time $s = 0$ from the root of the tree and then explores the tree from the left to the right, moving continuously along the edges at unit speed, until all edges have been explored and the particle has come back to the root. Then the total time needed to explore the tree is $\zeta(\mathbf{t}) := 2(\#\mathbf{t} - 1)$.

The contour function of \mathbf{t} is the function $(C(\mathbf{t}, s), 0 \leq s \leq \zeta(\mathbf{t}))$ whose value at time $s \in [0, \zeta(\mathbf{t})]$ is the distance (on the tree) between the position of the particle at time s and the root. We set $C(\mathbf{t}, s) = 0$ for $s \in [\zeta(\mathbf{t}), 2\#\mathbf{t}]$.

Given a probability distribution $p = \{p_n : n = 0, 1, \dots\}$ with $p_1 < 1$, following [17] and [4], call a \mathbf{T}^∞ -valued random variable \mathcal{G}^p a Galton-Watson tree with offspring distribution p if

i) the number of children of \emptyset has distribution p :

$$\mathbb{P}(k_\emptyset \mathcal{G}^p = n) = p_n, \quad \forall n \geq 0;$$

ii) for each $h = 1, 2, \dots$, conditionally given $r_h \mathcal{G}^p = \mathbf{t}$ with $|\mathbf{t}| \leq h$, for $\nu \in \mathbf{t}$ with $|\nu| = h$, $k_\nu \mathcal{G}^p$ are i.i.d. random variables distributed according to p .

The second property is called branching property. From the definition we have for $a \in \mathbb{N}^*$ and any $\mathbf{t} \in \mathbf{T}$ such that $|\mathbf{t}| \leq a$,

$$\mathbb{P}(r_a \mathcal{G}^p = \mathbf{t}) = \prod_{\nu \in \mathbf{t}: |\nu| < a} p_{k_\nu \mathbf{t}}. \quad (2.1)$$

Define $m(p) = \sum_{k \geq 0} k p_k$. We say \mathcal{G}^p is sub-critical (resp. critical, super-critical) if $m(p) < 1$ (resp. $m(p) = 1, m(p) > 1$).

Given a tree $\mathbf{t} \in T^\infty$, then $r_a \mathbf{t}$ is a finite tree whose contour function is denoted by $\{C_a(\mathbf{t}, s) : s \geq 0\}$. For $k \geq 0$, we denote by $Y_k(\mathbf{t})$ the number of individuals in generation k :

$$Y_k(\mathbf{t}) = \#\{\nu \in \mathbf{t} : |\nu| = k\}, \quad k \geq 0.$$

Given a probability measure $p = \{p_k : k \geq 0\}$ with $\sum_{k \geq 0} k p_k > 1$. Let \mathcal{G}^p be a super-critical Galton-Watson tree with offspring distribution p . Then $\mathbb{P}(\#\mathcal{G}^p < \infty) = f(p)$, where $f(p)$ is the minimal solution of the following equation of s :

$$g^p(s) := \sum_{k \geq 0} s^k p_k = s, \quad 0 \leq s \leq 1.$$

Let $q = \{q_k : k \geq 0\}$ be another probability distribution such that

$$q_k = f(p)^{k-1} p_k, \text{ for } k \geq 1, \text{ and } q_0 = 1 - \sum_{k \geq 1} q_k.$$

Then $\sum_{k \geq 0} k q_k < 1$. Let \mathcal{G}^q be a subcritical GW tree with offspring distribution q . Note that $(Y_k(\mathcal{G}^q), k \geq 0)$ is a Galton-Watson process starting from a single ancestor with offspring distribution q . We first present a simple lemma.

Lemma 2.1. *Let F be any nonnegative measurable function on \mathbf{T} . Then for $\mathbf{t} \in \mathbf{T}$,*

$$\mathbb{P}[\mathcal{G}^p = \mathbf{t}] = f(p) \mathbb{P}[\mathcal{G}^q = \mathbf{t}] \quad (2.2)$$

and for any $a \in \mathbb{N}$,

$$\mathbb{E}[F(r_a \mathcal{G}^p)] = \mathbb{E}\left[f(p)^{1-Y_a(\mathcal{G}^q)} F(r_a \mathcal{G}^q)\right]. \quad (2.3)$$

Proof. (2.2) is just (4.8) in [2]. The proof of (2.3) is straightforward. Fix $a \in \mathbb{N}^*$ and $\mathbf{t} \in T$ such that $|\mathbf{t}| \leq a$. By (2.1), we have

$$\mathbb{P}(r_a \mathcal{G}^q = \mathbf{t}) = \prod_{\nu \in \mathbf{t}: |\nu| < a} f(p)^{k_\nu \mathbf{t} - 1} p_{k_\nu \mathbf{t}} = f(p)^{Y_a(\mathbf{t}) - 1} \prod_{\nu \in \mathbf{t}: |\nu| < a} p_{k_\nu \mathbf{t}} = f(p)^{Y_a(\mathbf{t}) - 1} \mathbb{P}(r_a \mathcal{G}^p = \mathbf{t})$$

since $Y_a(\mathbf{t}) - 1 = \sum_{\nu \in \mathbf{t}: |\nu| < a} (k_\nu \mathbf{t} - 1)$. We have completed the proof. \square

Remark 2.2. It is easy to see that $(f(p)^{-Y_n(\mathcal{G}^q)}, n \geq 0)$ is a martingale with respect to $\mathcal{F}_n = \sigma(r_n \mathcal{G}^q)$. In fact, by the branching property, we have for all $0 \leq m \leq n$,

$$\mathbb{E} \left[f(p)^{-Y_n(\mathcal{G}^q)} \middle| r_m \mathcal{G}^q \right] = \left[\mathbb{E} \left[f(p)^{-Y_{n-m}(\mathcal{G}^q)} \right] \right]^{Y_m(\mathcal{G}^q)} = f(p)^{-Y_m(\mathcal{G}^q)}. \quad (2.4)$$

Since contour functions code finite trees in \mathbf{T} , we immediately get the following result.

