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On the convergence of random tridiagonal matrices to stochastic semigroups

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Abstract. We develop an improved version of the stochastic semigroup approach to study the edge of β-ensembles pioneered by Gorin and Shkolnikov (*Ann. Probab.* **46** (2018) 2287–2344), and later extended to rank-one additive perturbations by the author and Shkolnikov (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 1402–1438). Our method is applicable to a significantly more general class of random tridiagonal matrices than that considered in (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 1402–1438; *Ann. Probab.* **46** (2018) 2287–2344), including some non-symmetric cases that are not covered by the stochastic operator formalism of Bloemendal, Ramírez, Rider, and Virág (*Probab. Theory Related Fields* **156** (2013) 795–825; *J. Amer. Math. Soc.* **24** (2011) 919–944).

We present two applications of our main results: Firstly, we prove the convergence of β-Laguerre-type (i.e., sample covariance) random tridiagonal matrices to the stochastic Airy semigroup and its rank-one spiked version. Secondly, we prove the convergence of the eigenvalues of a certain class of non-symmetric random tridiagonal matrices to the spectrum of a continuum Schrödinger operator with Gaussian white noise potential.

Résumé. Nous développons une version améliorée de l'approche de *stochastic semigroup* pour étudier l'extrémité des ensembles bêta introduits par Gorin et Shkolnikov (*Ann. Probab.* **46** (2018) 2287–2344), ensuite étendue aux ensembles bêta gaussiens avec perturbation de rang un par l'auteur et Shkolnikov (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 1402–1438). Notre méthode est applicable à une classe nettement plus générale de matrices tridiagonales aléatoires que celles dans (*Ann. Inst. Henri Poincaré Probab. Stat.* **55** (2019) 1402–1438; *Ann. Probab.* **46** (2018) 2287–2344), y compris certains cas non symétriques qui ne sont pas couverts par la méthode de *stochastic operators* introduite par Bloemendal, Ramírez, Rider et Virág (*Probab. Theory Related Fields* **156** (2013) 795–825; *J. Amer. Math. Soc.* **24** (2011) 919–944).

Nous présentons deux applications de nos principaux résultats : Premièrement, nous prouvons la convergence de matrices tridiagonales aléatoires de type β-Laguerre (c.-à-d., matrices de covariances empiriques) vers le semi-groupe du *stochastic Airy operator* et sa perturbation de rang un. Deuxièmement, nous prouvons la convergence des valeurs propres d'une certaine classe de matrices tridiagonales aléatoires non symétriques vers le spectre d'opérateurs de Schrödinger avec bruit blanc gaussien.

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1. Introduction

1.1. *Operator limits of random matrices*

This paper, which is a direct sequel of [14,20], is concerned with operator limits of random matrices. The theory of operator limits was initiated in [10,11,28] and eventually gave rise to a vast literature on the subject. We refer to the survey article [32] for a recent historical account of these early developments.

A fundamental object in this theory is the *stochastic Airy operator*, formally defined as

$$\mathrm{SAO}_\beta f(x) := -f''(x) + xf(x) + W_\beta'(x)f(x), \qquad f : \mathbb{R}_+ \to \mathbb{R},$$

where $\beta > 0$ is fixed parameter, W_β is a Brownian motion with variance $4/\beta$, $\mathbb{R}_+ := [0, \infty)$, and f obeys a Dirichlet or Robin boundary condition at the origin. We refer to [6, Section 2.3], [26, Section 2], and [28, Section 2] for a rigorous definition.

The interest of studying SAO_β comes from the fact that its eigenvalue point process captures the asymptotic edge fluctuations of a large class of random matrices and interacting particle systems. In [6,28], this was proved for the β-Hermite ensemble, the β-Laguerre ensemble (for the right edge), as well as rank-one perturbations of the β-Hermite and β-Laguerre ensembles (the spiked models). Then, [23] established operator limits as a means of proving edge universality for general β-ensembles (cf., [8]). More generally, [6,28] proved the eigenvalue and eigenvector convergence of a wide class of symmetric random tridiagonal matrices to the spectrum of Schrödinger operators of the form $-\Delta + Y'$, where Y is a random function.

1.2. *Stochastic semigroups*

More recently, Gorin and Shkolnikov introduced in [20] a new method of studying edge fluctuations of β-ensembles. Their main result was that high powers of a generalized version of the β-Hermite ensemble converge to a random Feynman-Kac-type semigroup that was dubbed the *stochastic Airy semigroup* ([20, Theorem 2.1]), which we denote by $\mathrm{SAS}_\beta(t)$ for $\beta, t > 0$ (see Definition 2.7 and Notation 2.9).

Combining their result with the fact that the edge-rescaled β-Hermite ensemble converges to SAO_β, Gorin and Shkolnikov concluded that $\mathrm{SAS}_\beta(t) = \mathrm{e}^{-t\mathrm{SAO}_\beta/2}$ for all $t > 0$ ([20, Corollary 2.2]), thus providing a new tool with which SAO_β's spectrum can be studied. As a demonstration of this, it was shown in [20, Corollary 2.3 and Proposition 2.6] that certain statistics of $\mathrm{SAS}_\beta(t)$ admit an especially simple form when $\beta = 2$. Among other things, this provided the first manifestation of the special integrable structure in the β-ensembles when $\beta \in \{1, 2, 4\}$ at the level of the operator limits describing edge fluctuations. These results were extended to rank-one spiked β-Hermite models in [14]. Feynman-Kac formulas for general one-dimensional Schrödinger operators with multiplicative Gaussian noise were obtained more recently in [13].

1.3. *Overview of main results*

In this paper, we introduce a modification of the formalism developed in [14,20]. Our main results (Theorems 2.20 and 2.21) establish the convergence of high powers of a large class of random tridiagonal matrices to the semigroups of continuum Schrödinger operators with Gaussian white noise. Our results improve on [14,20] and [6,28] in two significant ways.

Firstly, a main technical achievement of [20] was to show that the moment method can be used to study edge fluctuations of β-ensembles for $\beta \notin \{1, 2, 4\}$. The key to achieving this is to relate the combinatorics of traces of high powers of tridiagonal matrices to strong invariance principles for random walks and their occupation measures ([20, Section 3] and [14, Section 3.1]). A notable feature of the combinatorial analysis in [20] is that it requires the tridiagonal matrices under consideration to have diagonal entries of smaller order than their super/sub-diagonal entries (see Section 4.3 for details). In particular, this argument is not directly applicable to the β-Laguerre ensemble. In this context, one contribution of this paper is to develop an improved version of the stochastic semigroup formalism that does not have restrictions on the relative size of diagonal/off-diagonal entries. As a demonstration of this, we prove that our main results apply to every matrix model considered in [14,20], as well as generalized β-Laguerre ensembles (Section 3.2).

Secondly, a notable feature of our results is that they appear to be the first to apply to non-symmetric matrices. As a consequence, we prove new limit laws for the eigenvalues of certain non-symmetric random tridiagonal matrices (Propositions 3.1 and 3.5). In particular, we identify a new matrix model whose edge fluctuations are in the Tracy–Widom universality class (Corollary 3.13). These results complement previous investigations on the spectrum of non-symmetric random tridiagonal matrices, such as [16–19].

Several features of the strategy of proof in [14,20] for analyzing the combinatorics of large powers of tridiagonal matrices carry over to this paper. For instance, strong invariance principles for occupation measures of random walks also play a fundamental role in our proofs. That being said, the differences are significant enough that many nontrivial modifications and new ideas need to be introduced. Most notably, several results in the literature concerning strong approximations of Brownian local time that are used without modification in [14,20] require significant work to be applicable to our setting (Sections 5 and 6).

1.4. *Organization*

In Section 2, we introduce our random matrix models, their continuum limits, and we state our main results. In Section 3, we discuss applications of our main results to random matrices. In Section 4, we explain the main idea in our strategy of proof, and we make a brief comparison with the method of [14,20]. In Sections 5 and 6, we prove two local time strong invariance results that lie at the heart of our proof. Finally, in Sections 7, 8, and 9, we complete the proofs of our main results.

2. Setup and main results

2.1. *Random matrix models*

We begin by introducing our random matrix models. Let $(m_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers and $(w_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that the following holds:

Assumption 2.1. There exists $0 < C \leq 1$ and $1/13 < \mathfrak{d} < 1/2$ such that

$$Cn^{\mathfrak{d}} \leq m_n \leq C^{-1}n^{\mathfrak{d}}, \quad n \in \mathbb{N}. \tag{2.1}$$

Assumption 2.2. There exists some $w \in \mathbb{R}$ such that

$$\lim_{n\to\infty} m_n(1 - w_n) = w. \tag{2.2}$$

For every $n \in \mathbb{N}$, let us define the $(n + 1) \times (n + 1)$ tridiagonal matrices Δ_n, Δ_n^w, and Q_n as

$$\Delta_n := m_n^2 \begin{bmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}, \qquad Q_n := \begin{bmatrix} D_n(0) & U_n(0) & & \\ L_n(0) & \ddots & \ddots & \\ & \ddots & \ddots & U_n(n-1) \\ & & L_n(n-1) & D_n(n) \end{bmatrix},$$

$$\Delta_n^w := \Delta_n + \mathrm{diag}_n\big(m_n^2 w_n, 0, \ldots, 0\big),$$

where $D_n(a), U_n(a), L_n(a)$ are real-valued random variables for every $n \in \mathbb{N}$ and $0 \leq a \leq n$ (or $0 \leq a \leq n - 1$).

Notation 2.3. Throughout, we index the entries of a $(n + 1) \times (n + 1)$ matrix M as $M(a, b)$ for $0 \leq a, b \leq n$. Similarly, $v \in \mathbb{R}^{n+1}$ is indexed as $v(a)$ for $0 \leq a \leq n$. We use $\mathrm{diag}_n(d_0, \ldots, d_n)$ to denote the $(n + 1) \times (n + 1)$ diagonal matrix M with entries $M(a, a) = d_a$ for $0 \leq a \leq n$.

Notation 2.4. For simplicity, we often state properties of $D_n(a), U_n(a), L_n(a)$ for $0 \leq a \leq n$, with the understanding that $a \leq n - 1$ for $U_n(a)$ and $L_n(a)$.

We assume that the entries of Q_n satisfy the following decomposition: For $E \in \{D, U, L\}$,

$$E_n(a) = V_n^E(a) + \xi_n^E(a), \quad 0 \leq a \leq n, \tag{2.3}$$

where the $V_n^E(a)$ are deterministic and the $\xi_n^E(a)$ are random. We call V_n^E the *potential terms* and ξ_n^E the *noise terms*. The random matrix models studied in this paper are as follows.

Definition 2.5 (Random matrix models). For every $n \in \mathbb{N}$ and $t > 0$, we define

$$\hat{K}_n(t) := \left(I_n - \frac{-\Delta_n + Q_n}{3m_n^2}\right)^{\lfloor m_n^2(3t/2)\rfloor}, \qquad \hat{K}_n^w(t) := \left(I_n - \frac{-\Delta_n^w + Q_n}{3m_n^2}\right)^{\lfloor m_n^2(3t/2)\rfloor}. \tag{2.4}$$

2.2. *Continuum limit*

We now describe the continuum limits of (2.4). In order to describe these objects, we need some notations:

Notation 2.6. We use B to denote a standard Brownian motion on $\mathbb{R}$, and X to denote a standard reflected Brownian motion on $\mathbb{R}_+$.

Let $Z = B$ or X. For every $t > 0$ and $x, y \geq 0$, we denote

$$Z^x := \big(Z|Z(0) = x\big) \quad \text{and} \quad Z_t^{x,y} := \big(Z|Z(0) = x \text{ and } Z(t) = y\big),$$

and we use $\mathbf{E}^x$ and $\mathbf{E}_t^{x,y}$ to denote the expected value with respect to the law of Z^x and $Z_t^{x,y}$ respectively.

For any $t > 0$, we use $x \mapsto L_t^x(Z)$ to denote the continuous version of the local time process of Z on $[0, t]$, which we characterize by the requirement that for every measurable function f, one has

$$\int_0^t f\big(Z(s)\big)\,\mathrm{d}s = \int_{\mathbb{R}} L_t^x(Z) f(x)\,\mathrm{d}x. \tag{2.5}$$

As a matter of convention, in the case where $Z = X$, we distinguish the boundary local time $\mathfrak{L}_t^0(Z)$ from the above, which we define as

$$\mathfrak{L}_t^0(Z) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{0 \leq Z(s) < \varepsilon\}}\,\mathrm{d}s. \tag{2.6}$$

Finally, we let $\tau_0(B)$ denote the first hitting time of zero by B.

Definition 2.7 (Continuum limits). Let Q be the diffusion process

$$\mathrm{d}Q(x) = V(x)\mathrm{d}x + \mathrm{d}W(x), \quad x \geq 0,$$

where $V \geq 0$ is a deterministic locally integrable function on $\mathbb{R}_+$, and W is a Brownian motion with variance $\sigma^2 > 0$. For every $t > 0$, we let $\hat{K}(t)$ and $\hat{K}^w(t)$ be the integral operators on $L^2(\mathbb{R}_+)$ with random kernels

$$\hat{K}(t; x, y) := \frac{\mathrm{e}^{-(x-y)^2/2t}}{\sqrt{2\pi t}} \mathbf{E}_t^{x,y}\big[\mathbf{1}_{\{\tau_0(B) > t\}} \mathrm{e}^{-\langle L_t(B), Q'\rangle}\big], \tag{2.7}$$

$$\hat{K}^w(t; x, y) := \bigg(\frac{\mathrm{e}^{-(x-y)^2/2t}}{\sqrt{2\pi t}} + \frac{\mathrm{e}^{-(x+y)^2/2t}}{\sqrt{2\pi t}}\bigg) \mathbf{E}_t^{x,y}\big[\mathrm{e}^{-\langle L_t(X), Q'\rangle - w\mathfrak{L}_t^0(X)}\big] \tag{2.8}$$

for $x, y \geq 0$, where

1. we assume that B and X are independent of W, and that $\mathbf{E}_t^{x,y}$ is the conditional expected value of $B_t^{x,y}$ or $X_t^{x,y}$ given W; and
2. for any piecewise continuous and compactly supported function f,

$$\langle f, Q'\rangle := \int_{\mathbb{R}} f(x)\,\mathrm{d}Q(x)$$

denotes $\mathrm{d}Q$ pathwise stochastic integration (see [13, Remark 2.18]).

Remark 2.8. Consider the operator $\hat{H} := -\frac{1}{2}\Delta + V + W'$ acting on $\mathbb{R}_+$ with Dirichlet boundary condition at zero, and let $\hat{H}^w$ be the same operator but with Robin boundary condition $f'(0) = wf(0)$. If the function V satisfies

$$\lim_{x \to \infty} V(x)/\log x = \infty, \tag{2.9}$$

then $\hat{H}$ and $\hat{H}^w$ can be rigorously defined as self-adjoint operators with compact resolvent (and thus discrete spectrum) using quadratic forms ([13, Proposition 2.9 and Corollary 2.12]; see also [6,26,28]). According to [13, Theorem 2.23], for every $t > 0$, it holds with probability one that $\hat{K}(t)$ and $\hat{K}^w(t)$ are self-adjoint Hilbert–Schmidt operators on $L^2(\mathbb{R}_+)$, and $\hat{K}(t) = \mathrm{e}^{-t\hat{H}}$ and $\hat{K}^w(t) = \mathrm{e}^{-t\hat{H}^w}$. We also have the trace formula $\mathrm{Tr}[\hat{K}(t)] = \int_0^\infty \hat{K}(t; x, x)\,\mathrm{d}x < \infty$.

Notation 2.9. Let $\beta > 0$. If $V(x) = x/2$ and $\sigma^2 = 1/\beta$ in Definition 2.7, then we use the notation $\mathrm{SAS}_\beta(t) := \hat{K}(t)$ and $\mathrm{SAS}_\beta^w(t) := \hat{K}^w(t)$, since in this case we recover the stochastic Airy semigroup defined in [14,20], which is the semigroup of the stochastic Airy operator.

2.3. *Technical assumptions*

We are now finally in a position to state our main results and the assumptions under which they apply. We begin with the assumptions on the random entries of Q_n in (2.3); our theorems are stated in Section 2.4.

2.3.1. *Assumptions on the potential terms* V_n^E

Assumption 2.10 (Potential convergence). There exists nonnegative continuous functions $V^D, V^U, V^L : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\lim_{n\to\infty} V_n^E\big(\lfloor m_n x\rfloor\big) = V^E(x), \quad x \geq 0$$

uniformly on compact sets for every $E \in \{D, U, L\}$. Moreover, the function

$$V := \frac{1}{2}\big(V^D + V^U + V^L\big), \quad x \geq 0 \tag{2.10}$$

satisfies (2.9).

Assumption 2.11 (Growth upper bounds). For every $E \in \{D, U, L\}$ we have the following: For large enough n,

$$0 \leq V_n^E(a) \leq 2m_n^2, \quad 0 \leq a \leq n, \tag{2.11}$$

and if $C_n = o(n)$ as $n \to \infty$, then

$$\max_{a\leq C_n} V_n^E(a) = o\big(m_n^2\big), \quad n \to \infty. \tag{2.12}$$

Assumption 2.12 (Growth lower bounds). At least one of $E \in \{D, U, L\}$ satisfies the following: For every $\theta > 0$, there exists $c = c(\theta) > 0$ and $N = N(\theta) \in \mathbb{N}$ such that for every $n \geq N$,

$$\theta \log(1 + a/m_n) - c \leq V_n^E(a) \leq m_n^2, \quad 0 \leq a \leq n. \tag{2.13}$$

Moreover, at least one of $E \in \{D, U, L\}$ (not necessarily the same as (2.13)) satisfies the following: With $\mathfrak{d}$ as in (2.1), there exists $\mathfrak{d}/2(1-\mathfrak{d}) < \alpha \leq 2\mathfrak{d}/(1-\mathfrak{d})$, $\varepsilon > 0$, and positive constants κ and $C > 0$ such that

$$\kappa(a/m_n)^\alpha \leq V_n^E(a) \leq m_n^2, \quad Cn^{1-\varepsilon} \leq a \leq n \tag{2.14}$$

for n large enough.

