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Hyperbolic scaling limit of non-equilibrium fluctuations for a weakly anharmonic chain*

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Abstract

We consider a chain of n coupled oscillators placed on a one-dimensional lattice with periodic boundary conditions. The interaction between particles is determined by a weakly anharmonic potential $V_n = r^2/2 + \sigma_n U(r)$, where U has bounded second derivative and σ_n vanishes as $n \to \infty$. The dynamics is perturbed by noises acting only on the positions, such that the total momentum and length are the only conserved quantities. With relative entropy technique, we prove for dynamics out of equilibrium that, if σ_n decays sufficiently fast, the fluctuation field of the conserved quantities converges in law to a linear p-system in the hyperbolic space-time scaling limit. The transition speed is spatially homogeneous due to the vanishing anharmonicity. We also present a quantitative bound for the speed of convergence to the corresponding hydrodynamic limit.

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1 Introduction

One of the central topics in statistical physics is to derive macroscopic equations in scaling limits of microscopic dynamics. For Hamiltonian lattice field, Euler equations can be formally obtained in the limit, under a generic assumption of local equilibrium. However, to prove this for deterministic dynamics is known as a difficult task. In particular when nonlinear interaction exists, the appearance of shock waves in the Euler equations complicates further the problem. In that case, the convergence to the entropy solution is expected.

The situation is better understood when the microscopic dynamics is perturbed stochastically. Proper noises can provide the dynamics with enough ergodicity, in

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the sense that the only conserved quantities are those evolving with the macroscopic equations [13]. The deduction of partial differential equations from the limit of properly rescaled conserved quantities in these dynamics is called *hydrodynamic limit*. For Hamiltonian dynamics with noises conserving volume, momentum and energy, Euler equations are obtained under the hyperbolic space-time scale [23, 4]. They are proved by relative entropy technique and restricted to the smooth regime of Euler equations.

As hydrodynamic limit can be viewed as the law of large numbers in functional spaces, we can go one step further towards the corresponding central limit theorem. More precisely, we can investigate the macroscopic time evolution of the fluctuations of the conserved quantities around its hydrodynamic centre. If the dynamics is in its equilibrium, these fluctuations are Gaussian and evolve following linearized equations, known as *equilibrium fluctuation*. To prove it requires to approximate the space-time variance of the currents associated to the conserved quantities by their linear functions. This step is usually called the *Boltzmann–Gibbs principle* [5, 18]. For gradient, reversible systems, a general proof of the Boltzmann–Gibbs principle is established in [6] using entropy method. In other cases, such as anharmonic Hamiltonian dynamics, the proof usually relies on model-dependent arguments, such as the spectral gap [22, 24].

Our main interest is *non-equilibrium fluctuation*, namely the central limit theorem associated to the corresponding hydrodynamic limit for dynamics out of equilibrium. Compared to the equilibrium case, the non-equilibrium fluctuation field exhibits long-range space-time correlations, which turns out to be the main difficulty. For some dynamics such as symmetric exclusion process (SSEP) and reaction-diffusion model, duality method can be used to control the correlations and obtain the non-equilibrium version of the Boltzmann–Gibbs principle [21, 10, 3, 25]. For one-dimensional weakly asymmetric exclusion process (WASEP), a microscopic Cole–Hopf transformation [14] can be applied, instead of the Boltzmann–Gibbs principle, to linearize the currents [8, 26, 2]. While most works deal with the diffusive space-time scale, the totally asymmetric exclusion process (TASEP) is the only model in which non-equilibrium fluctuation is proved under the hyperbolic scale [27]. Note that all these works are restricted to models with *stochastic integrability* and *single conservation law*.

In the absence of stochastic integrability, non-equilibrium fluctuations are understood for only few models. In [7], an Ornstein–Uhlenbeck process is obtained from non-equilibrium fluctuations for one-dimensional Ginzburg–Landau model using logarithmic Sobolev inequality. A general derivation of non-equilibrium fluctuations for conservative systems has been largely open for a long period of time since then. Recently in [16, 17], a new approach is developed and applied to spatially inhomogeneous WASEP in dimensions $d < 4$. Their main tool is relative entropy technique. Briefly speaking, Yau's relative entropy inequality [30] says that the derivative of the relative entropy with respect to a given local Gibbs measure is bounded by a dissipative term and an entropy production term. In [16, 17], the authors obtain an estimate allowing them to control the entropy production term by the dissipative term, which they called the key lemma. An entropy estimate then follows directly from this lemma. Using both the lemma and the entropy estimate as input, Boltzmann–Gibbs principle can be proved by a generalized Feyman–Kac inequality [16, Lemma 3.5].

In the present article we study non-equilibrium fluctuations for a Hamiltonian lattice field under the hyperbolic scale. Observe that part of the ideas in [16, 17] is robust enough to be applied to our model, cf. Section 5. Meanwhile, the proof of the key lemma relies heavily on the particular basis of the local functions on the configuration space of WASEP. In Section 3 we establish a similar estimate for Hamiltonian dynamics. The main tools we used are the Poisson equation and the equivalence of ensembles, see Section 8 and 9 for details.

The microscopic model we study is a noisy Hamiltonian system on one-dimensional lattice space with *vanishing anharmonicity* and *two conservation laws*. Precisely speaking, consider a chain of n coupled oscillators, each of them has mass 1. For $i = 0, 1, \dots, n$, denote by $(p_i, q_i) \in \mathbb{R}^2$ the momentum and position of the particle i. The periodic boundary condition $(p_0, q_0) = (p_n, q_n)$ is applied to the chain.

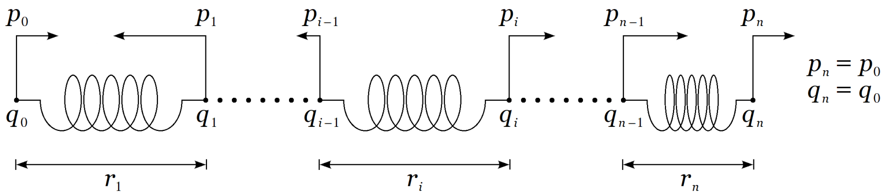

Figure 1: Chain of oscillators with periodic boundary

Each pair of consecutive particles $i-1$ and i is connected by a spring with potential defined by $V(q_i - q_{i-1})$, where V is a nice function on $\mathbb{R}$. With $r_i = q_i - q_{i-1}$ being the relative position, the energy of the chain is given by the Hamiltonian

$$H_n(\mathbf{p}, \mathbf{r}) = \sum_{i=1}^{n} \frac{p_i^2}{2} + V(r_i).$$

When V is quadratic, the corresponding Hamiltonian dynamics is harmonic, and the macroscopic behaviour is known to be purely ballistic. We add an anharmonic perturbation to the quadratic potential and define

$$V_\sigma(r) = \frac{r^2}{2} + \sigma U(r), \quad \forall r \in \mathbb{R},$$

where U is a smooth function with good properties, and $\sigma > 0$ is a small parameter which regulates the nonlinearity. When $\sigma > 0$ is fixed we say the potential is *anharmonic*, whereas $\sigma \to 0$ is the *weakly anharmonic* case.

The deterministic Hamiltonian dynamics is perturbed by random, continuous exchange of volume stretch (r_i, r_{i+1}) for each i, such that $r_i + r_{i+1}$ is conserved. The corresponding micro canonical surface is a line, where we add a Wiener process. This stochastic perturbation is generated by a symmetric second order differential operator $\mathcal{S}_{n,\sigma}$ defined later in (2.1). The noise does not conserve $V(r_i) + V(r_{i+1})$, thus breaks the conservation law of energy. Notice that the total momentum is naturally another conserved quantity, which is untouched by the noise. Similar noise that destroys the energy conservation is also adopted in [12]. Note that the noise in [12] includes also the exchange of momentum between the nearest neighbour particles. In our case the noise on momentum can be dropped, thanks to the linear construction of the momentum fluctuation in the microscopic level. We choose the noise in such way that the momentum and volume are the only conserved quantities, hence the equilibrium states are given by canonical Gibbs measures at a fixed temperature $\beta^{-1} > 0$.

For the anharmonic case, the hydrodynamic equation is

$$\partial_t \mathfrak{p}(t,x) = \partial_x \boldsymbol{\tau}_\sigma(\mathfrak{r}(t,x)), \quad \partial_t \mathfrak{r}(t,x) = \partial_x \mathfrak{p}(t,x),$$

where $\boldsymbol{\tau}_\sigma$ is the equilibrium tension defined later in (2.4). It is proved in [23] in smooth regime. Denote by $(\mathfrak{p}_\sigma, \mathfrak{r}_\sigma)$ the solution of the equation above. Consider the fluctuation field of the conserved quantities along the hydrodynamic equation, given by

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \begin{pmatrix} p_i(t) - \mathfrak{p}_\sigma(t, i/n) \\ r_i(t) - \mathfrak{r}_\sigma(t, i/n) \end{pmatrix} \delta\left(x - \frac{i}{n}\right).$$

Formally, it is expected to converge to a solution of the linearized system

$$\partial_t \tilde{\mathfrak{p}}_\sigma(t,x) = \boldsymbol{\tau}'_\sigma(\mathfrak{r}_\sigma)\partial_x \tilde{\mathfrak{r}}_\sigma(t,x), \quad \partial_t \tilde{\mathfrak{r}}_\sigma(t,x) = \partial_x \tilde{\mathfrak{p}}_\sigma(t,x).$$

Particularly for the equilibrium system, $(\mathfrak{p}_\sigma, \mathfrak{r}_\sigma)$ degenerates to constants and the fluctuation equation is proved in [24], even with the energy conservation and boundary conditions. Non-equilibrium fluctuations for anharmonic dynamics remain an open problem.

We work with the weakly anharmonic case that $\sigma = \sigma_n$ depends on the scaling parameter n in such way that $\sigma_n = o(1)$. Similar model with vanishing anharmonicity is also considered in [1], where the authors take the FPU-type perturbation $U = r^4$ and the flip-type noise conserving the total energy as well as the sum of the total volume and momentum. Although the main interest of [1] lies in the anomalous diffusion of energy fluctuation, they also prove that under the hyperbolic scale, the time evolution of the fluctuation field of the equilibrium dynamics is governed by a p-system. Our main result, Theorem 2.4, shows that non-equilibrium fluctuations evolve following a linear p-system with spatially homogeneous sound speed, provided that U has bounded second order derivative and σ_n decays fast enough. This is the first rigorous result obtained for non-equilibrium fluctuations for a Hamiltonian dynamics presenting some level of nonlinearity. We also prove a quantitative version of the corresponding hydrodynamic limit in Corollary 2.3.

We believe that the macroscopic fluctuation equation proved in this work should be valid with noises acting only on momentum, but the answer is unclear even when the dynamics is in equilibrium. Another interesting problem concerns the presentation of boundary conditions in the fluctuation. Boundary driven non-equilibrium fluctuations are studied for one-dimensional SSEP in [19, 11] and for WASEP in [15]. However for Hamiltonian dynamics, it is only studied for equilibrium dynamics [24].

The article is organized as follows. In Section 2 we present the precise definition of the microscopic dynamics and state our main results. In Section 3 we prove the technical lemma, relying on the equivalence of ensembles under inhomogeneous canonical measures and a gradient estimate for the solution of the Poisson equation. In Section 4 we prove the relative entropy estimate Theorem 2.2, based on the technical lemma. We also prove the quantitative hydrodynamics limit Corollary 2.3 as an application of Theorem 2.2. In Section 5 we prove the Boltzmann–Gibbs principle out of equilibrium, along the approach introduced in [16, 17]. In Section 6 and 7 we prove the two aspects of the weak convergence of non-equilibrium fluctuations in Theorem 2.4, namely the finite-dimensional convergence and the tightness. In Section 8 and 9 we establish the equivalence of ensembles and the gradient estimate for the Poisson equation, respectively. Both of them play an important role in the proof of the technical lemma. Finally, some auxiliary estimates are collected in the appendix.

We close this section with some notations used through the article. Let $\mathbb{T} \sim [0,1)$ be the one-dimensional torus. For a bounded function $f : \mathbb{T} \to \mathbb{R}^d$, define

$$|f|_{\mathbb{T}} = \sup_{\mathbb{T}} |f(x)|_{\mathbb{R}^d}, \quad \|f\|^2 = \int_{\mathbb{T}} |f(x)|^2_{\mathbb{R}^d} dx.$$

Let $\{\varphi_m, m \in \mathbb{Z}\}$ be the Fourier basis on $\mathbb{T}$ given by $\varphi_m(x) = e^{2mx\pi i}$. For a smooth function $f \in C^\infty(\mathbb{T};\mathbb{R}^2)$ and $k \in \mathbb{R}$, define

$$\|f\|_k^2 = \sum_{m\in\mathbb{Z}} \frac{\left|\hat{f}(m)\right|^2_{\mathbb{C}^2}}{(1+m^2)^k}, \quad \hat{f}(m) = \int_{\mathbb{T}} f(x)\overline{\varphi_m(x)}dx.$$

Define the Sobolev space $\mathcal{H}_k(\mathbb{T})$ as the closure of $C^\infty(\mathbb{T};\mathbb{R}^2)$ with respect to the norm $\|\cdot\|_k$. By a standard dual argument, we can identify $\mathcal{H}_{-k}(\mathbb{T})$ with the space of linear functionals on $C^\infty(\mathbb{T};\mathbb{R}^2)$ which is continuous with respect to $\|\cdot\|_k$. For $T>0$, $C([0,T];\mathcal{H}_{-k})$ denotes the set of all continuous trajectories on $[0,T]$ taking values in $\mathcal{H}_{-k}$, equipped with the uniform topology. Also let $C^\alpha([0,T];\mathcal{H}_{-k})$ be the subset of $C([0,T];\mathcal{H}_{-k})$, consisting of Hölder continuous trajectories with order $\alpha>0$.

2 Microscopic model and main results

For $n\in\mathbb{N}_+$, denote by $\mathbb{T}_n=\mathbb{Z}/n\mathbb{Z}$ the one-dimensional discrete n-torus, and let $\Omega_n=(\mathbb{R}^2)^{\mathbb{T}_n}$ be the configuration space. Elements in Ω_n are denoted by $\vec{\eta}=\{\eta_i;i\in\mathbb{T}_n\}$, where $\eta_i=(p_i,r_i)\in\mathbb{R}^2$. Let U be a smooth function on $\mathbb{R}$ with bounded second order derivative. To simplify the arguments, we assume that

$$U(0)=U'(0)=0,\quad U''(r)\in[-1,1],\quad \forall r\in\mathbb{R}.$$

For $\sigma\in[0,1)$, which is supposed to be small eventually, define

$$V_\sigma(r)=\frac{r^2}{2}+\sigma U(r),\quad \forall r\in\mathbb{R}.$$

Note that V_σ is a smooth function with quadratic growth:

$$\inf V_\sigma''\geq 1-\sigma>0,\quad \sup V_\sigma''\leq 1+\sigma<\infty.$$

Define the Hamiltonian $H_{n,\sigma}=\sum_{i\in\mathbb{T}_n}p_i^2/2+V_\sigma(r_i)$. The corresponding Hamiltonian system is generated by the following Liouville operator

$$\begin{aligned}\mathcal{A}_{n,\sigma}&=\sum_{i\in\mathbb{T}_n}(p_i-p_{i-1})\frac{\partial}{\partial r_i}+\big(V_\sigma'(r_{i+1})-V_\sigma'(r_i)\big)\frac{\partial}{\partial p_i}\\&=\sum_{i\in\mathbb{T}_n}(p_i-p_{i-1})\frac{\partial}{\partial r_i}+(r_{i+1}-r_i)\frac{\partial}{\partial p_i}+\sigma\big(U'(r_{i+1})-U'(r_i)\big)\frac{\partial}{\partial p_i}\end{aligned}$$

At each bond $(i,i+1)$, the deterministic system is contact with a thermal bath at fixed temperature. More precisely, fix some $\beta>0$ and define

$$\mathcal{Y}_i=\frac{\partial}{\partial r_{i+1}}-\frac{\partial}{\partial r_i},\quad \mathcal{Y}_{i,\sigma}^*=\beta\big(V_\sigma'(r_{i+1})-V_\sigma'(r_i)\big)-\mathcal{Y}_i.$$

Notice that β is fixed through this article, thus we omit the dependence on it in most cases. For $\gamma>0$, consider the operator $\mathcal{L}_{n,\sigma,\gamma}$, given by

$$\mathcal{L}_{n,\sigma,\gamma}=n\big(\mathcal{A}_{n,\sigma}+\gamma\mathcal{S}_{n,\sigma}\big),\quad \mathcal{S}_{n,\sigma}=-\frac{1}{2}\sum_{i\in\mathbb{T}_n}\mathcal{Y}_{i,\sigma}^*\mathcal{Y}_i,\tag{2.1}$$

where γ regulates the strength of the noise. With an infinite system of independent, standard Brownian motions $\{B^i;i\geq 1\}$, the Markov process generated by $\mathcal{L}_{n,\sigma,\gamma}$ can be expressed by the solution of the following system of stochastic differential equations:

$$\begin{cases}dp_i(t)=n\big(V_\sigma'(r_{i+1})-V_\sigma'(r_i)\big)dt,\\dr_i(t)=n(p_{i+1}-p_i)dt+\dfrac{n\beta\gamma}{2}\big(V_\sigma'(r_{i+1})+V_\sigma'(r_{i-1})-2V_\sigma'(r_i)\big)dt\\\qquad\qquad+\sqrt{n\gamma}\big(dB_t^{i-1}-dB_t^i\big),\qquad \forall i\in\mathbb{T}_n.\end{cases}$$

It can be treated as the dynamics of the chain of oscillators illustrated in Section 1, rescaled hyperbolically and perturbed with the noise conserving the total momentum $\sum p_i$ as well as the total length $\sum r_i$. The total energy $H_{n,\sigma}$ is no longer conserved.

For $\tau \in \mathbb{R}$ and $0 \le \sigma < 1$, define the probability measure $\pi_{\tau,\sigma}$ by

$$\pi_{\tau,\sigma}(dr) = \frac{1}{Z_\sigma(\tau)} e^{-\beta(V_\sigma(r)-\tau r)} dr, \tag{2.2}$$

where $Z_\sigma(\tau)$ is the normalization constant given by

$$Z_\sigma(\tau) = \int_{\mathbb{R}} e^{-\beta(V_\sigma(r)-\tau r)} dr = \int_{\mathbb{R}} \exp\left\{ -\frac{\beta r^2}{2} - \beta\sigma U(r) + \beta\tau r \right\} dr.$$

The Gibbs potential G_σ and the free energy F_σ are then given for each $\tau \in \mathbb{R}$, $r \in \mathbb{R}$ by the following Legendre transform

$$G_\sigma(\tau) = \frac{1}{\beta} \log Z_\sigma(\tau), \quad F_\sigma(r) \triangleq \sup_{\tau \in \mathbb{R}} \big\{ \tau r - G_\sigma(\tau) \big\}. \tag{2.3}$$

Denote by $\bar{r}_\sigma$ and $\boldsymbol{\tau}_\sigma$ the corresponding convex conjugate variables

$$\bar{r}_\sigma(\tau) = E_{\pi_{\tau,\sigma}}[r] = G'_\sigma(\tau), \quad \boldsymbol{\tau}_\sigma(r) = F'_\sigma(r). \tag{2.4}$$

Observe that given any finite interval $[r_-, r_+] \in \mathbb{R}$,

$$\big|\boldsymbol{\tau}_\sigma(r) - r\big| \le C\sigma, \quad \big|\boldsymbol{\tau}'_\sigma(r) - 1\big| \le C\sigma, \quad \big|\boldsymbol{\tau}''_\sigma(r)\big| \le C\sigma \tag{2.5}$$

holds with a uniform constant C for all $r \in [r_-, r_+]$ and sufficiently small $\sigma \ge 0$. The details of these asymptotic properties are discussed in Appendix A.