Corollary 2.3. For any nonnegative measurable function F on $C(\mathbb{R}^+, \mathbb{R}^+)$ and $a \in \mathbb{N}$,

$$\mathbb{E} [F(C_a(\mathcal{G}^p, \cdot))] = \mathbb{E} \left[f(p)^{1-Y_a(\mathcal{G}^q)} F(C_a(\mathcal{G}^q, \cdot)) \right]. \quad (2.5)$$

Lemma 2.1 could be regarded as a discrete counterpart of the martingale transformation for Lévy trees in Section 4 of [1]; see also (2.11) below in this paper. To see this, we need to introduce continuous state branching processes and Lévy trees.

2.2 Continuous state branching processes

Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and π be a σ -finite measure on $(0, +\infty)$ such that $\int_{(0, +\infty)} (1 \wedge r^2) \pi(dr) < +\infty$. The branching mechanism ψ with characteristics (α, β, π) is defined by:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0, +\infty)} (e^{-\lambda r} - 1 + \lambda r 1_{\{r < 1\}}) \pi(dr). \quad (2.6)$$

A càd-làg \mathbb{R}^+ -valued Markov process $Y^{\psi, x} = (Y_t^{\psi, x}, t \geq 0)$ started at $x \geq 0$ is called ψ -continuous state branching process (ψ -CSBP in short) if its transition kernels satisfy

$$\mathbb{E}[e^{-\lambda Y_t^{\psi, x}}] = e^{-x u_t(\lambda)}, \quad t \geq 0, \lambda > 0,$$

where $u_t(\lambda)$ is the unique nonnegative solution of

$$\frac{\partial u_t(\lambda)}{\partial t} = -\psi(u_t(\lambda)), \quad u_0(\lambda) = \lambda.$$

ψ and $Y^{\psi, x}$ are said to be sub-critical (resp. critical, super-critical) if $\psi'(0+) \in (0, +\infty)$ (resp. $\psi'(0+) = 0, \psi'(0+) \in [-\infty, 0)$). We say that ψ and $Y^{\psi, x}$ are (sub)critical if they are critical or sub-critical.

In the sequel of this paper, we will assume that the following assumptions on ψ are in force:

(H1) The Grey condition holds:

$$\int^{+\infty} \frac{d\lambda}{\psi(\lambda)} < +\infty. \quad (2.7)$$

The Grey condition is equivalent to the a.s. finiteness of the extinction time of the corresponding CSBP. This assumption is used to ensure that the corresponding height process is continuous.

(H2) The branching mechanism ψ is conservative: for all $\varepsilon > 0$,

$$\int_{(0, \varepsilon]} \frac{d\lambda}{|\psi(\lambda)|} = +\infty.$$

The conservative assumption is equivalent to the finiteness of the corresponding CSBP at all time.

Let us remark that (H1) implies $\beta > 0$ or $\int_{(0,1)} r\pi(dr) = +\infty$. And if ψ is (sub)critical, then we must have $\alpha - \int_{(1,+\infty)} r\pi(dr) \in [0, +\infty)$. We end this subsection by collecting some results from [1].

Let $(X_t, t \geq 0)$ denote the canonical process of $\mathcal{D} := D(\mathbb{R}^+, \mathbb{R})$. Let P_x^ψ be the probability measure on $D(\mathbb{R}^+, \mathbb{R})$ such that $P_x^\psi(X_0 = x) = 1$ and $(X_t, t \geq 0)$ is a ψ -CSBP under P_x^ψ . It is well-known that under (H1) and (H2), P_x^ψ -a.s. $X_\infty = \lim_{t \rightarrow \infty} X_t$ exists with $X_\infty \in \{0, \infty\}$ and

$$P_x^\psi(X_\infty = 0) = e^{-\gamma x},$$

where γ is the largest root of $\psi(\lambda) = 0$.

Lemma 2.4. (Lemma 2.4 in [1]) Assume that ψ is supercritical satisfying (H1) and (H2). Then for any nonnegative random variable measurable w.r.t. $\sigma(X_t, t \geq 0)$, we have

$$E_x^\psi[W | X_\infty = 0] = E_x^{\psi_\gamma}[W],$$

where $\psi_\gamma(\cdot) = \psi(\cdot + \gamma)$.

2.3 Height processes

To code the genealogy of the ψ -CSBP, Le Gall and Le Jan [15] introduced the so-called height process, which is a functional of the Lévy process with Laplace exponent ψ ; see also Duquesne and Le Gall [6].

Assume that ψ is (sub)critical satisfying (H1). Let \mathbb{P}^ψ be a probability measure on \mathcal{D} such that under \mathbb{P}^ψ , $X = (X_t, t \geq 0)$ is a Lévy process with nonnegative jumps and with Laplace exponent ψ :

$$\mathbb{E}^\psi \left[e^{-\lambda X_t} \right] = e^{t\psi(\lambda)}, \quad t \geq 0, \lambda \geq 0.$$

The so-called continuous-time height process denoted by H is defined for every $t \geq 0$ by:

$$H_t = \liminf_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t 1_{\{X_s < I_t^s + \epsilon\}} ds,$$

where the limit exists in \mathbb{P}^ψ -probability and $I_t^s = \inf_{s \leq r \leq t} X_r$; see [6]. Under \mathbb{P}^ψ , the process H has a continuous modification. From now on we only consider this modification. Under \mathbb{P}^ψ , for $a \geq 0$, the local time of the height process at level a is the continuous increasing process $(L_s^a, s \geq 0)$ which can be characterized via the approximation

$$\limsup_{\epsilon \rightarrow 0} \mathbb{E}^\psi \left[\sup_{a \geq \epsilon} \left[\sup_{s \leq t} \left| \epsilon^{-1} \int_0^s 1_{\{a - \epsilon < H_r \leq a\}} dr - L_s^a \right| \right] \right] = 0. \tag{2.8}$$

Furthermore, for any nonnegative measurable function g on \mathbb{R}^+ ,

$$\int_0^s g(H_r) dr = \int_{\mathbb{R}^+} g(a) L_s^a da, \quad s \geq 0. \tag{2.9}$$

For any $x > 0$, define

$$T_x = \inf\{t \geq 0 : I_t \leq -x\},$$

where $I_t = \inf_{0 \leq r \leq t} X_r$.