2.3.2. *Assumptions on the noise terms* ξ_n^E

Assumption 2.13 (Independence). For every $n \in \mathbb{N}$, the variables $\xi_n^D(0), \ldots, \xi_n^D(n)$ are independent, and likewise for $\xi_n^U(0), \ldots, \xi_n^U(n-1)$ and $\xi_n^L(0), \ldots, \xi_n^L(n-1)$. We emphasize, however, that the random vectors ξ_n^D, ξ_n^U, and ξ_n^L need not be independent of each other (for instance, if Q_n is symmetric, then $\xi_n^U = \xi_n^L$).

Assumption 2.14 (Moment asymptotics). For every $E \in \{D, U, L\}$, we have:

$$\big|\mathbf{E}\big[\xi_n^E(a)\big]\big| = o\big(m_{n-a}^{-1/2}\big) \quad \text{as } (n-a) \to \infty, \tag{2.15}$$

and there exists constants $C > 0$ and $0 < \gamma < 2/3$ such that

$$\mathbf{E}\big[\big|\xi_n^E(a)\big|^q\big] \leq m_n^{q/2} C^q q^{\gamma q} \tag{2.16}$$

for every $0 \leq a \leq n$, integer $q \in \mathbb{N}$, and n large enough.

Assumption 2.15 (Noise convergence). There exists Brownian motions W^D, W^U, and W^L such that

$$\lim_{n\to\infty}\left(\frac{1}{m_n}\sum_{a=0}^{\lfloor m_n x\rfloor} \xi_n^E(a)\right)_{E=D,U,L} = \big(W^E(x)\big)_{E=D,U,L}, \quad x \geq 0 \tag{2.17}$$

in joint distribution with respect to the Skorokhod topology. We assume that

$$W := \frac{1}{2}\big(W^D + W^U + W^L\big) \tag{2.18}$$

is also a Brownian motion with some variance $\sigma^2 > 0$. Furthermore, if $\varphi_1, \dots, \varphi_k$ are continuous and compactly supported functions and $(\varphi_1^{(n)})_{n\in\mathbb{N}}, \dots, (\varphi_k^{(n)})_{n\in\mathbb{N}}$ are such that $\varphi_i^{(n)} \to \varphi_i$ uniformly for every $1 \le i \le k$, then

$$\lim_{n\to\infty}\left(\sum_{a\in\mathbb{N}_0}\varphi_i^{(n)}(a/m_n)\frac{\xi_n^E(a)}{m_n}\right)_{E=D,U,L;1\le i\le k} = \left(\int_{\mathbb{R}_+}\varphi_i(a)\,\mathrm{d}W^E(a)\right)_{E=D,U,L;1\le i\le k} \tag{2.19}$$

in joint distribution, and also jointly with (2.17).

2.3.3. *Assumptions for the Robin boundary condition*

The following assumption will only be made when considering $\hat{K}_n^w(t)$:

Definition 2.16. We say that a sequence $(X_n)_{n\in\mathbb{N}}$ is uniformly sub-Gaussian if there exists $C, c > 0$ independent of n such that

$$\sup_{n\in\mathbb{N}}\mathbf{E}\big[\mathrm{e}^{y|X_n|}\big] \le C\mathrm{e}^{cy^2}, \quad y \ge 0. \tag{2.20}$$

Assumption 2.17. $(D_n(0)/m_n^{1/2})_{n\in\mathbb{N}}$ is uniformly sub-Gaussian.

Remark 2.18. If $\gamma < 1/2$ in (2.16), then Assumption 2.17 is satisfied.

2.4. *Main theorems*

Notation 2.19. In order to make sense of the claim that $\hat{K}_n(t) \to \hat{K}(t)$ and $\hat{K}_n^w(t) \to \hat{K}^w(t)$, we need to ensure that the discrete and continuous objects act on the same space. For this purpose, we note that the action of the matrices (2.4) on $\mathbb{R}^{n+1}$ can naturally be extended to step functions on $\mathbb{R}_+$ of the form

$$\sum_{a=0}^{n} v(a)\mathbf{1}_{[a/m_n,(a+1)/m_n)} \quad \text{for some } v \in \mathbb{R}^{n+1}.$$

This can then be further extended to any locally integrable $f : \mathbb{R}_+ \to \mathbb{R}$ via

$$\pi_n f := m_n^{1/2}\sum_{a=0}^{n}\int_{a/m_n}^{(a+1)/m_n} f(x)\,\mathrm{d}x\,\mathbf{1}_{[a/m_n,(a+1)/m_n)}. \tag{2.21}$$

Thus, for any $(n+1)\times(n+1)$ matrix M and locally integrable functions f, g, we define Mf as the vector/step function $M(\pi_n f)$, and we define

$$\langle f, Mg\rangle := m_n\sum_{0\le a,b\le n}\left(\int_{a/m_n}^{(a+1)/m_n} f(x)\,\mathrm{d}x\right)M(a,b)\left(\int_{b/m_n}^{(b+1)/m_n} g(x)\,\mathrm{d}x\right).$$

Our limit results are as follows.

Theorem 2.20. *Suppose that Assumptions 2.1 and 2.10–2.15 hold. Let $\hat{K}(t)$ be defined as in (2.7), where V is given by (2.10) and W is given by (2.18). Then, $\hat{K}_n(t) \to \hat{K}(t)$ as $n \to \infty$ in the following two senses*:

1. *For every $t_1, \dots, t_k > 0$ and $f_1, g_1, \dots, f_k, g_k : \mathbb{R}_+ \to \mathbb{R}$ uniformly continuous and bounded,*

$$\lim_{n\to\infty}\big(\langle f_i, \hat{K}_n(t_i)g_i\rangle\big)_{1\le i\le k} = \big(\langle f_i, \hat{K}(t_i)g_i\rangle\big)_{1\le i\le k}$$

in joint distribution and mixed moments.

2. *For every* $t_1, \ldots, t_k > 0$,

$$\lim_{n\to\infty} \big(\mathrm{Tr}\big[\hat{K}_n(t_i)\big]\big)_{1\le i\le k} = \big(\mathrm{Tr}\big[\hat{K}(t_i)\big]\big)_{1\le i\le k}$$

in joint distribution and mixed moments.

Theorem 2.21. *Suppose that Assumptions 2.1, 2.2, and 2.10–2.17 hold. Let* $\hat{K}^w(t)$ *be defined as in (2.8), where* V *is given by (2.10) and* W *is given by (2.18). Then,* $\hat{K}^w_n(t) \to \hat{K}^w(t)$ *as* $n \to \infty$ *in the following sense: For every* $t_1, \ldots, t_k > 0$ *and* $f_1, g_1, \ldots, f_k, g_k : \mathbb{R}_+ \to \mathbb{R}$ *uniformly continuous and bounded,*

$$\lim_{n\to\infty} \big(\langle f_i, \hat{K}^w_n(t_i) g_i\rangle\big)_{1\le i\le k} = \big(\langle f_i, \hat{K}^w(t_i) g_i\rangle\big)_{1\le i\le k}$$

in joint distribution and mixed moments.

Remark 2.22. Unlike Theorem 2.20, Theorem 2.21 contains no statement on the convergence of traces. Similarly to the lack of trace convergence in [14], this is due to the fact that we were unable to construct a strong coupling of a certain Markov chain and its occupation measures with the reflected Brownian bridge $X^{x,x}_t$ and its local time process. Throughout this paper, we make several remarks and conjectures concerning this trace convergence, its consequences, and the related strong invariance result (see Conjectures 2.23 and 6.11, and Remark 3.2).

Conjecture 2.23. *In the setting of Theorem 2.21, for every* $t_1, \ldots, t_k > 0$,

$$\lim_{n\to\infty} \big(\mathrm{Tr}\big[\hat{K}^w_n(t_i)\big]\big)_{1\le i\le k} = \big(\mathrm{Tr}\big[\hat{K}^w(t_i)\big]\big)_{1\le i\le k}$$

in joint distribution and mixed moments.

Remark 2.24. The conclusions of Theorems 2.20 and 2.21 remain valid if we define

$$\hat{K}_n(t) = \left(I_n - \frac{-\Delta_n + Q_n}{3m_n^2}\right)^{\vartheta(n,t)}, \qquad \hat{K}^w_n(t) = \left(I_n - \frac{-\Delta^w_n + Q_n}{3m_n^2}\right)^{\vartheta(n,t)}$$

for $\vartheta(n,t) := \lfloor m_n^2(3t/2)\rfloor \pm 1$, instead of (2.4). Thus, up to making this minor change, there is no loss of generality in assuming that $\lfloor m_n^2(3t/2)\rfloor$ is always even or odd if that is more convenient (this distinction comes in handy in the proof of Proposition 3.1 below). We refer to Remark 7.2 for more details.

3. Applications to random matrices

In this section, we provide applications of our main results to the study of random matrices and β-ensembles. We begin by stating our results in Sections 3.1–3.3, and then provide their proofs in Sections 3.4–3.9.

3.1. *Application* 1. *Convergence of eigenvalues*

Throughout Section 3.1, we assume that $-\Delta_n + Q_n$ satisfies the hypotheses of Theorem 2.20, and we denote by $-\infty < \lambda_1(\hat{H}) \le \lambda_2(\hat{H}) \le \cdots$ the eigenvalues of the operator $\hat{H} = -\frac{1}{2}\Delta + V + W'$ (as per Remark 2.8), where W is given by (2.18), and V by (2.10). The main result of Section 3.1 is the following:

Proposition 3.1. *Suppose that* $-\Delta_n + Q_n$ *is diagonalizable with real eigenvalues* $\lambda_{n;1} \le \lambda_{n;2} \le \cdots \le \lambda_{n;n+1}$ *for large enough* n, *and that there exists* $\delta > 0$ *such that*

$$\mathbf{P}\big[\lambda_{n;n+1} \ge (6-\delta)m_n^2 \text{ for infinitely many } n\big] = 0. \tag{3.1}$$

Then for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{1}{2}(\lambda_{n;1}, \ldots, \lambda_{n;k}) = \big(\lambda_1(\hat{H}), \ldots, \lambda_k(\hat{H})\big) \quad \textit{in joint distribution.} \tag{3.2}$$

Remark 3.2. Proposition 3.1 is only stated for the Dirichlet boundary condition since it depends on the trace convergence of Theorem 2.20-(2). If Conjecture 2.23 holds, then the same argument used to prove Proposition 3.1 would imply that the eigenvalues of $\frac{1}{2}(-\Delta_n^w + Q_n)$ converge to that of $\hat{H}^w$.

Question 3.3. It would be interesting to see if some analog of Proposition 3.1 can be proved in the case where $-\Delta_n + Q_n$ is diagonalizable with complex eigenvalues. We leave this as an open question.

We have the following convenient sufficient condition for (3.1), which is easily seen to be satisfied for every example considered in Sections 3.2 and 3.3 below.

Proposition 3.4. *Suppose that there exists $\bar{\delta} > 0$ and $N \in \mathbb{N}$ such that*

$$\max_{0\leq a\leq n}\left(2+\frac{V_n^D(a)}{m_n^2}+\left|\frac{V_n^U(a)}{m_n^2}-1\right|+\left|\frac{V_n^L(a-1)}{m_n^2}-1\right|\right)\leq 6-\bar{\delta} \tag{3.3}$$

for every $n \geq N$. Then, (3.1) *holds.*

Finally, the following result provides a simple sufficient condition that allows to apply Proposition 3.1 to a very general class of non-symmetric matrices.

Proposition 3.5. *Suppose that there exists $N \in \mathbb{N}$ large enough so that Q_n's off-diagonal entries satisfy*

$$\big(U_n(a)-m_n^2\big)\big(L_n(a)-m_n^2\big)>0, \quad 0\leq a\leq n-1, n\geq N. \tag{3.4}$$

Then, $-\Delta_n + Q_n$ is diagonalizable with real eigenvalues for $n \geq N$.

Propositions 3.1, 3.4, and 3.5 are proved in Sections 3.4–3.6. See Section 3.3 for an example of how these three results can be combined to prove new eigenvalue limit laws for non-symmetric tridiagonal matrices.

3.2. *Application* 2. *Classical β-ensembles*

In Section 3.2 we show that our main results apply to the edge-rescaled β-Hermite ensemble, the right-edge-rescaled β-Laguerre ensemble, as well as their rank-one spiked versions. In all cases, the limits we obtain are the stochastic Airy semigroups $\mathrm{SAS}_\beta(t)$ and $\mathrm{SAS}_\beta^w(t)$ respectively, thus extending the results of [14,20].

3.2.1. *Generalized β-Hermite ensembles*

Definition 3.6. Let $\xi_n^D \in \mathbb{R}^{n+1}$ and $\xi_n^U = \xi_n^L \in \mathbb{R}^n$ be random vectors that satisfy Assumptions 2.13–2.15 with $m_n = n^{1/3}$. Let $\beta > 0$ be such that the Brownian motion W in (2.18) has variance $1/\beta$. Let us denote $\chi_n(a) := \sqrt{n-a} - \xi_n^U(a)/n^{1/6}$ for all $0 \leq a \leq n$. We define the *generalized β-Hermite ensemble* as

$$H_n := \begin{bmatrix} -\xi_n^D(0)/n^{1/6} & \chi_n(0) & & \\ \chi_n(0) & \ddots & \ddots & \\ & \ddots & \ddots & \chi_n(n-1) \\ & & \chi_n(n-1) & -\xi_n^D(n)/n^{1/6} \end{bmatrix}.$$

Definition 3.7. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} n^{-1/6}(\sqrt{n}-\mu_n) = w \in \mathbb{R}. \tag{3.5}$$

Let ξ_n^E and H_n be as in Definition 3.6, assuming further that $(\xi_n^D(0)/n^{1/6})_{n\in\mathbb{N}}$ is uniformly sub-Gaussian. The *generalized spiked β-Hermite ensemble* is defined as $H_n^w := H_n + \mathrm{diag}_n(\mu_n, 0, \ldots, 0)$.

H_n and H_n^w are slight generalizations of the random matrix models studied in [14,20]. As shown in [20, Lemma 2.1], the β-Hermite ensemble studied in [10,11,28] is a special case of H_n. Similarly, as noted in [14, Remarks 1.3 and 1.8],

H_n^w generalizes the spiked β-Hermite ensemble with a critical (i.e., of size $\sqrt{n}$) rank-one additive perturbation introduced in [6, (1.5)] (see also [27]). As per classical theory, the edge fluctuations of H_n and H_n^w are captured by the rescalings

$$R_n := n^{1/6}(2\sqrt{n}I_n - H_n) \quad \text{and} \quad R_n^w := n^{1/6}\big(2\sqrt{n}I_n - H_n^w\big). \tag{3.6}$$

We have the following result regarding (3.6), which we prove in Section 3.7.

Corollary 3.8. *We can define Q_n so that $R_n = -\Delta_n + Q_n$ and $R_n^w = -\Delta_n^w + Q_n$ satisfy the hypotheses of Theorems 2.20 and 2.21 respectively, where $m_n = n^{1/3}$, $w_n = \mu_n/\sqrt{n}$, W in (2.18) has variance $1/\beta$, and $V(x)$ in (2.10) equals $x/2$.*

3.2.2. *Generalized β-Laguerre ensembles*

Definition 3.9. Suppose that $\tilde{\xi}_n^D$ and $\tilde{\xi}_n^U = \tilde{\xi}_n^L$ satisfy Assumptions 2.13 and 2.14 with $m_n = n^{1/3}$, and that $\tilde{\xi}_n^E$ satisfy (2.17) and (2.19) with $m_n = n^{1/3}$. Denoting the limits in distribution

$$\tilde{W}^E(x) := \lim_{n\to\infty} \frac{1}{n^{1/3}} \sum_{a=0}^{\lfloor n^{1/3}x\rfloor} \tilde{\xi}_n^E(a), \qquad E \in \{D, U, L\},$$

we further assume that $\tilde{W}^D + \tilde{W}^U = \tilde{W}^D + \tilde{W}^L$ is a Brownian motion with variance $1/\beta$ for some $\beta > 0$. Let $p = p(n) > n$ be an increasing sequence such that $n/p \to \nu \in [0, 1]$ as $n \to \infty$. Denote $\chi_n(a) := \sqrt{n-a} - \tilde{\xi}_n^U(a)/n^{1/6}$ and $\chi_{n;p}(a) := \sqrt{p-a} - \tilde{\xi}_n^D(a)/n^{1/6}$. We define the *generalized β-Laguerre ensemble* as $L_n := (L_n^*)^\top L_n^*$, where

$$L_n^* := \begin{bmatrix} \chi_{n;p}(0) & & & & \\ \chi_n(0) & \chi_{n;p}(1) & & & \\ & \ddots & \ddots & & \\ & & \chi_n(n-1) & \chi_{n;p}(n) & \end{bmatrix}.$$

Definition 3.10. Let $\tilde{\xi}_n^E$, $p(n)$, and L_n^* be as in Definition 3.9, with the additional assumption that $(\tilde{\xi}_n^D(0)/n^{1/6})_{n\in\mathbb{N}}$ and $(\tilde{\xi}^L(0)/n^{1/6})_{n\in\mathbb{N}}$ are uniformly sub-Gaussian. Let $(\ell_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that

$$\lim_{n\to\infty} \left(\frac{\sqrt{np}}{\sqrt{n}+\sqrt{p}}\right)^{2/3} \big(1 - \sqrt{p/n}(\ell_n - 1)\big) = w \in \mathbb{R}. \tag{3.7}$$

The *generalized spiked β-Laguerre ensemble* is defined as $L_n^w := (L_n^*)^\top \mathrm{diag}_n(\ell_n, 1, \dots, 1)L_n^*$.