For $n \ge 1$, the (grand) Gibbs states of the generator $\mathcal{L}_{n,\sigma,\gamma}$ are given by the family of product measures $\{\nu^n_{\bar{p},\tau,\sigma}; (\bar{p},\tau) \in \mathbb{R}^2\}$ on Ω_n, defined as

$$\nu^n_{\bar{p},\tau,\sigma}(d\vec{\eta}) = \prod_{i \in \mathbb{T}_n} \sqrt{\frac{\beta}{2\pi}} \exp\left\{ -\frac{\beta(p_i - \bar{p})^2}{2} \right\} dp_i \otimes \pi_{\tau,\sigma}(dr_i). \tag{2.6}$$

It is easy to see that $\mathcal{A}_{n,\sigma}$ is anti-symmetric, while $\mathcal{S}_{n,\sigma}$ is symmetric with respect to the Gibbs states, and for all smooth functions f, g on Ω_n,

$$\int_{\Omega_n} f\big(\mathcal{S}_{n,\sigma} g\big)\, d\nu^n_{\bar{p},\tau,\sigma} = -\frac{1}{2} \int_{\Omega_n} \sum_{i \in \mathbb{T}_n} \mathcal{Y}_i f \mathcal{Y}_i g \, d\nu^n_{\bar{p},\tau,\sigma}.$$

In particular, $\nu^n_{\bar{p},\tau,\sigma}$ is invariant with respect to $\mathcal{L}_{n,\sigma,\gamma}$.

2.1 Weakly anharmonic oscillators

Pick two positive sequences $\{\sigma_n\}$, $\{\gamma_n\}$ and consider the Markov process in Ω_n associated to the infinitesimal generator

$$\mathcal{L}_n = \mathcal{L}_{n,\sigma_n,\gamma_n}, \quad \forall n \ge 1.$$

Basically, we demand that $\sigma_n \to 0$, $\gamma_n \ge 1$ and $\gamma_n = o(n)$. These conditions correspond to a weakly anharmonic interaction and assure that the noise would not appear in the hyperbolic scaling limit. From here on, we denote

$$V_n = V_{\sigma_n}, \quad \mathcal{S}_n = \mathcal{S}_{n,\sigma_n}, \quad \bar{r}_n = \bar{r}_{\sigma_n}, \quad \boldsymbol{\tau}_n = \boldsymbol{\tau}_{\sigma_n} \tag{2.7}$$

for short. For any fixed $T > 0$, denote by

$$\big\{ \vec{\eta}(t) = (\eta_i(t); i \in \mathbb{T}_n) \in \Omega_n; t \in [0,T] \big\}$$

the Markov process generated by $\mathcal{L}_n$ and initial distribution ν_n on Ω_n. This is the main subject treated in this article. Denote by $\mathbb{P}_n$, $\mathbb{E}_n$ the corresponding distribution and expectation on the trajectory space $C([0,T]; \Omega_n)$ of $\vec{\eta}(\cdot)$, respectively.

2.2 Hydrodynamic limit

We start from the anharmonic case $\sigma_n \equiv \sigma \in (0,1)$. Let $\mathbb{P}_{n,\sigma}$ denote the law of the Markov process generated by $\mathcal{L}_{n,\sigma,1}$ and ν_n. Assume some profile $\mathfrak{v} \in C^2(\mathbb{T};\mathbb{R}^2)$, such that for any smooth function g on $\mathbb{T}$,

$$\lim_{n\to\infty} \nu_n \left\{ \left| \frac{1}{n} \sum_{i\in\mathbb{T}_n} g\left(\frac{i}{n}\right) \eta_i(0) - \int_{\mathbb{T}} g(x)\mathfrak{v}(x)dx \right| > \epsilon \right\} = 0, \quad \forall \epsilon > 0.$$

The hydrodynamic limit is then given by the following convergence

$$\lim_{n\to\infty} \mathbb{P}_{n,\sigma} \left\{ \left| \frac{1}{n} \sum_{i\in\mathbb{T}_n} g\left(\frac{i}{n}\right) \eta_i(t) - \int_{\mathbb{T}} g(x) \begin{pmatrix} \mathfrak{p}_\sigma \\ \mathfrak{r}_\sigma \end{pmatrix} (t,x)dx \right| > \epsilon \right\} = 0, \tag{2.8}$$

for all $\epsilon > 0$. Here $(\mathfrak{p}_\sigma, \mathfrak{r}_\sigma)$ solves the *quasi-linear p-system*:

$$\partial_t \mathfrak{p}_\sigma = \partial_x \boldsymbol{\tau}_\sigma(\mathfrak{r}_\sigma), \quad \partial_t \mathfrak{r}_\sigma = \partial_x \mathfrak{p}_\sigma, \quad (\mathfrak{p}_\sigma, \mathfrak{r}_\sigma)(0,\cdot) = \mathfrak{v}, \tag{2.9}$$

where $\boldsymbol{\tau}_\sigma = \boldsymbol{\tau}_\sigma(r)$ is the *equilibrium tension* given in (2.4). Note that the Lagrangian material coordinate is considered as the space variable. It is well known that even with smooth initial data, (2.9) generates shock wave in finite time T_σ. With the arguments in [4], (2.8) can be proved in its smooth regime, that is, for any $t < T_\sigma$.

Now we return to the weakly anharmonic case. To simplify the notations, denote by $(\mathfrak{p}_n, \mathfrak{r}_n)$ the solution of (2.9) with $\sigma = \sigma_n$. The next proposition allows us to consider only the smooth regime of $(\mathfrak{p}_n, \mathfrak{r}_n)$ for any $T > 0$.

Proposition 2.1. $\lim_{\sigma\downarrow 0} T_\sigma = +\infty$. *In particular, for any fixed time $T > 0$, we can choose n_0 sufficiently large, such that $(\mathfrak{p}_n, \mathfrak{r}_n)$ is smooth on $[0,T]$ for all $n \geq n_0$.*

Proposition 2.1 follows directly from (2.5) and Lemma B.1 in Appendix B. It is not hard to observe that the hydrodynamic equation associated to the weakly anharmonic chain turns out to be the linear p-system

$$\partial_t \mathfrak{p} = \partial_x \mathfrak{r}, \quad \partial_t \mathfrak{r} = \partial_x \mathfrak{p}, \quad (\mathfrak{p}, \mathfrak{r})(0,\cdot) = \mathfrak{v}. \tag{2.10}$$

We prove a quantitative convergence in Corollary 2.3 later.

2.3 Relative entropy

For a probability measure μ on a measurable space Ω, and a density function f with respect to μ, its *relative entropy* is defined by

$$H(f;\mu) = \int_\Omega f \log f d\mu. \tag{2.11}$$

Given $T > 0$, let $(\mathfrak{p}_i^n, \mathfrak{r}_i^n)$ be the interpolation of $(\mathfrak{p}_n, \mathfrak{r}_n)$ in (2.9):

$$(\mathfrak{p}_i^n, \mathfrak{r}_i^n)(t) = (\mathfrak{p}_n, \mathfrak{r}_n)\left(t, \frac{i}{n}\right), \quad t \in [0,T],\ i \in \mathbb{T}_n.$$

As discussed before, we assume without loss of generality that $(\mathfrak{p}_n, \mathfrak{r}_n)$ is smooth for $t \in [0,T]$. Denote by $\mu_{t,n}$ the *local Gibbs measure* on Ω_n associated to the smooth profiles $\mathfrak{p}_n(t,\cdot)$ and $\boldsymbol{\tau}_n(\mathfrak{r}_n(t,\cdot))$:

$$\mu_{t,n}(d\vec{\eta}) = \prod_{i\in\mathbb{T}_n} \nu^1_{\mathfrak{p}_i^n, \boldsymbol{\tau}_i^n, \sigma_n}(d\eta_i), \quad \boldsymbol{\tau}_i^n = \boldsymbol{\tau}_n(\mathfrak{r}_i^n).$$

Let $f_{t,n}$ be the density of the dynamics $\vec{\eta}(t)$ with respect to $\mu_{t,n}$, and

$$H_n(t) \triangleq H(f_{t,n};\mu_{t,n}).$$

Our first theorem is an estimate on $H_n(t)$, which improves the classical upper bound $H_n(t) \le C(H_n(0)+n)$ for all $t \in [0,T]$.

Theorem 2.2. *There exists a constant $C = C_{\beta,\mathfrak{v},T}$, such that*

$$H_n(t) \le C(H_n(0)+K_n), \quad \forall t \in [0,T],\ n \ge 1,$$

where K_n is the deterministic sequence given by

$$K_n = \max\left\{\sigma_n^{\frac{6}{5}}\gamma_n^{-\frac{1}{5}}n^{\frac{4}{5}}, \gamma_n\right\}.$$

From Theorem 2.2, if $\{\sigma_n\}$, $\{\gamma_n\}$ satisfy that

$$\lim_{n\to\infty} \gamma_n^2 n^{-1} = 0, \quad \lim_{n\to\infty} \sigma_n^6 \gamma_n^{-1} n^{\frac{3}{2}} = 0, \tag{2.12}$$

then $K_n = o(\sqrt{n})$ as $n \to \infty$. As an application of this observation, we have the following quantitative version of hydrodynamic limit.

Corollary 2.3. *Assume* (2.12) *and a constant C_0 such that $H_n(0) \le C_0\sqrt{n}$ for all n. For any $1 \le p < 2$, $t \in [0,T]$ and smooth function $h : \mathbb{T} \to \mathbb{R}^2$,*

$$\mathbb{E}_n\left[\left|\frac{1}{n}\sum_{i\in\mathbb{T}_n} h\left(\frac{i}{n}\right)\cdot\begin{pmatrix} p_i(t) - \mathfrak{p}_i^n(t) \\ r_i(t) - \mathfrak{r}_i^n(t)\end{pmatrix}\right|^p\right] \le \frac{C\|h\|^p}{n^{\frac{p}{4}}}$$

holds with some constant $C = C(\beta,\mathfrak{v},T,C_0,p)$.

Theorem 2.2 and Corollary 2.3 are proved in Section 4.

2.4 Fluctuation field

By non-equilibrium fluctuation, we mean the fluctuation field of the conserved quantities around its hydrodynamic limit. Define the empirical distribution of these fluctuations as

$$Y_t^n(h) = \frac{1}{\sqrt{n}}\sum_{i\in\mathbb{T}_n} h\left(\frac{i}{n}\right)\cdot\begin{pmatrix} p_i(t) - \mathfrak{p}_i^n(t) \\ r_i(t) - \mathfrak{r}_i^n(t)\end{pmatrix}, \tag{2.13}$$

for $t \in [0,T]$, $n \ge 1$ and smooth function $h : \mathbb{T} \to \mathbb{R}^2$. Notice that the conserved quantities are centred with solutions of (2.9) instead of (2.10). Observe that as $n \to \infty$,

$$\|\mathfrak{p}_n(t,\cdot) - \mathfrak{p}(t,\cdot)\| + \|\mathfrak{r}_n(t,\cdot) - \mathfrak{r}(t,\cdot)\| = O(\sigma_n).$$

Therefore, $(\mathfrak{p}_n,\mathfrak{r}_n)$ and $(\mathfrak{p},\mathfrak{r})$ are indistinguishable in (2.13) only if $\sqrt{n}\sigma_n = o(1)$. This is not necessarily satisfied in our setting, see (2.12) and (2.14) later.

By duality, (2.13) defines a process $\{Y_t^n \in \mathcal{H}_{-k}(\mathbb{T}); t \in [0,T]\}$ for $k > 1/2$. The major goal of this article is to derive the macroscopic equation of Y_t^n. Suppose that there is a random variable $Y_0 \in \mathcal{H}_{-k}$, such that Y_0^n converges weakly to Y_0 as $n \to \infty$. In the following theorem, we prove that $Y_\cdot^n$ converges weakly to the solution of the a linear p-system with homogeneous sound speed under some additional assumptions.

Theorem 2.4. *Assume* (2.12) *and some $\epsilon > 0$, such that*

$$\limsup_{n\to\infty} \sigma_n^2 K_n^{3-2\epsilon} n^{\epsilon-1} < \infty, \quad \sup_n H_n(0) < \infty, \tag{2.14}$$

where K_n is the sequence appeared in Theorem 2.2 before. For every $T > 0$, $\{(Y_t^n)_{0\le t\le T};\ n \ge 1\}$ converges in law to the unique solution of

$$\partial_t Y(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \partial_x Y(t), \quad Y(0) = Y_0, \tag{2.15}$$

with respect to the topology of $C([0,T];\mathcal{H}_{-k})$ for $k > 9/2$.

Remark 2.5. The additional assumptions in (2.14) are necessary only for the proof of tightness, see Section 7. For the convergence of finite-dimensional laws of Y_t^n proved in Section 6, it is sufficient to assume that $H_n(0) = o(\sqrt{n})$ and (2.12).

Remark 2.6. In the particular case that $\sigma_n = n^{-a}$, $\gamma_n = n^b$ with $a > 0$, $b \ge 0$, the conditions (2.12) and (2.14) are equivalent to

$$a > \frac{1}{5}, \quad b \in \big(f_-(a), f_+(a)\big) \cap \left[0, \frac{1}{2}\right),$$

where $f_\pm(a)$ are respectively given by

$$f_-(a) = \frac{7-28a}{3} \quad \text{and} \quad f_+(a) = \frac{2a+1}{3}.$$

Hence, if σ_n decays strictly faster than $n^{-1/5}$, then the result in Theorem 2.4 holds with some properly chosen sequence γ_n.

The proof of Theorem 2.4 is divided into two parts. In Section 6 we show the convergence of finite-dimensional distribution, based on the Boltzmann–Gibbs principle proved in Section 5. In Section 7 we show the tightness of the laws of $Y_\cdot^n$. The weak convergence in Theorem 2.4 then follows from the uniqueness of the solution of (2.15).

3 The main lemma

Fix some C^1-smooth function $\tau = \tau(\cdot)$ on $\mathbb{T}$. For each $n \ge 1$, define a product measure μ_n (dependent on $\tau(\cdot)$, σ_n) on $\mathbb{R}^n$ by

$$\mu_n(d\mathbf{r}) = \prod_{i\in\mathbb{T}_n} \pi_{\tau_i^n,\sigma_n}(dr_i), \quad \tau_i^n = \tau\left(\frac{i}{n}\right).$$

Note that μ_n is the $(r_1,\ldots,r_n)$-marginal distribution of a local Gibbs measure. To simplify the notations, let $\langle\,\cdot\,\rangle_{\tau,\sigma}$ denote the integral with respect to $\pi_{\tau,\sigma}$. Define

$$\begin{aligned} \Phi_i^n(r_i) &= V_n'(r_i) - \langle V_n'\rangle_{\tau_i^n,\sigma_n} - \frac{d}{dr}\langle V_n'\rangle_{\boldsymbol{\tau}_n(r),\sigma_n}\Big|_{r=r_i^n}(r_i - r_i^n) \\ &= V_n'(r_i) - \tau_i^n - \boldsymbol{\tau}_n'(r_i^n)\big(r_i - r_i^n\big), \end{aligned} \tag{3.1}$$

where $r_i^n = \bar{r}_n(\tau_i^n)$ and $\bar{r}_n$, $\boldsymbol{\tau}_n$ are functions given by (2.4), (2.7). In this section, we prove an estimate for the space variance associated to Φ_i^n.

For a probability measure μ on $\mathbb{R}^n$ and a density function f with respect to μ, define the Dirichlet form associated to $\mathcal{S}_{n,\sigma}$ by

$$D(f;\mu) = \frac{1}{2}\sum_{i\in\mathbb{T}_n}\int_{\mathbb{R}^n}(\mathcal{Y}_i f)^2 d\mu.$$

For $g \in C^1(\mathbb{T})$, define the random local functional

$$W_n(g) = \sum_{i\in\mathbb{T}_n} g_i^n \Phi_i^n, \quad g_i^n = g\left(\frac{i}{n}\right). \tag{3.2}$$

Lemma 3.1. *For any $\delta > 0$ and density function f with respect to μ_n, there exists a random functional $W_{n,\delta}(g)$ (depending on $\tau(\cdot)$ and f), such that*

$$\int f\big[W_n(g) - W_{n,\delta}(g)\big] d\mu_n \le \delta n \gamma_n D\left(\sqrt{f}; \mu_n\right), \tag{3.3}$$

where $W_n(g)$ is defined through (3.1) *and* (3.2) *above, and*

$$\int f\big|W_{n,\delta}(g)\big| d\mu_n \le C(1 + M_g)\left[H(f;\mu_n) + \left(1 + \frac{1}{\delta}\right)\kappa_n\right], \tag{3.4}$$

where C is a constant dependent on β, $|\tau|_{\mathbb{T}}$ and $|\tau'|_{\mathbb{T}}$, and

$$M_g = |g|_{\mathbb{T}}^2 + |g'|_{\mathbb{T}}, \quad \kappa_n = \max\left\{\sigma_n^{\frac{6}{5}} \gamma_n^{-\frac{1}{5}} n^{\frac{4}{5}}, \sigma_n \sqrt{n}\right\}.$$

In particular, if the second limit in (2.12) *is satisfied, then $\kappa_n = o(\sqrt{n})$.*

To prove Lemma 3.1, we make use of the sub-Gaussian property of the local function Φ_i^n. A real-valued random variable X is sub-Gaussian of order $C > 0$, if

$$\log E\big[e^{sX}\big] \le \frac{Cs^2}{2}, \quad \forall s \in \mathbb{R}. \tag{3.5}$$

Recall that $V_n = r^2/2 + \sigma_n U$ with U'' bounded. We have the following lemma.

Lemma 3.2. *For all $n \ge 1$, $i \in \mathbb{T}_n$, $U'(r_i) - \langle U' \rangle_{\tau_i^n, \sigma_n}$ and $r_i - r_i^n$ are sub-Gaussian of a uniform order dependent only on β and $|\tau|_{\mathbb{T}}$.*

The proof of the sub-Gaussian property is direct and is postponed to the end of this section. Some general properties of sub-Gaussian variables used hereafter are summarized in Appendix E. Now we state the proof of Lemma 3.1.

Proof of Lemma 3.1. Pick some $\ell = \ell(n) \ll n$ which grows with n. Let

$$g_{i,\ell}^n = g_i^n - \frac{1}{\ell} \sum_{j=0}^{\ell-1} g_{i-j}^n, \quad \Phi_{i,\ell}^n = E_{\mu_n}\left[\frac{1}{\ell} \sum_{j=0}^{\ell-1} \Phi_{i+j}^n \,\middle|\, \sum_{j=0}^{\ell-1} r_{i+j}\right].$$

For each $i \in \mathbb{T}_n$, denote by $\mathcal{Y}_{i,n}^*$ the adjoint of $\mathcal{Y}_i$ with respect to the inhomogeneous measure μ_n. It is easy to see that for smooth F,

$$\mathcal{Y}_{i,n}^* F = \beta\big(V_n'(r_{i+1}) - V_n'(r_i) - \tau_{i+1}^n + \tau_i^n\big)F - \mathcal{Y}_i F. \tag{3.6}$$

Let $\psi_{i,\ell}^n = \psi_{i,\ell}^n(r_i, \ldots, r_{i+\ell-1})$ solve the Poisson equation

$$\sum_{j=0}^{\ell-2} \mathcal{Y}_{i+j,n}^* \mathcal{Y}_{i+j} \psi_{i,\ell}^n = \Psi_{i,\ell}^n, \quad \Psi_{i,\ell}^n = \frac{1}{\ell} \sum_{j=0}^{\ell-1} \Phi_{i+j}^n - \Phi_{i,\ell}^n. \tag{3.7}$$

By Proposition 9.1, $\psi_{i,\ell}^n \in C_b^1(\mathbb{R}^\ell)$. Define the auxiliary functionals

$$W_{n,\ell}^{(1)}(g) = \sum_{i \in \mathbb{T}_n} g_{i,\ell}^n \Phi_i^n, \quad W_{n,\ell}^{(2)}(g) = \sum_{i \in \mathbb{T}_n} g_i^n \Phi_{i,\ell}^n,$$

$$W_{n,\ell}^{(3)}(g) = \frac{2(\ell - 1)}{n\gamma_n} \sum_{i \in \mathbb{T}_n} (g_i^n)^2 \sum_{j=0}^{\ell-2} \big(\mathcal{Y}_{i+j} \psi_{i,\ell}^n\big)^2,$$

for each $n \ge 1$, ℓ and $i \in \mathbb{T}_n$.