Let $\mathcal{C} := C(\mathbb{R}^+, \mathbb{R}^+)$ be the space of nonnegative continuous functions with compact support on \mathbb{R}^+ equipped with the usual topology of the uniform convergence on every compact subsets. Denote by $(e_t, t \geq 0)$ the canonical process of \mathcal{C} . Denote by \mathbb{P}_x^ψ the law of $(H_t \wedge T_x, t \geq 0)$ under \mathbb{P}^ψ . Then \mathbb{P}_x^ψ is a probability distribution on \mathcal{C} . Set $Z_a = L_{T_x}^a$ under \mathbb{P}_x^ψ . Then $(Z_a, a \geq 0)$ has the same finite dimensional marginals as a CSBP with branching mechanism ψ and initial value x ; see Theorem 1.4.1 of [6]. In the following we will work with the càd-làg modification of Z without changing notation.

2.4 Super-critical Lévy trees

Height processes code the genealogy of (sub)critical CSBPs. However, for super-critical CSBPs, it is not so convenient to introduce the height process since the super-critical CSBPs may have infinite mass. Abraham and Delmas in [1] studied the distributions of trees cut at a fixed level, where a super-critical Lévy trees was constructed via a Girsanov transformation. We recall their construction here.

For any $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $a > 0$, define

$$\Gamma_{f,a}(x) = \int_0^x 1_{\{f(t) \leq a\}} dt, \quad \Pi_{f,a}(x) = \inf\{r \geq 0 : \Gamma_{f,a}(r) > x\}, \quad x \geq 0.$$

Note that since f has compact support, $\Pi_{f,a}(x)$ is finite for every $x \geq 0$. Then we define

$$\pi_a(f)(x) = f(\Pi_{f,a}(x)), \quad f \in C(\mathbb{R}^+, \mathbb{R}^+), \quad x \geq 0.$$

One can check that $\pi_a(f) \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $\pi_a \circ \pi_b = \pi_a$ for $0 \leq a \leq b$. Let ψ be a super-critical branching mechanism satisfying (H2). Denote by q^* the unique (positive) root of $\psi'(q) = 0$. Then the branching mechanism $\psi_q(\cdot) = \psi(\cdot + q) - \psi(q)$ is critical for $q = q^*$ and sub-critical for $q > q^*$. We also have $\gamma > q^*$. Because super-critical branching processes may have infinite mass, in [1] it was cut at a given level to construct the corresponding genealogical continuum random tree. Define

$$M_a^{\psi_q, -q} = \exp \left\{ -qZ_0 + qZ_a + \psi(q) \int_0^a Z_s ds \right\}, \quad a \geq 0.$$

Define a filtration $\mathcal{H}_a = \sigma(\pi_a(e)) \vee \mathcal{N}$, where \mathcal{N} is the class of $\mathbb{P}_x^{\psi_q}$ negligible sets. By (2.8), we have $M^{\psi_q, -q}$ is \mathcal{H} -adapted.

Theorem 2.5. (Theorem 2.2 in [1]) For each $q \geq q^*$, $M^{\psi_q, -q}$ is an \mathcal{H} -martingale under $\mathbb{P}_x^{\psi_q}$.

Proof. See Theorem 2.2 and arguments in Section 4 in [1]. \square

Define the distribution $\mathbb{P}_x^{\psi, a}$ of the ψ -CRT cut at level a with initial mass x , as the distribution of $\pi_a(e)$ under $M_a^{\psi_q, -q} d\mathbb{P}_x^{\psi_q}$: for any non-negative measurable function F on $C(\mathbb{R}^+, \mathbb{R}^+)$,

$$\mathbb{E}_x^{\psi, a}[F(e)] = \mathbb{E}_x^{\psi_q} \left[M_a^{\psi_q, -q} F(\pi_a(e)) \right], \quad (2.10)$$

which do not depend on the choice of $q \geq q^*$; see Lemma 4.1 of [1]. Taking $q = \gamma$ in (2.10), we see

$$\mathbb{E}_x^{\psi, a}[F(e)] = \mathbb{E}_x^{\psi, \gamma} \left[e^{-\gamma x + \gamma Z_a} F(\pi_a(e)) \right] \quad (2.11)$$

and $(e^{-\gamma x + \gamma Z_a}, a \geq 0)$ under $\mathbb{P}_x^{\psi, \gamma}$ is an \mathcal{H} -martingale with mean 1.