L_n is a generalization of the β-Laguerre ensemble studied in [10,11,28]; L_n^w is a generalization of the critical (i.e., of size $1 + \sqrt{\nu}$) rank-one spiked model of the β-Laguerre ensemble (cf., [2] and [6, (1.2)]). The right-edge (i.e., largest eigenvalues) fluctuations of these matrices are captured by the rescalings

$$\Sigma_n := \frac{m_n^2}{\sqrt{np}}\big((\sqrt{n}+\sqrt{p})^2 I_n - L_n\big), \quad \text{where } m_n := \left(\frac{\sqrt{np}}{\sqrt{n}+\sqrt{p}}\right)^{2/3}, \tag{3.8}$$

and $\Sigma_n^w := (m_n^2/\sqrt{np})((\sqrt{n}+\sqrt{p})^2 I_n - L_n^w)$ with the same m_n. The following is proved in Section 3.8:

Corollary 3.11. *We can define Q_n so that $\Sigma_n = -\Delta_n + Q_n$ and $\Sigma_n^w = -\Delta_n^w + Q_n$ satisfy the hypotheses of Theorems 2.20 and 2.21 respectively, where m_n is as in (3.8), $w_n = \sqrt{p/n}(\ell_n - 1)$, W in (2.18) has variance $1/\beta$, and $V(x)$ in (2.10) equals $x/2$.*

3.3. *Application* 3. *Non-symmetric ensemble*

We now provide an example of a non-symmetric matrix model for which we can prove a new limit law. The following model is inspired by the β-Hermite ensemble:

Definition 3.12. Suppose that ξ_n^D and $\xi_n^U \neq \xi_n^L$ satisfy Assumptions 2.13–2.15 with $m_n = n^{1/3}$. Let us denote $\chi_n^U(a) := \sqrt{n-a} - \xi_n^U(a)/n^{1/6}$ and $\chi_n^L(a) := \sqrt{n-a} - \xi_n^L(a)/n^{1/6}$, and assume that $\chi_n^U(a), \chi_n^L(a) > 0$ (or, equivalently, $\xi_n^U(a), \xi_n^L(a) < n^{1/6}\sqrt{n-a}$) for every $0 \le a \le n-1$. Define the random matrix

$$\tilde{H}_n := \begin{bmatrix} -\xi_n^D(0)/n^{1/6} & \chi_n^U(0) & & \\ \chi_n^L(0) & \ddots & \ddots & \\ & \ddots & \ddots & \chi_n^U(n-1) \\ & & \chi_n^L(n-1) & -\xi_n^D(n)/n^{1/6} \end{bmatrix}. \tag{3.9}$$

In order to capture the edge fluctuations of $\tilde{H}_n$, we consider the rescaled version

$$\tilde{R}_n := n^{1/6}(2\sqrt{n} I_n - \tilde{H}_n).$$

The following result is proved in Section 3.9:

Corollary 3.13. *For every $k \in \mathbb{N}$, the k smallest eigenvalues of $\tilde{R}_n$ converge in joint distribution to the k smallest eigenvalues of* SAO_β *with Dirichlet boundary condition.*

3.4. *Proof of Proposition* 3.1

As argued in [20, Section 6] and [30, Section 5], it suffices to prove the convergence of Laplace transforms

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n+1} \mathrm{e}^{-t_i\lambda_{n;j}/2}\right)_{0\le i\le k} = \left(\sum_{j=1}^{\infty} \mathrm{e}^{-t_i\lambda_j(\hat{H})}\right)_{0\le i\le k}, \quad t_1,\ldots,t_k > 0$$

in joint distribution. On the one hand, if $-\Delta_n + Q_n$ is diagonalizable, then

$$\mathrm{Tr}\big[\hat{K}_n(t)\big] = \sum_{j=1}^{n+1}\left(1 - \frac{\lambda_{n;j}}{3m_n^2}\right)^{\lfloor m_n^2(3t/2)\rfloor}$$

for every $t > 0$. On the other hand, by [13, Theorem 2.23], for every $t > 0$,

$$\mathrm{Tr}\big[\hat{K}(t)\big] = \sum_{j=1}^{\infty} \mathrm{e}^{-t\lambda_j(\hat{H})} < \infty \quad \text{almost surely.}$$

Consequently, by Theorem 2.20-(2), we need only prove that

$$\lim_{n\to\infty}\left(\sum_{j=1}^{n+1} \mathrm{e}^{-t_i\lambda_{n;j}/2} - \left(1 - \frac{\lambda_{n;j}}{3m_n^2}\right)^{\lfloor m_n^2(3t_i/2)\rfloor}\right)_{0\le i\le k} = (0,\ldots,0) \tag{3.10}$$

in joint distribution.

By the Skorokhod representation theorem, if $\hat{K}_n(t) \to \hat{K}(t)$ in the sense of Theorem 2.20-(2), then there exists a coupling of the sequence $(\lambda_{n;j})_{1\le j\le n+1, n\in\mathbb{N}}$ and $(\lambda_j(\hat{H}))_{j\in\mathbb{N}}$ such that

$$\lim_{n\to\infty}\sum_{j=1}^{n+1}\left(1 - \frac{\lambda_{n;j}}{3m_n^2}\right)^{\lfloor m_n^2(3t_i/2)\rfloor} = \sum_{j=1}^{\infty} \mathrm{e}^{-t_i\lambda_j(\hat{H})} < \infty \tag{3.11}$$

almost surely for $1 \le i \le k$. By Remark 2.24, there is no loss of generality in assuming that $\lfloor m_n^2(3t_i/2)\rfloor$ is even for all n; hence

$$\sum_{j=1}^{n+1}\left(1 - \frac{\lambda_{n;j}}{3m_n^2}\right)^{\lfloor m_n^2(3t_i/2)\rfloor} = \sum_{j=1}^{n+1}\left|1 - \frac{\lambda_{n;j}}{3m_n^2}\right|^{\lfloor m_n^2(3t_i/2)\rfloor}.$$

Let us fix $0 < \delta < 1$ and $0 < \varepsilon < \mathfrak{d}$, where $\mathfrak{d}$ is as in (2.1). We consider four different regimes of eigenvalues of $-\Delta_n + Q_n$:

1. $J_{n;1} := \{j : \lambda_{n;j} < -n^{\varepsilon}\}$;
2. $J_{n;2} := \{j : -n^{\varepsilon} \leq \lambda_{n;j} < n^{\varepsilon}\}$;
3. $J_{n;3} := \{j : n^{\varepsilon} \leq \lambda_{n;j} < (6-\delta)\lfloor m_n^2(3t_i/2)\rfloor\}$; and
4. $J_{n;4} := \{j : (6-\delta)\lfloor m_n^2(3t_i/2)\rfloor \leq \lambda_{n;j}\}$.

Firstly, note that

$$\sum_{j\in J_{n;1}} \left|1-\frac{\lambda_{n;j}}{3m_n^2}\right|^{\lfloor m_n^2(3t_i/2)\rfloor} \geq |J_{n;1}|\left(1+\frac{n^{\varepsilon}}{3m_n^2}\right)^{\lfloor m_n^2(3t_i/2)\rfloor},$$

where $|J_{n;1}|$ denotes the cardinality of $J_{n;1}$. If $|J_{n;1}| > 0$ for infinitely many n, then this quantity diverges, contradicting the convergence of (3.11). Hence $J_{n;1}$ does not contribute to (3.10).

Secondly, recall the elementary inequalities

$$0 < e^{z} - \left(1+\frac{z}{m}\right)^{m} < \left(1+\frac{z}{m}\right)^{m}\left(\left(1+\frac{z}{m}\right)^{z} - 1\right), \quad \forall z, m > 0$$

and

$$0 < e^{-z} - \left(1-\frac{z}{m}\right)^{m} < \left(1-\frac{z}{m}\right)^{m}\left(\left(1-\frac{z}{m}\right)^{-z} - 1\right), \quad \forall m > z > 0,$$

which imply that

$$\left|\sum_{j\in J_{n;2}} \mathrm{e}^{-t_i\lambda_{n;j}/2} - \left(1-\frac{\lambda_{n;j}}{3m_n^2}\right)^{\lfloor m_n^2(3t_i/2)\rfloor}\right| \leq \left(\left(1+\frac{n^{\varepsilon}}{3m_n^2}\right)^{n^{\varepsilon}} - 1\right)\sum_{j\in J_{n;2}}\left|1-\frac{\lambda_{n;j}}{3m_n^2}\right|^{\lfloor m_n^2(3t_i/2)\rfloor}.$$

Since $n^{2\varepsilon} = o(m_n^2)$, we have $(1+\frac{n^{\varepsilon}}{3m_n^2})^{n^{\varepsilon}} = 1+o(1)$, and thus (3.11) implies that the contribution of $J_{n;2}$ to (3.10) vanishes.

Thirdly, one the one hand, we have that

$$\sum_{j\in J_{n;3}} \mathrm{e}^{-t_i\lambda_{n;j}/2} \leq |J_{n;3}|\mathrm{e}^{-t_i n^{\varepsilon}/2},$$

and on the other hand, since $|1-z| \leq \max\{\mathrm{e}^{-z}, \mathrm{e}^{z-2}\}$ ($z\in\mathbb{R}$), we see that

$$\begin{aligned}&\sum_{j\in J_{n;3}}\left|1-\frac{\lambda_{n;j}}{3m_n^2}\right|^{\lfloor m_n^2(3t_i/2)\rfloor}\\ &\quad\leq |J_{n;3}| \max_{j\in J_{n;3}} \max\left\{\exp\left(-\frac{\lfloor m_n^2(3t_i/2)\rfloor\lambda_{m;j}}{3m_n^2}\right), \exp\left(\frac{\lfloor m_n^2(3t_i/2)\rfloor\lambda_{m;j}}{3m_n^2} - 2\lfloor m_n^2(3t_i/2)\rfloor\right)\right\}.\end{aligned}$$

Note that $|J_{n;3}| \leq n+1$ and that there exists a constant $C > 0$ independent of n such that for every $j \in J_{n;3}$,

$$\exp\left(-\frac{\lfloor m_n^2(3t_i/2)\rfloor\lambda_{m;j}}{3m_n^2}\right) \leq \mathrm{e}^{-Cn^{\varepsilon}},$$

$$\exp\left(\frac{\lfloor m_n^2(3t_i/2)\rfloor\lambda_{m;j}}{3m_n^2} - 2\lfloor m_n^2(3t_i/2)\rfloor\right) \leq \exp\left(-\frac{\delta}{3}\lfloor m_n^2(3t_i/2)\rfloor\right).$$

Consequently, the contribution of $J_{n;3}$ to (3.10) vanishes.

Finally, we know from (3.1) that there is eventually no eigenvalue in $J_{n;4}$, and thus it has no contribution to (3.10), completing the proof Proposition 3.1.

3.5. *Proof of Proposition* 3.4

According to the Gershgorin disc theorem (e.g., [33, Corollary 9.11]),

$$\frac{\lambda_{n;n+1}}{m_n^2} \leq \max_{0\leq a\leq n}\left(2+\frac{D_n(a)}{m_n^2}+\left|-1+\frac{U_n(a)}{m_n^2}\right|+\left|-1+\frac{L_n(a-1)}{m_n^2}\right|\right).$$

By combining this with (3.3) and the triangle inequality, we get

$$\frac{\lambda_{n;n+1}}{m_n^2} \leq 6-\bar{\delta}+\sum_{E=D,U,L}\max_{0\leq a\leq n}\frac{|\xi_n^E(a)|}{m_n^2}$$

for large enough n. By a union bound, (2.16), and Markov's inequality, we see that

$$\mathbf{P}\left[\max_{0\leq a\leq n}\frac{|\xi_n^E(a)|}{m_n^2}\geq\tilde{\delta}\right]=O\left(\frac{n}{m_n^{3q/2}}\right)$$

for any $\tilde{\delta}\in(0,\bar{\delta})$ and $q\in\mathbb{N}$. By (2.1), we can take q large enough so that $\sum_n n/m_n^{3q/2}<\infty$; the result then follows by the Borel–Cantelli lemma.

3.6. *Proof of Proposition* 3.5

This is a direct consequence of the following classical result in matrix theory:

Lemma 3.14 ([21, 3.1.P22; see also p. 585]). *Let M be a $(n+1)\times(n+1)$ real-valued tridiagonal matrix. If $M(a,a+1)M(a+1,a)>0$ for every $0\leq a\leq n-1$, then M is similar to a Hermitian matrix.*

3.7. *Proof of Corollary* 3.8

Thanks to (3.6), straightforward computations reveal that we can write $R_n=-\Delta_n+Q_n$ and $R_n^w=-\Delta_n^w+Q_n$ with $m_n=n^{1/3}$, where the noise terms ξ_n^E are as in Definition 3.6, and the potential terms are

$$V_n^D(a)=0 \quad\text{and}\quad V_n^U(a)=V_n^L(a)=n^{1/6}(\sqrt{n}-\sqrt{n-a}) \tag{3.12}$$

for $0\leq a\leq n$. By Definitions 3.6 and 3.7, ξ_n^E satisfy Assumptions 2.13–2.15, and Assumptions 2.2 and 2.17 hold for H_n^w with $w_n=\mu_n/\sqrt{n}$. Thus, it only remains to prove that (3.12) satisfies Assumptions 2.10–2.12 with $V(x)$ in (2.10) equal to $x/2$.

Note that $n^{1/6}(\sqrt{n}-\sqrt{n-a})=n^{2/3}(1-\sqrt{1-a/n})$; hence Assumption 2.11 is met. Elementary calculus shows that for any $0<\kappa<1/2$ and $c>0$, the function

$$x\mapsto c^2\big(1-\sqrt{1-x/c^3}\big)-\kappa x/c$$

is nonnegative on $x\in[0,c^3]$. Taking $c=m_n$, this implies that Assumption 2.12 is met with $E=U,L$ in both (2.13) and (2.14). Finally, for $E=U,L$ and $x\geq 0$,

$$V^E(x):=\lim_{n\to\infty}V_n^E\big(\lfloor n^{1/3}x\rfloor\big)=\lim_{n\to\infty}n^{2/3}\big(1-\sqrt{1-\lfloor n^{1/3}x\rfloor/n}\big)=x/2\quad\text{pointwise.}$$

Since $V_n^E(\lfloor n^{1/3}x\rfloor)$ is nondecreasing in x for every n, the convergence is uniform on compacts. Then, we are led to $V(x)=\frac{1}{2}(V^U(x)+V^L(x))=x/2$, as desired.

3.8. *Proof of Corollary* 3.11

Remark 3.15. Unless otherwise stated, m_n in this proof refers to the quantity $(\frac{\sqrt{np}}{\sqrt{n}+\sqrt{p}})^{2/3}$ defined in (3.8). If we invoke statements regarding quantities that satisfy Assumptions 2.13–2.15 with other values of m_n, we will explicitly state so.

By definition of p and ν, $n^{-1/3}m_n=(1+\sqrt{\nu})^{-2/3}(1+o(1))$, and thus (2.1) holds with $\mathfrak{d}=1/3$. With this in hand, straightforward computations using (3.8) reveal that we can write $\Sigma_n=-\Delta_n+Q_n$ with the potential terms

$$V_n^D(a)=2\frac{m_n^2}{\sqrt{np}}a,\qquad V_n^U(a)=V_n^L(a)=m_n^2\big(1-\sqrt{(1-a/n)\big(1-(a-1)/p\big)}\big)$$

and the noise terms

$$\xi_n^D(a)=\frac{m_n^2}{\sqrt{np}}\left(2\left(\sqrt{p-a}\frac{\tilde{\xi}_n^D(a)}{n^{1/6}}+\sqrt{n-a}\frac{\tilde{\xi}_n^U(a)}{n^{1/6}}\right)-\frac{\tilde{\xi}_n^D(a)^2}{n^{2/3}}-\frac{\tilde{\xi}_n^U(a)^2}{n^{2/3}}\right),$$

$$\xi_n^U(a)=\xi_n^L(a)=\frac{m_n^2}{\sqrt{np}}\left(\left(\sqrt{n-a}\frac{\tilde{\xi}_n^D(a+1)}{n^{1/6}}+\sqrt{p-a-1}\frac{\tilde{\xi}_n^U(a)}{n^{1/6}}\right)-\frac{\tilde{\xi}_n^D(a+1)\tilde{\xi}_n^U(a)}{n^{2/3}}\right).$$

We can similarly write $\Sigma_n^w=-\Delta_n^w+Q_n$ with $w_n=\sqrt{p/n}(\ell_n-1)$, the only difference in Q_n being in the $(0,0)$ entry, which has $V^D(0)=0$ and

$$\xi_n^D(0)=\frac{m_n^2}{\sqrt{np}}\left(2\left(\sqrt{p}\ell_n\frac{\tilde{\xi}_n^D(0)}{n^{1/6}}+\sqrt{n}\frac{\tilde{\xi}^U(0)}{n^{1/6}}\right)-\ell_n\frac{\tilde{\xi}_n^D(0)^2}{n^{2/3}}-\frac{\tilde{\xi}_n^U(0)^2}{n^{2/3}}\right). \tag{3.13}$$

We now check that the hypotheses of Theorems 2.20 and 2.21 are met.