Our first step is to observe that for any $\delta > 0$,

$$
\begin{aligned}
&\int f\left[W_n(g) - W^{(1)}_{n,\ell}(g) - W^{(2)}_{n,\ell}(g) - \frac{1}{\delta}W^{(3)}_{n,\ell}(g)\right]d\mu_n \\
&= \int \sum_{i\in\mathbb{T}_n} g_i^n \sum_{j=0}^{\ell-2}\big(\mathcal{Y}_{i+j}\psi^n_{i,\ell}\big)\big(\mathcal{Y}_{i+j}f\big)d\mu_n - \frac{1}{\delta}\int fW^{(3)}_{n,\ell}(g)d\mu_n \\
&\le \frac{\delta n\gamma_n}{8(\ell-1)}\int f^{-1}\sum_{i\in\mathbb{T}_n}\sum_{j=0}^{\ell-2}(\mathcal{Y}_{i+j}f)^2 d\mu_n = \delta n\gamma_n D\left(\sqrt{f};\mu_n\right).
\end{aligned}
$$

Hence, the strategy is to bound the integrals of the auxiliary functionals by relative entropy together with terms of ℓ and n, and then optimize the order of ℓ.

For the first functional $W^{(1)}_{n,\ell}$, note that $\Phi^n_i = \sigma_n\phi^n_i$, where

$$
\begin{aligned}
\phi^n_i(r_i) &= U'(r_i) - \langle U'\rangle_{\tau^n_i,\sigma_n} - \frac{d}{dr}\langle U'\rangle_{\boldsymbol{\tau}_n(r^n_i),\sigma_n}\big(r_i - r^n_i\big), \\
&= U'(r_i) - \frac{\tau^n_i - r^n_i}{\sigma_n} - \frac{\boldsymbol{\tau}'_n(r^n_i)-1}{\sigma_n}\big(r_i - r^n_i\big).
\end{aligned}
$$

In view of (2.5), there is a constant $C_{\beta,|\tau|_{\mathbb{T}}}$, such that

$$
\left|\frac{\boldsymbol{\tau}'_n(r^n_i)-1}{\sigma_n}\right| \le C_{\beta,|\tau|_{\mathbb{T}}}, \quad \forall n\ge 1,\ i\in\mathbb{T}_n.
$$

As $\{\phi^n_i; i\in\mathbb{T}_n\}$ is an independent family, by the entropy inequality (D.5),

$$
\int f\big|W^{(1)}_{n,\ell}\big|d\mu_n \le \frac{1}{\alpha}\left(H(f;\mu_n) + \sum_{i\in\mathbb{T}_n}\log\int e^{\alpha\sigma_n|g^n_{i,\ell}\phi^n_i|}d\mu_n\right),
$$

for any $\alpha > 0$. By Lemma 3.2 and direct computation, ϕ^n_i is sub-Gaussian of a uniform order $c = c_{\beta,|\tau|_{\mathbb{T}}}$. Choosing $\alpha_{n,\ell} = (2\sigma_n|g^n_{i,\ell}|)^{-1}$ and applying Lemma E.2,

$$
\int f\big|W^{(1)}_{n,\ell}\big|d\mu_n \le \frac{1}{\alpha_{n,\ell}}\left[H(f;\mu_n) + \sum_{i\in\mathbb{T}_n}\left(\log 3 + \frac{c}{4}\right)\right].
$$

As $|g^n_{i,\ell}| \le C|g'|_{\mathbb{T}}\ell n^{-1}$ with some universal constant C, therefore,

$$
\begin{aligned}
\int f\big|W^{(1)}_{n,\ell}\big|d\mu_n &\le \frac{C|g'|_{\mathbb{T}}\sigma_n\ell}{n}\big(H(f;\mu_n) + C_1 n\big) \\
&\le C|g'|_{\mathbb{T}}\big(H(f;\mu_n) + C_2\sigma_n\ell\big).
\end{aligned}
\tag{3.8}
$$

The second functional $W^{(2)}_{n,\ell}$ is the variance of a canonical ensemble. Indeed, $\Phi^n_{i,\ell} = \sigma_n\phi^n_{i,\ell}$, where $\phi^n_{i,\ell}$ is the conditional expectation on the box $(r_i,\ldots,r_{i+\ell-1})$:

$$
\phi^n_{i,\ell} = E_{\mu_n}\left[\frac{1}{\ell}\sum_{j=0}^{\ell-1}\phi^n_{i+j}\ \middle|\ \sum_{j=0}^{\ell-1}r_{i+j}\right].
$$

The definition of ϕ^n_i suggests that this term can be estimated by the theory of *equivalence of ensembles* presented in Section 8. First notice that $\{\phi^n_{i,\ell}, i\in\mathbb{T}_n\}$ is an ℓ-independent class. With (D.5) we obtain that for any $\alpha > 0$,

$$
\int f\big|W^{(2)}_{n,\ell}\big|d\mu_n \le \frac{1}{\alpha}\left(H(f;\mu_n) + \frac{1}{\ell}\sum_{i\in\mathbb{T}_n}\log\int e^{\alpha\ell\sigma_n|g^n_i\phi^n_{i,\ell}|}d\mu_n\right).
$$

Since $\phi^n_{i,\ell}$ is sub-Gaussian of order c, in view of Lemma E.2,

$$\int e^{s|\phi^n_i|}d\mu_n \le \frac{1+s}{1-s}e^{\frac{cs}{2}} \le e, \quad \forall |s| \le A = A(c).$$

Hence, Proposition 8.3 yields that if $\ell \le O(n^{2/3})$,

$$\int e^{s|\ell\phi^n_{i,\ell}|}d\mu_n \le C_1, \quad \forall |s| \le A' = A'(c),$$

with some universal constant C_1. Choosing $\alpha_n = A'(|g|_{\mathbb{T}}\sigma_n)^{-1}$,

$$\int f|W^{(2)}_{n,\ell}|d\mu_n \le \frac{1}{\alpha_n}\left(H(f;\mu_n) + \frac{C_1 n}{\ell}\right) \le C_2|g|_{\mathbb{T}}\left(H(f;\mu_n) + \frac{C_1\sigma_n n}{\ell}\right). \tag{3.9}$$

For the third functional $W^{(3)}_{n,\ell}$, recall the Poisson equation (3.6)–(3.7). Using the C^1 estimate of the Poisson equation in Proposition 9.1,

$$\sum_{j=0}^{\ell-2}\big(\mathcal{Y}_{i+j}\psi^n_{i,\ell}\big)^2 \le C_\beta \ell^4 \sup_{\mathbb{R}^\ell}\left\{\sum_{j=0}^{\ell-2}\big(\mathcal{Y}_{i+j}\Psi^n_{i,\ell}\big)^2\right\},$$

where the supremum is taken over all $(r_i, r_{i+1}, \dots, r_{i+\ell-1}) \in \mathbb{R}^\ell$. From the definition of $\Psi^n_{i,\ell}$, with $b^n_i = \boldsymbol{\tau}'_n(r^n_i) = \boldsymbol{\tau}'_n(\bar{r}_n(\tau^n_i))$,

$$\begin{aligned}\mathcal{Y}_{i+j}\Psi^n_{i,\ell} &= \frac{1}{\ell}\big(V''_n(r_{i+j-1}) - V''_n(r_{i+j}) - b^n_{i+j-1} + b^n_{i+j}\big)\\ &= \frac{1}{\ell}\big(\sigma_n U''(r_{i+j-1}) - \sigma_n U''(r_{i+j}) - b^n_{i+j-1} + b^n_{i+j}\big).\end{aligned}$$

In view of the condition $|U''(r)| \le 1$ and (2.5),

$$\big|\mathcal{Y}_{i+j}\Psi^n_{i,\ell}\big| \le \frac{C\sigma_n}{\ell}\left(1 + \frac{1}{n}\right),$$

with some constant C dependent on $|\tau'|_{\mathbb{T}}$. Therefore,

$$\int fW^{(3)}_{n,\ell}d\mu_n \le \frac{2CC_\beta(\ell-1)\sigma_n^2}{n\gamma_n}\sum_{i\in\mathbb{T}_n}(g^n_i)^2\ell^3 \le \frac{C_1|g|^2_{\mathbb{T}}\sigma_n^2\ell^4}{\gamma_n}. \tag{3.10}$$

Combining (3.8)–(3.10), we obtain that if $\ell \le O(n^{2/3})$,

$$\begin{aligned}&\int f\left|W^{(1)}_{n,\delta}(g) + W^{(2)}_{n,\delta}(g) + \frac{1}{\delta}W^{(3)}_{n,\delta}(g)\right|d\mu_n\\ &\le C\big(1 + |g|^2_{\mathbb{T}} + |g'|_{\mathbb{T}}\big)\left(H(f;\mu_n) + \sigma_n\ell + \frac{\sigma_n n}{\ell} + \frac{\sigma_n^2\ell^4}{\delta\gamma_n}\right),\end{aligned}$$

holds with some constant $C = C(\beta, |\tau|_{\mathbb{T}}, |\tau'|_{\mathbb{T}})$. The optimal choice of ℓ is

$$\ell(n) = \min\left\{(\sigma_n^{-1}\gamma_n n)^{\frac{1}{5}},\ n^{\frac{1}{2}}\right\}.$$

Indeed, if $\sigma_n^{-1}\gamma_n > n^{3/2}$, we take $\ell = \sqrt{n}$, and

$$\sigma_n\ell + \frac{\sigma_n n}{\ell} + \frac{\sigma_n^2\ell^4}{\delta\gamma_n} = \left(2 + \frac{n^{\frac{3}{2}}}{\delta}\frac{\sigma_n}{\gamma_n}\right)\sigma_n\sqrt{n} < \left(2 + \frac{1}{\delta}\right)\sigma_n\sqrt{n}.$$

On the other hand, if $\sigma_n^{-1}\gamma_n \le n^{3/2}$, we take $\ell = (\sigma_n^{-1}\gamma_n n)^{1/5}$, and

$$\sigma_n\ell + \frac{\sigma_n n}{\ell} + \frac{\sigma_n^2\ell^4}{\delta\gamma_n} = \sigma_n^{\frac{6}{5}}\gamma_n^{-\frac{1}{5}}n^{\frac{4}{5}}\left(\sigma_n^{-\frac{2}{5}}\gamma_n^{\frac{2}{5}}n^{-\frac{3}{5}} + 1 + \frac{1}{\delta}\right) \\ \le \left(2 + \frac{1}{\delta}\right)\sigma_n^{\frac{6}{5}}\gamma_n^{-\frac{1}{5}}n^{\frac{4}{5}}.$$

In consequence, (3.3), (3.4) are in force by defining

$$W_{n,\delta}(g) = W_{n,\ell}^{(1)}(g) + W_{n,\ell}^{(2)}(g) + \frac{1}{\delta}W_{n,\ell}^{(3)}(g),$$

with $\ell = \ell(n)$ chosen above. □

Before proceeding to the proof of Lemma 3.2, we discuss the anharmonic case briefly. If $\sigma_n \equiv \sigma$ and $\gamma_n = o(n)$, similar argument yields the estimate with κ_n replaced by $\kappa_n' = n^{3/5+}$. Apparently, it is insufficient for deriving the macroscopic fluctuation, which demands at least $\kappa_n' = o(\sqrt{n})$. By computing explicitly under Gaussian canonical measure, the upper bounds presented for the first and second auxiliary functionals in the proof of Lemma 3.1 turn out to be sharp. Meanwhile, (3.10) should be improvable. Indeed, by using (D.5), the left-hand side of (3.10) is bounded from above with

$$2H(f;\mu_n) + \frac{2}{\ell}\sum_{i\in\mathbb{T}_n}\log\int\exp\left\{\frac{\ell^2(g_i^n)^2}{n\gamma_n}\sum_{j=0}^{\ell-2}\left(\mathcal{Y}_{i+j}\psi_{i,\ell}^n\right)^2\right\}d\mu_n,$$

Therefore, we guess that a nice upper bound of the exponential moment term above could help us take the advantage of the entropy and improve (3.10).

Lemma 3.2 is a special case of the next result.

Lemma 3.3. *Let $V \in C(\mathbb{R})$ satisfy $c_-r^2 \le 2V(r) \le c_+r^2$ with two positive constants $c_\pm$. For $\tau \in \mathbb{R}$, let π_τ be a probability measure on $\mathbb{R}$ given by*

$$\pi_\tau = e^{-V(r)+\tau r-G(r)}dr, \quad G(r) = \log\int_{\mathbb{R}} e^{-V(r)+\tau(r)}dr.$$

If F is a measurable function on $\mathbb{R}$ such that $|F(r)| \le c|r|$ with constant c, then $F - E_{\pi_\tau}[F]$ is sub-Gaussian of order $C = C(\tau, c, c_\pm)$ under π_τ. Furthermore, C is uniformly bounded for all the coefficients in any compact intervals.

Proof. Notice that for all $\tau \in \mathbb{R}$,

$$e^{G(\tau)} \ge \int_{\mathbb{R}}\exp\left\{-\frac{c_+r^2}{2} + \tau r\right\} = \frac{\sqrt{2\pi}}{2c_+}\exp\left\{\frac{\tau^2}{2c_+}\right\}.$$

For any t such that $0 < t < c_-/(2c^2)$,

$$E_{\pi_\tau}\left[\exp(tF^2)\right] \le e^{-G(\tau)}\int_{\mathbb{R}}\exp\left\{-\frac{(c_- - 2tc^2)r^2}{2} + \tau r\right\}dr \\ \le \frac{c_+}{c_- - 2tc^2}\exp\left\{\frac{\tau^2}{2}\left(\frac{1}{c_- - 2tc^2} - \frac{1}{c_+}\right)\right\}.$$

Denote $F_* = F - E_{\pi_\tau}[F]$. By convexity, for all $t \ge 0$,

$$E_{\pi_\tau}\left[\exp(tF_*^2)\right] \le \exp\left(2tE_{\pi_\tau}^2[F]\right)E_{\pi_\tau}\left[\exp(2tF^2)\right] \le E_{\pi_\tau}\left[\exp(4tF^2)\right].$$

Therefore, we obtain that

$$E_{\pi_\tau}\left[\exp\left(\frac{c_-}{16c^2}F_*^2\right)\right] \le E_{\pi_\tau}\left[\exp\left(\frac{c_-}{4c^2}F^2\right)\right] \le \frac{2c_+}{c_-}\exp\left\{\frac{\tau^2}{2}\left(\frac{2}{c_-} - \frac{1}{c_+}\right)\right\}.$$

Using the ϕ_2-condition (see Lemma E.1), we can conclude that F_* is a sub-Gaussian random variable of the order given by

$$C(\tau, c, c_\pm) = \frac{64cc_+}{c_-^2} \exp\left\{\frac{\tau^2}{2}\left(\frac{2}{c_-} - \frac{1}{c_+}\right)\right\}.$$

The lemma then follows directly. □

4 Entropy estimate

In this section we prove Theorem 2.2 and Corollary 2.3. They are direct results of Lemma 3.1 and the relative entropy inequality established in [30].

Proof of Theorem 2.2. Recall that $(\mathfrak{p}_i^n, \mathfrak{r}_i^n) = (\mathfrak{p}_n, \mathfrak{r}_n)(t, i/n)$ and $\boldsymbol{\tau}_i^n = \boldsymbol{\tau}_n(\mathfrak{r}_i^n)$. We start from Yau's entropy inequality stated in Appendix C:

$$H_n'(t) \le -2n\gamma_n D\left(\sqrt{f_{t,n}}; \mu_{t,n}\right) + \beta \int f_{t,n} J_t^n d\mu_{t,n} + C\gamma_n, \tag{4.1}$$

where $C = C_{\beta,\mathfrak{v},T}$. The remainder J_t^n can be expressed by

$$J_t^n = W_n(h_n) + E_t^n, \quad h_n = \partial_t \mathfrak{r}_n(t, \cdot), \tag{4.2}$$

where the functional W_n is defined through (3.2), and

$$E_t^n = \sum_{i \in \mathbb{T}_n} \epsilon_i^n \cdot \begin{pmatrix} p_i - \mathfrak{p}_i^n \\ V_n'(r_i) - \boldsymbol{\tau}_i^n \end{pmatrix}, \quad \epsilon_i^n = -\partial_t \begin{pmatrix} \mathfrak{p}_n \\ \mathfrak{r}_n \end{pmatrix}\left(t, \frac{i}{n}\right) + n \begin{pmatrix} \boldsymbol{\tau}_{i+1}^n - \boldsymbol{\tau}_i^n \\ \mathfrak{p}_i^n - \mathfrak{p}_{i-1}^n \end{pmatrix}. \tag{4.3}$$

For the integral of E_t^n, (D.5) yields that

$$\int f_{t,n} E_t^n d\mu_{t,n} \le H_n(t) + \sum_{i \in \mathbb{T}_n} \log \int \exp\left\{\epsilon_i^n \cdot \begin{pmatrix} p_i - \mathfrak{p}_i^n \\ V_n'(r_i) - \boldsymbol{\tau}_i^n \end{pmatrix}\right\} d\mu_{t,n}.$$

Notice that under $\mu_{t,n}$, $p_i - \mathfrak{p}_i^n$ is a Gaussian variable, while due to Lemma 3.2, $V_n'(r_i) - \boldsymbol{\tau}_i^n$ is sub-Gaussian of order $C = C_{\beta,\mathfrak{v},T}$, so that

$$\int f_{t,n} E_t^n d\mu_{t,n} \le H_n(t) + C \sum_{i \in \mathbb{T}_n} |\epsilon_i^n|^2 \le H_n(t) + \frac{C_{\beta,\mathfrak{v},T}'}{n}. \tag{4.4}$$

For $W_n(h_n)$, denote by $\mu_{t,n}^*$ the marginal distribution of $\mu_{t,n}$ on positions $(r_1, \ldots, r_n)$, and by $f_{t,n}^*$ the density of $(r_1, \ldots, r_n)(t)$ with respect to $\mu_{t,n}^*$. Applying Lemma 3.1 with $\delta = 2/\beta$, and using the relation $H(f_{t,n}^*; \mu_{t,n}^*) \le H(f_{t,n}; \mu_{t,n})$ (see (D.2)),

$$\begin{aligned} \int f_{t,n} W_n(h_n) d\mu_{t,n} &\le \frac{2n\gamma_n}{\beta} D\left(\sqrt{f_{t,n}}; \mu_{t,n}\right) + C\big(H(f_{t,n}^*; \mu_{t,n}^*) + \kappa_n\big) \\ &\le \frac{2n\gamma_n}{\beta} D\left(\sqrt{f_{t,n}}; \mu_{t,n}\right) + C\big(H_n(t) + \kappa_n\big). \end{aligned}$$

Hence, we obtain from (4.1) that for all $t \in [0, T]$,

$$H_n'(t) \le C_{\beta,\mathfrak{v},T}\big(H_n(t) + \max\{\kappa_n, \gamma_n\}\big) = C_{\beta,\mathfrak{v},T}(H_n(t) + K_n).$$

Theorem 2.2 then follows from the Grönwall's inequality. □

Corollary 2.3 is a special case of the following result.

Corollary 4.1. *Let $F \in C(\mathbb{R})$ satisfy that $|F(r)| \le c|r|$. For all $p \in [1,2)$, there is a constant $C = C(\beta, \mathfrak{v}, T, c, p)$, such that for all $h \in C(\mathbb{T})$, $t \in [0,T]$,*

$$\mathbb{E}_n\left[\left|\frac{1}{n}\sum_{i\in\mathbb{T}_n} h\left(\frac{i}{n}\right)\big(F(r_i) - E_{\mu_{t,n}}[F(r_i)]\big)\right|^p\right] \le \frac{C(1+H_n(t))^{\frac{p}{2}}\|h\|^p}{n^{\frac{p}{2}}},$$

In particular, if (2.12) *holds and $H_n(0) \le C_0\sqrt{n}$, then*

$$\mathbb{E}_n\left[\left|\frac{1}{n}\sum_{i\in\mathbb{T}_n} h\left(\frac{i}{n}\right)\big(F(r_i) - E_{\mu_{t,n}}[F(r_i)]\big)\right|^p\right] \le \frac{C\|h\|^p}{n^{\frac{p}{4}}},$$

with some constant $C = C(\beta, \mathfrak{v}, T, c, p, C_0)$. Similar result holds for $F(p_i)$.