Remark 2.6. $\mathbb{P}_x^{\psi, a}$ gives the law of super-critical Lévy trees truncated at height a . Then the law of the whole tree could be defined as a projective limit. To be more precise, let \mathcal{W} be the set of $C(\mathbb{R}^+, \mathbb{R}^+)$ -valued functions endowed with the σ -field generated by the coordinate maps. Let $(w^a, a \geq 0)$ be the canonical process on \mathcal{W} . Proposition 4.2 in [1] proved that there exists a probability measure $\bar{\mathbb{P}}_x^\psi$ on \mathcal{W} such that for every $a \geq 0$, the distribution of w^a under $\bar{\mathbb{P}}_x^\psi$ is $\mathbb{P}_x^{\psi, a}$ and for $0 \leq a \leq b$

$$\pi_a(w^b) = \pi_a \quad \bar{\mathbb{P}}_x^\psi - a.s.$$

Remark 2.7. The above definitions of $\mathbb{E}_x^{\psi, a}$ and $\bar{\mathbb{P}}_x^\psi$ are also valid for (sub)critical branching mechanisms.

3 From Galton-Watson forests to Lévy forests

Comparing (2.5) with (2.11), one can see that the l.h.s. are similar. The super-critical trees (discrete or continuum) truncated at height a are connected to sub-critical trees, via a martingale transformation. Motivated by Duquesne and Le Gall's work [6], which studied the scaling limit of (sub)critical trees, one may hope that the laws of suitably rescaled super-critical Galton-Watson trees truncated at height a could converge to the law defined in (2.11). Our main result, Theorem 3.3, will show it is true.

For each integer $n \geq 1$ and real number $x > 0$,

- Let $[x]$ denote the integer part of x and let $\lceil x \rceil$ denote the minimal integer which is larger than x .
- Let $p^{(n)} = \{p_k^{(n)} : k = 0, 1, 2, \dots\}$ be a probability measure on \mathbb{N} .
- Let $\mathcal{G}_1^{p^{(n)}}, \mathcal{G}_2^{p^{(n)}}, \dots, \mathcal{G}_{[nx]}^{p^{(n)}}$ be independent Galton-Watson trees with the same offspring distribution $p^{(n)}$.
- Define $Y_k^{p^{(n)},x} = \sum_{i=1}^{[nx]} Y_k(\mathcal{G}_i^{p^{(n)}})$. Then $Y^{p^{(n)},x} = (Y_k^{p^{(n)},x}, k = 0, 1, \dots)$ is a Galton-Watson process with offspring distribution $p^{(n)}$ starting from $[nx]$.
- For $a \in \mathbb{N}$, define the contour function of trees cut at level a , $C_a^{p^{(n)},x} = (C_a^{p^{(n)},x}(t), t \geq 0)$, by concatenating the contour functions $(C(r_a \mathcal{G}_1^{p^{(n)}}), t) \in [0, 2\#r_a \mathcal{G}_1^{p^{(n)}}], \dots, (C(r_a \mathcal{G}_{[nx]}^{p^{(n)}}), t) \in [0, 2\#r_a \mathcal{G}_{[nx]}^{p^{(n)}}]$ and setting $C_a^{p^{(n)},x}(t) = 0$ for $t \geq 2 \sum_{i=1}^{[nx]} \#r_a \mathcal{G}_i^{p^{(n)}}$.
- For $a \in \mathbb{R}^+$, define $C_a^{p^{(n)},x} = \pi_a(C_{[a]}^{p^{(n)},x})$.
- If $\sum_{k \geq 0} k p_k^{(n)} \leq 1$, then we define the contour function $C^{p^{(n)},x} = (C^{p^{(n)},x}(t), t \geq 0)$ by concatenating the contour functions $(C(\mathcal{G}_1^{p^{(n)}}), t) \in [0, 2\#\mathcal{G}_1^{p^{(n)}}], \dots, (C(\mathcal{G}_{[nx]}^{p^{(n)}}), t) \in [0, 2\#\mathcal{G}_{[nx]}^{p^{(n)}}]$ and setting $C^{p^{(n)},x}(t) = 0$ for $t \geq 2 \sum_{i=1}^{[nx]} \#\mathcal{G}_i^{p^{(n)}}$.

Let $(\gamma_n, n = 1, 2, \dots)$ be a nondecreasing sequence of positive numbers converging to ∞ . Define

$$G^{(n)}(\lambda) = n\gamma_n [g^{p^{(n)}}(e^{-\lambda/n}) - e^{-\lambda/n}],$$

where $g^{p^{(n)}}$ is the generating function of $p^{(n)}$, and define a probability measure on $[0, \infty)$ by

$$\mu^{(n)}\left(\frac{k-1}{n}\right) = p_k^{(n)}, \quad k \geq 0.$$

We then present the following statements. By $\xrightarrow{(d)}$ we mean convergence in distribution.

(A1) $G^{(n)}(\lambda) \rightarrow \psi(\lambda)$ as $n \rightarrow \infty$ uniformly on any bounded interval.

(A2)

$$\left(\frac{1}{n} Y_{[\gamma_n t]}^{p^{(n)},x}, t \geq 0\right) \xrightarrow{(d)} (Y_t^{\psi,x}, t \geq 0), \quad \text{as } n \rightarrow \infty, \quad (3.1)$$

in $D(\mathbb{R}^+, \mathbb{R}^+)$.

(A3) There exists a probability measure μ on $(-\infty, +\infty)$ such that $(\mu^{(n)})^{*\lceil n\gamma_n \rceil} \rightarrow \mu$ as $n \rightarrow \infty$, where $\int e^{-\lambda x} \mu(dx) = e^{\psi(\lambda)}$.

The following lemma is a variant of Theorem 3.4 in [9].

Lemma 3.1. *Let ψ be a branching mechanism satisfying (H1) and (H2). Then (A1), (A2) and (A3) are equivalent.*

Remark 3.2. (A3) is just the condition (i) in Theorem 3.4 of [9]. Under our assumption on ψ , we do not need condition (b) there. (A3) is also equivalent to the convergence of random walks to ψ -Lévy processes; see Theorem 2.1.1 of [6] for (sub)critical case.