Regarding the potential terms, (2.11) and (2.12) are immediate from the definition of V_n^E above. Given that $(1-\sqrt{(1-a/n)(1-(a-1)/p)})\geq(1-\sqrt{(1-a/n)})$, the same argument used in the proof of Corollary 3.8 implies that (2.13) and (2.14) both hold with $E=U,L$. Next, by writing $n=\nu p(1+o(1))$, we observe that we have the following pointwise limits in $x\geq 0$:

$$V^D(x):=\lim_{n\to\infty}V_n^D\big(\lfloor m_nx\rfloor\big)=\frac{2\sqrt{\nu}x}{(1+\sqrt{\nu})^2},$$

$$V^E(x):=\lim_{n\to\infty}V_n^E\big(\lfloor m_nx\rfloor\big)=\frac{(1+\nu)x}{2(1+\sqrt{\nu})^2},\quad E=U,L.$$

Once again the monotonicity in x of the functions involved implies uniform convergence on compacts, and we have $V(x):=\frac{1}{2}(V^D(x)+V^U(x)+V^L(x))=x/2$.

We now prove that the noise terms ξ_n^E satisfy Assumptions 2.13–2.15. Since

$$m_n=O\big(n^{1/3}\big)\quad\text{and}\quad m_n^2/\sqrt{n},m_n^2/\sqrt{p}=O\big(m_n^{1/2}\big)=O\big(n^{1/6}\big), \tag{3.14}$$

the fact that $\tilde{\xi}_n^E$ satisfies Assumptions 2.13 and 2.14 with $m_n=n^{1/3}$ implies that ξ_n^E satisfy Assumptions 2.13 and 2.14 as well. Recall that, by definition, $\tilde{\xi}_n^E$ satisfy Assumption 2.15 with $m_n=n^{1/3}$ (and we denote the corresponding limiting Brownian motions as $\tilde{W}^D$, $\tilde{W}^U=\tilde{W}^L$). Since $m_n/n^{1/3}\to(1+\sqrt{\nu})^{-2/3}$ converges to a constant, it then follows from a straightforward Brownian scaling that $\frac{1}{m_n}\sum_{a=0}^{\lfloor m_nx\rfloor}\tilde{\xi}_n^E(a)\to(1+\sqrt{\nu})^{1/3}\tilde{W}^E(x)$ in distribution. Combining this with the fact that for every $a=o(n)$, one has

$$\lim_{n\to\infty}\frac{m_n^2\sqrt{p-a}}{n^{1/6}\sqrt{np}}=\frac{1}{(1+\sqrt{\nu})^{4/3}}\quad\text{and}\quad\lim_{n\to\infty}\frac{m_n^2\sqrt{n-a}}{n^{1/6}\sqrt{np}}=\frac{\sqrt{\nu}}{(1+\sqrt{\nu})^{4/3}}$$

we then obtain that ξ_n^E satisfy Assumption 2.15 with

$$W^D(x):=\lim_{n\to\infty}\frac{1}{m_n}\sum_{a=0}^{\lfloor m_nx\rfloor}\xi_n^D(a)=\left(\frac{2}{1+\sqrt{\nu}}\right)\tilde{W}^D(x)+\left(\frac{2\sqrt{\nu}}{1+\sqrt{\nu}}\right)\tilde{W}^U(x),$$

and for $E=L,U$,

$$W^E(x):=\lim_{n\to\infty}\frac{1}{m_n}\sum_{a=0}^{\lfloor m_nx\rfloor}\xi_n^E(a)=\left(\frac{1}{1+\sqrt{\nu}}\right)\tilde{W}^U(x)+\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)\tilde{W}^D(x).$$

From this we immediately obtain that $W:=\frac{1}{2}(W^D+W^U+W^L)=\tilde{W}^D+\tilde{W}^L$ is a Brownian motion with variance $1/\beta$, as desired.

We conclude the proof by checking the assumptions related to the rank-one spike in L_n^w. That Assumption 2.2 is satisfied with $w_n = \sqrt{p/n}(\ell_n - 1)$ is an immediate consequence of (3.7). As for (3.13) satisfying Assumption 2.17, this is immediate from the fact that $\tilde{\xi}_n^E(0)/n^{1/3}$ are uniformly sub-Gaussian, the estimates (3.14), and the fact that $\ell_n = 1 + \sqrt{\nu} + O(m_n^{-1})$ (by (3.7)).

3.9. *Proof of Corollary* 3.13

It is easy to see that $\tilde{R}_n$ is of the form $-\Delta_n + Q_n$ (with $m_n = n^{1/3} = n^{1/6}\sqrt{n}$), where, for $E = U, L$, one has

$$U_n(a) = n^{1/6}\big(\sqrt{n} - \sqrt{n-a} + \xi_n^U(a)/n^{1/6}\big),$$

and $D_n(a) = \xi_n^D(a)$. Given that $-\sqrt{n-a} + \frac{\xi_n^U(a)}{n^{1/6}}, -\sqrt{n-a} + \frac{\xi_n^L(a)}{n^{1/6}} < 0$ (by Definition 3.12), $\tilde{R}_n$ satisfies (3.4). We can prove that $\tilde{R}_n$ satisfies Assumptions 2.1 and and 2.10–2.15 in the same way as Corollary 3.8; hence the result follows from Propositions 3.1, 3.4, and 3.5 ((3.3) is easily seen to hold here).

4. From matrices to Feynman–Kac functionals

In this section, we derive probabilistic representations for $\langle f, \hat{K}_n(t)g\rangle$, $\mathrm{Tr}[K_n(t)]$, and $\langle f, \hat{K}_n^w(t)g\rangle$ that serve as finite-dimensional analogs of (2.7) and (2.8).

4.1. *Dirichlet boundary condition*: *Lazy random walk*

Definition 4.1 (Lazy random walk). Let $S = (S(u))_{u\in\mathbb{N}_0}$ ($\mathbb{N}_0 := \{0, 1, 2, \ldots\}$) be a lazy random walk, i.e., the increments $S(u) - S(u-1)$ are i.i.d. uniform random variables on $\{-1, 0, 1\}$. For every $a, b, u \in \mathbb{N}_0$, we denote $S^a := (S|S(0) = a)$ and $S_u^{a,b} := (S|S(0) = a \text{ and } S(u) = b)$.

4.1.1. *Inner product*

Let M be a $(n+1) \times (n+1)$ random tridiagonal matrix, let $v \in \mathbb{R}^{n+1}$ be a vector, and let $\vartheta \in \mathbb{N}$ be a fixed integer. By definition of matrix product, for every $0 \le a \le n$,

$$\left(\left(\frac{1}{3}M\right)^{\vartheta} v\right)(a) = \frac{1}{3^{\vartheta}} \sum_{a_1,\ldots,a_{\vartheta-1}} M(a, a_1)M(a_1, a_2)\cdots M(a_{\vartheta-1}, a_{\vartheta})v(a_{\vartheta}), \tag{4.1}$$

where the sum is taken over all $a_1, \ldots, a_{\vartheta} \in \mathbb{N}_0$ such that $(a, a_1, \ldots, a_{\vartheta})$ forms a path on the lattice $\{0, 1, 2, \ldots, n\}$ with self-edges (i.e., $|a_i - a_{i-1}| \in \{0, 1\}$). The probability that S^a is equal to any such path is $3^{-\vartheta}$, and thus we see that

$$\left(\left(\frac{1}{3}M\right)^{\vartheta} v\right)(a) = \mathbf{E}^a\left[\mathbf{1}_{\{\tau^{(n)}(S)>\vartheta\}}\left(\prod_{u=0}^{\vartheta-1} M\big(S(u), S(u+1)\big)\right)v\big(S(\vartheta)\big)\right], \tag{4.2}$$

where the random walk S is independent of the randomness in M, $\mathbf{E}^a$ denotes the expected value with respect to the law of S^a conditional on M, and

$$\tau^{(n)}(S) := \min\big\{u \ge 0 : S(u) = -1 \text{ or } n+1\big\}.$$

We can think of the contribution of M to (4.2) as a type of random walk in random scenery process on the edges of $\{0, 1, 2, \ldots, n\}$, that is, each passage of S on an edge contributes to the multiplication of the corresponding entry in M. In particular, if we define the *edge-occupation measures*

$$\Lambda_{\vartheta}^{(a,b)}(S) := \sum_{u=0}^{\vartheta-1} \mathbf{1}_{\{S(u)=a \text{ and } S(u+1)=b\}}, \quad 0 \le a, b \le n, \tag{4.3}$$

then we have that

$$\prod_{u=0}^{\vartheta-1} M\big(S(u), S(u+1)\big) = \prod_{a,b\in\mathbb{Z}} M(a, b)^{\Lambda_{\vartheta}^{(a,b)}(S)}. \tag{4.4}$$

We now apply the above discussion to the study of $\hat{K}_n(t)$. We observe that

$$\left(I_n - \frac{-\Delta_n + Q_n}{3m_n^2}\right)(a,a) = \frac{1}{3}\left(1 - \frac{D_n(a)}{m_n^2}\right), \quad 0 \le a \le n, \tag{4.5}$$

$$\left(I_n - \frac{-\Delta_n + Q_n}{3m_n^2}\right)(a,a+1) = \frac{1}{3}\left(1 - \frac{U_n(a)}{m_n^2}\right), \quad 0 \le a \le n-1, \tag{4.6}$$

$$\left(I_n - \frac{-\Delta_n + Q_n}{3m_n^2}\right)(a+1,a) = \frac{1}{3}\left(1 - \frac{L_n(a)}{m_n^2}\right), \quad 0 \le a \le n-1. \tag{4.7}$$

Let $t > 0$ and $n \in \mathbb{N}$ be fixed, and let us denote $\vartheta = \vartheta(n,t) := \lfloor m_n^2(3t/2)\rfloor$. By combining (4.5)–(4.7), the combinatorial analysis in (4.1)–(4.4), and the embedding π_n in (2.21), we see that

$$\langle f, \hat{K}_n(t)g\rangle = \int_0^{(n+1)/m_n} f(x)\mathbf{E}^{\lfloor m_n x\rfloor}\left[F_{n,t}(S)m_n \int_{S(\vartheta)/m_n}^{(S(\vartheta)+1)/m_n} g(y)\,\mathrm{d}y\right]\mathrm{d}x, \tag{4.8}$$

where S is independent of Q_n, we define the random functional

$$F_{n,t}(S) := \mathbf{1}_{\{\tau^{(n)}(S)>\vartheta\}} \prod_{a\in\mathbb{N}_0}\left(1 - \frac{D_n(a)}{m_n^2}\right)^{\Lambda_\vartheta^{(a,a)}(S)}\left(1 - \frac{U_n(a)}{m_n^2}\right)^{\Lambda_\vartheta^{(a,a+1)}(S)}\left(1 - \frac{L_n(a)}{m_n^2}\right)^{\Lambda_\vartheta^{(a+1,a)}(S)}, \tag{4.9}$$

and for any $x \ge 0$, $\mathbf{E}^{\lfloor m_n x\rfloor}$ denotes the expected value with respect to $S^{\lfloor m_n x\rfloor}$, conditional on Q_n.

4.1.2. *Trace*

Letting M be as in the previous section, it is easy to see that

$$\mathrm{Tr}\left[\left(\frac{1}{3}M\right)^\vartheta\right] = \sum_{a=0}^n \mathbf{P}\big[S^a(\vartheta) = a\big]\mathbf{E}_\vartheta^{a,a}\left[\mathbf{1}_{\{\tau^{(n)}(S)>\vartheta\}}\prod_{u=0}^{\vartheta-1} M\big(S(u), S(u+1)\big)\right],$$

where S is independent of M, and $\mathbf{E}_\vartheta^{a,a}$ denotes the expected value with respect to the law of $S_\vartheta^{a,a}$, conditional on M. Given that $\mathbf{P}[S^a(\vartheta) = a] = \mathbf{P}[S^0(\vartheta) = 0]$ is independent of a, if we apply a Riemann sum on the grid $m_n^{-1}\mathbb{Z}$ to the previous expression for $\mathrm{Tr}[(\frac{1}{3}M)^\vartheta]$, we note that

$$\mathrm{Tr}\left[\left(\frac{1}{3}M\right)^\vartheta\right] = m_n\mathbf{P}\big[S^0(\vartheta) = 0\big]\int_0^{(n+1)/m_n} \mathbf{E}_\vartheta^{\lfloor m_n x\rfloor,\lfloor m_n x\rfloor}\left[\mathbf{1}_{\{\tau^{(n)}(S)>\vartheta\}}\prod_{u=0}^{\vartheta-1} M\big(S(u), S(u+1)\big)\right]\mathrm{d}x.$$

Applying this to the model of interest $\hat{K}_n(t)$, we then see that

$$\mathrm{Tr}\big[\hat{K}_n(t)\big] = m_n\mathbf{P}\big[S^0(\vartheta) = 0\big]\int_0^{(n+1)/m_n} \mathbf{E}_\vartheta^{\lfloor m_n x\rfloor,\lfloor m_n x\rfloor}\big[F_{n,t}(S)\big]\,\mathrm{d}x, \tag{4.10}$$

where $\vartheta = \vartheta(n,t) = \lfloor m_n^2(3t/2)\rfloor$, S is independent of Q_n, $E_\vartheta^{\lfloor m_n x\rfloor,\lfloor m_n x\rfloor}$ denotes the expected value of $S_\vartheta^{\lfloor m_n x\rfloor,\lfloor m_n x\rfloor}$ conditional on Q_n, and $F_{n,t}$ is as in (4.9).

4.2. *Robin boundary condition*: *"Reflected" random walk*

Definition 4.2. Let $T = (T(u))_{u\in\mathbb{N}_0}$ be the Markov chain on the state space $\mathbb{N}_0$ with the following transition probabilities:

$$\mathbf{P}\big[T(u+1) = a+b | T(u) = a\big] = \frac{1}{3} \quad \text{if } a \in \mathbb{N}_0 \setminus \{0\} \text{ and } b \in \{-1,0,1\},$$

$$\mathbf{P}\big[T(u+1) = 0 | T(u) = 0\big] = \frac{2}{3}, \quad \text{and} \quad \mathbf{P}\big[T(u+1) = 1 | T(u) = 0\big] = \frac{1}{3}.$$

We denote $T^a := (T | T(0) = a)$ and $T_u^{a,b} := (T | T(0) = a \text{ and } T(u) = b)$.

Let M be a $(n+1)\times(n+1)$ tridiagonal matrix, and let $\tilde{M}$ be defined as

$$\tilde{M}(a,b)=\begin{cases}\frac{2}{3}M(a,b) & \text{if } a=b=0,\\ \frac{1}{3}M(a,b) & \text{otherwise.}\end{cases}$$

For any $\vartheta\in\mathbb{N}$, $0\leq a\leq n$, and vector $v\in\mathbb{R}^{n+1}$,

$$\big(\tilde{M}^{\vartheta}v\big)(a)=\mathbf{E}^{a}\left[\mathbf{1}_{\{\tau^{(n)}(T)>\vartheta\}}\left(\prod_{a,b\in\mathbb{N}_0}M(a,b)^{\Lambda_{\vartheta}^{(a,b)}(T)}\right)v\big(T(\vartheta)\big)\right] \tag{4.11}$$

with T independent of M, $\mathbf{E}^a$ denoting the expected value of T^a conditioned on M, and we define $\Lambda_{\vartheta}^{(a,b)}(T)$ in the same way as (4.3).

We now apply this to the study of the matrix model $\hat{K}_n^w(t)$. The entries of $I_n-(-\Delta_n^w+Q_n)/3m_n^2$ are the same as (4.5)–(4.7) except for the $(0,0)$ entry, which is equal to

$$\begin{aligned}\left(I_n-\frac{-\Delta_n^w+Q_n}{3m_n^2}\right)(0,0)&=\frac{1}{3}\left(1+w_n-\frac{D_n(0)}{m_n^2}\right)\\ &=\frac{1}{3}\left(2-(1-w_n)-\frac{D_n(0)}{m_n^2}\right)=\frac{2}{3}\left(1-\frac{(1-w_n)}{2}-\frac{D_n(0)}{2m_n^2}\right).\end{aligned}$$

Therefore, if we let $\vartheta=\vartheta(n,t):=\lfloor m_n^2(3t/2)\rfloor$, then

$$\big\langle f,\hat{K}_n^w(t)g\big\rangle=\int_0^{(n+1)/m_n}f(x)\mathbf{E}^{\lfloor m_nx\rfloor}\left[F_{n,t}^w(T)m_n\int_{T(\vartheta)/m_n}^{(T(\vartheta)+1)/m_n}g(y)\,\mathrm{d}y\right]\mathrm{d}x, \tag{4.12}$$

where T is independent of Q_n, we define the random functional

$$F_{n,t}^w(T):=\mathbf{1}_{\{\tau^{(n)}(T)>\vartheta\}}\left(1-\frac{(1-w_n)}{2}-\frac{D_n(0)}{2m_n^2}\right)^{\Lambda_{\vartheta}^{(0,0)}(T)} \tag{4.13}$$

$$\cdot\left(\prod_{a\in\mathbb{N}}\left(1-\frac{D_n(a)}{m_n^2}\right)^{\Lambda_{\vartheta}^{(a,a)}(T)}\right) \tag{4.14}$$

$$\cdot\left(\prod_{a\in\mathbb{N}_0}\left(1-\frac{U_n(a)}{m_n^2}\right)^{\Lambda_{\vartheta}^{(a,a+1)}(T)}\left(1-\frac{L_n(a)}{m_n^2}\right)^{\Lambda_{\vartheta}^{(a+1,a)}(T)}\right), \tag{4.15}$$

and $\mathbf{E}^{\lfloor m_nx\rfloor}$ is the expected value of $T^{\lfloor m_nx\rfloor}$ conditional on Q_n.