Proof. Denote by $F_i = F(r_i) - E_{\mu_{t,n}}[F(r_i)]$ for short. By (D.3),

$$\mathbb{P}_n\left\{\left|\frac{1}{n}\sum_{i\in\mathbb{T}_n} h_i^n F_i\right| > \lambda\right\} \le \frac{H_n(t) + \log 2}{-\log \mu_{t,n}\{|\sum_i h_i^n F_i| > \lambda n\}}.$$

In view of Lemma 3.3, $\{F_i, i \in \mathbb{T}_n\}$ is an independent family of sub-Gaussian variables of a uniform order under $\mu_{t,n}$. Then, with a constant $C = C_{\beta,\mathfrak{v},T,c}$,

$$E_{\mu_{t,n}}\left[\exp\left\{s\sum_{i\in\mathbb{T}_n} h_i^n F_i\right\}\right] \le \exp\left\{\frac{Cs^2}{2}\sum_{i\in\mathbb{T}_n}(h_i^n)^2\right\}, \quad \forall s \in \mathbb{R}.$$

Therefore, $\sum_i h_i^n F_i$ is sub-Gaussian of order $Cn\|h\|^2$, and

$$\mu_{t,n}\left\{\left|\sum_{i\in\mathbb{T}_n} h_i^n F_i\right| > \lambda n\right\} \le 2\exp\left\{-\frac{\lambda^2 n}{2C\|h\|^2}\right\}.$$

From this and the estimate above, we obtain that for any $\lambda > 0$,

$$\mathbb{P}_n\left\{\left|\frac{1}{n}\sum_{i\in\mathbb{T}_n} h_i^n F_i\right| > \lambda\right\} \le \frac{C(1+H_n(t))\|h\|^2}{\lambda^2 n}.$$

By the moment estimate in Lemma D.4, for all $p \in [1,2)$,

$$\mathbb{E}_n\left[\left|\frac{1}{n}\sum_{i\in\mathbb{T}_n} h_i^n F_i\right|^p\right] \le \frac{C(1+H_n(t))^{\frac{p}{2}}\|h\|^p}{n^{\frac{p}{2}}}.$$

The second inequality in Corollary 4.1 follows directly from Theorem 2.2. The parallel result for $F(p_i)$ can be proved in the same way. □

5 Boltzmann–Gibbs principle

In this section, we prove the proposition which is known in the literature as the *Boltzmann–Gibbs principle*, firstly established for the equilibrium dynamics of zero range jump process in [5]. It aims at determining the space-time variance of a local observation of conserved field by its linear approximation.

Proposition 5.1. *Suppose* (2.12) *and in additional*

$$\lim_{n\to\infty} \frac{H_n(0)}{\sqrt{n}} = 0. \tag{5.1}$$

Let $g_n = g_n(t,x)$ be a sequence of functions on $[0,T] \times \mathbb{T}$, such that $|g_n|_{\mathbb{T}}$ and $|\partial_x g_n|_{\mathbb{T}}$ are uniformly bounded for $t \in [0,T]$ and $n \geq 1$. For any $0 \leq t < t' \leq T$,

$$\lim_{n\to\infty} \mathbb{P}_n \left\{ \left| \frac{1}{\sqrt{n}} \int_t^{t'} \sum_{i\in\mathbb{T}_n} g_i^n \Phi_i^n ds \right| > \lambda \right\} = 0, \quad \forall \lambda > 0, \tag{5.2}$$

where $g_i^n = g_n(t, i/n)$ and Φ_i^n is given by (3.1).

We prove it along the approach in [17, Theorem 5.1].

Proof. As we can consider $-g_n$ instead of g_n, it suffices to prove that

$$\lim_{n\to\infty} \mathbb{P}_n \left\{ \int_t^{t'} \sum_{i\in\mathbb{T}_n} g_i^n \Phi_i^n ds > \lambda\sqrt{n} \right\} = 0.$$

Recall the auxiliary functional $W_{n,\delta}$ defined in Lemma 3.1, and the expressions E_t^n, $W_n(h_n)$ in (4.2). Define for any $\alpha > 0$ and $n \geq 1$

$$U_{n,\alpha}(g_n) = W_{n,\frac{1}{2\alpha}}(g_n) + \frac{\beta}{2\alpha}\big[E_t^n + W_{n,\frac{1}{\beta}}(h_n)\big]. \tag{5.3}$$

Note that the parameter α is not needed here, but would be used in Section 7. Let $\mathbb{P}_{t,n}$ and $\mathbb{P}_{t,n}^*$ be the law of the dynamics generated by $\mathcal{L}_n$, respectively with initial distributions $\mu_{t,n}$ and $f_{t,n} d\mu_{t,n}$. By the Markov property,

$$H\left(\frac{d\mathbb{P}_{t,n}^*}{d\mathbb{P}_{t,n}}; \mathbb{P}_{t,n}\right) = H_n(f_{t,n}; \mu_{t,n}) = H_n(t).$$

Therefore, we can apply (D.3) to the trajectory space to get

$$\begin{aligned} &\mathbb{P}_n \left\{ \int_t^{t'} \big[W_n(g) - U_{n,\alpha}(g)\big] ds > \lambda\sqrt{n} \right\} \\ &\leq \frac{H(t) + \log 2}{-\log \mathbb{P}_{t,n}\{\int_0^{t'-t} [W_n(g) - U_{n,\alpha}(g)] ds > \lambda\sqrt{n}\}}. \end{aligned}$$

Applying [16, Lemma 3.5] (see also [17, Lemma A.2]) to the reference measures $\{\mu_{t+s,n}; s \in [0, t'-t]\}$,

$$\begin{aligned} &\log \mathbb{P}_{t,n} \left\{ \int_0^{t'-t} \big[W_n(g_n) - U_{n,\alpha}(g_n)\big] ds > \lambda\sqrt{n} \right\} \\ &\leq -\alpha\lambda\sqrt{n} + \int_0^{t'-t} \sup_f \left\{ -n\gamma_n D\left(\sqrt{f}; \mu_{t+s,n}\right) + \right. \\ &\left. \int f \left[\alpha\big(W_n(g_n) - U_{n,\alpha}(g_n)\big) + \frac{\beta J_{t+s}^n}{2} \right] d\mu_{t+s,n} \right\} ds, \end{aligned}$$

where the supremum runs over all the density functions f with respect to $\mu_{t+s,n}$. Since $J_t^n = E_t^n + W_n(h_n)$, (3.3) in Lemma 3.1 yields that

$$\int f \left[\alpha\big(W_n(g_n) - U_{n,\alpha}(g_n)\big) + \frac{\beta J_{t+s}^n}{2} \right] d\mu_{t+s,n} \leq n\gamma_n D\left(\sqrt{f}; \mu_{t+s,n}\right).$$

Hence, we can conclude that for all $\lambda > 0$,

$$\log \mathbb{P}_{t,n} \left\{ \int_0^{t'-t} \big[W_n(g_n) - U_{n,\alpha}(g_n)\big] ds > \lambda\sqrt{n} \right\} \leq -\alpha\lambda\sqrt{n}.$$

As the conditions and Theorem 2.2 assure that $H_n(t) = o(\sqrt{n})$,

$$\lim_{n\to\infty} \mathbb{P}_n \left\{ \int_t^{t'} \big[W_n(g_n) - U_{n,\alpha}(g_n)\big] ds > \lambda\sqrt{n} \right\} \le \lim_{n\to\infty} \frac{H_n(t) + \log 2}{\alpha\lambda\sqrt{n}} = 0. \tag{5.4}$$

For the integral of $U_{n,\alpha}$, notice that by Chebyshev's inequality,

$$\mathbb{P}_n \left\{ \int_t^{t'} U_{n,\alpha}(g_n) ds > \lambda\sqrt{n} \right\} \le \frac{1}{\lambda\sqrt{n}} \mathbb{E}_n \left[\left| \int_t^{t'} U_{n,\alpha}(g_n) ds \right| \right].$$

By (3.4) in Lemma 3.1 and (4.4), with a constant $C = C_{\beta,\mathfrak{v},T}$,

$$\mathbb{E}_n \big[|U_{n,\alpha}(g_n)| \big] \le C\big(\alpha + \alpha^{-1}\big)\big(1 + |g_n(t)|_{\mathbb{T}}^2 + |\partial_x g_n(t)|_{\mathbb{T}}\big)\big(H_n(t) + K_n\big).$$

From the conditions on g_n and Theorem 2.2,

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}_n \left\{ \int_t^{t'} U_{n,\alpha}(g_n) ds > \lambda\sqrt{n} \right\} \\ &\le \frac{C|t'-t|(1+C_g)}{\lambda} \left(\alpha + \frac{1}{\alpha}\right) \lim_{n\to\infty} \frac{H_n(0) + K_n}{\sqrt{n}} = 0, \end{aligned} \tag{5.5}$$

where $C_g = \sup_{n\ge 1, t\in[0,T]} \{|g_n(t)|_{\mathbb{T}}^2 + |\partial_x g_n(t)|_{\mathbb{T}}\}$. By summing up (5.4) and (5.5) together we prove the result. □

6 Convergence of finite-dimensional laws

In this section we prove that every possible weak limit point of Y_t^n in (2.13) satisfies (2.15). Let $H : [0,T] \times \mathbb{T} \to \mathbb{R}^2$ be a smooth function, and write $H = (H_1, H_2)$. By Itô's formula, there is a square integrable martingale $M_t^n(H)$, such that

$$Y_t^n(H(t)) - Y_0^n(H(0)) = \int_0^t \left(\frac{d}{ds} + \mathcal{L}_n \right) Y_s^n(H(s)) ds + M_t^n(H),$$

and the quadratic variation of $M_t^n(H)$ is given by

$$\big\langle M_t^n(H) \big\rangle = n\gamma_n \int_0^t \Gamma_n \big[Y_t^n(H(s))\big] ds, \quad \Gamma_n f = \mathcal{S}_n[f^2] - 2f\mathcal{S}_n f. \tag{6.1}$$

Recall that $\mathfrak{u}_n = (\mathfrak{p}_n, \mathfrak{r}_n)$ denotes the solution to (2.9) with $\sigma = \sigma_n$, $(\mathfrak{p}_i^n, \mathfrak{r}_i^n) = \mathfrak{u}_n(s, i/n)$ and $\boldsymbol{\tau}_i^n = \boldsymbol{\tau}_n(\mathfrak{r}_i^n)$. We also write $\mathfrak{b}_i^n = \boldsymbol{\tau}'_n(\mathfrak{r}_i^n)$. Through direct computation,

$$\begin{aligned} \frac{d}{ds} Y_s^n(H) &= \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{T}_n} \big(\partial_s H_i^n - L_i^n H\big) \cdot \big(\eta_i - \mathfrak{u}_i^n\big) \\ &\quad - \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{T}_n} H_i^n \cdot \big(\partial_s \mathfrak{u}_i^n - \Lambda_i^n\big) \\ &\quad + \frac{1}{\sqrt{n}} \sum_{i\in\mathbb{T}_n} \begin{pmatrix} \nabla_{i-1}^n H_1 \\ \nabla_i^n H_2 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\tau}_i^n + \mathfrak{b}_i^n (r_i - \mathfrak{r}_i^n) \\ p_i \end{pmatrix}, \\ \mathcal{A}_n[Y_s^n(H)] &= n^{-\frac{3}{2}} \sum_{i\in\mathbb{T}_n} \begin{pmatrix} \nabla_{i-1}^n H_1 \\ \nabla_i^n H_2 \end{pmatrix} \cdot \begin{pmatrix} -V_n'(r_i) \\ -p_i \end{pmatrix}, \\ \mathcal{S}_n[Y_s^n(H)] &= 2^{-1} n^{-\frac{5}{2}} \sum_{i\in\mathbb{T}_n} \Delta_i^n H_2 V_n'(r_i). \end{aligned}$$

Here ∇_i^n and Δ_i^n are discrete derivatives given by

$$\nabla_i^n f = n\left[f\left(\frac{i+1}{n}\right) - f\left(\frac{i}{n}\right)\right], \quad \Delta_i^n f = n\big(\nabla_i^n f - \nabla_{i-1}^n f\big),$$

while the operator $L_i^n = L_i^n(s)$ and approximate field $\Lambda_i^n = \Lambda_i^n(s)$ are

$$L_i^n H = \begin{bmatrix} 0 & 1 \\ \mathfrak{b}_i^n & 0 \end{bmatrix} \begin{pmatrix} \nabla_{i-1}^n H_1 \\ \nabla_i^n H_2 \end{pmatrix}, \quad \Lambda_i^n = n \begin{pmatrix} \boldsymbol{\tau}_{i+1}^n - \boldsymbol{\tau}_i^n \\ \mathfrak{p}_i^n - \mathfrak{p}_{i-1}^n \end{pmatrix}.$$

With the notations above, $Y_t^n(H(t))$ is split into

$$Y_t^n(H(t)) = Y_0^n(H(0)) + \mathscr{R}_t^n(H) + \mathscr{A}_t^n(H) + \mathscr{S}_t^n(H) + \mathscr{W}_t^n(H) + M_t^n(H), \tag{6.2}$$

where $\mathscr{R}_t^n$, $\mathscr{A}_t^n$, $\mathscr{S}_t^n$ and $\mathscr{W}_t^n$ are given respectively by

$$\begin{aligned}
\mathscr{R}_t^n(H) &= \frac{1}{\sqrt{n}} \int_0^t \sum_{i \in \mathbb{T}_n} \left[\partial_s H\left(s, \frac{i}{n}\right) - L_i^n H(s)\right] \cdot \big(\eta_i - \mathfrak{u}_i^n\big) ds, \\
\mathscr{A}_t^n(H) &= \frac{1}{\sqrt{n}} \int_0^t \sum_{i \in \mathbb{T}_n} H\left(s, \frac{i}{n}\right) \cdot \left[-\partial_s \mathfrak{u}_n\left(s, \frac{i}{n}\right) + \Lambda_i^n\right] ds, \\
\mathscr{S}_t^n(H) &= \frac{\gamma_n}{2\sqrt{n}} \frac{1}{n} \int_0^t \sum_{i \in \mathbb{T}_n} \Delta_i^n H_2(s) V_n'(r_i) ds, \\
\mathscr{W}_t^n(H) &= \frac{1}{\sqrt{n}} \int_0^t \sum_{i \in \mathbb{T}_n} \nabla_{i-1}^n H_1(s) \big[- V_n'(r_i) + \boldsymbol{\tau}_i^n + \mathfrak{b}_i^n (r_i - \mathfrak{r}_i^n)\big] ds.
\end{aligned}$$

The finite-dimensional convergence is stated below.

Proposition 6.1. *Assume* (2.12) *and* (5.1). *Define* $h_n = h_n(t,x)$ *be the solution to the following adjoint equation on* $(t,x) \in [0,T] \times \mathbb{T}$*:*

$$\partial_t h_n - \begin{bmatrix} 0 & 1 \\ \boldsymbol{\tau}'_n(\mathfrak{r}_n) & 0 \end{bmatrix} \partial_x h_n = 0, \quad h_n(0,\cdot) = H, \tag{6.3}$$

with some fixed initial condition $H \in C^\infty(\mathbb{T})$*. For any* $\lambda > 0$*,*

$$\lim_{n\to\infty} \mathbb{P}_n \left\{ \sup_{t \in [0,T]} |Y_t^n(h_n(t)) - Y_0^n(H)| > \lambda \right\} = 0, \quad \forall \lambda > 0.$$

Proof. We investigate each term in (6.2) respectively. The martingale term M_t^n is the easiest. From (6.1), for all $t \in [0,T]$,

$$\mathbb{E}_n\big[|M_t^n(h_n)|^2\big] = \gamma_n \int_0^t \sum_{i \in \mathbb{T}_n} \frac{1}{n^2} \big(\nabla_i^n h_n\big)^2 ds \le \frac{\gamma_n t}{n} \sup_{s \in [0,t]} |\partial_x h_n(s)|_{\mathbb{T}}^2. \tag{6.4}$$

From Doob's inequality,

$$\mathbb{E}_n\left[\sup_{t \in [0,T]} |M_t^n(h_n)|^2\right] \le 4\mathbb{E}_n\big[|M_T^n(h_n)|^2\big] \le \frac{C\gamma_n}{n},$$

which vanishes as $n \to \infty$. For the integral $\mathscr{R}_t^n$, by (6.3),

$$\left|\partial_s h_n\left(s, \frac{i}{n}\right) - L_i^n h_n(s)\right| \le \frac{C|\partial_x^2 h_n(s)|_{\mathbb{T}}}{n}.$$

Using this estimate and Corollary 2.3, for all $p \in [1,2)$,

$$\mathbb{E}_n\left[\left|\frac{1}{\sqrt{n}}\sum_{i\in\mathbb{T}_n}\left[\partial_s h_n\left(s,\frac{i}{n}\right) - L_i^n h_n(s)\right]\cdot\begin{pmatrix} p_i - \mathfrak{p}_i^n \\ r_i - \mathfrak{r}_i^n \end{pmatrix}\right|^p\right] \le \frac{C_p|\partial_x^2 h_n(s)|_{\mathbb{T}}^p}{n^{p(1+\frac{1}{4}-\frac{1}{2})}}. \tag{6.5}$$

Taking $p = 1$ and with Cauchy–Schwarz inequality,

$$\lim_{n\to\infty}\mathbb{E}_n\left[\sup_{t\in[0,T]}\left|\mathscr{R}_t^n(h_n)\right|\right] \le \lim_{n\to\infty} CTn^{-\frac{3}{4}} = 0.$$

For the integral $\mathscr{A}_t^n$, since we assume that the quasi-linear system (2.9) has smooth solution at least up to time T, therefore

$$\sup_{s\in[0,T]}\left|\partial_t\mathfrak{u}_n\left(s,\frac{i}{n}\right) - \Lambda_i^n(s)\right| \le \frac{C_{\beta,\mathfrak{v},T}}{n}.$$

This gives us the uniform estimate that

$$\left|\frac{1}{\sqrt{n}}\sum_{i\in\mathbb{T}_n} h_n\left(s,\frac{i}{n}\right)\left[\partial_s\mathfrak{u}_n\left(s,\frac{i}{n}\right) - \Lambda_i^n\right]\right| \le \frac{C|h_n(s)|_{\mathbb{T}}}{\sqrt{n}}. \tag{6.6}$$

Hence, $|\mathscr{A}_t^n|$ vanishes uniformly on $t \in [0,T]$ as $n \to \infty$. For the integral $\mathscr{S}_t^n$, observe that the integrand can be bounded from above by

$$\left|\sum_{i\in\mathbb{T}_n}\Delta_i^n h_{n,2}(s)\big(V_n'(r_i) - \boldsymbol{\tau}_i^n\big)\right| + n\big|\partial_x^2 h_{n,2}(s)\boldsymbol{\tau}_n(\mathfrak{r}_n)\big|_{\mathbb{T}}.$$

Again, by Corollary 2.3, for all $p \in [1,2)$,

$$\mathbb{E}_n\left[\left|\frac{1}{n}\sum_{i\in\mathbb{T}_n}\Delta_i^n h_{n,2}(s)V_n'(r_i)\right|^p\right] \le C_p|\partial_x^2 h_n(s)|_{\mathbb{T}}^p\left(n^{-\frac{p}{4}} + 1\right). \tag{6.7}$$

Taking $p = 1$ and using Cauchy–Schwarz inequality,

$$\lim_{n\to\infty}\mathbb{E}_n\left[\sup_{t\in[0,T]}\left|\mathscr{S}_i^n(h_n)\right|\right] \le \lim_{n\to\infty}\frac{C\gamma_n}{2\sqrt{n}} = 0.$$

Finally, we apply Proposition 5.1 to $\mathscr{W}_t^n$ to get that

$$\lim_{n\to\infty}\mathbb{P}_n\left\{\sup_{t\in[0,T]}\left|\mathscr{W}_t^n(h_n)\right| > \lambda\right\} = 0, \quad \forall\lambda > 0.$$

The proof is then completed. □

7 Tightness

In this section, we prove that the laws of $\{Y_t^n; t \in [0,T]\}$ forms a tight sequence in proper trajectory space. We start with two lemmas. Suppose that for $n \ge 1$ and $f \in C^2(\mathbb{T})$, $\{X_t^n = X_t^n(f); t \in [0,T]\}$ is a random field on Ω_n. Define

$$\mathscr{X}_t^n(f) = \int_0^t X_s^n(f)ds, \quad \forall t \in [0,T].$$

By Kolmogorov–Prokhorov's tightness criterion, to show the tightness of $\mathscr{X}^n_\cdot$ on the α-Hölder continuous path space $C^\alpha([0,T];\mathcal{H}_{-k})$, one need to estimate

$$\big\| \mathscr{X}^n_{t'} - \mathscr{X}^n_t \big\|^2_{-k} = \sum_{m\in\mathbb{Z}} \frac{1}{(1+m^2)^k} \big| \mathscr{X}^n_{t'}(\varphi_m) - \mathscr{X}^n_t(\varphi_m) \big|^2,$$

with the Fourier basis φ_m defined in Section 1. Since Corollary 2.3 and 4.1 only hold with powers $p < 2$, the next result is helpful here.