Proof. We shall show that (A2) \Leftrightarrow (A3) and (A3) \Leftrightarrow (A1).

i): If (A2) holds, then ψ is conservative implies that $\mathbb{P}(Y_t < \infty) = 1$ for all $t \geq 0$. Then Theorem 3.3 in [9] gives (A2) \Rightarrow (A3). Meanwhile, Theorem 3.1 in [9] implies (A3) \Rightarrow (A2).

ii): We first show (A3) \Rightarrow (A1). Denote by $L^{(n)}(\lambda)$ the Laplace transform of $(\mu^{(n)})^{*[n\gamma_n]}$. Then Theorem 2.1 in [9], together with (A3), gives that for every real number $d > 0$

$$\log L^{(n)}(\lambda) = [n\gamma_n] \log \left(\frac{e^{\lambda/n}}{n\gamma_n} G^{(n)}(\lambda) + 1 \right) \rightarrow \psi(\lambda), \text{ as } n \rightarrow \infty, \quad (3.2)$$

uniformly in $\lambda \in [0, d]$, which implies that for any $\epsilon > 0$, all $n > n(d, \epsilon)$ and $\lambda \in [0, d]$,

$$n\gamma_n \left(e^{\frac{\psi(\lambda) - \epsilon}{[n\gamma_n]}} - 1 \right) < e^{\lambda/n} G^{(n)}(\lambda) < n\gamma_n \left(e^{\frac{\psi(\lambda) + \epsilon}{[n\gamma_n]}} - 1 \right).$$

Then by $|e^x - 1 - x| < e^{|x|}|x|^2/2$,

$$-\frac{2(\psi(\lambda) - \epsilon)^2 e^{\frac{|\psi(\lambda) - \epsilon|}{[n\gamma_n]}}}{n\gamma_n} - 2\epsilon < e^{\lambda/n} G^{(n)}(\lambda) - \psi(\lambda) < \frac{(\psi(\lambda) + \epsilon)^2 e^{\frac{|\psi(\lambda) + \epsilon|}{[n\gamma_n]}}}{n\gamma_n} + \epsilon.$$

Note that ψ is locally bounded. Thus as $n \rightarrow \infty$, $G^{(n)}(\lambda) \rightarrow \psi(\lambda)$, uniformly on any bounded interval, which is just (A1). Similarly, one can deduce that if (A1) holds, then $L^{(n)}(\lambda) \rightarrow e^{\psi(\lambda)}$ as $n \rightarrow \infty$, which implies (A3). \square

Now, we are ready to present our main theorem. Define $\mathcal{E}^{p^{(n)}, x} = \inf\{k \geq 0 : Y_k^{p^{(n)}, x} = 0\}$ and $\mathcal{E}^{\psi, x} = \inf\{t \geq 0 : Y_t^{\psi, x} = 0\}$ with the convention that $\inf \emptyset = +\infty$. Denote by $g_k^{p^{(n)}}$ the k -th iterate of $g^{p^{(n)}}$.

Theorem 3.3. Let ψ be a branching mechanism satisfying (H1) and (H2). Assume that (A1) or (A2) holds. Suppose in addition that for every $\delta > 0$,

$$\liminf_{n \rightarrow \infty} g_{[\delta\gamma_n]}^{p^{(n)}}(0)^n > 0. \quad (3.3)$$

Then for $x > 0$,

$$\frac{1}{\gamma_n} \mathcal{E}^{p^{(n)}, x} \xrightarrow{(d)} \mathcal{E}^{\psi, x} \text{ on } [0, +\infty] \quad (3.4)$$

and for any bounded continuous function F on $C(\mathbb{R}^+, \mathbb{R}^+)$ and every $a \geq 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[F \left(\gamma_n^{-1} C_{\gamma_n a}^{p^{(n)}, x} (2n\gamma_n \cdot) \right) \right] = \mathbb{E}_x^{\psi, a} [F(e)]. \quad (3.5)$$

Before proving the theorem, we would like to give some remarks.

Remark 3.4. (3.3) is essential to (3.5); see the comments following Theorem 2.3.1 in [6]. In fact under our assumptions (H1), (H2) and (A1), (3.3) is equivalent to (3.4). To see (3.4) implies (3.3), note that

$$g_{[\delta\gamma_n]}^{p^{(n)}}(0)^{[n\delta]} = \mathbb{P} \left[Y_{[\delta\gamma_n]}^{p^{(n)}, x} = 0 \right] = \mathbb{P}[\mathcal{E}^{p^{(n)}, x} / \gamma_n < \delta]$$

which, together with (3.4), gives

$$\liminf_{n \rightarrow \infty} g_{[\delta\gamma_n]}^{p^{(n)}}(0)^{[n\delta]} = \liminf_{n \rightarrow \infty} \mathbb{P}[\mathcal{E}^{p^{(n)}, x} / \gamma_n < \delta] \geq \mathbb{P}[\mathcal{E}^{\psi, x} < \delta] > 0,$$

where the last inequality follows from our assumption (H1); see Chapter 10 in [12] for details.

Remark 3.5. Some related work on the convergence of discrete Galton-Watson trees has been done in [6] and [8]. In [6], only the (sub)critical case was considered; see Theorem 3.6 below. In Theorem 4.15 of [8], a similar work was done using a quite different formalism. The assumptions there are same as our assumptions in the Theorem 3.3. But the convergence holds for locally compact rooted real trees in the sense of the pointed Gromov-Hausdorff distance, which is a weaker convergence. Thus Theorem 3.3 implies that the super-critical Lévy trees constructed in [1] coincides with the one studied in [8]; see also [3] and [7].

We then present a variant of Theorem 2.3.1 and Corollary 2.5.1 in [6] which is essential to our proof of Theorem 3.3.