4.3. *A brief comparison with other matrix models*

The assumptions made in Section 2 suggest that $\frac{1}{2}(-\Delta_n+Q_n)\to\hat{H}$ and $\frac{1}{2}(-\Delta_n^w+Q_n)\to\hat{H}^w$ as $n\to\infty$. Thus, by Remark 2.8, we expect that for any sequence of functions $(f_{n;t})_{n\in\mathbb{N}}$ such that $f_{n;t}(x)\to\mathrm{e}^{-tx/2}$ in a suitable sense, one has $f_{n;t}(-\Delta_n+Q_n)\to\hat{K}(t)$ and $f_{n;t}(-\Delta_n^w+Q_n)\to\hat{K}^w(t)$. The difficulty involved in carrying this out rigorously in the generality aimed in this paper is to choose $f_{n;t}$'s that are both amenable to combinatorial analysis and applicable to general tridiagonal models. The main insight of this paper is that the matrix models $\hat{K}_n(t)$ and $\hat{K}_n^w(t)$ (which correspond to the choice $f_{n;t}(x):=(1-x/3m_n^2)^{\lfloor m_n^2(3t/2)\rfloor}$) are in this sense better suited than arguably more "obvious" choices of $f_{n;t}$.

In order to illustrate this claim, we compare our matrix models with $f_{n;t}(x):=(1-x/2m_n^2)^{\lfloor m_n^2t/2\rfloor}$, which is what was used in [14,20], and $f_{n;t}(x):=\mathrm{e}^{-tx/2}$, which is arguably the most straightforward matrix model one could use in order to obtain semigroup limits. We begin with the latter: If Q_n is diagonal, then we can express the matrix exponential $\mathrm{e}^{-t(-\Delta_n+Q_n)/2}$ in terms of a Feynman-Kac formula involving the continuous-time simple random walk on $\mathbb{Z}$ with exponential jump times. This formula is very similar to (2.7) and (2.8) and is arguably easier to work with than (4.9) or (4.13). However, for general tridiagonal Q_n, the Feynman-Kac formula becomes much more unwieldy. In particular, the generator of the associated random walk depends on the entries of Q_n, making a general unified treatment more difficult.

As for the matrix model used in [14,20], we note that

$$\left(I_n - \frac{-\Delta_n + Q_n}{2m_n^2}\right)(a,a) = \frac{-D_n(a)}{m_n^2}, \quad 0 \le a \le n,$$
$$\left(I_n - \frac{-\Delta_n + Q_n}{2m_n^2}\right)(a,a+1) = \frac{1}{2}\left(1 - \frac{U_n(a)}{m_n^2}\right), \quad 0 \le a \le n-1,$$
$$\left(I_n - \frac{-\Delta_n + Q_n}{2m_n^2}\right)(a+1,a) = \frac{1}{2}\left(1 - \frac{L_n(a)}{m_n^2}\right), \quad 0 \le a \le n-1.$$

If $D_n(a) = 0$ for all n and a, then a combinatorial analysis similar to the one performed earlier in this section can relate the above to a functional of simple symmetric random walks on $\mathbb{Z}$ (i.e., i.i.d. uniform ± 1 increments). More generally, if $D_n(a)$ is of smaller order than $m_n^2 - U_n(a)$ and $m_n^2 - L_n(a)$ for large n (e.g., for β-Hermite), then a similar analysis holds, but with additional technical difficulties (see [14, Section 3.1] and [20, Section 3] for the details). However, if $D_n(a)$ is allowed to be of the same order as $m_n^2 - U_n(a)$ and $m_n^2 - L_n(a)$ (e.g., for β-Laguerre), then the analysis of [20] and [14] no longer applies.

5. Strong couplings for Theorem 2.20

Equations (4.8) and (4.10) suggest Theorem 2.20 relies on understanding how Brownian motion and its local time arises as the limit of the lazy random walk and its edge-occupation measures. This is the subject of this section.

Definition 5.1. For every $x \ge 0$, let $\tilde{B}^x$ be a Brownian motion started at x with variance $2/3$, and for every $t > 0$, let $\tilde{B}_t^{x,x} := (\tilde{B}^x | \tilde{B}^x(t) = x)$. We define the local time process for $\tilde{B}$ in the same way as in (2.5).

The main result of this section is the following.

Theorem 5.2. *Let $t > 0$ and $x \ge 0$ be fixed. For every $0 \le s \le t$ and $n \in \mathbb{N}$, let $\vartheta_s = \vartheta_s(n) := \lfloor m_n^2 s \rfloor$ and $x^n := \lfloor m_n x \rfloor$. We use the shorthand $\vartheta := \vartheta_t$. For every $y \in \mathbb{R}$, let $(y_n, \bar{y}_n)_{n\in\mathbb{N}}$ be equal to one of the three sequences*

$$\big(\lfloor m_n y \rfloor, \lfloor m_n y \rfloor\big)_{n\in\mathbb{N}}, \qquad \big(\lfloor m_n y \rfloor, \lfloor m_n y \rfloor + 1\big)_{n\in\mathbb{N}}, \quad \text{or} \quad \big(\lfloor m_n y \rfloor + 1, \lfloor m_n y \rfloor\big)_{n\in\mathbb{N}}. \tag{5.1}$$

Finally, suppose that $(Z_n, Z) = (S^{x^n}, \tilde{B}^x)$, or $(S_\vartheta^{x^n,x^n}, \tilde{B}_t^{x,x})$ for each $n \in \mathbb{N}$. For every $0 < \varepsilon < 1/5$, there exists a coupling of Z_n and Z such that the following holds almost surely as $n \to \infty$

$$\sup_{0\le s\le t} \left| \frac{Z_n(\vartheta_s)}{m_n} - Z(s) \right| = O\big(m_n^{-1} \log m_n\big), \tag{5.2}$$

$$\sup_{0\le s\le t, y\in\mathbb{R}} \left| \frac{\Lambda_{\vartheta_s}^{(y_n,\bar{y}_n)}(Z_n)}{m_n} - \frac{L_s^y(Z)}{3} \right| = O\big(m_n^{-1/5+\varepsilon} \log m_n\big). \tag{5.3}$$

Classical results on strong couplings of local time (such as [3]) concern the *vertex-occupation measures* of a random walk:

$$\Lambda_u^a(S) := \sum_{j=0}^{u} \mathbf{1}_{\{S(j)=a\}}, \quad a \in \mathbb{Z}, u \in \mathbb{N}. \tag{5.4}$$

Indeed, for any measurable $f : \mathbb{R} \to \mathbb{R}$, the vertex-occupation measures satisfy

$$\sum_{j=0}^{u} f\big(S(j)\big) = \sum_{a\in\mathbb{Z}} \Lambda_u^a(S) f(a), \tag{5.5}$$

making a direct comparison with local time more convenient by (2.5). Thus, our strategy of proof for Theorem 5.2 has two steps: We first use standard methods to construct a strong coupling of the vertex-occupation measures of S^{x^n} and $S_\vartheta^{x^n,x^n}$ with the local time of their corresponding continuous processes. Then, we prove that the occupation measure of a given edge (a,b) is very close to a multiple of the occupation measure of the vertices a and b. More precisely:

Proposition 5.3. *For every* $0 < \varepsilon < 1/5$, *there exists a coupling such that*

$$\sup_{0\le s\le t, y\in\mathbb{R}} \left| \frac{\Lambda_{\vartheta_s}^{\lfloor m_n y\rfloor}(Z_n)}{m_n} - L_s^y(Z)\right| = O\big(m_n^{-1/5+\varepsilon}\log m_n\big) \tag{5.6}$$

and (5.2) *hold almost surely as* $n\to\infty$.

Proposition 5.4. *Almost surely, as* $n\to\infty$, *one has*

$$\sup_{\substack{0\le u\le\vartheta\\ a,b\in\mathbb{Z},|a-b|\le 1}} \frac{1}{m_n}\left|\Lambda_u^{(a,b)}(Z_n) - \frac{\Lambda_u^a(Z_n)}{3}\right| = O\big(m_n^{-1/2}\log m_n\big).$$

Notation 5.5. In Propositions 5.3 and 5.4, and the remainder of Section 5, whenever we state a result for Z_n and Z, we mean that the result in question applies to $(Z_n, Z) = (S^{x^n}, \tilde{B}^x)$ and $(S_\vartheta^{x^n,x^n}, \tilde{B}_t^{x,x})$.

5.1. *Condition for strong local time coupling*

We begin with a criterion for local time couplings. The following lemma is essentially the content of the proof of [3, Theorem 3.2]; we provide a full proof since we need a modification of the result in Section 6.

Lemma 5.6. *For any* $0<\delta<1$, *the following holds almost surely as* $n\to\infty$:

$$\begin{aligned}
&\sup_{0\le u\le\vartheta, a\in\mathbb{Z}}\left|\frac{\Lambda_u^a(Z_n)}{m_n} - L_{u/m_n^2}^{a/m_n}(Z)\right| \\
&\quad= O\Bigg(\sup_{0\le s\le t,|y-z|\le m_n^{-\delta}} \big|L_s^y(Z) - L_s^z(Z)\big| \\
&\qquad + m_n^{2\delta}\sup_{0\le s\le t}\left|\frac{Z_n(\vartheta_s)}{m_n} - Z(s)\right| + \sup_{0\le u\le\vartheta,|a-b|\le m_n^{1-\delta}} \frac{|\Lambda_u^a(Z_n)-\Lambda_u^b(Z_n)|}{m_n} + \sup_{0\le u\le\vartheta, a\in\mathbb{Z}} \frac{\Lambda_u^a(Z_n)}{m_n^{2-\delta}}\Bigg).
\end{aligned}$$

Proof. Let $n\in\mathbb{N}$ and $a\in\mathbb{Z}$ be fixed, and for each $\varepsilon>0$, define the function $f_\varepsilon:\mathbb{R}\to\mathbb{R}$ as follows.

1. $f_\varepsilon(a/m_n) = 1/\varepsilon$;
2. $f_\varepsilon(z) = 0$ whenever $|z - a/m_n| > \varepsilon$; and
3. define $f_\varepsilon(z)$ by linear interpolation for $|z-a/m_n|\le\varepsilon$.

Since f_ε integrates to one, for every $0\le u\le\vartheta$, we have that

$$\begin{aligned}
\left|\int_0^{u/m_n^2} f_\varepsilon\big(Z(s)\big)\,\mathrm{d}s - L_{u/m_n^2}^{a/m_n}(Z)\right| &= \left|\int_{\mathbb{R}} f_\varepsilon(y)\big(L_{u/m_n^2}^y(Z) - L_{u/m_n^2}^{a/m_n}(Z)\big)\,\mathrm{d}y\right| \\
&\le \sup_{|y-a/m_n|\le\varepsilon}\big|L_{u/m_n^2}^y(Z) - L_{u/m_n^2}^{a/m_n}(Z)\big|.
\end{aligned}$$

Note that $|f_\varepsilon(z)-f_\varepsilon(y)|/|z-y|\le\frac{1}{\varepsilon^2}$ for all $z,y\in\mathbb{R}$; hence, for every $0\le u\le\vartheta$,

$$\begin{aligned}
\left|\int_0^{u/m_n^2} f_\varepsilon\big(Z(s)\big)\,\mathrm{d}s - \frac{1}{m_n^2}\sum_{j=1}^u f_\varepsilon\big(Z_n(j)/m_n\big)\right| &= \left|\int_0^{u/m_n^2} f_\varepsilon\big(Z(s)\big) - f_\varepsilon\big(Z_n(\vartheta_s)/m_n\big)\,\mathrm{d}s\right| \\
&\le \frac{t}{\varepsilon^2}\sup_{0\le s\le u/m_n^2}\left|\frac{Z_n(\vartheta_s)}{m_n} - Z(s)\right|.
\end{aligned}$$

Finally,

$$\frac{1}{m_n^2}\sum_{j=1}^{u} f_\varepsilon\big(Z_n(j)/m_n\big) - \frac{\Lambda_u^a(Z_n)}{m_n} = \frac{1}{m_n^2}\sum_{b\in\mathbb{Z}} f_\varepsilon(b/m_n)\Lambda_u^b(Z_n) - \frac{\Lambda_u^a(Z_n)}{m_n}$$

$$= \frac{1}{m_n}\sum_{b\in\mathbb{Z}} f_\varepsilon(b/m_n)\frac{(\Lambda_u^b(Z_n)-\Lambda_u^a(Z_n))}{m_n} + \frac{\Lambda_u^a(Z_n)}{m_n}\bigg(\frac{1}{m_n}\sum_{b\in\mathbb{Z}} f_\varepsilon(b/m_n) - 1\bigg).$$

By a Riemann sum approximation,

$$\bigg|\frac{1}{m_n}\sum_{b\in\mathbb{Z}} f_\varepsilon(b/m_n) - 1\bigg| = O\bigg(\frac{1}{\varepsilon m_n}\bigg),$$

and thus we conclude that

$$\frac{1}{m_n^2}\sum_{j=1}^{u} f_\varepsilon\big(Z_n(j)/m_n\big) - \frac{\Lambda_u^a(Z_n)}{m_n} = O\bigg(\sup_{|a-b|\le\varepsilon m_n}\frac{|\Lambda_u^b(Z_n)-\Lambda_u^a(Z_n)|}{m_n} + \frac{\Lambda_u^a(Z_n)}{\varepsilon m_n^2}\bigg).$$

The result then follows by taking a supremum over $0\le u\le\vartheta$ and $a\in\mathbb{Z}$, and taking $\varepsilon=\varepsilon(n)=m_n^{-\delta}$. □

5.2. *Proof of Proposition* 5.3

We begin with the proof of (5.2):

Lemma 5.7. *There exists a coupling such that* (5.2) *holds. In particular, for any* $0<\delta<1/2$, *it holds almost surely as* $n\to\infty$ *that*

$$m_n^{2\delta}\sup_{0\le s\le t}\bigg|\frac{Z_n(\vartheta_s)}{m_n} - Z(s)\bigg| = O\big(m_n^{-1+2\delta}\log m_n\big).$$

Proof. Suppose first that $x=0$ so that $(Z_n,Z)=(S^0,\tilde B^0)$ or $(S_\vartheta^{0,0},\tilde B_t^{0,0})$. According to the classical KMT coupling (e.g., [24, Section 7]) for Brownian motion and its extension to the Brownian bridge (e.g., [7, Theorem 2]), it holds that

$$\sup_{0\le u\le\vartheta}\bigg|\frac{Z_n(u)}{m_n} - Z\big(u/m_n^2\big)\bigg| = O\big(m_n^{-1}\log m_n\big)$$

almost surely. Thus it only remains to prove that

$$\sup_{0\le s\le t}\big|Z\big(\vartheta_s/m_n^2\big) - Z(s)\big| = O\big(m_n^{-1}\log m_n\big).$$

For $Z=\tilde B^0$, this is Lévy's modulus of continuity theorem. For $Z=\tilde B_t^{0,0}$, we note that the laws of $(\tilde B_t^{0,0}(s))_{s\in[0,t/2]}$ and $(\tilde B_t^{0,0}(t-s))_{s\in[0,t/2]}$ are absolutely continuous with respect the the law of $(\tilde B^0(s))_{s\in[0,t/2]}$.

Suppose now that $x>0$. We can define $S^{x^n}:=x^n+S^0$ and $S_\vartheta^{x^n,x^n}:=x^n+S_\vartheta^{0,0}$, and similarly for $\tilde B$. Since $x^n/m_n = x+O(m_n^{-1})$, our proof in the case $x=0$ yields

$$\sup_{0\le s\le t}\bigg|\frac{S^{x^n}(\vartheta_s)}{m_n} - \tilde B^x(s)\bigg| = O\big(m_n^{-1}\log n + m_n^{-1}\big)$$

and similarly for the bridge, as desired. □

With (5.2) established, the proof (5.6) is a straightforward application of Lemma 5.6:

Lemma 5.8. *For every* $\delta>0$ *and* $0<\varepsilon<\delta/2$,

$$\sup_{0\le s\le t,\,|y-z|\le m_n^{-\delta}}\big|L_s^y(Z) - L_s^z(Z)\big| = O\big(m_n^{-\delta/2+\varepsilon}\log m_n\big)$$

almost surely as $n\to\infty$.

Proof. The result for $\tilde{B}^x$ is a direct application of [3, Equation (3.7)] (see also [31, (2.1)]). We obtain the same result for $\tilde{B}_t^{x,x}$ by the absolute continuity of $(\tilde{B}_t^{x,x}(s))_{s\in[0,t/2]}$ and $(\tilde{B}_t^{x,x}(t-s))_{s\in[0,t/2]}$ with respect to $(\tilde{B}^x(s))_{s\in[0,t/2]}$, and the fact that local time is additive and invariant under time reversal. □

Lemma 5.9. *For every* $\delta > 0$ *and* $0 < \varepsilon < \delta/2$,

$$\sup_{0\le u\le\vartheta,|a-b|\le m_n^{1-\delta}} \frac{|\Lambda_u^a(Z_n) - \Lambda_u^b(Z_n)|}{m_n} = O\big(m_n^{-\delta/2+\varepsilon}\log m_n\big)$$

almost surely as $n\to\infty$.