Lemma 7.1. *Assume some $p > 1$ and $a > 0$, such that*

$$\mathbb{E}_n\big[|X^n_t(\varphi_m)|^p\big] \le C|m|^{ap}, \quad \forall m \in \mathbb{Z},\ t \in [0,T].$$

Then there exists a constant C_p, such that for $k > a + 3/2$,

$$\mathbb{P}_n\left\{ \big\| \mathscr{X}^n_{t'} - \mathscr{X}^n_t \big\|_{-k} > \lambda \right\} \le \frac{C_p|t'-t|^p}{\lambda^p}, \quad \forall 0 \le t < t' \le T.$$

In particular, $\mathscr{X}^n_\cdot$ is tight in $C^\alpha([0,T];\mathcal{H}_{-k})$ for $\alpha < 1 - 1/p$.

The proof of Lemma 7.1 is direct and we postpone it to the end of this section. In order to use Lemma 7.1, we need the following priori moment estimate.

Lemma 7.2. *Assume* (2.12) *and* (2.14) *with some $\epsilon \in \mathbb{R}$. For all $1 \le p < p_\epsilon = (4-2\epsilon)/(3-2\epsilon) \vee 2$ and $f \in C^2(\mathbb{T})$,*

$$\mathbb{E}_n\big[|Y^n_t(f)|^p\big] \le C\big(1 + |f|^p_{\mathbb{T}} + |f'|^{\delta p}_{\mathbb{T}} + |f''|^p_{\mathbb{T}}\big),$$

where $\delta = \delta_\epsilon = (3-2\epsilon)/(2-\epsilon) \wedge 1$ and $C = C_{\beta,\mathfrak{v},T,\epsilon,p}$.

Note that if we apply Corollary 2.3 to the left-hand side above, the upper bound could diverse. The additional condition (2.14) helps to avoid this.

Proof of Lemma 7.2. Fix some $t \in [0,T]$, and define $f_n = f_n(s,x)$ to be the solution of the following *backward equation* on $(s,x) \in [0,t] \times \mathbb{T}$:

$$\partial_s f_n - \begin{bmatrix} 0 & 1 \\ \boldsymbol{\tau}'_n(\mathfrak{r}_n) & 0 \end{bmatrix} \partial_x f_n = 0, \quad f_n(t,\cdot) = f.$$

Apply (6.2) with $H = f_n$ and estimate each term in the right-hand side.

For $Y^n_0(f_n(0))$, use (D.3) to get that for all $\lambda > 0$,

$$\mathbb{P}_n\big\{|Y^n_0(f_n(0))| > \lambda\big\} \le \frac{H_n(0) + \log 2}{-\log \mu_{0,n}\{|Y^n_0(f_n(0))| > \lambda\}}.$$

By Lemma 3.3 and the independence, with some $C = C_{\beta,\mathfrak{v}}$,

$$\mu_{0,n}\big\{|Y^n_0(f_n(0))| > \lambda\big\} \le 2\exp\left\{ -\frac{\lambda^2}{2C\|f_n(0)\|^2} \right\}.$$

As $H_n(0)$ is assumed to be bounded,

$$\mathbb{P}_n\big\{|Y^n_0(f_n(0))| > \lambda\big\} \le \frac{C\|f_n(0)\|^2(H_n(0)+1)}{\lambda^2} \le \frac{C'\|f_n(0)\|^2}{\lambda^2}.$$

Using Lemma D.4, for all $p \in [1,2)$,

$$\mathbb{E}_n\big[|Y^n_0(f_n(0))|^p\big] \le C_p\|f_n(0)\|^p \le C'_p|f|^p_{\mathbb{T}}.$$

For M^n_t, apply an interpolation of (6.4) with $p \in [1,2]$:

$$\mathbb{E}_n\big[|M^n_t(f_n)|^p\big] \le \big(\gamma_n t n^{-1}\big)^{\frac{p}{2}} \sup_{s\in[0,t]} |\partial_x f_n(s)|^p_{\mathbb{T}} \le C_{T,p}|f'|^p_{\mathbb{T}}.$$

For $\mathscr{R}_t^n$, it is easy to obtain from (6.5) that, for $p \in [1, 2)$,

$$\mathbb{E}_n\left[\left|\mathscr{R}_t^n(f_n)\right|^p\right] \le t^{p-1}\int_0^t Cn^{-\frac{3p}{4}}\|\partial_x^2 f_n(s)\|_{\mathbb{T}}^p ds \le C'|f''|_{\mathbb{T}}^p.$$

For $\mathscr{A}_t^n$, the upper bound of p-moment follows directly from the uniform estimate in (6.6). For $\mathscr{S}_t^n$, the estimate can be obtained from (6.7) similarly to $\mathscr{R}_t^n$.

The only term needs extra effort is $\mathscr{W}_t^n$. Rewrite this term as

$$\begin{aligned}\mathscr{W}_t^n(f) = \frac{\sigma_n}{\sqrt{n}}\int_0^t \sum_{i\in\mathbb{T}_n}\left(-U'(r_i) + \frac{\boldsymbol{\tau}_i^n - \mathfrak{t}_i^n}{\sigma_n}\right)\nabla_{i-1}^n f ds \\ + \frac{\mathfrak{b}_i^n - 1}{\sqrt{n}}\int_0^t \sum_{i\in\mathbb{T}_n}(r_i - \mathfrak{r}_i^n)\nabla_{i-1}^n f ds.\end{aligned}$$

By (2.5), $|\mathfrak{b}_i^n - 1| = O(\sigma_n)$, so we obtain from Theorem 2.2 and Corollary 4.1 that

$$\mathbb{E}_n\left[\left|\mathscr{W}_t^n(f)\right|^q\right] \le C(q)t^q\sigma_n^q(H_n(0) + K_n + 1)^{\frac{q}{2}}|f'|_{\mathbb{T}}^q,$$

for all $q \in [1, 2)$ with some $C(q) = C_{\beta,\mathfrak{v},T}(q)$. Thus, for all $\lambda > 0$,

$$\mathbb{P}_n\left\{\left|\mathscr{W}_t^n(f)\right| > \lambda\right\} \le \frac{Ct^q\sigma_n^q(H_n(0) + K_n + 1)^{\frac{q}{2}}|f'|_{\mathbb{T}}^q}{\lambda^q}.$$

In view of Lemma D.4, if $\sigma_n^2 K_n$ is bounded, or equivalently $\epsilon \ge 1$,

$$\mathbb{E}_n\left[\left|\mathscr{W}_t^n(f)\right|^p\right] \le C_{\beta,\mathfrak{v},T,p}t^p|f'|_{\mathbb{T}}^p$$

for all $p \in [1, 2)$ and we obtain the desired estimate. On the other hand, using (5.4) and (5.5) with $\alpha = t^{-1/2}$, we get the same probability bounded by

$$\mathbb{P}_n\left\{\left|\mathscr{W}_t^n(f)\right| > \lambda\right\} \le \frac{C\sqrt{t}(H_n(0) + K_n + 1)}{\lambda\sqrt{n}}\big(1 + |f'|_{\mathbb{T}}^2 + |f''(s)|_{\mathbb{T}}\big).$$

Note that the expression above vanishes for large n. Therefore, in case that $0 < \epsilon < 1$, we can apply the following interpolation for $\theta \in (0, 1)$ that

$$\begin{aligned}\mathbb{P}_n\left\{\left|\mathscr{W}_t^n(f)\right| > \lambda\right\} \le \frac{C(q,\theta)M_f(q,\theta)}{\lambda^{q\theta+1-\theta}}t^{q\theta+\frac{1-\theta}{2}} \times \\ \sigma_n^{q\theta}\big(H_n(0) + K_n + 1\big)^{\frac{q\theta}{2}+1-\theta}n^{\frac{\theta-1}{2}},\end{aligned}$$

where $C(q,\theta) = C_{\beta,\mathfrak{v},T}(q,\theta)$ and

$$M_f(q,\theta) = |f'|_{\mathbb{T}}^{q\theta}\big(1 + |f'|_{\mathbb{T}}^2 + |f''|_{\mathbb{T}}\big)^{1-\theta} \le C'(q,\theta)\big(1 + |f'|_{\mathbb{T}}^{q\theta+2(1-\theta)} + |f''|_{\mathbb{T}}\big).$$

To assure that the second line above is bounded in n, choose

$$\theta = \theta(\epsilon, q) = \frac{1}{1 + (1-\epsilon)q}.$$

The estimate above becomes

$$\mathbb{P}_n\left\{\left|\mathscr{W}_t^n(f)\right| > \lambda\right\} \le C(\epsilon, q)\lambda^{-q'}\big(1 + |f'|_{\mathbb{T}}^{q_*} + |f''|_{\mathbb{T}}\big)t^{q_{**}}, \tag{7.1}$$

where

$$q' = \frac{(2-\epsilon)q}{1 + (1-\epsilon)q}, \quad q_* = \frac{(3-2\epsilon)q}{1 + (1-\epsilon)q}, \quad q_{**} = \frac{(3-\epsilon)q}{2 + 2(1-\epsilon)q}.$$

The dependence on t is not important here. Notice that $1 < q' \leq 2$ for $q \in [1,2)$ and $\epsilon < 1$, thus we get from Lemma D.4 that for all $p \in [1, q')$,

$$\mathbb{E}_n\left[\left|\mathscr{W}_t^n(f)\right|^p\right] \leq C_{\beta,\mathfrak{v},T}(p,q')\big(1 + |f'|_{\mathbb{T}}^{\delta p} + |f''|_{\mathbb{T}}^{p/q'}\big),$$

where $\delta = q_*/q' = (3-2\epsilon)/(2-\epsilon)$ is independent of q. Since q can be taken arbitrarily close to 2, the inequality above holds for all $1 < p < p_\epsilon$. Finally, the lemma is proved by collecting all the moment estimate together. □

With these lemmas, we can prove the tightness of $Y_\cdot^n$ stated below.

Proposition 7.3. *Assume* (2.12) *and* (2.14) *with some* $\epsilon > 0$. *The laws of* $\{Y_t^n; t \in [0,T]\}$ *is tight with respect to the topology of* $C([0,T]; \mathcal{H}_{-k})$ *for* $k > 9/2$.

Proof. We need to investigate the tightness for each term in (6.2). Similar with (6.4), it is easy to observe that M_t^n is tight on $C([0,T]; \mathcal{H}_{-k})$ for $k > 3/2$:

$$\mathbb{E}_n\left[\|M_{t'}^n - M_t^n\|_{-k}^2\right] \leq \frac{\gamma_n|t'-t|}{n}\sum_{m\in\mathbb{Z}}\frac{|\varphi_m'|_{\mathbb{T}}^2}{(1+m^2)^k} \to 0.$$

The computations for $\mathscr{R}_\cdot^n$, $\mathscr{A}_\cdot^n$ and $\mathscr{S}_\cdot^n$ are also direct. For $\mathscr{R}_t^n$, note that

$$\mathscr{R}_t^n(\varphi_m) = -\int_0^t Y_s^n(L_i^n\varphi_m)ds.$$

From Lemma 7.2, for $1 \leq p < p_\epsilon$,

$$E\left[|Y_t^n(L_i^n\varphi_m)|^p\right] \leq C|m|^{3p}.$$

By Lemma 7.1, $\mathscr{R}_\cdot^n$ is tight on $C^\alpha([0,T]; \mathcal{H}_{-k})$ for $k > 9/2$, $\alpha < \alpha_\epsilon = 1 - 1/p_\epsilon$. For $\mathscr{A}_t^n$, observe that from (6.6), for $k > 1/2$,

$$\left\|\mathscr{A}_{t'}^n - \mathscr{A}_t^n\right\|_{-k}^2 \leq \frac{C|t'-t|^2}{n}\sum_{m\in\mathbb{Z}}\frac{|\varphi_m|_{\mathbb{T}}^2}{(1+m^2)^k} \to 0.$$

Therefore, it is tight in $C^1([0,T]; \mathcal{H}_{-k}(\mathbb{T}))$ for $k > 1/2$. For $\mathscr{S}_t^n$, notice that $\varphi_m'' = Cm^2\varphi_m$. Substituting this into (6.7), we obtain that

$$\mathbb{E}_n\left[\left|\frac{1}{n}\sum_{i\in\mathbb{T}_n}\Delta_i^n\varphi_m V_n'(r_i)\right|^p\right] \leq C|m|^{2p}, \quad \forall p \in [1,2).$$

By Lemma 7.1, it is tight on $C^\alpha([0,T]; \mathcal{H}_{-k}(\mathbb{T}))$ for $k > 7/2$, $\alpha < 1/2$.

We are left with $\mathscr{W}_t^n$. In order to prove its tightness, we need to track the power of t in (7.1). Repeat the computation, we obtain that for $1 \leq p < p_\epsilon$,

$$\mathbb{E}_n\left[\left|\mathscr{W}_{t'}^n(f) - \mathscr{W}_t^n(f)\right|^p\right] \leq C\big(1 + |f'|_{\mathbb{T}}^{\delta p} + |f''|_{\mathbb{T}}\big)|t'-t|^{q_{**}p/p_\epsilon}.$$

As $\epsilon > 0$, $q_{**} > 1$ when $2/(1+\epsilon) < q < 2$. Therefore, there exists some $p > 1$, smaller than but close to p_ϵ, such that

$$\mathbb{E}_n\left[\left|\mathscr{W}_{t'}^n(f) - \mathscr{W}_t^n(f)\right|^p\right] \leq C\big(1 + |f'|_{\mathbb{T}}^{\delta p} + |f''|_{\mathbb{T}}\big)|t'-t|^{p'},$$

where $p' > 1$. Applying the estimate to $f = \varphi_m$ and noticing that $\delta < 3/2$, by Lemma 7.1 we know that $\mathscr{W}_t^n$ is tight in $C^\alpha([0,T], \mathcal{H}_{-k})$ for $\alpha < 1 - 1/p$ and $k > 9/2$. In conclusion, the laws of $Y_\cdot^n$ is tight with respect to the topology of $C([0,T]; \mathcal{H}_{-k})$ with $k > 9/2$. □

Proof of Lemma 7.1. For any t, $t' \in [0,T]$,

$$\mathbb{E}_n\left[\left|\mathscr{X}^n_{t'}(\varphi_m) - \mathscr{X}^n_t(\varphi_m)\right|^p\right] \le C|t'-t|^p|m|^{ap}.$$

For any $\epsilon > 0$, with $C(\epsilon) = \sum_{m\in\mathbb{Z}}(1+m^2)^{-\frac{1}{2}-\epsilon}$,

$$\begin{aligned}
&\mathbb{P}_n\left\{\left\|\mathscr{X}^n_{t'} - \mathscr{X}^n_t\right\|_{-k} > \lambda\right\} \\
\le &\sum_{m\in\mathbb{Z}} \mathbb{P}_n\left\{\left|\mathscr{X}^n_{t'}(\varphi_m) - \mathscr{X}^n_t(\varphi_m)\right| \ge \frac{\lambda}{\sqrt{C(\epsilon)}}(1+m^2)^{\frac{1}{2}(k-\frac{1}{2}-\epsilon)}\right\} \\
\le &\frac{C(p,\epsilon)}{\lambda^p}\sum_{m\in\mathbb{Z}}(1+m^2)^{-\frac{p}{2}(k-\frac{1}{2}-\epsilon)}C|t'-t|^p|m|^{ap}.
\end{aligned}$$

Hence, for any $k > a + 3/2$, the probability is bounded from above by

$$\frac{C'(p,\epsilon)|t'-t|^p}{\lambda^p}\sum_{m\in\mathbb{Z}}\frac{1}{(1+m^2)^{\frac{p}{2}(1-\epsilon)}}.$$

By fixing some ϵ such that $p(1-\epsilon) > 1$, we obtain the desired estimate. For the tightness, only note that by Lemma D.4,

$$\mathbb{E}_n\left[\left\|\mathscr{X}^n_{t'} - \mathscr{X}^n_t\right\|^q_{-k}\right] \le C_{p,q}|t'-t|^q, \quad \forall q \in (1,p),$$

and invoke Kolmogorov–Prokhorov's tightness criterion. □

8 Equivalence of ensembles

In this section we prove the equivalence of ensembles for inhomogeneous canonical measure, which is used in Section 3. Our main result, Proposition 8.3, is valid not only for the weakly anharmonic case, but also for the general anharmonic case.

Recall that for $\tau \in \mathbb{R}$, $\sigma \in [0,1)$, we have the probability measure

$$\pi_{\tau,\sigma}(dr) = \exp\left\{-\frac{\beta r^2}{2} - \beta\sigma U(r) + \beta\tau r - \beta G_\sigma(\tau)\right\}dr.$$

For simplicity, we fix $\beta = 1$ in this section, but the arguments apply to any fixed β naturally. For $\vec{\tau} = (\tau_1, \ldots, \tau_n)$, define $\mu_{\vec{\tau},\sigma}$ as the product measure $\otimes_{j=1}^n \pi_{\tau_j,\sigma}(dr_j)$ on $\mathbb{R}^n$. For bounded continuous function F on $\mathbb{R}^n$, define

$$\langle F|u\rangle_{\vec{\tau},\sigma} = E_{\mu_{\vec{\tau},\sigma}}\left[F|r_{(n)} = u\right], \quad r_{(n)} = \frac{1}{n}\sum_{j=1}^n r_j.$$

The conditioned probability distribution $\langle\,\cdot\,|u\rangle_{\vec{\tau},\sigma}$ is called the *micro canonical ensemble*, while $\mu_{\vec{\tau},\sigma}$ is called the *canonical ensemble*.

First of all, we present a basic property of the micro canonical ensemble, which would be frequently used hereafter in this section. Note that as U is smooth, we can define the regular conditional expectation $\langle F|u\rangle_{\vec{\tau},\sigma}$ point-wisely for all $u \in \mathbb{R}$.

Proposition 8.1. *For all $u \in \mathbb{R}$, $\vec{\tau} \in \mathbb{R}^n$ and $\tau \in \mathbb{R}$,*

$$\langle F|u\rangle_{\vec{\tau},\sigma} = \langle F|u\rangle_{\vec{\tau}-\tau,\sigma}, \tag{8.1}$$

where $\vec{\tau} - \tau \triangleq (\tau_1 - \tau, \ldots, \tau_n - \tau)$. Moreover, there is $\vec{\nu} = \vec{\nu}(u;\vec{\tau},\sigma)$, such that

$$E_{\mu_{\vec{\nu},\sigma}}[r_{(n)}] = u, \quad \langle\,\cdot\,|u\rangle_{\vec{\nu},\sigma} = \langle\,\cdot\,|u\rangle_{\vec{\tau},\sigma}. \tag{8.2}$$

In particular when $n = 1$, $\nu(u;\tau,\sigma) = \boldsymbol{\tau}_\sigma(u)$.