Theorem 3.6. (Theorem 2.3.1 and Corollary 2.5.1 of [6]) Let ψ be a (sub)critical branching mechanism satisfying (H1). Assume that (A1) or (A2) holds. Suppose in addition that for every $\delta > 0$,

$$\liminf_{n \rightarrow \infty} g_{[\delta\gamma_n]}^{p^{(n)}}(0)^n > 0. \quad (3.6)$$

Then

$$\frac{1}{\gamma_n} \mathcal{E}^{p^{(n)},x} \xrightarrow{(d)} \mathcal{E}^{\psi,x} \text{ on } [0, +\infty) \quad (3.7)$$

and for any bounded continuous function F on $C(\mathbb{R}^+, \mathbb{R}^+) \times D(\mathbb{R}^+, \mathbb{R}^+)$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[F \left(\pi_a \left(\gamma_n^{-1} C^{p^{(n)},x}(2n\gamma_n \cdot) \right), \left(\frac{1}{n} Y_{[\gamma_n a]}^{p^{(n)},x} \right)_{a \geq 0} \right) \right] = \mathbb{E}_x^\psi [F(\pi_a(e), (Z_a)_{a \geq 0})]. \quad (3.8)$$

Proof. The comments following Theorem 2.3.1 in [6] give (3.7). And by Corollary 2.5.1 in [6], we have

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[F \left(\left(\gamma_n^{-1} C^{p^{(n)},x}(2n\gamma_n \cdot) \right), \left(\frac{1}{n} Y_{[\gamma_n a]}^{p^{(n)},x} \right)_{a \geq 0} \right) \right] = \mathbb{E}_x^\psi [F(e, (Z_a)_{a \geq 0})]. \quad (3.9)$$

On the other hand, let \mathcal{C}_a be the set of discontinuities of π_a . (2.9) yields

$$\Gamma_{e,a}(x) = \int_0^x 1_{\{e_t \leq a\}} dt = \int_{\mathbb{R}^+} 1_{\{s \leq a\}} L_x^s ds = \int_0^x 1_{\{e_t < a\}} dt, \quad \mathbb{P}_x^\psi - a.s. \quad (3.10)$$

Then by arguments on page 746 in [14], $\mathbb{P}_x^\psi(\mathcal{C}_a) = 0$. Then (3.8) follows readily from Theorem 2.7 in [5]. \square

Recall that γ is the largest root of $\psi(\lambda) = 0$.

Lemma 3.7. Let ψ be a branching mechanism satisfying (H1) and (H2). Assume that (A1) or (A2) holds. Suppose in addition that for every $\delta > 0$,

$$\liminf_{n \rightarrow \infty} g_{[\delta\gamma_n]}^{p^{(n)}}(0)^n > 0. \quad (3.11)$$

Then as $n \rightarrow \infty$,

$$f(p^{(n)})^{[nx]} \rightarrow e^{-\gamma x}, \quad x > 0. \quad (3.12)$$

Proof. Recall that $f(p^{(n)})$ denotes the minimal solution of $g^{p^{(n)}}(s) = s$. For each $n \geq 1$, define

$$q_k^{(n)} = f(p^{(n)})^{k-1} p_k^{(n)}, \quad k \geq 1 \quad \text{and} \quad q_0^{(n)} = 1 - \sum_{k \geq 1} q_k^{(n)}.$$

Then $q^{(n)} = \{q_k^{(n)} : k \geq 0\}$ is a probability distribution with generating function given by

$$g^{q^{(n)}}(s) = g^{p^{(n)}}(sf(p^{(n)})) / f(p^{(n)}), \quad 0 \leq s \leq 1. \tag{3.13}$$

Thus $g^{q^{(n)}}(0) = g^{p^{(n)}}(0) / f(p^{(n)})$ and by induction we further have for all $k \geq 1$,

$$g_{k+1}^{q^{(n)}}(0) = g^{q^{(n)}}(g_k^{q^{(n)}}(0)) = g^{p^{(n)}}(g_k^{q^{(n)}}(0)f(p^{(n)})) / f(p^{(n)}) = g_{k+1}^{p^{(n)}}(0) / f(p^{(n)}). \tag{3.14}$$

With (3.11), we see that for any $\delta > 0$,

$$1 \geq \liminf_{n \rightarrow \infty} g_{[\delta\gamma_n]}^{q^{(n)}}(0)^n = \liminf_{n \rightarrow \infty} g_{[\delta\gamma_n]}^{p^{(n)}}(0)^n / f(p^{(n)})^n \geq \liminf_{n \rightarrow \infty} g_{[\delta\gamma_n]}^{p^{(n)}}(0)^n > 0. \tag{3.15}$$

Let $\gamma_0 \in [0, \infty)$ be such that $e^{-\gamma_0} = \liminf_{n \rightarrow \infty} f(p^{(n)})^n > 0$. Since $f(p^{(n)}) \leq 1$, we may write $f(p^{(n)}) = e^{-a_n/n}$ for some $a_n \geq 0$. We further have $\limsup_{n \rightarrow \infty} a_n = \gamma_0$. We shall show that $\gamma_0 = \gamma$ and $\{a_n : n \geq 1\}$ is a convergent sequence. To this end, let $\{a_{n_k} : k \geq 1\}$ be a convergent subsequence of $\{a_n : n \geq 1\}$ with $\lim_{k \rightarrow \infty} a_{n_k} =: \tilde{\gamma} \leq \gamma_0$. Then by (A1),

$$0 = n_k \gamma_{n_k} [g^{p^{(n_k)}}(e^{-a_{n_k}/n_k}) - e^{-a_{n_k}/n_k}] \rightarrow \psi(\tilde{\gamma}), \quad \text{as } k \rightarrow \infty.$$