Proof. According to [3, Proposition 3.1], for every $0<\eta<1/2$, it holds that

$$\mathbf{P}\left[\sup_{0\le u\le\vartheta,a,b\in\mathbb{Z}} \frac{m_n^{-1}|\Lambda_u^a(S^{x^n}) - \Lambda_u^b(S^{x^n})|}{(|a/m_n - b/m_n|^{1/2-\eta}\wedge 1)} \ge \lambda\right] = O\big(\mathrm{e}^{-c\lambda} + m_n^{-14}\big) \tag{5.7}$$

for every $\lambda>0$, where $c>0$ is independent of n and λ. We recall that $m_n \asymp n^{\mathfrak{d}}$ with $1/13<\mathfrak{d}$, which implies in particular that m_n^{-14} is summable in n. Thus, if we take $\lambda=\lambda(n)=C\log m_n$ for a large enough $C>0$, then Borel–Cantelli yields

$$\sup_{0\le u\le\vartheta,|a-b|\le m_n^{1-\delta}} \frac{|\Lambda_u^a(S^{x^n}) - \Lambda_u^b(S^{x^n})|}{m_n} = O\big(m_n^{\delta(\eta-1/2)}\log m_n\big)$$

almost surely, proving the result for $Z_n = S^{x^n}$. In order to extend the result to $Z_n = S_\vartheta^{x^n,x^n}$ we apply the local CLT (i.e., $\mathbf{P}[S^{x^n}(\vartheta)=x^n]^{-1} = O(m_n)$; e.g., [15, §49]) with the elementary inequality $\mathbf{P}[E_1|E_2]\le\mathbf{P}[E_1]/\mathbf{P}[E_2]$ to (5.7):

$$\mathbf{P}\left[\sup_{0\le u\le\vartheta,a,b\in\mathbb{Z}} \frac{m_n^{-1}|\Lambda_u^a(S_\vartheta^{x^n,x^n}) - \Lambda_u^b(S_\vartheta^{x^n,x^n})|}{(|a/m_n - b/m_n|^{1/2-\eta}\wedge 1)} \ge \lambda\right] = O\big(m_n\mathrm{e}^{-c_2\lambda} + m_n^{-13}\big)$$

for all $\lambda>0$. Since $\sum_n m_n^{-13}<\infty$ the result follows by Borel–Cantelli. □

Lemma 5.10. *For every* $0<\delta<1$, *it holds almost surely as* $n\to\infty$ *that*

$$\sup_{0\le u\le\vartheta,a\in\mathbb{Z}} \frac{\Lambda_u^a(Z_n)}{m_n^{2-\delta}} = O\big(m_n^{-1+\delta}\log m_n\big).$$

Proof. Note that, for any $n,u\in\mathbb{N}$, $|S^{x^n}(u)|\le|x^n|+O(m_n^2)$ Therefore, by taking a large b in (5.7) (i.e., large enough so that $\Lambda_\vartheta^b(S^{x^n})=0$ surely), we see that

$$\mathbf{P}\left[\sup_{0\le u\le\vartheta,a\in\mathbb{Z}} \frac{\Lambda_u^a(S^{x^n})}{m_n} \ge \lambda\right] = O\big(\mathrm{e}^{-C\lambda} + m_n^{-14}\big) \tag{5.8}$$

for all $\lambda>0$. The proof then follows from the same arguments as in Lemma 5.9. □

By combining Lemmas 5.6–5.10, we obtain that

$$\sup_{0\le u\le\vartheta,a\in\mathbb{Z}}\left|\frac{\Lambda_u^a(Z_n)}{m_n} - L_{u/m_n^2}^{a/m_n}(Z)\right| = O\big(m_n^{\mathfrak{t}}\log m_n\big),$$

where, for every $0<\delta<1/2$ and $0<\varepsilon<\delta/2$, we have

$$\mathfrak{t}=\mathfrak{t}(\delta,\varepsilon) := \max\{-1+2\delta, -\delta/2+\varepsilon\}.$$

For any fixed $\varepsilon>0$, the smallest possible $\mathfrak{t}(\delta,\varepsilon)$ occurs at the intersection of the lines $\delta\mapsto -1+2\delta$ and $\delta\mapsto -\delta/2+\varepsilon$. This is attained at $\delta=2(1+\varepsilon)/5$, in which case $\mathfrak{t}=-1/5+4\varepsilon/5$. At this point, in order to get the statement of Proposition 5.3, we must show that

$$\sup_{0\le s\le t,y\in\mathbb{R}}\big|L_{\vartheta_s/m_n^2}^{\lfloor m_n y\rfloor/m_n}(Z) - L_s^y(Z)\big| = O\big(m_n^{-1/5+\varepsilon}\log m_n\big)$$

as $n \to \infty$ for any $\varepsilon > 0$. This follows by a combination of Lemma 5.8 with $\delta = 1$ and the estimate [31, (2.3)], which yields

$$\sup_{0 \le s, \bar{s} \le t, |s-\bar{s}| \le m_n^{-2}, y \in \mathbb{R}} \left| L_s^y(Z) - L_{\bar{s}}^y(Z) \right| = O\left(m_n^{-2/3} (\log m_n)^{2/3}\right).$$

(The results of [31] are only stated for the Brownian motion, but this can be extended to the Bridge by the absolute continuity argument used in Lemma 5.8.)

5.3. *Proof of Proposition* 5.4

We may assume without loss of generality that $x = 0$. We begin with the case of the unconditioned random walk $Z_n = S^0$.

Let $(\zeta_n^a)_{n \in \mathbb{N}_0, a \in \mathbb{Z}}$ be a collection of i.i.d. random variables with uniform distribution on $\{-1, 0, 1\}$. We can define the random walk S^0 as follows: For every $u, v \in \mathbb{N}$ and $a \in \mathbb{Z}$, if $S^0(u) = a$ and $\Lambda_u^a(S^0) = v$, then $S^0(u+1) = S^0(u) + \zeta_v^a$. In doing so, up to an error of at most 1, it holds that

$$\Lambda_n^{(a,b)}(S^0) = \sum_{j=1}^{\Lambda_n^a(S^0)} \mathbf{1}_{\{\zeta_j^a = b-a\}}, \quad a, b \in \mathbb{Z}.$$

Hence, by the Borel–Cantelli lemma, it is enough to show that for any $z \in \{-1, 0, 1\}$,

$$\sum_{n \in \mathbb{N}} \mathbf{P}\left[\sup_{0 \le u \le \vartheta, a \in \mathbb{Z}} \left| \sum_{j=1}^{\Lambda_u^a(S^0)} \mathbf{1}_{\{\zeta_j^a = z\}} - \frac{\Lambda_u^a(S^0)}{3} \right| \ge C m_n^{1/2} \log m_n \right] < \infty \tag{5.9}$$

for some suitable finite constant $C > 0$. In order to prove this, we need two auxiliary estimates. Let us denote the range of a random walk by

$$\mathcal{R}_u(S) := \max_{0 \le j \le u} S(j) - \min_{0 \le j \le u} S(j), \quad u \in \mathbb{N}_0. \tag{5.10}$$

Lemma 5.11. *For every* $\varepsilon > 0$,

$$\sum_{n \in \mathbb{N}} \mathbf{P}\left[\mathcal{R}_\vartheta(S^0) \ge m_n^{1+\varepsilon}\right] < \infty.$$

Proof. According to [9, (6.2.3)], there exists $C > 0$ independent of n such that

$$\mathbf{E}\left[\mathcal{R}_\vartheta(S^0)^q\right] \le (C m_n)^q \sqrt{q!}, \quad q \in \mathbb{N}_0. \tag{5.11}$$

Consequently, for every $r < 2$ and $C > 0$,

$$\sup_{n \in \mathbb{N}} \mathbf{E}\left[\mathrm{e}^{C(\mathcal{R}_\vartheta(S^0)/m_n)^r}\right] < \infty, \tag{5.12}$$

The result then follows from Markov's inequality. □

Lemma 5.12. *If* $C > 0$ *is large enough*,

$$\sum_{n \in \mathbb{N}} \mathbf{P}\left[\sup_{a \in \mathbb{Z}} \Lambda_\vartheta^a(S^0) \ge C m_n \log m_n\right] < \infty.$$

Proof. This follows directly from (5.8) since $\sum_n m_n^{-14} < \infty$. □

According to Lemmas 5.11 and 5.12, to prove (5.9), it is enough to consider the sum of probabilities in question intersected with the events

$$D_n := \left\{ \mathcal{R}_\vartheta(S^0) \le m_n^{1+\varepsilon}, \sup_{0 \le u \le \vartheta, a \in \mathbb{Z}} \Lambda_n^a(S^0) \le C m_n \log m_n \right\}$$

for some large enough $C > 0$. By a union bound,

$$\begin{aligned}
&\mathbf{P}\left[\left\{\sup_{0\le u\le\vartheta, a\in\mathbb{Z}}\left|\sum_{u=1}^{\Lambda_u^a(S^0)}\mathbf{1}_{\{\zeta_j^a=z\}}-\frac{\Lambda_u^a(S^0)}{3}\right|\ge cm_n^{1/2}\log m_n\right\}\cap D_n\right]\\
&\quad\le \mathbf{P}\left[\left\{\max_{\substack{-m_n^{1+\varepsilon}\le a\le m_n^{1+\varepsilon}\\ 0\le h\le Cm_n\log m_n}}\left|\sum_{j=1}^{h}\mathbf{1}_{\{\zeta_j^a=z\}}-\frac{h}{3}\right|\ge cm_n^{1/2}\log m_n\right\}\cap D_n\right]\\
&\quad\le \sum_{\substack{-m_n^{1+\varepsilon}\le a\le m_n^{1+\varepsilon}\\ 0\le h\le Cm_n\log m_n}}\mathbf{P}\left[\left|\sum_{j=1}^{h}\mathbf{1}_{\{\zeta_j^a=z\}}-\frac{h}{3}\right|\ge cm_n^{1/2}\log m_n\right]. \qquad (5.13)
\end{aligned}$$

By Hoeffding's inequality,

$$\mathbf{P}\left[\left|\sum_{j=1}^{h}\mathbf{1}_{\{\zeta_j^a=z\}}-\frac{h}{3}\right|\ge \bar{C}m_n^{1/2}\log m_n\right]\le 2\mathrm{e}^{-2\bar{C}^2\log m_n/C}$$

uniformly in $0\le h\le Cm_n\log m_n$. Since the sum in (5.13) involves a polynomially bounded number of summands in m_n and the latter grows like a power of n,

$$\text{for any } q>0,\quad \text{we can choose } \bar{C}>0 \text{ so that (5.9) is of order } O(n^{-q}). \qquad (5.14)$$

This concludes the proof of Proposition 5.4 in the case $Z_n = S^0$ by Borel–Cantelli.

In order to extend the result to the case $Z_n = S_\vartheta^{x^n,x^n}$, it suffices to prove that (5.9) holds with the additional conditioning $\{S^0(\vartheta)=0\}$. The same local limit theorem argument used at the end of the proof of Lemma 5.9 implies that

$$\begin{aligned}
&\mathbf{P}\left[\sup_{0\le u\le\vartheta, a\in\mathbb{Z}}\left|\sum_{j=1}^{\Lambda_u^a(S_\vartheta^{x^n,x^n})}\mathbf{1}_{\{\zeta_j^a=z\}}-\frac{\Lambda_u^a(S_\vartheta^{x^n,x^n})}{3}\right|\ge Cm_n^{1/2}\log m_n\right]\\
&\quad= O\left(m_n\mathbf{P}\left[\sup_{0\le u\le\vartheta, a\in\mathbb{Z}}\left|\sum_{j=1}^{\Lambda_u^a(S^0)}\mathbf{1}_{\{\zeta_j^a=z\}}-\frac{\Lambda_u^a(S^0)}{3}\right|\ge Cm_n^{1/2}\log m_n\right]\right).
\end{aligned}$$

The result then follows from (5.14) by taking a large enough q.

6. Strong coupling for Theorem 2.21

We now provide the counterpart of Theorem 5.2 for the Markov chain T in Definition 4.2 that is needed for Theorem 2.21.

Definition 6.1. Let $\tilde{X}$ be a reflected Brownian motion on $\mathbb{R}_+$ with variance $2/3$. For every $x\ge 0$, we denote $\tilde{X}^x := (\tilde{X}|\tilde{X}(0)=x)$, and we define the local time and the boundary local time of $\tilde{X}$ as in (2.5) and (2.6), respectively.

Our main result in this section is the following.

Theorem 6.2. *Let $t>0$ and $x\ge 0$ be fixed. Let ϑ, ϑ_s $(0\le s\le t)$, x^n, and $(y_n,\bar{y}_n)$ $(y>0)$ be as in Theorem 5.2. For every $0<\varepsilon<1/5$, there exists a coupling of T^{x^n} and $\tilde{X}^x$ such that*

$$\sup_{0\le s\le t}\left|\frac{T^{x^n}(\vartheta_s)}{m_n}-\tilde{X}^x(s)\right| = O\big(m_n^{-1}\log m_n\big), \qquad (6.1)$$

$$\left|\frac{\Lambda_\vartheta^{(0,0)}(T^{x^n})}{m_n}-\frac{4\mathfrak{L}_t^0(\tilde{X}^x)}{3}\right| = O\big(m_n^{-1/2}(\log m_n)^{3/4}\big), \qquad (6.2)$$

$$\sup_{0\le s\le t, y>0}\left|\frac{\Lambda_{\vartheta_s}^{(y_n,\bar{y}_n)}(T^{x^n})}{m_n}\left(1-\frac{1}{2}\mathbf{1}_{\{(y_n,\bar{y}_n)=(0,0)\}}\right)-\frac{L_s^y(\tilde{X}^x)}{3}\right| = O\big(m_n^{-1/5+\varepsilon}\log m_n\big) \qquad (6.3)$$

almost surely as $n\to\infty$.

Remark 6.3. In contrast with Theorem 5.2, Theorem 6.2 does not include a strong invariance result for the T's bridge process $T_\vartheta^{x^n,x^n}$. We discuss this omission (and state a related conjecture) in Section 6.5 below.

The first step in the proof for Theorem 6.2 is to use a modification of the Skorokhod reflection trick developed in [14, Section 2] to reduce (6.1) to the KMT coupling stated in (5.2). As it turns out, this step also provides a proof of (6.2). The second step is to introduce a suitable modification of Lemma 5.6 that provides a criterion for the strong convergence of the vertex-occupation measures of T with the local time of $\tilde{X}$. The third step is to prove an analog of Proposition 5.4. We summarize the last two steps in the following propositions:

Proposition 6.4. *Almost surely, as $n \to \infty$, one has*

$$\sup_{\substack{0 \le u \le \vartheta \\ (a,b) \in \mathbb{N}_0^2 \setminus \{(0,0)\}, |a-b| \le 1}} \frac{1}{m_n} \left| \Lambda_u^{(a,b)}\big(T^{x^n}\big) - \frac{\Lambda_u^a(T^{x^n})}{3} \right| = O\big(m_n^{-1/2} \log m_n\big)$$

and

$$\sup_{0 \le u \le \vartheta} \frac{1}{m_n} \left| \Lambda_u^{(0,0)}\big(T^{x^n}\big) - \frac{2\Lambda_u^0(T^{x^n})}{3} \right| = O\big(m_n^{-1/2} \log m_n\big).$$

Proposition 6.5. *For every $0 < \varepsilon < 1/5$, under the same coupling as (6.1), it holds almost surely as $n \to \infty$ that*

$$\sup_{0 \le s \le t, y > 0} \left| \frac{\Lambda_{\vartheta_s}^{\lfloor m_n y \rfloor}(T^{x^n})}{m_n} - L_s^y\big(\tilde{X}^x\big) \right| = O\big(m_n^{-1/5+\varepsilon} \log m_n\big).$$

6.1. *Proof of* (6.1)

Definition 6.6 (Skorokhod map). Let $Z = (Z(t))_{t \ge 0}$ be a continuous-time stochastic process. We define the Skorokhod map of Z, denoted Γ_Z, as the process

$$\Gamma_Z(t) := Z(t) + \sup_{s \in [0,t]} \big(-Z(s)\big)_+, \quad t \ge 0,$$

where $(\cdot)_+ := \max\{0, \cdot\}$ denotes the positive part of a real number.

Notation 6.7. In the sequel, whenever we discuss the Skorokhod map of the random walk S, Γ_S, we mean the Skorokhod map applied to the continuous-time process $s \mapsto S(\vartheta_s)$ for $0 \le s \le t$.

Note that $Z \mapsto \Gamma_Z$ is 2-Lipschitz with respect to the supremum norm on compact time intervals. Therefore, (6.1) is a direct consequence of (5.2) if we provide couplings (T, S) and $(\tilde{X}, \bar{B})$ such that $T^{x_n}(\vartheta_s) = \Gamma_{S^{x_n}}(s)$ and $\tilde{X}^x(s) = \Gamma_{\bar{B}^x}(s)$.