Proof. By direct computation, for $F = F(r_1, \dots, r_k)$ and $n \geq k$,

$$\langle F|u\rangle_{\vec{\tau},\sigma} = \frac{n}{n-k} \int_{\mathbb{R}^k} \frac{1}{f_{\vec{\tau},\sigma}(u)} f_{\vec{\tau}^*,\sigma}\left(\frac{nu - kr_{(k)}}{n-k}\right) F(r_1, \dots, r_k) \prod_{j=1}^{k} \pi_{n,j}(dr_j), \tag{8.3}$$

where $f_{\vec{\tau},\sigma}$ denotes the density of $r_{(n)}$ under $\mu_{\vec{\tau},\sigma}$ and $\vec{\tau}^* = (\tau_{k+1}, \dots, \tau_n)$. Observe that for bounded continuous function h on $\mathbb{R}$ and $\tau \in \mathbb{R}$,

$$\begin{aligned}
E_{\mu_{\vec{\tau},\sigma}}\big[h \circ r_{(n)}\big] &= \int h\left(\frac{1}{n}\sum_{j=1}^{n} r_j\right) \exp\left\{\sum_{j=1}^{n} \tau_j r_j - \frac{r_j^2}{2} - \sigma U(r_j) - G_\sigma(\tau_j)\right\} d\mathbf{r} \\
&= \exp\left\{\sum_{j=1}^{n} G_\sigma(\tau_j - \tau) - G_\sigma(\tau_j)\right\} E_{\mu_{\vec{\tau}-\tau,\sigma}}\big[e^{n\tau r_{(n)}} h \circ r_{(n)}\big].
\end{aligned}$$

Since h is arbitrary,

$$f_{\vec{\tau},\sigma}(u) = \exp\left\{n\tau u + \sum_{j=1}^{n} G_\sigma(\tau_j - \tau) - G_\sigma(\tau_j)\right\} f_{\vec{\tau}-\tau,\sigma}(u). \tag{8.4}$$

The relation (8.1) then follows from (8.3) and (8.4). In order to define $\vec{\nu}$ that fulfils (8.2), observe that as G_σ is strictly convex, there is a unique $\tau \in \mathbb{R}$ such that

$$E_{\mu_{\vec{\tau}-\tau,\sigma}}[r_{(n)}] = \frac{1}{n}\sum_{j=1}^{n} G'_\sigma(\tau_j - \tau) = u.$$

It suffices to define $\vec{\nu}(u; \vec{\tau}, \sigma) = \vec{\tau} - \tau$. □

Recall the functions F_σ, $\boldsymbol{\tau}_\sigma$ and $\bar{r}_\sigma$ defined in (2.3)–(2.4). For each pair of $(\tau, r) \in \mathbb{R}^2$, the rate function $I_\sigma(\tau, r)$ is defined as

$$\begin{aligned}
I_\sigma(\tau, r) &= G_\sigma(\tau) + F_\sigma(r) - r\tau \\
&= G_\sigma(\tau) - G_\sigma(\boldsymbol{\tau}_\sigma(r)) - G'_\sigma(\boldsymbol{\tau}_\sigma(r))(\tau - \boldsymbol{\tau}_\sigma(r)).
\end{aligned} \tag{8.5}$$

Taking advantage of (8.2) and (8.4), we can rewrite the density as

$$f_{\vec{\tau},\sigma}(u) = \exp\left\{-\sum_{j=1}^{n} I_\sigma\big(\tau_j, \bar{r}_\sigma(\nu_j)\big)\right\} f_{\vec{\nu},\sigma}(u), \quad \forall u \in \mathbb{R}, \tag{8.6}$$

where $\vec{\nu} = (\nu_1, \dots, \nu_n) = \vec{\nu}(u; \vec{\tau}, \sigma)$ is defined through (8.2).

The classical equivalence of ensembles (cf. [18, Appendix 2]) can be extended to the case that canonical measure is inhomogeneous. In order to cover the weakly anharmonic setting in Section 2, for each $n \geq 1$, pick $\sigma_n \in [0, 1)$, $\vec{\tau}_n = (\tau_{n,1}, \dots, \tau_{n,n}) \in \mathbb{R}^n$ and fix them. For sake of readability, in the following we write

$$\pi_{n,j} = \pi_{\tau_{n,j},\sigma_n}, \quad \mu_n = \mu_{\vec{\tau}_n,\sigma_n}, \quad E_n = E_{\mu_n}, \quad \langle \cdot |u\rangle_n = \langle \cdot |u\rangle_{\vec{\tau}_n,\sigma_n}.$$

Also denote that

$$u_n = E_n[r_{(n)}], \quad u_{n,2} = \sqrt{E_n\big[(r_{(n)} - u_n)^2\big]}.$$

We have the following result (cf. [18, Corollary A2.1.4, pp. 353]).

Proposition 8.2. *Assume some $\epsilon > 0$ and $K > 0$, such that*

$$\sup\{\sigma_n; n \geq 1\} < 1 - \epsilon, \quad \sup\{|\tau_{n,j}|; n \geq 1, 1 \leq j \leq n\} \leq K. \tag{8.7}$$

For any $F = F(r_1, \ldots, r_k)$ such that $E_n[F^2] < \infty$, we have

$$\big|\langle F|u_n\rangle_n - E_n[F]\big| \leq \frac{Ck}{n}\sqrt{E_n\big[(F - E_n[F])^2\big]},$$

with some constant $C = C_{\epsilon,K}$ for each $n \geq k$.

Proof. In view of (8.3), the key point is to understand the asymptotic behaviour of the density of $r_{(n)}$. To this end, we first check the conditions of the local central limit theorem in Appendix F. Similarly to Appendix A, for $0 \leq \ell \leq 4$, the ℓ-derivative of G_σ satisfies that

$$\big|G_\sigma^{(\ell)}(\tau) - G_0^{(\ell)}(\tau)\big| \leq C_1\sigma, \quad \forall \sigma \in [0, 1-\epsilon),\ \tau \in [-K, K],$$

with a uniform constant $C_1 = C_1(\epsilon, K)$. Let $\Phi_{\tau,\sigma}$ be the characteristic function

$$\Phi_{\tau,\sigma}(\xi) = \int_{\mathbb{R}} \exp\big\{i\xi(r - E_{\pi_{\tau,\sigma}}[r])\big\}\pi_{\tau,\sigma}(dr).$$

By the integration by parts formula,

$$i\xi\Phi_{\tau,\sigma}(\xi) = \int_{\mathbb{R}} \exp\big\{i\xi(r - E_{\pi_{\tau,\sigma}}[r])\big\}(r + \sigma U'(r) - \tau)\pi_{\tau,\sigma}(dr).$$

It is not hard to obtain with some $C_2 = C_2(\epsilon, K)$ that

$$|\Phi_{\tau,\sigma}(\xi)| \leq C_2(1 + |\xi|)^{-1}, \quad \forall \sigma \in [0, 1-\epsilon),\ \tau \in [-K, K].$$

Moreover, using the inequality $|e^x - 1| \leq e^{|x|}|x|$, for $0 \leq \ell \leq 4$,

$$\big|\Phi_{\tau,\sigma}^{(\ell)}(\xi) - \Phi_{\tau,0}^{(\ell)}(\xi)\big| \leq C_3\sigma, \quad \forall \sigma \in [0, 1-\epsilon),\ \tau \in [-K, K],$$

with some $C_3 = C_3(\epsilon, K)$. By the arguments above, the conditions (i), (ii), (iii) in Appendix F are fulfilled by $\pi_{0,\sigma}$ *uniformly* for $\sigma < 1 - \epsilon$. Hence, (8.7) assures that Lemma F.1 is applicable to μ_n, even when the reference measure π_{0,σ_n} is changing with n.

Fix some $k \geq 1$ and a function $F = F(r_1, \ldots, r_k)$. Denote by f_n the density of $r_{(n)}$ under μ_n. According to Lemma F.1, with a bounded sequence $C_{n,0}$,

$$\frac{1}{\sqrt{n}} f_n(u_n) = \frac{1}{u_{n,2}\sqrt{2\pi}}\left(1 + \frac{C_{n,0}}{n}\right) + o\left(\frac{1}{n}\right).$$

Similarly, denote by f_n^* the density of $r_{(n-k)}$ under $\mu_{\vec{\tau}_n^*,\sigma_n}$, $\vec{\tau}_n^* = (\tau_{n,k+1}, \ldots \tau_{n,n})$, then there are bounded sequences $C_{n,0}^*$, $C_{n,1}^*$, such that

$$\begin{aligned}
&\frac{1}{\sqrt{n-k}} f_n^*\left(\frac{nu_n - kr_{(k)}}{n-k}\right) \\
&= \frac{1}{u_{n,2}^*\sqrt{2\pi}} \exp\left\{-\frac{y_{(k)}^2}{2(n-k)}\right\}\left(1 + \frac{C_{n,0}^* + C_{n,1}^* y_{(k)}}{n-k}\right) + o\left(\frac{1}{n}\right),
\end{aligned}$$

where

$$u_{n,2}^* = \sqrt{\frac{1}{n}\sum_{j=1}^{n-k} G''_{\sigma_n}(\tau_{n,j+k})}, \quad y_{(k)} = \sum_{j=1}^{k} \frac{r_j - E_n[r_j]}{u_{n,2}^*}.$$

Therefore, the density in (8.3) satisfies the estimate

$$\begin{aligned}&\frac{n}{n-k}\frac{1}{f_n(u_n)}f_n^*\left(\frac{nu_n-kr_{(k)}}{n-k}\right)\\&\le\frac{u_{n,2}\sqrt{n}}{u_{n,2}^*\sqrt{n-k}}\left[1+\frac{C}{n}\big(1+y_{(k)}+y_{(k)}^2\big)\right]+o\left(\frac{1}{n}\right),\end{aligned}$$

where $C=C_{\epsilon,K}$ is a uniform constant. Furthermore,

$$\frac{u_{n,2}\sqrt{n}}{u_{n,2}^*\sqrt{n-k}}=\sqrt{1+\frac{G''_{\sigma_n}(\tau_{n,1})+\ldots+G''_{\sigma_n}(\tau_{n,k})}{G''_{\sigma_n}(\tau_{n,k+1})+\ldots+G''_{\sigma_n}(\tau_{n,n})}}\le 1+\frac{Ck}{2n}+o\left(\frac{1}{n}\right).$$

Therefore, with some constant $C'=C'_{\epsilon,K}$,

$$\left|\frac{n}{n-k}\frac{1}{f_n(u_n)}f_n^*\left(\frac{nu_n-kr_{(k)}}{n-k}\right)-1\right|\le\frac{C'}{n}\big(k+y_{(k)}+y_{(k)}^2\big)+o\left(\frac{1}{n}\right).$$

Proposition 8.2 then follows from (8.3) and Schwarz inequality. □

Proposition 8.2 is valid only for cylinder functions $F=F(r_1,\ldots,r_k)$. In Section 3, it is required to control the exponential moment of the micro canonical expectation of a particular extensive observation. Next, we give the corresponding result.

Recall that $\bar{r}_n(\tau)=\bar{r}_{\sigma_n}(\tau)$, $\tau_n(r)=\boldsymbol{\tau}_{\sigma_n}(r)$. Given $F:\mathbb{R}\to\mathbb{R}$, let

$$\mathcal{F}_{n,j}(r)=F(r)-E_{\pi_{n,j}}[F]-\frac{d}{dr}E_{\pi_{\tau_n(r),\sigma_n}}[F]\Big|_{r=\bar{r}_n(\tau_{n,j})}\big(r-\bar{r}_n(\tau_{n,j})\big),$$

for $j=1,\ldots,n$, and define $\mathcal{F}=\sum_{j=1}^n\mathcal{F}_{n,j}(r_j)$.

Proposition 8.3. *Assume* (8.7)*, and a constant M such that*

$$|\tau_{n,j}-\tau_{n,j+1}|\le Mn^{-\frac{3}{2}},\quad\forall n\ge 1,\ 1\le j\le n.\tag{8.8}$$

Suppose that for each n, $\tau\mapsto\int F d\pi_{\tau,\sigma_n}$ is twice continuously differentiable, and there is some constant $A>0$, such that for all (n,j),

$$E_n\big[\exp(s|\mathcal{F}_{n,j}|)\big]\le e,\quad\forall|s|\le A.$$

Then, we can find $A_1<\infty$ and $A_2>0$, such that for all $n\ge 1$,

$$E_n\big[\exp(s|\langle\mathcal{F}|\cdot\rangle_n|)\big]\le A_1,\quad\forall|s|\le A_2.$$

Remark 8.4. Proposition 8.3 is stated for function F on $\mathbb{R}$, but the parallel result for F on $\mathbb{R}^k$ for each $k\ge 1$ can be proved without additional efforts. Furthermore, the Euler's constant e in the condition is not sensible.

Proof of Proposition 8.3. Fix an F fulfilling the conditions. Recall that $u_n=E_n[r_{(n)}]$, and let $A_{n,\delta}=\{\mathbf{r}\in\mathbb{R}^n;r_{(n)}\in(u_n-\delta,u_n+\delta)\}$ for $\delta>0$. Note that

$$E_n\big[e^{s|\langle\mathcal{F}|u\rangle_n|}\big]=E_n\big[e^{s|\langle\mathcal{F}|u\rangle_n|}\mathbf{1}_{A_{n,\delta}^c}\big]+E_n\big[e^{c|\langle\mathcal{F}|u\rangle_n|}\mathbf{1}_{A_{n,\delta}}\big].$$

We estimate the two terms respectively.

For the integral on $A_{n,\delta}^c$, recall the rate function I_σ in (8.5). As G_{σ_n} is strictly convex and $\tau_{n,j}$, σ_n are bounded, for δ sufficiently small we have that

$$I_{\sigma_n}(\tau_{n,j},r)\ge C\delta^2,\quad\forall|r-\bar{r}_n(\tau_{n,j})|\ge\delta,\tag{8.9}$$

with some $C = C(\delta)$. By (8.6), (8.9) and Lemma F.1, for δ small but fixed,

$$f_n(u) \le \exp\left\{-M\delta^2 n + \frac{\log n}{2}\right\}, \quad \forall |u - u_n| \ge \delta.$$

Hence, by Hölder's inequality, for p, $q > 1$ such that $1/p + 1/q = 1$,

$$\begin{aligned} E_n\big[e^{s|\langle \mathcal{F}|u\rangle_n|}\mathbf{1}_{A_{n,\delta}^c}\big] &\le \big(\mu_n\{|r_{(n)} - u_n| \ge \delta\}\big)^{\frac{1}{p}} \left(E_n\big[e^{sq|\langle \mathcal{F}|u\rangle_n|}\big]\right)^{\frac{1}{q}} \\ &\le \exp\left\{-\frac{M\delta^2 n}{p} + \frac{\log n}{2p}\right\} \prod_{j=1}^n E_n\big[e^{sq|\mathcal{F}_{n,j}|}\big]. \end{aligned}$$

Choose some $p < M\delta^2 + 1$, we have that for any $|s| < q^{-1}A$ that

$$E_n\big[e^{c|\langle \mathcal{F}|u\rangle_n|}\mathbf{1}_{A_{n,\delta}^c}\big] \le \exp\left\{-\frac{M\delta^2 n}{p} + \frac{n}{q} + \frac{\log n}{2p}\right\} \to 0.$$

To deal with the integral on $A_{n,\delta}$, divide $\langle \mathcal{F}_{n,j}(r_j)|u\rangle_n$ into two parts:

$$\begin{aligned} K_{n,j} &= \langle F(r_j)|u\rangle_n - E_{\pi_{\nu_{n,j},\sigma_n}}[F] \\ &\quad - \frac{d}{dr}E_{\pi_{\tau_n(r),\sigma_n}}[F]\Big|_{r=\bar{r}_n(\tau_{n,j})}\big(\langle r_j|u\rangle_n - \bar{r}_n(\nu_{n,j})\big); \\ K'_{n,j} &= E_{\pi_{\nu_{n,j},\sigma_n}}[F] - E_{\pi_{\tau_{n,j},\sigma_n}}[F] \\ &\quad - \frac{d}{dr}E_{\pi_{\tau_n(r),\sigma_n}}[F]\Big|_{r=\bar{r}_n(\tau_{n,j})}\big(\bar{r}_n(\nu_{n,j}) - \bar{r}_n(\tau_{n,j})\big), \end{aligned}$$

where $(\nu_{n,1}, \dots \nu_{n,n}) = \vec{\nu}_n = \vec{\nu}(u; \vec{\tau}_n, \sigma_n)$ is the vector defined through (8.2). The definition of $\vec{\nu}_n$ together with Proposition 8.2 yields that

$$\langle F(r_j)|u\rangle_n = E_{\nu_{n,j}}[F] + O\left(\frac{1}{n}\right), \quad \langle r_j|u\rangle_n = \bar{r}_n(\nu_{n,j}) + O\left(\frac{1}{n}\right),$$

uniformly in $A_{n,\delta}$. Therefore, $\sum_j |K_{n,j}|$ is uniformly bounded. Meanwhile,

$$|K'_{n,j}| \le \frac{1}{2}\sup_{|u-u_n|<\delta}\left|\frac{d^2}{du^2}E_{\pi_{\tau_n(r),\sigma_n}}[F]\Big|_{r=\bar{r}_n(\tau_{n,j})}\right|\big(\bar{r}_n(\nu_{n,j}) - \bar{r}_n(\tau_{n,j})\big)^2.$$

Hence, it suffices to prove that

$$E_n\left[\exp\left\{s\sum_{j=1}^n \big(\bar{r}_n(\nu_{n,j}) - \bar{r}_n(\tau_{n,j})\big)^2\right\}\right] \le A_1, \quad \forall |s| \le A_2,$$

with some $A_1 < \infty$ and $A_2 > 0$. To this end, note that

$$\big(\bar{r}_n(\nu_{n,j}) - \bar{r}_n(\tau_{n,j})\big)^2 \le 3(r_{(n)} - u_n)^2 + 3(u_n - \bar{r}_n(\tau_{n,j}))^2 + 3(\bar{r}_n(\nu_{n,j}) - r_{(n)})^2.$$

We estimate the three terms in the right-hand side respectively. For the first term, it is easy to see from central limit theorem that, for $|s| < u_{n,2}/2$,

$$\lim_{n\to\infty} E_n\big[\exp\big\{cn(r_{(n)} - E_n[r_{(n)}])^2\big\}\big] = \frac{1}{u_{n,2}\sqrt{2\pi}}\int_{\mathbb{R}} e^{cx^2}e^{-\frac{x^2}{2u_{n,2}}}dx < \infty$$

For the second term, taking advantage of (8.8), we obtain that

$$\begin{aligned} \sum_{j=1}^n \big(u_n - \bar{r}_n(\tau_{n,j})\big)^2 &= \sum_{j=1}^n \left(\frac{1}{n}\sum_{j'=1}^n \bar{r}_n(\tau_{n,j'}) - u(\tau_{n,j})\right)^2 \\ &\le \frac{1}{n}\sum_{j,j'}\big(\bar{r}_n(\tau_{n,j'}) - \bar{r}_n(\tau_{n,j})\big)^2 \le O(1). \end{aligned}$$

For the third term, observe that by the definition of $\vec{\nu}_n$,

$$\frac{1}{n}\sum_{j=1}^{n}\bar{r}_n(\nu_{n,j}) = r_{(n)}, \quad \nu_{n,j'} - \nu_{n,j} = \tau_{n,j'} - \tau_{n,j},$$

so, it can be estimated similarly to the second term. □

9 Gradient estimate for the Poisson equation

In this section, we present a gradient-type estimate for the solution to the Poisson equation (3.7), which is used in the proof of Lemma 3.1.