Thus $\psi(\tilde{\gamma}) = 0$. On the other hand, note that ψ is a convex function with $\psi(0) = 0$ and γ is the largest root of $\psi(\lambda) = 0$. Then we have $\psi(\lambda_1) < 0$ and $\psi(\lambda_2) > 0$ for $0 < \lambda_1 < \gamma < \lambda_2$. If $\tilde{\gamma} \neq \gamma$, then $\tilde{\gamma} = 0$. In this case, we may find a sequence $\{b_{n_k} : k \geq 1\}$ with $b_{n_k} > a_{n_k}$ for all $k \geq 1$ such that $b_{n_k} \rightarrow \gamma$ and for k sufficiently large

$$g^{p^{(n_k)}}(e^{-b_{n_k}/n_k}) - e^{-b_{n_k}/n_k} = 0.$$

This contradicts the fact that $f(p^{(n)}) = e^{-a_n/n}$ is the minimal solution of $g^{p^{(n)}}(s) = s$. Thus $\tilde{\gamma} = \gamma$ which implies that $\lim_{n \rightarrow \infty} a_n = \gamma$ and $\lim_{n \rightarrow \infty} f(p^{(n)})^{[nx]} = e^{-\gamma x}$ for any $x > 0$. \square

We are in the position to prove Theorem 3.3.

Proof of Theorem 3.3: With Theorem 3.6 in hand, we only need to prove the result when ψ is super-critical. The proof will be divided into three steps.

First step: One can deduce from (A1) and (3.12) that

$$\begin{aligned} & n\gamma_n [g^{q^{(n)}}(e^{-\lambda/n}) - e^{-\lambda/n}] \\ &= n\gamma_n [g^{p^{(n)}}(e^{-\lambda/n} f(p^{(n)})) - e^{-\lambda/n} f(p^{(n)})] / f(p^{(n)}) \\ &\rightarrow \psi(\lambda + \gamma), \quad \text{as } n \rightarrow \infty, \end{aligned} \tag{3.16}$$

uniformly on any bounded interval. Then Lemma 3.1 and Theorem 3.6, together with (3.15) and (3.16), imply that

$$\frac{1}{\gamma_n} \mathcal{E}^{q^{(n)},x} \xrightarrow{(d)} \mathcal{E}^{\psi_\gamma,x} \text{ on } [0, +\infty) \tag{3.17}$$

and for any bounded continuous function F on $C(\mathbb{R}^+, \mathbb{R}^+) \times D(\mathbb{R}^+, \mathbb{R})$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[F \left(\pi_a \left((\gamma_n^{-1} C^{q^{(n)},x}(2n\gamma_n \cdot)), \left(\frac{1}{n} Y_{[\gamma_n a]}^{q^{(n)},x} \right)_{a \geq 0} \right) \right) \right] = \mathbb{E}_x^{\psi_\gamma} [F(\pi_a(e), (Z_a)_{a \geq 0})] \tag{3.18}$$

Second step: We shall prove (3.4). Note that

$$\{\mathcal{E}^{p^{(n)},x} < \infty\} = \{\mathcal{G}_i^{p^{(n)}}, i = 1, \dots, [nx] \text{ are finite trees}\}.$$

Then by Corollary 2.3, for $f \in C(\mathbb{R}^+, \mathbb{R}^+)$,

$$\mathbb{E} \left[f \left(\mathcal{E}^{p^{(n)},x} / \gamma_n \right) 1_{\{\mathcal{E}^{p^{(n)},x} < \infty\}} \right] = f(p^{(n)})^{[nx]} \mathbb{E} \left[f \left(\mathcal{E}^{q^{(n)},x} / \gamma_n \right) \right]$$

which, by (3.12), (3.17) and Lemma 2.4, converges to

$$e^{-\gamma x} \mathbb{E} \left[f \left(\mathcal{E}^{\psi_\gamma, x} \right) \right] = \mathbb{E} \left[f \left(\mathcal{E}^{\psi, x} \right) 1_{\{\mathcal{E}^{\psi, x} < \infty\}} \right],$$

as $n \rightarrow \infty$. We also have that

$$\mathbb{P}[\mathcal{E}^{p^{(n)},x} = \infty] = 1 - f(p^{(n)})^{[nx]} \rightarrow 1 - e^{-\gamma x} = \mathbb{P}[\mathcal{E}^{\psi, x} = \infty], \quad \text{as } n \rightarrow \infty,$$

which gives (3.4).

Third step: We shall prove (3.5). By Corollary 2.3, for any continuous bounded nonnegative function F on $C(\mathbb{R}^+, \mathbb{R}^+)$ and $a \geq 0$,

$$\mathbb{E} \left[F(C_{[a]}^{p^{(n)},x}(\cdot)) \right] = \mathbb{E} \left[f(p^{(n)})^{[nx] - Y_{[a]}^{q^{(n)},x}} F(C_{[a]}^{q^{(n)},x}(\cdot)) \right]. \tag{3.19}$$

Note that

$$C_a^{q^{(n)},x} = \pi_a C_{[a]}^{q^{(n)},x} \quad \text{and} \quad \pi_a(\gamma_n^{-1} C^{q^{(n)},x}) = \gamma_n^{-1} C_{\gamma_n a}^{q^{(n)},x}.$$

Then by (3.19) we have for $a \in \mathbb{R}^+$

$$\mathbb{E} \left[F(C_a^{p^{(n)},x}(\cdot)) \right] = \mathbb{E} \left[f(p^{(n)})^{[nx] - Y_{[a]}^{q^{(n)},x}} F(C_a^{q^{(n)},x}(\cdot)) \right] \tag{3.20}$$

and

$$\mathbb{E} \left[F \left(\gamma_n^{-1} C_{\gamma_n a}^{p^{(n)},x}(2n\gamma_n \cdot) \right) \right] = \mathbb{E} \left[f(p^{(n)})^{[nx] - Y_{[\gamma_n a]}^{q^{(n)},x}} F \left(\pi_a \left(\gamma_n^{-1} C^{q^{(n)},x}(2n\gamma_n \cdot) \right) \right) \right].$$