Let us begin with the coupling of $\tilde{X}^x$ and $\bar{B}^x$. Note that we can define $\tilde{X}^x := |\bar{B}^x|$, where $\bar{B}$ is a Brownian motion with variance $2/3$. Since the quadratic variation of $\bar{B}^x$ is $t \mapsto (2/3)t$, it follows from Tanaka's formula that

$$\tilde{X}^x(t) = x + \int_0^t \operatorname{sgn}\big(\bar{B}^x(s)\big)\, \mathrm{d}\bar{B}^x(s) + \frac{2\mathfrak{L}_t^0(\bar{B}^x)}{3}, \quad t \ge 0$$

(e.g., [29, Chapter VI, Theorem 1.2 and Corollary 1.9]), where

$$\mathfrak{L}_t^0\big(\bar{B}^x\big) := \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{-\varepsilon < \bar{B}^x(s) < \varepsilon\}}\, \mathrm{d}s = \mathfrak{L}_t^0\big(\tilde{X}^x\big).$$

If we define

$$\tilde{B}_t^x := x + \int_0^t \operatorname{sgn}\big(\bar{B}^0(s)\big)\, \mathrm{d}\bar{B}^0(s), \quad t \ge 0,$$

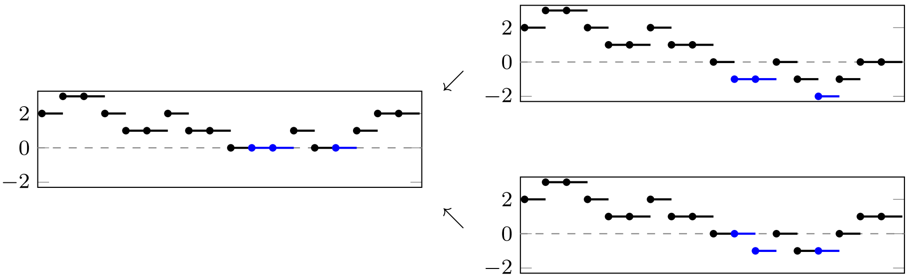

Fig. 1. On the left is a step function $A \in \mathcal{C}_+$ (where $x^n = 2$). the segments contributing to $\mathcal{H}_0(A)$ are blue. On the right are two (out of $2^{\mathcal{H}_0(S)} = 8$) step functions $\tilde{A} \in \mathcal{C}$ such that $\Gamma_{\tilde{A}} = A$.

which is a Brownian motion with variance $2/3$ started at x, then we get from [29, Chapter VI, Lemma 2.1 and Corollary 2.2] that $\tilde{X}_t^x = \Gamma_{\tilde{B}^x}(t)$ and

$$(2/3)\mathfrak{L}_t^0(\tilde{X}^x) = \sup_{s\in[0,t]} \big(-\tilde{B}^x(s)\big)_+ \tag{6.4}$$

for every $t \geq 0$, as desired.

We now provide the coupling of T^{x^n} and S^{x^n}. (See Figure 1 for an illustration of the procedure we are about to describe.) Let $\mathcal{C}$ be the set of step functions of the form

$$A(s) = \sum_{u=0}^{\vartheta} A_u \mathbf{1}_{[u,u+1)}(s), \tag{6.5}$$

where $A_0, A_1, \dots, A_\vartheta \in \mathbb{Z}$ are such that $A_0 = x^n$ and $A_{u+1} - A_u \in \{-1, 0, 1\}$ for all u. Let $\mathcal{C}_+ \subset \mathcal{C}$ be the subset of such functions that are nonnegative. For every $A \in \mathcal{C}$, let us define

$$\mathcal{H}_0(A) := \sum_{u=0}^{\vartheta-1} \mathbf{1}_{\{A_u = A_{u+1} = 0\}}. \tag{6.6}$$

By definition of S and T, we see that for any $A \in \mathcal{C}$,

$$\mathbf{P}\big[\big(S^{x^n}(\vartheta_s)\big)_{0\leq s\leq t} = \big(A(\vartheta_s)\big)_{0\leq s\leq t}\big] = \frac{1}{3^\vartheta},$$
$$\mathbf{P}\big[\big(T^{x^n}(\vartheta_s)\big)_{0\leq s\leq t} = \big(A(\vartheta_s)\big)_{0\leq s\leq t}\big] = \frac{2^{\mathcal{H}_0(A)}}{3^\vartheta}\mathbf{1}_{\{A\in\mathcal{C}_+\}}.$$

It is clear that $A \mapsto \Gamma_A$ maps $\mathcal{C}$ to $\mathcal{C}_+$ and that this map is surjective since $\Gamma_A = A$ for any $A \in \mathcal{C}_+$. Thus, in order to construct a coupling such that $T^{x_n}(\vartheta_s) = \Gamma_{S^{x_n}}(s)$, it suffices to show that for every $A \in \mathcal{C}_+$, there are exactly $2^{\mathcal{H}_0(A)}$ distinct functions $\tilde{A} \in \mathcal{C}$ such that $\Gamma_{\tilde{A}} = A$.

Let $A \in \mathcal{C}_+$. If $\mathcal{H}_0(A) = 0$, then there is no $\tilde{A} \neq A$ such that $\Gamma_{\tilde{A}} = A$, as desired. Suppose then that $\mathcal{H}_0(A) = h > 0$. Let $0 \leq u_1, \dots, u_h \leq \vartheta - 1$ be the integer coordinates such that $A_{u_j} = A_{u_j+1} = 0$, $1 \leq j \leq h$. Then, $\Gamma_{\tilde{A}} = A$ if and only if the following conditions hold:

1. $\tilde{A}_{u_j+1} - \tilde{A}_{u_j} = 0$ or $\tilde{A}_{u_j+1} - \tilde{A}_{u_j} = -1$ for all $1 \leq j \leq h$, and
2. $\tilde{A}_{u+1} - \tilde{A}_u = A_{u+1} - A_u$ for all integers u such that $u \notin \{u_1, \dots, u_h\}$.

Note that, up to choosing whether the increments $\tilde{A}_{u_j+1} - \tilde{A}_{u_j}$ $(1 \leq j \leq h)$ are equal to 0 or -1, the above conditions completely determine $\tilde{A}$. Moreover, there are 2^h ways of choosing these increments, each of which yields a different $\tilde{A}$. Therefore, there are $2^{\mathcal{H}_0(A)}$ distinct functions $\tilde{A} \in \mathcal{C}$ such that $\Gamma_{\tilde{A}} = A$, as desired.

6.2. *Proof of* (6.2)

Since the map $Z \mapsto \sup_{s\in[0,t]}(-Z(s))_+$ is Lipschitz with respect to the supremum norm on $[0,t]$, if we prove that the coupling of T and S introduced in Section 6.1 is such that

$$\left|\frac{\Lambda_\vartheta^{(0,0)}(T^{x^n})}{m_n} - 2\max_{0\le s\le t}\frac{(-S^{x^n}(\vartheta_s))_+}{m_n}\right| = O\big(m_n^{-1/2}(\log m_n)^{3/4}\big) \tag{6.7}$$

almost surely as $n\to\infty$, then (6.2) is proved by a combination of (5.2) and (6.4).

Note that if $T^{x^n}(\vartheta_s)=A(\vartheta_s)$ for $s\le t$, where $A\in\mathcal{C}_+$ is a step function of the form (6.5), then $\Lambda_\vartheta^{(0,0)}(T^{x^n})=\mathcal{H}_0(A)$, as defined in (6.6). By analyzing the construction of the coupling of T and S in Section 6.1, we see that, conditional on the event $\{\Lambda_\vartheta^{(0,0)}(T^{x^n})=h\}$ ($h\in\mathbb{N}_0$), the quantity $\max_{0\le s\le t}(-S^{x^n}(\vartheta_s))_+$ is a binomial random variable with h trials and probability $1/2$. With this in mind, our strategy is to prove (6.7) using a binomial concentration bound similar to (5.13). For this, we need a good control on the tails of $\Lambda_\vartheta^{(0,0)}(T^{x^n})$:

Proposition 6.8. *There exists constants $C,c>0$ independent of n such that for every $y\ge 0$,*

$$\sup_{n\in\mathbb{N},x\ge 0}\mathbf{P}\big[\Lambda_\vartheta^{(0,0)}\big(T^{x^n}\big)\ge m_n y\big]\le C\mathrm{e}^{-cy^2}.$$

In particular, there exists $C>0$ large enough so that

$$\sum_{n\in\mathbb{N}}\mathbf{P}\big[\Lambda_\vartheta^{(0,0)}\big(T^{x^n}\big)\ge Cm_n\sqrt{\log m_n}\big]<\infty. \tag{6.8}$$

Indeed, with this result in hand, we obtain by Hoeffding's inequality that

$$\mathbf{P}\Big[\Big|(h/2)-\max_{0\le s\le t}\big(-S^{x^n}(\vartheta_s)\big)_+\Big|\ge \tilde{C}m_n^{1/2}(\log m_n)^{3/4}/2\big|\Lambda_\vartheta^{(0,0)}\big(T^{x^n}\big)=h\Big]\le 2\mathrm{e}^{-\tilde{C}^2\log m_n/2C}$$

uniformly in $0\le h\le Cm_n\sqrt{\log m_n}$. By taking $\tilde{C}>0$ large enough, we conclude that (6.2) holds by an application of the Borel–Cantelli lemma combined with (6.8).

Proof of Proposition 6.8. Let T and S be coupled as in Section 6.1, and let

$$\mu_\vartheta(S):=\sum_{u=0}^{\vartheta-1}\mathbf{1}_{\{S(u+1)\le\min\{S(0),S(1),\dots,S(u)\}\}},$$

that is, the number of times that S is smaller or equal to its running minimum over the first ϑ steps. Then, we see that

$$\Lambda_\vartheta^{(0,0)}(T)=\sum_{u=0}^{\vartheta-1}\mathbf{1}_{\{S(u)\le 0,S(u+1)\le\min\{S(0),S(1),\dots,S(u)\}\}}\le\mu_\vartheta(S).$$

Given that $\mu_\vartheta(S)$ is independent of S's starting point, it suffices to prove that

$$\sup_{n\in\mathbb{N}}\mathbf{P}\big[\mu_\vartheta\big(S^0\big)\ge m_n y\big]\le C\mathrm{e}^{-cy^2},\quad y\ge 0 \tag{6.9}$$

for some constants $C,c>0$.

If $y>m_n t$, then $m_n y\ge\vartheta$, hence $\mathbf{P}[\mu_\vartheta(S^0)\ge m_n y]=0$. Thus, it suffices to prove (6.9) for $y\le m_n t$. Our proof of this is inspired by [25, Lemma 7]: Let $0=\mathfrak{t}_0<\mathfrak{t}_1<\mathfrak{t}_2<\cdots$ be the weak descending ladder epochs of S^0, that is,

$$\mathfrak{t}_{u+1}:=\min\big\{v>\mathfrak{t}_u:S^0(v)\le S^0(\mathfrak{t}_u)\big\},\quad u\in\mathbb{N}_0.$$

Then, for any $\nu>0$,

$$\begin{aligned}\mathbf{P}\big[\mu_\vartheta\big(S^0\big)\ge m_n y\big]&=\mathbf{P}[\mathfrak{t}_{\lceil m_n y\rceil}\le\vartheta]\le\mathbf{P}\Big[S^0(\mathfrak{t}_{\lceil m_n y\rceil})\ge\min_{0\le u\le\vartheta}S^0(u)\Big]\\&\le\mathbf{P}\big[S^0(\mathfrak{t}_{\lceil m_n y\rceil})\ge-\nu m_n y\big]+\mathbf{P}\Big[\min_{0\le u\le\vartheta}S^0(u)<-\nu m_n y\Big].\end{aligned}$$

On the one hand, we note that $S^0(\mathrm{t}_{\lceil m_n y\rceil})$ is equal in distribution to the sum of $\lceil m_n y\rceil$ i.i.d. copies of $S^0(\mathrm{t}_1)$, which we call the the ladder height of S^0. Moreover, it is easily seen that the ladder height has distribution $\mathbf{P}[S^0(\mathrm{t}_1)=0]=2/3$ and $\mathbf{P}[S^0(\mathrm{t}_1)=-1]=1/3$. In particular, $\mathbf{E}[S^0(\mathrm{t}_{\lceil m_n y\rceil})]=-\lceil m_n y\rceil/3$. Thus, if we choose ν small enough (namely $\nu<1/3$), then by combining Hoeffding's inequality with $m_n\geq y/t$, we obtain

$$\mathbf{P}\big[S^0(\mathrm{t}_{\lceil m_n y\rceil})\geq-\nu m_n y\big]=\mathbf{P}\bigg[S^0(\mathrm{t}_{\lceil m_n y\rceil})+\frac{\lceil m_n y\rceil}{3}\geq-\nu m_n y+\frac{\lceil m_n y\rceil}{3}\bigg]\leq C_1\mathrm{e}^{-c_1 m_n y}\leq C_1\mathrm{e}^{-c_1 y^2/t}$$

for some $C_1,c_1>0$ independent of n.

On the other hand, by Etemadi's and Hoeffding's inequalities,

$$\mathbf{P}\Big[\min_{0\leq u\leq\vartheta}S^0(u)<-\nu m_n y\Big]\leq\mathbf{P}\Big[\max_{0\leq u\leq\vartheta}\big|S^0(u)\big|>\nu m_n y\Big]\leq 3\max_{0\leq u\leq\vartheta}\mathbf{P}\big[\big|S^0(u)\big|>\nu m_n y/3\big]\leq C_2\mathrm{e}^{-c_2 y^2}$$

for some $C_2,c_2>0$ independent of n, concluding the proof of (6.9) for $y\leq m_n t$. □

6.3. *Proof of Proposition* 6.4

By replicating the binomial concentration argument in the proof of Proposition 5.4, it suffices to prove that

$$\sum_{n\in\mathbb{N}}\mathbf{P}\big[\mathcal{R}_\vartheta\big(T^{x^n}\big)\geq m_n^{1+\varepsilon}\big]<\infty \tag{6.10}$$

for every $\varepsilon>0$, where we define $\mathcal{R}_\vartheta(T^{x^n})$ as in (5.10), and

$$\sum_{n\in\mathbb{N}}\mathbf{P}\Big[\sup_{a\in\mathbb{Z}}\Lambda^a_\vartheta\big(T^{x^n}\big)\geq Cm_n\log m_n\Big]<\infty \tag{6.11}$$

provided $C>0$ is large enough. In order to prove this, we introduce another coupling of S and T, which will also be useful later in the paper:

Definition 6.9. Let $a\in\mathbb{N}_0$ be fixed. Given a realization of T^a, let us define the time change $(\tilde\varrho^a(u))_{u\in\mathbb{N}_0}$ as follows:

1. $\tilde\varrho^a(0)=0$.
2. If $T^a(\tilde\varrho^a(u))\neq 0$ or $T^a(\tilde\varrho^a(u)+1)\neq 0$, then $\tilde\varrho^a(u+1)=\tilde\varrho^a(u)+1$.
3. If $T^a(\tilde\varrho^a(u))=0$ and $T^a(\tilde\varrho^a(u)+1)=0$ then we sample

$$\mathbf{P}\big[\tilde\varrho^a(u+1)=\tilde\varrho^a(u)+1\big]=\frac{1}{4}\quad\text{and}\quad\mathbf{P}\big[\tilde\varrho^a(u+1)=\tilde\varrho^a(u)+2\big]=\frac{3}{4},$$

independently of the increments in T^a.

In words, we go through the path of T^a and skip every visit to the self-edge $(0,0)$ independently with probability $3/4$. Then, we define ϱ^a as the inverse of $\tilde\varrho^a$, which is well defined since the latter is strictly increasing.

By a straightforward geometric sum calculation, it is easy to see that we can couple S and T in such a way that

$$T^{x^n}(u)=\big|S^{x^n}\big(\varrho^{x^n}(u)\big)\big|,\quad u\in\mathbb{N}_0. \tag{6.12}$$

For the remainder of the proof of Proposition 6.4 we adopt this coupling.

On the one hand, $\mathcal{R}_\vartheta(T^{x^n})=\mathcal{R}_{\varrho^{x^n}(\vartheta)}(|S^{x^n}|)\leq\mathcal{R}_\vartheta(S^{x^n})$. Thus (6.10) follows directly from Lemma 5.11. On the other hand, for every $a\neq 0$,

$$\Lambda^a_u\big(T^{x^n}\big)=\Lambda^a_{\varrho^{x^n}(\vartheta)}\big(S^{x^n}\big)+\Lambda^{-a}_{\varrho^{x^n}(\vartheta)}\big(S^{x^n}\big)\leq\Lambda^a_\vartheta\big(S^{x^n}\big)+\Lambda^{-a}_\vartheta\big(S^{x^n}\big) \tag{6.13}$$

and

$$\Lambda^0_u\big(T^{x^n}\big)=\Lambda^{(0,0)}_u\big(T^{x^n}\big)+\Lambda^{(0,-1)}_{\varrho^{x^n}(u)}\big(S^{x^n}\big)+\Lambda^{(0,1)}_{\varrho^{x^n}(u)}\big(S^{x^n}\big)\leq\Lambda^{(0,0)}_u\big(T^{x^n}\big)+\Lambda^0_u\big(S^{x^n}\big).$$

Thus (6.11) follows from (6.8) and Lemma 5.12.

6.4. *Proof of Proposition* 6.5

The following extends Lemma 5.6 to T.