We work under the following case with general anharmonic potential function. Let V be a given C^2-smooth, uniformly convex function:

$$V''(x) \geq c > 0, \quad \forall x \in \mathbb{R}.$$

With a given vector $\mathbf{a} = (a_1, \ldots, a_n)$, define $U : \mathbb{R}^n \to \mathbb{R}$ by

$$U(\mathbf{x}) = \sum_{j=1}^{n} V(x_j) - \mathbf{a}\cdot\mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n.$$

Then, $D_j U = V'(x_{j+1}) - V'(x_j) - a_{j+1} + a_j$, where the operator D_j is

$$D_j = \frac{\partial}{\partial x_{j+1}} - \frac{\partial}{\partial x_j}, \quad \forall j = 1, \ldots, n-1.$$

For $x \in \mathbb{R}$, let $\Sigma_x = \{\mathbf{x} \in \mathbb{R}^n; x_1 + \ldots + x_n = x\}$ be the $(n-1)$-dimensional hyperplane. Suppose a differentiable function Ψ to satisfy the following conditions:

$$\sup_{\mathbb{R}^n}\sum_{j=1}^{n-1}|D_j\Psi| < \infty, \quad \int_{\Sigma_x} e^{-U(\mathbf{x})}\Psi(\mathbf{x}) = 0,$$

for all $x \in \Sigma_x$. Consider the following partial differential equation:

$$-e^U\sum_{j=1}^{n-1} D_j\big(e^{-U}D_j\psi\big) = \Psi.$$

Note that the Poisson equation (3.7) discussed in the proof of Lemma 3.1 can be obtained by taking $n = \ell$, $V = \beta V_n$ and $a = \beta(\tau_i, \ldots, \tau_{i+\ell-1})$. A sharp gradient-type estimate for the solution ψ is obtained in [29, Theorem 1.1]. By investigating the constant in their estimate, we get the following result.

Proposition 9.1. *There is a constant C dependent on $c = \inf V''$, such that*

$$\big|\mathbf{D}\psi(\mathbf{x})\big|^2 \leq Cn^4 \sup_{\mathbb{R}^n}\big|\mathbf{D}\Psi\big|^2, \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

where $\mathbf{D} = (D_1, \ldots, D_{n-1})$.

Proof. Rewrite the equation with the new coordinates:

$$y_j = -\sum_{i=1}^{j} x_i, \quad \forall j = 1, \ldots, n-1, \quad y_* = -\sum_{j=1}^{n} x_j.$$

Notice that for $1 \leq j \leq n-1$, $D_j = \partial_{y_j}$. The new equation is

$$\nabla_{\mathbf{y}}\widetilde{U}(\mathbf{y}; y_*) \cdot \nabla_{\mathbf{y}}\widetilde{\psi} - \Delta_{\mathbf{y}}\widetilde{\psi} = \widetilde{\Psi}(\mathbf{y}; y_*),$$

where y_* is viewed as a parameter, $\mathbf{y} = (y_1, \dots, y_{n-1})$, and

$$\widetilde{\psi}(\mathbf{y}; y_*) = \psi(\mathbf{x}), \quad \widetilde{\Psi}(\mathbf{y}; y_*) = \Psi(\mathbf{x}),$$
$$\widetilde{U} = V(-y_1) + \sum_{j=1}^{n-2} V(y_j - y_{j+1}) + V(y_{n-1} - y_*) + \sum_{j=1}^{n-1} (a_j - a_{j+1}) y_j.$$

Denote by $\lambda_n = \lambda_{\min}(H_n)$ the smallest eigenvalue of

$$H_n = \operatorname{hess}_{\mathbf{y}} \widetilde{U}(\cdot, y_*) = \begin{pmatrix} b_1 + b_2 & -b_2 & 0 & \dots & 0 \\ -b_2 & b_2 + b_3 & -b_3 & \dots & 0 \\ 0 & -b_3 & b_3 + b_4 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & b_{n-1} + b_n \end{pmatrix}, \tag{9.1}$$

where we write $b_j = V''(x_j)$ for $1 \le j \le n$. As each $b_j > 0$, it is easy to observe that $\lambda_n > 0$. Applying [29, Theorem 1.1] for each fixed y_*,

$$\left| \nabla_{\mathbf{y}} \widetilde{\psi}(\mathbf{y}; y_*) \right| \le \lambda_n^{-1} \sup_{\mathbf{y}} \left| \nabla_{\mathbf{y}} \widetilde{\Psi}(\mathbf{y}; y_*) \right|, \quad \forall (\mathbf{y}, y_*) \in \mathbb{R}^n.$$

In Lemma 9.2, we show that $\lambda_n \ge C n^{-2}$ with some constant $C = C(c)$. By returning to the original variables $\mathbf{x}$, we get the desired estimate. □

The proof of Proposition 9.1 is completed by the following lower bound of λ_n.

Lemma 9.2. *In* (9.1)*, suppose that* $b_j \ge c > 0$ *for all* j*, then*

$$\lambda_n = \lambda_{\min}(H_n) \ge \frac{6c}{(n-1)(n+1)}, \quad \forall n \ge 2. \tag{9.2}$$

Proof. Let I_n be the $n \times n$ identity matrix, and define $Q_1(\lambda) = -b_1^{-1}$,

$$Q_n(\lambda) = (-1)^n \det\big(\lambda I_{n-1} - H_n\big) \prod_{j=1}^{n} \frac{1}{b_n}, \quad \forall n \ge 2. \tag{9.3}$$

Notice that $Q_2(0) = -(b_1^{-1} + b_2^{-1})$, and

$$\frac{Q_n(0) - Q_{n-1}(0)}{Q_{n-1}(0) - Q_{n-2}(0)} = \frac{b_{n-1}}{b_n}, \quad \forall n \ge 3.$$

By a simple inductive argument, we obtain that

$$Q_n(0) = -\sum_{j=1}^{n} \frac{1}{b_j}, \quad \forall n \ge 1.$$

Similarly, we have $Q_1'(0) = 0$, $Q_2'(0) = (b_1 b_2)^{-1}$, and

$$b_n\big(Q_n'(0) - Q_{n-1}'(0)\big) - b_{n-1}\big(Q_{n-1}'(0) - Q_{n-2}'(0)\big) = -Q_{n-1}(0) > 0, \quad \forall n \ge 3.$$

By using this relation recurrently, we have the expression

$$Q_n'(0) = -\sum_{j'=2}^{n} \frac{1}{b_{j'}} \sum_{j=1}^{j'-1} Q_j(0) = \sum_{j'=1}^{n-1} \left(\sum_{j=1}^{j'} \frac{1}{b_j} \right) \left(\sum_{j=j'+1}^{n} \frac{1}{b_j} \right).$$

Observing that for each $1 \leq j' \leq n-1$,

$$\left(\sum_{j=1}^{j'} \frac{1}{b_j} - \frac{j'}{n} \sum_{j=1}^{n} \frac{1}{b_j} \right) \left(\sum_{j=j'+1}^{n} \frac{1}{b_j} - \frac{n-j'}{n} \sum_{j=1}^{n} \frac{1}{b_j} \right) \leq 0.$$

Therefore, with the condition $b_j \geq c > 0$ for each j, we get

$$\begin{aligned} \left(\sum_{j=1}^{j'} \frac{1}{b_j} \right) \left(\sum_{j=j'+1}^{n} \frac{1}{b_j} \right) &\leq \left[\frac{(j')^2}{n^2} \sum_{j=j'+1}^{n} \frac{1}{b_j} + \frac{(n-j')^2}{n^2} \sum_{j=1}^{j'} \frac{1}{b_j} \right] \sum_{j=1}^{n} \frac{1}{b_j} \\ &\leq \frac{(j')^2(n-j') + (n-j')^2 j'}{cn^2} \sum_{j=1}^{n} \frac{1}{b_j} \\ &= \frac{j'(n-j')}{cn} \sum_{j=1}^{n} \frac{1}{b_j}. \end{aligned}$$

Summing up the estimate above for $j' = 1$ to $n-1$,

$$0 < Q_n'(0) \leq \sum_{j'=1}^{n-1} \frac{j'(n-j')}{nc} \sum_{j=1}^{n} \frac{1}{b_j} = \frac{(n-1)(n+1)}{6c} \sum_{j=1}^{n} \frac{1}{b_j}. \tag{9.4}$$

Note that all the roots of Q_n are real and positive, so $\lambda_{\min}$ is the first root to the right of the origin. With this observation, (9.3) and (9.4) assure that

$$\lambda_{\min}(H_n) \geq -\frac{Q_n(0)}{Q_n'(0)} \geq \frac{6c}{(n-1)(n+1)}.$$

The lower bound for $\lambda_{\min}$ then follows. □

A Equilibrium tension

Recall the probability measure $\pi_{\tau,\sigma}$ defined in (2.2), and the normalization constant $Z_\sigma(\tau)$ appeared in it. Note that for $\beta > 0$ and $\sigma = 0$,

$$Z_0(\tau) = \sqrt{\frac{2\pi}{\beta}} \exp\left\{ \frac{\beta\tau^2}{2} \right\}, \quad \forall \tau \in \mathbb{R}.$$

Denote by $\langle \cdot \rangle_{\tau,\sigma}$ the integral with respect to $\pi_{\tau,\sigma}$. For any $\epsilon > 0$ and $\sigma \in [0, 1-\epsilon]$, with the elementary inequality $|e^x - 1 - x| \leq e^{|x|} x^2/2$ we can get that

$$\begin{aligned} &\big| Z_\sigma(\tau) - Z_0(\tau)\big(1 - \sigma\beta\langle U \rangle_{\tau,0}\big) \big| \leq C\sigma^2, \\ &\big| \beta^{-1} Z_\sigma'(\tau) - Z_0(\tau)\big(\tau - \sigma\beta\langle rU \rangle_{\tau,0}\big) \big| \leq C\sigma^2, \\ &\big| \beta^{-1} Z_\sigma''(\tau) - Z_0(\tau)\big(\beta\tau^2 + 1 - \sigma\beta^2\langle r^2 U \rangle_{\tau,0}\big) \big| \leq C\sigma^2, \\ &\big| \beta^{-2} Z_\sigma'''(\tau) - Z_0(\tau)\big(\beta\tau^3 + 3\tau - \sigma\beta^2\langle r^3 U \rangle_{\tau,0}\big) \big| \leq C\sigma^2, \end{aligned}$$

with some constant $C = C_{\beta,\tau,\epsilon}$. Furthermore, the constant C can be taken uniformly for τ in any compact intervals in $\mathbb{R}$.

Recall the functions $\bar{r}_\sigma$ and $\boldsymbol{\tau}_\sigma$ defined through (2.3)–(2.4). From the definition and the estimate above, we obtain that as $\sigma \to 0^+$,

$$\begin{aligned} \bar{r}_\sigma(\tau) &= \tau - \sigma\beta\big\langle (r-\tau)U \big\rangle_{\tau,0} + o_{\beta,\tau}(\sigma), \\ \bar{r}_\sigma'(\tau) &= 1 - \sigma\beta^2\big\langle [(r-\tau)^2 - \beta^{-1}]U \big\rangle_{\tau,0} + o_{\beta,\tau}(\sigma), \\ \bar{r}_\sigma''(\tau) &= -\sigma\beta^3\big\langle [(r-\tau)^3 - 3\beta^{-1}(r-\tau)]U \big\rangle_{\tau,0} + o_{\beta,\tau}(\sigma), \end{aligned}$$

uniformly for τ in any compact interval. As the *macroscopic tension function* $\boldsymbol{\tau}_\sigma$ is the inverse of $\bar{r}_\sigma$, we can conclude the following asymptotic behaviours

$$\begin{aligned}\boldsymbol{\tau}_\sigma(r) &= r + C_0(\beta, r)\sigma + o_{\beta,r}(\sigma),\\ \boldsymbol{\tau}'_\sigma(r) &= 1 + C_1(\beta, r)\sigma + o_{\beta,r}(\sigma),\\ \boldsymbol{\tau}''_\sigma(r) &= C_2(\beta, r)\sigma + o_{\beta,r}(\sigma),\end{aligned}$$

holds uniformly for r in any compact intervals in $\mathbb{R}$. Moreover, the constants C_0, C_1 and C_2 are continuously dependent on β and r.

B Quasi-linear p-system

In this appendix we present a lower bound for the life span of the classical solution of a quasi-linear p-system with smooth initial data. The result is necessary for the proof of Proposition 2.1.

Suppose that f is a positive function in $C^1(\mathbb{R})$. Consider the following partial differential equations for $t \geq 0$ and $x \in \mathbb{T}$:

$$\partial_t p(t,x) = f(r)\partial_x r(t,x), \quad \partial_t r(t,x) = \partial_x p(t,x), \tag{B.1}$$

with some given smooth initial data

$$p(0,\cdot) = p_0 \in C^1(\mathbb{T}), \quad r(0,\cdot) = r_0 \in C^1(\mathbb{T}).$$

Note that by taking $f = \boldsymbol{\tau}'_\sigma$, (B.1) coincides the hydrodynamic equation (2.9) for anharmonic potential. It is well-known that if $f \neq const$, (B.1) would produce shocks in finite time. Recall that $|\cdot|_\mathbb{T}$ represents the uniform norm on $\mathbb{T}$, and define

$$K = |p_0|_\mathbb{T} + |r_0|_\mathbb{T} \sup\left\{\sqrt{f(r)};\ |r| \leq |r_0|_\mathbb{T}\right\}.$$

The next lemma is a special case of the classical result in [20].

Lemma B.1. *Smooth solution of* (B.1) *exists on* $t \in [0,T]$ *for any*

$$T < T_* = 4\left|p_0'\sqrt{f(r_0)} + r_0' f(r_0)\right|_\mathbb{T}^{-1} \left(\sup_{|r|\leq K}\left|f^{-\frac{5}{4}}(r)f'(r)\right|\right)^{-1}.$$

Remark B.2. For the readers not familiar to the hyperbolic systems, it is worth mentioning that the bound we obtained above is not as sharp as the case of scalar equation, for instance the inviscid Burger's equation.

Proof. We briefly state the proof. Define an antiderivative of $\sqrt{f}$:

$$F(s) = \int_0^s \sqrt{f(r)}dr, \quad \forall s \in \mathbb{R}.$$

The equation can be rewritten in Riemann invariants as

$$\partial_t u = \lambda(u,v)\partial_x u, \quad \partial_t v = -\lambda(u,v)\partial_x v, \quad (u,v)(0,\cdot) = (u_0, v_0),$$

where $u = p + F(r)$, $v = p - F(r)$ and $\lambda(u,v) = \sqrt{f(r)}$.

Consider the characteristic lines $(t, x_{\pm,t})$, given by the ODEs

$$\frac{dx_t}{dt} = \pm\lambda\big(u(t,x_t), v(t,x_t)\big), \quad x_0 = x \in \mathbb{T}.$$

Within the life span of the smooth solution, u is constant along $(t, x_{+,t})$, thus

$$\sup_{x \in \mathbb{T}} |u(t,x)| \le \sup_{x \in \mathbb{T}} |u_0(x)| \le K. \tag{B.2}$$

Similarly, we have a priori bound for $v(t,x)$.

Suppose that the smooth solution of (B.1) exists on time interval $[0,T]$ for some $T > 0$. Taking spatial derivative on the equation of u,

$$\partial_{tx} u - \lambda \partial_{xx} u = \partial_u \lambda (\partial_x u)^2 + \partial_v \lambda \partial_x u \partial_x v, \quad t \in [0,T].$$

In order to investigating the continuity, let $z(t,x) = \sqrt{\lambda(u,v)}\partial_x u$. From the equation above, for $t \in [0,T]$, z solves the Riemann problem given by

$$\begin{cases} \partial_t z - \lambda \partial_x z = \Lambda z^2, \quad \Lambda = 2\partial_u \sqrt{\lambda}, \\ z(0,\cdot) = \sqrt{\lambda(u_0,v_0)} u_0'. \end{cases}$$

By (B.2), before the generation of shocks, $|\Lambda|$ is bounded from above by

$$K' \triangleq 2 \sup \left\{ \partial_u \sqrt{\lambda(u,v)};\ |u|, |v| \le K \right\}.$$

Via a comparison argument, one obtains that $|z(t,x)| < \infty$ for

$$(t,x) \in \left[0,\ \frac{1}{K' \sup_{x \in \mathbb{T}} |z(0,x)|}\right) \times \mathbb{T},$$

which guarantees that $|\partial_x u| < \infty$, so shock cannot form. Since

$$2\partial_u \sqrt{\lambda} = \frac{\partial_r \sqrt{\lambda}}{F'(r)} = 4^{-1} f^{-\frac{5}{4}}(r) f'(r),$$

the estimate in Lemma B.1 then follows. □

C Yau's entropy method

In this appendix, we apply Yau's relative entropy method to obtain the formulas (4.1)–(4.3) in the proof of Theorem 2.2.

Fix $n \ge 1$ and $\sigma \in (0,1)$. Take a smooth function $(p,r) = (p,r)(t,x)$ on $[0,T] \times \mathbb{T}$, and define $p_i^n = p(t, i/n)$, $r_i^n = r(t, i/n)$, $\tau_i^n = \boldsymbol{\tau}_\sigma(r_i^n)$ for each $i \in \mathbb{T}_n$, where $\boldsymbol{\tau}_\sigma$ is given by (2.4). Recall the Gibbs states defined in (2.6) and choose $\nu = \nu_{0,0,\sigma}^n$ as the reference measure on Ω_n. Consider the local Gibbs measure $d\mu_t = \exp(\beta \varphi_t) d\nu$, where

$$\varphi_t(\vec{\eta}) = \sum_{i \in \mathbb{T}_n} \left(p_i^n p_i + \tau_i^n r_i\right) + \sum_{i \in \mathbb{T}_n} \left[-\frac{(p_i^n)^2}{2} + G_\sigma(0) - G_\sigma(\tau_i^n) \right].$$

Let $\vec{\eta}(t)$ be the Markov process generated by $\mathcal{L}_{n,\sigma,\gamma}$ in (2.1) with some fixed $\gamma > 0$, and denote by f_t the density of $\vec{\eta}(t)$ with respect to μ_t.

From the definition of the relative entropy in (2.11),

$$\frac{d}{dt} H(f_t; \mu_t) = -4n\gamma D\left(\sqrt{f_t}, \mu_t\right) + \int \left(\mathcal{L}_{n,\sigma,\gamma} f_t - \beta f_t \frac{d}{dt} \varphi_t \right) d\mu_t,$$

where the Dirichlet form $D(f,\mu)$ is defined as

$$D(f,\mu) = \int \Gamma_n f d\mu, \quad \Gamma_n f = \frac{1}{2} \sum_{i \in \mathbb{T}_n} (\mathcal{Y}_i f)^2,$$

for probability measure μ and density function f on Ω_n. Since

$$\int \mathcal{A}_{n,\sigma} f_t d\mu_t = -\int f_t \mathcal{A}_{n,\sigma}\big[e^{\beta\varphi_t}\big] d\nu = -\beta \int f_t \mathcal{A}_{n,\sigma}\varphi_t d\mu_t,$$

$$\begin{aligned}\int \mathcal{S}_{n,\sigma} f_t d\mu_t &= -\frac{1}{2}\sum_{i\in\mathbb{T}_n}\int \mathcal{Y}_i f_t \cdot \mathcal{Y}_i\big[e^{\beta\varphi_t}\big] d\nu \\ &\le \frac{1}{4}\sum_{i\in\mathbb{T}_n}\int \frac{1}{f_t}(\mathcal{Y}_i f_t)^2 d\mu_t + \frac{\beta^2}{4}\sum_{i\in\mathbb{T}_n}\int f_t(\mathcal{Y}_i\varphi_t)^2 d\mu_t,\end{aligned}$$

we obtain that with $J_t^n = -(n\mathcal{A}_{n,\sigma} + d/dt)\varphi_t$,

$$\frac{d}{dt}H(f_t;\mu_t) \le -2n\gamma D\left(\sqrt{f_t};\mu_t\right) + \beta\int f_t J_t^n d\mu_t + \frac{\beta^2 n\gamma}{2}\int f_t(\Gamma_n\varphi_t)d\mu_t. \tag{C.1}$$

Using the explicit formula of φ_t,

$$\Gamma_n\varphi_t = \frac{1}{2}\sum_{i\in\mathbb{T}_n}\left(\tau_{i+1}^n(t) - \tau_i^n(t)\right)^2 \le \frac{1}{n}\int_{\mathbb{T}} |\partial_x\tau(r(t,x))|^2 dx,$$

so that $\Gamma_n\varphi_t \le C_T/n$. Also, by the formula of φ_t,

$$\begin{aligned}\mathcal{A}_{n,\sigma}\varphi_t &= \sum \tau_i^n(p_i - p_{i-1}) + p_i^n\left(V_\sigma'(r_{i+1}) - V_\sigma'(r_i)\right) \\ &= -\sum_{i\in\mathbb{T}_n}\begin{pmatrix}\tau_{i+1}^n - \tau_i^n \\ p_i^n - p_{i-1}^n\end{pmatrix}\cdot\begin{pmatrix} p_i - p_i^n \\ V_\sigma'(r_i) - \tau_i^n\end{pmatrix}, \\ \frac{d}{dt}\varphi_t &= \sum_{i\in\mathbb{T}_n}\frac{dp_i^n}{dt}(p_i - p_i^n) + \sum_{i\in\mathbb{T}_n}\frac{d\tau_i^n}{dt}(r_i - r_i^n) \\ &= \sum_{i\in\mathbb{T}_n}\frac{d}{dt}\begin{pmatrix}p_i^n \\ r_i^n\end{pmatrix}\cdot\begin{pmatrix} p_i - p_i^n \\ \boldsymbol{\tau}'_\sigma(r_i^n)(r_i - r_i^n)\end{pmatrix}.\end{aligned}$$

Therefore, we obtain the explicit form of J_t^n as

$$\begin{aligned}J_t^n = &\sum_{i\in\mathbb{T}_n}\left[-\frac{d}{dt}\begin{pmatrix}p_i^n \\ r_i^n\end{pmatrix} + n\begin{pmatrix}\tau_{i+1}^n - \tau_i^n \\ p_i^n - p_{i-1}^n\end{pmatrix}\right]\cdot\begin{pmatrix} p_i - p_i^n \\ V_\sigma'(r_i) - \tau_i^n\end{pmatrix} \\ &+ \sum_{i\in\mathbb{T}_n}\frac{dr_i^n}{dt}\cdot\left[V_\sigma'(r_i) - \tau_i^n - \boldsymbol{\tau}'_\sigma(r_i^n)(r_i - r_i^n)\right].\end{aligned} \tag{C.2}$$

In particular, the formulas (4.1)–(4.3) follow from (C.1), (C.2) by taking $\sigma = \sigma_n$, $\gamma = \gamma_n$ and (p,r) to be the solution $(\mathfrak{p}_n, \mathfrak{r}_n)$ of the hydrodynamic equation (2.9) for $\sigma = \sigma_n$.