We shall show that $\{f(p^{(n)})^{[nx] - Y_{[\gamma_n a]}^{q^{(n)},x}}, n \geq 1\}$ is uniformly integrable. Write $Y_a^n = Y_{[\gamma_n a]}^{q^{(n)},x} / n$ for simplicity. First, note that $\mathbb{E} [f(p^{(n)})^{[nx] - nY_a^n}] = 1$. Then with (3.12) and (3.18) in hand, we have

$$\lim_{l \rightarrow \infty} \lim_{n \rightarrow \infty} \mathbb{E} \left[f(p^{(n)})^{[nx] - n(l \wedge Y_a^n)} \right] = \lim_{l \rightarrow \infty} \mathbb{E}_x^{\psi_\gamma} \left[e^{-\gamma x + \gamma(l \wedge Z_a)} \right] = \mathbb{E}_x^{\psi_\gamma} \left[e^{-\gamma x + \gamma Z_a} \right] = 1,$$

by bounded convergence theorem for the limit in n and by monotone convergence for the limit in l . Note that both $\mathbb{E}_x^{\psi_\gamma} [e^{-\gamma x + \gamma(l \wedge Z_a)}]$ and $\mathbb{E} [f(p^{(n)})^{[nx] - n(l \wedge Y_a^n)}]$ are increasing in l . Thus for every $\epsilon > 0$, there exist l_0 and n_0 such that for all $l > l_0$ and $n > n_0$,

$$1 - \epsilon/2 < \mathbb{E} \left[f(p^{(n)})^{[nx] - n(l \wedge Y_a^n)} \right] \leq 1.$$

Meanwhile, since

$$\lim_{l \rightarrow \infty} \mathbb{E} \left[f(p^{(n)})^{[nx] - n(l \wedge Y_a^n)} \right] = \mathbb{E} \left[f(p^{(n)})^{[nx] - nY_a^n} \right] = 1,$$

there exists $l_1 > 0$ such that for all $n \geq 1$,

$$1 - \epsilon/2 < \mathbb{E} \left[f(p^{(n)})^{[nx] - n(l_1 \wedge Y_a^n)} \right] \leq \mathbb{E} \left[f(p^{(n)})^{[nx] - nY_a^n} \right] = 1. \tag{3.21}$$

Set

$$A_n = \mathbb{E} \left[F \left(\gamma_n^{-1} C_{\gamma_n a}^{p^{(n)},x}(2n\gamma_n \cdot) \right) \right]$$

and

$$A_{n,l} = \mathbb{E} \left[f(p^{(n)})^{[nx] - n(l \wedge Y_a^n)} F \left(\gamma_n^{-1} C_{\gamma_n a}^{q^{(n)}, x}(2n\gamma_n \cdot) \right) \right].$$

By (3.21), for any $\epsilon > 0$, there exists l_1 such that for any $l \geq l_1$ and $n \geq 1$,

$$0 \leq A_n - A_{n,l} \leq \|F\|_\infty \epsilon / 2.$$

Meanwhile, (3.18) implies that for any l ,

$$\lim_{n \rightarrow \infty} A_{n,l} = \mathbb{E}_x^{\psi_\gamma} \left[e^{-\gamma x + \gamma(l \wedge Z_a)} F(\pi_a e) \right].$$

Thus for any $l > l_1$,

$$\begin{aligned} \mathbb{E}_x^{\psi_\gamma} \left[e^{-\gamma x + \gamma(l \wedge Z_a)} F(\pi_a e) \right] &\leq \liminf_{n \rightarrow \infty} A_n \leq \limsup_{n \rightarrow \infty} A_n \\ &\leq \mathbb{E}_x^{\psi_\gamma} \left[e^{-\gamma x + \gamma(l \wedge Z_a)} F(\pi_a e) \right] + \|F\|_\infty \epsilon / 2, \end{aligned}$$

which implies (3.5) since

$$\lim_{l \rightarrow \infty} \mathbb{E}_x^{\psi_\gamma} \left[e^{-\gamma x + \gamma(l \wedge Z_a)} F(\pi_a e) \right] = \mathbb{E}_x^{\psi_\gamma} \left[e^{-\gamma x + \gamma Z_a} F(\pi_a e) \right]$$

by monotone convergence. We have completed the proof. \square

Remark 3.8. Write $C_t^n = \gamma_n^{-1} C_{\gamma_n a}^{p^{(n)}, x}(2n\gamma_n t)$ for simplicity and recall that $(w^a, a \geq 0)$ denotes the canonical process on \mathcal{W} . Suppose that the assumptions of Theorem 3.3 are satisfied. Then one can construct a sequence of probability measures $\bar{\mathbb{P}}_x^{p^n}$ on \mathcal{W} such that for every $a \geq 0$, the distribution of w^a under $\bar{\mathbb{P}}_x^{p^n}$ is the same as $\pi_a(C^n)$ and for $0 \leq a \leq b$,

$$\pi_a(w^b) = \pi_a \quad \bar{\mathbb{P}}_x^{p^{(n)}} - a.s.$$

We then have

$$\bar{\mathbb{P}}_x^{p^n} \rightarrow \bar{\mathbb{P}}_x^{\psi} \text{ as } n \rightarrow \infty. \quad (3.22)$$

Remark 3.9. In [1], an excursion measure ('distribution' of a single tree) was also defined. However, we could not find an easy proof of convergence of trees under such excursion measure.

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