D Entropy and moment inequalities

Recall the relative entropy $H(f;\mu)$ in (2.11) for probability measure μ and density function f on some measurable space Ω. In this appendix we give some classical inequalities related to $H(f;\mu)$. We begin from a variational formula of $H(f;\mu)$:

$$H(f;\mu) = \sup_{g\in B_b(\Omega)}\left\{\int_\Omega fg d\mu - \log\int_\Omega e^g d\mu\right\}, \tag{D.1}$$

where $B_b(\Omega)$ stands for the class of all bounded measurable functions on Ω. The proof of (D.1) can be found in [28, Theorem 4.1]. From (D.1) we immediately get the first lemma.

Lemma D.1. *Let $(\Omega_1, \mathscr{F}_1, \mu_1)$, $(\Omega_2, \mathscr{F}_2, \mu_2)$ be two probability spaces. Suppose f to be a density function on $\Omega = \Omega_1 \times \Omega_2$ with respect to $\mu = \mu_1 \otimes \mu_2$, then*

$$H(f_1; \mu_1) \le H(f; \mu), \tag{D.2}$$

where f_1 is the density of the marginal distribution of $f d\mu$ on Ω_1.

Next we give two inequalities frequently used in this article.

Lemma D.2. *For any measurable subset $A \subseteq \Omega$,*

$$\int_A f d\mu \le \frac{H(f; \mu) + \log 2}{-\log \mu(A)}. \tag{D.3}$$

If $X : \Omega \to \mathbb{R}$ is integrable under $f d\mu$, then for any $\alpha > 0$,

$$\int_\Omega f X d\mu \le \frac{1}{\alpha}\left[H(f; \mu) + \log \int_\Omega e^{\alpha X} d\mu\right]. \tag{D.4}$$

Proof. Taking $g = -\log(\mu(A))\mathbf{1}_A$ in (D.1), we obtain that

$$H(f; \mu) \ge -\log\big(\mu(A)\big) \int_A f d\mu - \log\big(2 - \mu(A)\big),$$

and (D.3) follows. For (D.4), if X is bounded, take $g = \alpha X$ to get

$$H(f; \mu) \ge \alpha \int_\Omega f X d\mu - \log \int_\Omega e^{\alpha X} d\mu.$$

We can obtain (D.4) via a standard approximating argument. □

A family of random variables $\{X_i; i = 1, \dots, m\}$ is said to be ℓ-independent for some $1 \le \ell \le m$, if for any subset $\Gamma \subseteq \{1, \dots, m\}$ such that $|i - j| \ge \ell$ for each $i \ne j \in \Gamma$, the sub family $\{X_i; i \in \Gamma\}$ is independent. From (D.4) we easily get the next lemma.

Lemma D.3. *If $\{X_1, X_2, \dots, X_m\}$ is ℓ-independent, then for any $\alpha > 0$,*

$$\left|\int f \sum_{i=1}^m X_i d\mu\right| \le \frac{1}{\alpha}\left[H(f; \mu) + \frac{1}{\ell}\sum_{i=1}^m \max\left\{\log \int e^{\pm\alpha\ell X_i} d\mu\right\}\right].$$

Proof. For $k = 0, 1, \dots, \ell - 1$, let $\Gamma_k = \{k + i\ell; 1 \le i \le (m-k)/\ell\}$. Since $\{X_i, i \in \Gamma_k\}$ is independent, (D.4) yields that

$$\int f \sum_{i \in \Gamma_k} X_i d\mu \le \frac{1}{\alpha\ell}\left[H(f; \mu) + \sum_{i \in \Gamma_k} \log \int e^{\alpha\ell X_i} d\mu\right].$$

Taking summation over $k \in \{0, 1, \dots, \ell - 1\}$, we get that

$$\int f \sum_{i=1}^m X_i d\mu \le \frac{1}{\alpha}\left[H(f; \mu) + \frac{1}{\ell}\sum_{i=1}^m \log \int e^{\alpha\ell X_i} d\mu\right]. \tag{D.5}$$

The proof is completed by repeating the argument with $-X_i$ instead of X_i. □

Taking $A = \{|X| > \lambda\}$ in (D.3) gives us tail estimates of X. The following result makes it possible to get moment bounds of X from tail estimates. It has been used in the proof of Corollary 4.1 and the tightness of the fluctuation field.

Lemma D.4. *Suppose a constant $C > 0$, some $q > 1$ such that*

$$P(|X| > \lambda) \le C\lambda^{-q}, \quad \forall \lambda > 0.$$

Then, for any $q \in [1, p)$, there exists a constant $K_{p,q} > 1$, such that

$$E\big[|X|^p\big] \le K_{p,q} C^{\frac{p}{q}}.$$

Proof. Using the integration-by-parts formula, for all $1 \le p < q$,

$$\begin{aligned} E\big[|X|^p\big] &\le \int_0^\infty \frac{d}{d\lambda}\big(\lambda^p\big) P(|X| \ge \lambda) d\lambda \\ &\le \int_{0 \le \lambda < C^{\frac{1}{q}}} p\lambda^{p-1} d\lambda + C \int_{\lambda \ge C^{\frac{1}{q}}} p\lambda^{p-1-q} d\lambda \le \frac{q}{q-p} C^{\frac{p}{q}}. \end{aligned}$$

Thus, the lemma holds with $K_{p,q} = q/(q-p) > 1$. □

E Sub-Gaussian random variable

Recall that a real random variable X, is called sub-Gaussian of order σ^2, if

$$\log E\big[e^{sX}\big] \le \frac{\sigma^2 s^2}{2}, \quad \forall s \in \mathbb{R}.$$

There is an elementary but useful condition for sub-Gaussian property.

Lemma E.1 (ϕ_2 condition)**.** *If $E[X] = 0$, and*

$$E\big[e^{cX^2}\big] \le C, \tag{E.1}$$

for some $c > 0$ and $C \ge 1$, then X is sub-Gaussian of order $2Cc^{-1}$.

Proof. Since $E[X] = 0$, we have for any $s \in \mathbb{R}$ that

$$E\big[e^{sX}\big] = 1 + \sum_{k=2}^\infty \frac{E[(sX)^k]}{k!} \le 1 + \frac{s^2}{2} \sum_{k=0}^\infty \frac{|s|^k E[|X|^{k+2}]}{k!}.$$

The summation in the right-hand side is bounded by

$$\frac{s^2}{2} E\big[X^2 e^{|sX|}\big] \le \frac{s^2}{2} E\left[X^2 \exp\left\{\frac{cX^2}{2} + \frac{s^2}{2c}\right\}\right]$$

for any $c > 0$. With the elementary inequality $ye^y \le e^{2y}$,

$$\frac{s^2}{2} E\left[X^2 \exp\left\{\frac{cX^2}{2} + \frac{s^2}{2c}\right\}\right] \le \frac{s^2}{c} \exp\left\{\frac{s^2}{2c}\right\} E\big[e^{cX^2}\big].$$

Hence, by the condition (E.1),

$$E\big[e^{sX}\big] \le 1 + \frac{Cs^2}{c} \exp\left\{\frac{s^2}{2c}\right\} \le \exp\left\{\frac{Cs^2}{c}\right\}.$$

As s is arbitrary, the proof is completed. □

Recall that in Lemma 3.1 we need to bound the exponential integral of the absolute value of a sub-Gaussian variable. The general estimate is as follows.

Lemma E.2. *If X is sub-Gaussian of order σ^2, then*

$$E\big[e^{s|X|}\big] \le \frac{1+|s|}{1-|s|} \exp\left\{\frac{\sigma^2 |s|}{2}\right\}, \quad \forall s \in (-1, 1).$$

Proof. By Chernoff's method, for any $\lambda > 0$,

$$P(X \geq \lambda) \leq E\left[\exp\left\{\frac{\lambda(X-\lambda)}{\sigma^2}\right\}\right] \leq \exp\left\{\frac{\lambda^2}{2\sigma^2} - \frac{\lambda^2}{\sigma^2}\right\} = \exp\left\{-\frac{\lambda^2}{2\sigma^2}\right\}.$$

Since similar estimate holds for $P(X \leq -\lambda)$,

$$P(|X| \geq \lambda) \leq 2\exp\left\{-\frac{\lambda^2}{2\sigma^2}\right\}.$$

For $0 \leq t < 1/(2\sigma^2)$, the integration-by-parts formula yields that

$$E\big[e^{tX^2}\big] \leq 1 + \int_0^\infty \frac{d}{d\lambda}\big(e^{t\lambda^2}\big)P(|X| \geq \lambda)d\lambda \leq \frac{1+2t\sigma^2}{1-2t\sigma^2}.$$

Hence, for any $s \in [0,1)$,

$$E\big[e^{s|X|}\big] \leq E\left[\exp\left\{\frac{sX^2}{2\sigma^2} + \frac{\sigma^2 s}{2}\right\}\right] \leq \frac{1+s}{1-s}\exp\left\{\frac{\sigma^2 s}{2}\right\}.$$

The case $s \in (-1,0)$ holds similarly. □

F Local central limit theorem

In this appendix, we state a local central limit theorem with expansions for the sum of independent, non-identically distributed random variables. It is used in the proof of equivalence of ensembles in Section 8.

We work under the following setting. Suppose that π is some Borel measure on $\mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is an integrable function. Assume for all $\tau \in \mathbb{R}$ that

$$G(\tau) \triangleq \log \int_{\mathbb{R}} e^{\tau f(r)}\pi(dr) < \infty.$$

Denote by π_τ the tilted probability measure on $\mathbb{R}$, given by

$$\pi_\tau(dr) = \exp\{\tau f - G(\tau)\}\pi(dr).$$

Let $\Phi_\tau(\xi) = \int \exp\{i\xi(f - \int f d\pi_\tau)\}\pi_\tau(dr)$ be the characteristic function of f. For all $K > 0$, we assume the following conditions with a constant M_K:

(i) G is four times differentiable on $\mathbb{R}$, and

$$G''(\tau) > M_K^{-1},\ \big|G^{(\ell)}(\tau)\big| < M_K,\ \forall \tau \in [-K,K],\ \ell = 0,1,2,3,4.$$

(ii) $|\Phi_\tau(\xi)| < M_K(1+|\xi|)^{-1}$ for all $\xi \in \mathbb{R}$ and $\tau \in [-K,K]$;

(iii) Φ_τ is four times differentiable on $\mathbb{R}$ for all $\tau \in \mathbb{R}$, and

$$\forall \epsilon > 0,\ \exists \delta = \delta(\epsilon, K) > 0,\ \text{s.t. } \big|\Phi_\tau^{(\ell)}(\xi) - \Phi_\tau^{(\ell)}(0)\big| < \epsilon,$$

for all $|\xi| < \delta$, $\tau \in [-K,K]$ and $\ell = 0$, 1, 2, 3, 4.

Given $\vec{\tau} = (\tau_1, \ldots, \tau_n)$, define the inhomogeneous product measure

$$\mu_n(d\mathbf{r}) = \mu_n(\vec{\tau}; d\mathbf{r}) = \prod_{j=1}^n \pi_{\tau_j}(dr_i), \quad \mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n.$$

Define $u_\ell = u_\ell(\vec{\tau}) > 0$ for $\ell = 1, 2, 3, 4$ via the formula

$$|u_\ell|^\ell = \frac{1}{n}\sum_{j=1}^{n} G^{(\ell)}(\tau_j).$$

Observe that $u_1 = E_{\mu_n}[\bar{r}]$ and $u_2^2 = E_{\mu_n}[(\bar{r}-u_1)^2]$, where $\bar{r} = n^{-1}\sum r_j$.

The local central limit theorem is stated as follows. Let ϕ be the standard Gaussian density, and $\{H_j; j \geq 0\}$ be the group of Hermite polynomials:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad H_j(x) = (-1)^j e^{\frac{x^2}{2}} \frac{d^j}{dx^j}\left[e^{-\frac{x^2}{2}}\right].$$

In particular, $H_3 = x^3 - 3x$, $H_4 = x^4 - 6x^2 + 3$ and $H_6 = x^6 - 15x^4 + 45x^2 - 15$.

Lemma F.1. *Assume that $\tau_j \in [-K, K]$ for all $1 \leq j \leq n$. Let $g_n(\vec{\tau}; \cdot)$ be the density function with respect of μ_n of the random variable*

$$\frac{1}{u_2(\vec{\tau})\sqrt{n}} \sum_{j=1}^{n} \big(r_j - u_1(\vec{\tau})\big).$$

For any $\epsilon > 0$, there exists $N = N(\epsilon, K, M_K)$ sufficiently large, such that if $n \geq N$, then the following estimate holds uniformly for $x \in \mathbb{R}$:

$$\left| g_n(\vec{\tau}; x) - \phi(x)\left[1 + \frac{1}{\sqrt{n}} Q_{n,1}(x) + \frac{1}{n} Q_{n,2}(x)\right]\right| < \frac{C}{n}\left(\epsilon + \frac{1}{\sqrt{n}}\right)$$

where $C = C(M_K)$ is a constant and $Q_{n,1}$, $Q_{n,2}$ are given by

$$Q_{n,1} = \frac{1}{3!}\left(\frac{u_3}{u_2}\right)^3 H_3, \quad Q_{n,2} = \frac{1}{4!}\left(\frac{u_4}{u_2}\right)^4 H_4 + \frac{1}{2(3!)^2}\left(\frac{u_3}{u_2}\right)^6 H_6.$$

Lemma F.1 can be proved following [9, Theorem XVI.2.2, pp. 535]. Here we briefly sketch the proof to emphasize the dependence of (N, C) on ϵ, K and M_K.

Proof. By the definition of characteristic function Φ_τ,

$$g_n(\vec{\tau}; x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ix\xi} \prod_{j=1}^{n} \Phi_{\tau_j}\left(\frac{\xi}{u_2\sqrt{n}}\right) d\xi.$$

Let us define $\Delta_n = \Delta_n(\vec{\tau}; \xi)$ for each $\xi \in \mathbb{R}$ by

$$\Delta_n(\vec{\tau}; \xi) = \prod_{j=1}^{n} \Phi_{\tau_j}\left(\frac{\xi}{u_2\sqrt{n}}\right) - \exp\left\{-\frac{\xi^2}{2}\right\}\left[1 + P_n(i\xi) + \frac{1}{2}P_n^2(i\xi)\right],$$

where $P_n = P_n(\vec{\tau}; \cdot)$ is the polynomial given by

$$P_n = \frac{1}{3!\sqrt{n}}\left(\frac{u_3}{u_2}\right)^3 x^3 + \frac{1}{4!n}\left(\frac{u_4}{u_2}\right)^4 x^4.$$

From the definition of Hermite polynomials, it suffices to prove that

$$\int_{\mathbb{R}} \big|\Delta_n(\vec{\tau}; \xi)\big| d\xi \leq \frac{C}{n}\left(\epsilon + \frac{1}{\sqrt{n}}\right).$$

For any $\epsilon > 0$, Taylor's theorem yields that there is $\delta = \delta(\epsilon, K) > 0$, such that

$$\left| \log \Phi_\tau(\xi) + \frac{G''(\tau)}{2}\xi^2 - \sum_{\ell=3}^{4} \frac{1}{\ell!} G^{(\ell)}(\tau)(i\xi)^\ell \right| < \epsilon\xi^4,$$

for all $|\xi| < \delta$ and $\tau \in [-K, K]$. Therefore, when $|\xi| < \delta u_2 \sqrt{n}$,

$$\left| \sum_{j=1}^{n} \log \Phi_{\tau_j} \left(\frac{\xi}{u_2 \sqrt{n}} \right) + \frac{\xi^2}{2} - P_n(i\xi) \right| < \frac{\epsilon \xi^4}{u_2^4 n}.$$

Without loss of generality we can choose $\delta < 1$, so that

$$|P_n(i\xi)| < \left(\frac{u_3^3}{3! u_2^3} + \frac{\delta u_4^4}{4! u_2^3} \right) \frac{|\xi|^3}{\sqrt{n}} < \frac{C_1 |\xi|^3}{\sqrt{n}},$$

with some $C_1 = C_1(M_K)$. Using the elementary inequality

$$\left| e^x - 1 - x' - \frac{(x')^2}{2} \right| \le e^{\max\{|x|, |x'|\}} \big(|x - x'| + |x'|^3 \big), \quad \forall x, x' \in \mathbb{R},$$

we obtain that when $|\xi| < \delta u_2 \sqrt{n}$,

$$\big| \Delta_n(\vec{\tau}, \xi) \big| < \exp \left\{ \frac{\epsilon \xi^4}{u_2^4 n} + \frac{C_1 |\xi|^3}{\sqrt{n}} - \frac{\xi^2}{2} \right\} \frac{1}{n} \left(\frac{\epsilon \xi^4}{u_2^4} + \frac{C_1^3 |\xi|^9}{\sqrt{n}} \right).$$

By furthermore choosing $\delta = \delta(\epsilon, K, M_K)$ sufficiently small, we get

$$\big| \Delta_n(\vec{\tau}, \xi) \big| < \frac{C_2}{n} \exp \left\{ -\frac{\xi^2}{4} \right\} \left(\epsilon \xi^4 + \frac{|\xi|^9}{\sqrt{n}} \right),$$

with some $C_2 = C_2(M_K)$ on the set $\{|\xi| < \delta u_2 \sqrt{n}\}$. From the estimate above, we have some constant $C = C(M_K)$, such that for all $n \ge 1$,

$$\int_{|\xi| < \delta u_2 \sqrt{n}} \big| \Delta_n(\vec{\tau}, \xi) \big| d\xi \le \frac{C}{n} \left(\epsilon + \frac{1}{\sqrt{n}} \right).$$

On the remaining set $\{|\xi| \ge \delta u_2 \sqrt{n}\}$, by (ii) we have that

$$\big| \Delta_n(\vec{\tau}, \xi) \big| < \frac{M_K^n}{(1 + |\xi|)^n} + \exp \left\{ -\frac{\xi^2}{2} \right\} \left[1 + P_n + \frac{1}{2} P_n^2 \right].$$

Hence, we can choose $N = N(\delta, M_K)$, such that for all $n \ge N$,

$$\int_{|\xi| \ge \delta u_2 \sqrt{n}} \big| \Delta_n(\vec{\tau}, \xi) \big| d\xi < \frac{1}{n^{3/2}}.$$

The proof is then completed. □

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