Lemma 6.10. *For any* $0<\delta<1$, *the following holds almost surely as* $n\to\infty$:

$$\sup_{0\le u\le\vartheta, a\in\mathbb{N}_0}\left|\frac{\Lambda_u^a(T^{x^n})}{m_n}-L_{u/m_n^2}^{a/m_n}(\tilde{X}^x)\right|$$
$$=O\biggl(\sup_{\substack{0\le s\le t\\ y,z\ge 0,|y-z|\le m_n^{-\delta}}}\bigl|L_s^y(\tilde{X}^x)-L_s^z(\tilde{X}^x)\bigr|$$
$$+m_n^{2\delta}\sup_{0\le s\le t}\left|\frac{T^{x^n}(\vartheta_s)}{m_n}-\tilde{X}^x(s)\right|+\sup_{\substack{0\le u\le\vartheta\\ a,b\in\mathbb{N}_0,|a-b|\le m_n^{1-\delta}}}\frac{|\Lambda_u^a(T^{x^n})-\Lambda_u^b(T^{x^n})|}{m_n}+\sup_{0\le u\le\vartheta,a\in\mathbb{N}_0}\frac{\Lambda_u^a(T^{x^n})}{m_n^{2-\delta}}\biggr).$$

Proof. Let $a\in\mathbb{N}$ be fixed. For every $\varepsilon>0$, let $f_\varepsilon:\mathbb{R}\to\mathbb{R}$ be defined as in the proof of Lemma 5.6, and let us define $g_\varepsilon:\mathbb{R}_+\to\mathbb{R}$ as

$$g_\varepsilon(z):=f_\varepsilon(z)\left(\int_0^\infty f_\varepsilon(z)\,\mathrm{d}z\right)^{-1},\quad z\ge 0.$$

g_ε integrates to one on $\mathbb{R}_+$, and $|g_\varepsilon(z)-g_\varepsilon(y)|/|z-y|\le\frac{2}{\varepsilon^2}$ for $y,z\ge 0$. By repeating the proof of Lemma 5.6 verbatim with g_ε instead of f_ε, we obtain the result. □

We now apply Lemma 6.10. (6.1) yields

$$m_n^{2\delta}\sup_{0\le s\le t}\left|\frac{T^{x^n}(\vartheta_s)}{m_n}-\tilde{X}^x(s)\right|=O\bigl(m_n^{1-2\delta}\log m_n\bigr).$$

As for the regularity of the vertex-occupation measures and local time of T^{x^n} and $\tilde{X}^x$, they follow directly from the proof of Proposition 5.3 using Lemma 5.6 by applying some carefully chosen couplings of T^{x_n} with S^{x_n}, and $\tilde{X}^x$ with $\tilde{B}^x$:

We begin with the latter. If we define $\tilde{X}^x(s)=|\tilde{B}^x(s)|$, then for every $y\ge 0$ and $s\ge 0$, we have that $L_s^y(\tilde{X}^x)=L_s^y(\tilde{B}^x)+L_s^{-y}(\tilde{B}^x)$. Consequently,

$$\bigl|L_s^y(\tilde{X}^x)-L_{\bar{s}}^z(\tilde{X}^x)\bigr|\le\bigl|L_s^y(\tilde{B}^x)-L_{\bar{s}}^z(\tilde{B}^x)\bigr|+\bigl|L_s^{-y}(\tilde{B}^x)-L_{\bar{s}}^{-z}(\tilde{B}^x)\bigr|.\tag{6.14}$$

The regularity estimates for $L_s^y(\tilde{X}^x)$ then follow from the same results for $L_s^y(\tilde{B}^x)$.

To prove the desired estimates on the occupation measures, we use the coupling introduced in Definition 6.9. This immediately yields an adequate control of the supremum of $\Lambda_\vartheta^a(T^{x^n})$ by (6.11). As for regularity, one the one hand, we note that

$$\bigl|\Lambda_u^a(T^{x^n})-\Lambda_u^b(T^{x^n})\bigr|\le\bigl|\Lambda_{\varrho^{x^n}(u)}^a(S^{x^n})-\Lambda_{\varrho^{x^n}(u)}^b(S^{x^n})\bigr|+\bigl|\Lambda_{\varrho^{x^n}(u)}^{-a}(S^{x^n})-\Lambda_{\varrho^{x^n}(u)}^{-b}(S^{x^n})\bigr|$$

for any $a,b\ne 0$. On the other hand, for any $a\ne 0$,

$$\bigl|\Lambda_u^0(T^{x^n})-\Lambda_u^a(T^{x^n})\bigr|\le\left|\frac{1}{2}\Lambda_u^0(T^{x^n})-\Lambda_{\varrho^{x^n}(u)}^a(S^{x^n})\right|+\left|\frac{1}{2}\Lambda_u^0(T^{x^n})-\Lambda_{\varrho^{x^n}(u)}^{-a}(S^{x^n})\right|.$$

Hence we get the desired estimate by Lemma 5.9 if we prove that

$$\sup_{0\le u\le\vartheta}\left|\frac{1}{2}\Lambda_u^0(T^{x^n})-\Lambda_{\varrho^{x^n}(u)}^0(S^{x^n})\right|=O\bigl(m_n^{-1/2}\log m_n\bigr)\tag{6.15}$$

almost surely as $n\to\infty$. By Propositions 5.4 and 6.4, (6.15) can be reduced to

$$\sup_{0\le u\le\vartheta}3\left|\frac{1}{4}\Lambda_u^{(0,0)}(T^{x^n})-\Lambda_{\varrho^{x^n}(u)}^{(0,0)}(S^{x^n})\right|=O\bigl(m_n^{-1/2}\log m_n\bigr).\tag{6.16}$$

By Definition 6.9, conditional on $\Lambda_u^{(0,0)}(T^{x^n})$, we note that $\Lambda_{\varrho^{x^n}(u)}^{(0,0)}(S^{x^n})$ is a binomial random variable with $\Lambda_u^{(0,0)}(T^{x^n})$ trials and probability $1/4$. Hence we obtain (6.16) by combining (6.8) with Hoeffding's inequality similarly to (5.13).

6.5. *Coupling of $T_\vartheta^{a,b}$*

In light of Theorems 5.2 and 6.2, the following conjecture is natural.

Conjecture 6.11. *The statement of Theorem 6.2 holds with every instance of T^{x^n} replaced by $T_{\vartheta_t}^{x^n,x^n}$, and every instance of $\tilde{X}^{x^n}$ replaced by $\tilde{X}_t^{x^n,x^n}$.*

However, if we couple T in S as in Section 6.1, then conditioning on the endpoint of T corresponds to an unwieldy conditioning of the path of S:

$$\mathbf{P}\big[T^a(\vartheta)=a\big]=\mathbf{P}\Big[S^a(\vartheta)=\max_{0\le u\le\vartheta}\big(-S^a(u)\big)_+ +a\Big].$$

There seems to be no existing strong invariance result (such as KMT) applicable to this conditioning. Consequently, it appears that a proof of Conjecture 6.11 relies on a strong invariance result for conditioned random walks that is outside the scope of the current literature, or that it requires an altogether different reduction to a classical coupling (which we were not able to find).

7. Proof of Theorem 2.20-(1)

For the remainder of this section, we fix some times $t_1,\dots,t_k>0$ and uniformly continuous and bounded functions $f_1,g_1,\dots,f_k,g_k$.

7.1. *Step* 1: *Convergence of mixed moments*

Consider a mixed moment

$$\mathbf{E}\Bigg[\prod_{i=1}^k\big\langle f_i,\hat{K}_n(t_i)g_i\big\rangle^{n_i}\Bigg],\quad n_1,\dots,n_k\in\mathbb{N}_0.$$

Up to making some f_i's, g_i's, and t_i's equal to each other and reindexing, there is no loss of generality in writing the above in the form

$$\mathbf{E}\Bigg[\prod_{i=1}^k\big\langle f_i,\hat{K}_n(t_i)g_i\big\rangle\Bigg]. \tag{7.1}$$

By applying Fubini's theorem to (4.8), we can write (7.1) as

$$\int_{[0,(n+1)/m_n)^k}\Bigg(\prod_{i=1}^k f_i(x_i)\Bigg)\mathbf{E}\Bigg[\prod_{i=1}^k F_{n,t_i}\big(S^{i;x_i^n}\big)m_n\int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n}g_i(y)\,\mathrm{d}y\Bigg]\mathrm{d}x_1\cdots\mathrm{d}x_k \tag{7.2}$$

and the corresponding limiting expression as

$$\mathbf{E}\Bigg[\prod_{i=1}^k\big\langle f_i,\hat{K}(t_i)g_i\big\rangle\Bigg]=\int_{\mathbb{R}_+^k}\Bigg(\prod_{i=1}^k f_i(x_i)\Bigg)\mathbf{E}\Bigg[\prod_{i=1}^k\mathbf{1}_{\{\tau_0(B^{i;x_i})>t\}}\mathrm{e}^{-\langle L_t(B^{i;x_i}),Q'\rangle}g_i\big(B^{i;x_i}(t)\big)\Bigg]\mathrm{d}x_1\cdots\mathrm{d}x_k, \tag{7.3}$$

where

1. $\vartheta_i=\vartheta_i(n,t_i):=\lfloor m_n^2(3t_i/2)\rfloor$ for every $n\in\mathbb{N}$ and $1\le i\le k$;
2. $x_i^n:=\lfloor m_nx_i\rfloor$ for every $n\in\mathbb{N}$ and $1\le i\le k$;
3. $S^{1;x_1^n},\dots,S^{k;x_k^n}$ are independent copies of S with respective starting points $x_1^n,\dots,x_k^n$; and
4. $B^{1;x_1},\dots,B^{k;x_k}$ are independent copies of B with respective starting points $x_1,\dots,x_k$.

We further assume that the $S^{i;x_i^n}$ are independent of Q_n, and that the $B^{i;x_i}$ are independent of Q. The proof of moment convergence is based on the following:

Proposition 7.1. *Let $x_1, \ldots, x_n \geq 0$ be fixed. There is a coupling of the $S^{i;x_i^n}$ and $B^{i;x_i}$ such that the following limits hold jointly in distribution over $1 \leq i \leq k$:*

1. $\lim_{n\to\infty} \sup_{0\leq s\leq t_i} |\frac{S^{i;x_i^n}(\lfloor m_n^2(3s/2)\rfloor)}{m_n} - B^{i;x_i}(s)| = 0.$
2. $\lim_{n\to\infty} \sup_{y\in\mathbb{R}} |\frac{\Lambda_{\vartheta_i}^{(y_n,\bar{y}_n)}(S^{i;x_i^n})}{m_n} - \frac{1}{2}L_{t_i}^y(B^{i;x_i})| = 0,$ *jointly in $(y_n, \bar{y}_n)_{n\in\mathbb{N}}$ equal to the three sequences in (5.1).*
3. $\lim_{n\to\infty} m_n \int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y = g_i(B^{i;x_i}(t)).$
4. *The convergences in (2.17).*
5. $\lim_{n\to\infty} \sum_{a\in\mathbb{N}_0} \frac{\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(S^{i;x_i^n})}{m_n}\frac{\xi_n^E(a)}{m_n} = \frac{1}{2}\int_{\mathbb{R}_+} L_{t_i}^y(B^{i;x_i})\,\mathrm{d}W^E(y)$ *jointly in $E \in \{D, U, L\}$, where for every $a \in \mathbb{N}_0$,*

$$(a_E, \bar{a}_E) := \begin{cases} (a, a) & \text{if } E = D, \\ (a, a+1) & \text{if } E = U, \\ (a+1, a) & \text{if } E = L. \end{cases} \tag{7.4}$$

Proof. According to Theorem 5.2 in the case of the lazy random walk, we can couple $S^{i;x_i^n}$ with a Brownian motion with variance $2/3$ started at x_i, $\tilde{B}^{i;x_i}$, in such a way that

$$\frac{S^{i;x_i^n}(\lfloor m_n^2(3s/2)\rfloor)}{m_n} \to \tilde{B}^{i;x_i}(3s/2) \quad \text{and} \quad \frac{\Lambda_{\vartheta_i}^{(y_n,\bar{y}_n)}(S^{i;x_i^n})}{m_n} \to \frac{1}{3}L_{3t_i/2}^y\big(\tilde{B}^{i;x_i}\big)$$

uniformly almost surely. Let $B^{i;x_i}(s) := \tilde{B}^{i;x_i}(3s/2)$. By the Brownian scaling property, $B^{i;x_i}$ is standard, and $L_{3t_i/2}^y(\tilde{B}^{i;x_i}) = \frac{3}{2}L_{t_i}^y(B^{i;x_i})$. Hence (1) and (2) hold almost surely. Since g_i is uniformly continuous, (3) holds almost surely by (1) and the Lebesgue differentiation theorem. With this given, (4) and (5) follow from Assumption 2.15. □

Remark 7.2. Since the strong invariance principles in Theorem 5.2 are uniform in the time parameter, it is clear that Proposition 7.1 remains valid of we take $\vartheta_i := \lfloor m_n^2(3t_i/2)\rfloor \pm 1$ instead of $\lfloor m_n^2(3t_i/2)\rfloor$. Referring back to Remark 2.24, there is no loss of generality in assuming that the ϑ_i have a particular parity. The same comment applies to our proof of Theorem 2.20-(2) and Theorem 2.21.

7.1.1. *Convergence inside the expected value*

We first prove that for every fixed $x_1, \ldots, x_k \geq 0$, there exists a coupling such that

$$\lim_{n\to\infty} \prod_{i=1}^k F_{n,t_i}\big(S^{i;x_i^n}\big) m_n \int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y = \prod_{i=1}^k \mathbf{1}_{\{\tau_0(B^{i;x_i})>t_i\}} \mathrm{e}^{-\langle L_{t_i}(B^{i;x_i}), Q'\rangle} g_i\big(B^{i;x_i}(t_i)\big) \tag{7.5}$$

in probability. According to the Skorokhod representation theorem (e.g., [5, Theorem 6.7]), there is a coupling such that Proposition 7.1 holds almost surely. For the remainder of Section 7.1.1, we adopt such a coupling.

Since $m_n^{-1}S^{i;x_i^n}(\lfloor m_n^2(3s/2)\rfloor) \to B^{i,x_i}(s)$ uniformly on $s \in [0, t_i]$, and $m_n^2 = o(n)$,

$$\lim_{n\to\infty} \mathbf{1}_{\{\tau^{(n)}(S^{i;x_i^n})>\vartheta_i\}} = \mathbf{1}_{\{\tau_0(B^{i;x_i})>t_i\}}$$

almost surely. By combining this with Proposition 7.1-(3), it only remains to prove that the terms involving the matrix entries D_n, U_n, and L_n in the functional F_{n,t_i} converge to $\mathrm{e}^{-\langle L_{t_i}(B^{i;x_i}), Q'\rangle}$. To this effect, we note that for $E \in \{D, U, L\}$,

$$\prod_{a\in\mathbb{N}_0} \left(1 - \frac{E_n(a)}{m_n^2}\right)^{\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(S^{i;x_i^n})} = \exp\left(\sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big) \log\left(1 - \frac{E_n(a)}{m_n^2}\right)\right),$$

where we recall that $(a_E, \bar{a}_E)$ are defined as in (7.4). By using the Taylor expansion $\log(1+z) = z + O(z^2)$, this is equal to

$$\exp\bigg(-\sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{E_n(a)}{m_n^2} + O\bigg(\sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{E_n(a)^2}{m_n^4}\bigg)\bigg). \tag{7.6}$$

We begin by analyzing the leading order term in (7.6). On the one hand, the uniform convergence of Proposition 7.1-(2) (which implies in particular that $y \mapsto \Lambda_{\vartheta_i}^{(y_n,\bar{y}_n)}(S^{i;x_i^n})/m_n$ and $y \mapsto L^y(B^{i;x_i})$ are supported on a common compact interval almost surely) together with the fact that $V_n^E(\lfloor m_n y\rfloor) \to V^E(y)$ uniformly on compacts (by Assumption 2.10) implies that

$$\lim_{n\to\infty} \sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{V_n^E(a)}{m_n^2} = \lim_{n\to\infty}\int_0^\infty \frac{\Lambda_{\vartheta_i}^{(y_n,\bar{y}_n)}(S^{i;x_i^n})}{m_n} V_n^E\big(\lfloor m_n y\rfloor\big)\,\mathrm{d}y = \frac{1}{2}\big\langle L_{t_i}\big(B^{x_i}\big), V^E\big\rangle \tag{7.7}$$

almost surely (where we choose the appropriate sequence $(y_n, \bar{y}_n)$ as defined in (5.1) depending on $(a_E, \bar{a}_E)$). By combining this with Proposition 7.1-(5), we get

$$\lim_{n\to\infty} \sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{E_n(a)}{m_n^2} = \frac{1}{2}\big\langle L_{t_i}\big(B^{x_i}\big), \big(Q^E\big)'\big\rangle \tag{7.8}$$

almost surely, where $\mathrm{d}Q^E(y) := V^E(y)\mathrm{d}y + \mathrm{d}W^E(y)$.

Next, we control the error term in (7.6). By using $(z+\bar{z})^2 \le 2(z^2+\bar{z}^2)$, for this it suffices control

$$\sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{V_n^E(a)^2}{m_n^4} \quad \text{and} \quad \sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{\xi_n^E(a)^2}{m_n^4}$$

separately. On the one hand, the argument used in (7.7) yields

$$\sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{V_n^E(a)^2}{m_n^4} = m_n^{-2}\frac{1}{2}\big(1+o(1)\big)\big\langle L_{t_i}\big(B^{x_i}\big), \big(V^E\big)^2\big\rangle.$$

Since V^E is continuous and $L_{t_i}(B^{x_i})$ is compactly supported with probability one, this converges to zero almost surely. On the other hand, by definition of (4.3),

$$\sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a,b)}\big(S^{i;x_i^n}\big) \le \vartheta_i = O\big(m_n^2\big)$$

uniformly in $b \in \mathbb{Z}$. Therefore, it follows from the tower property and (2.16) that

$$\mathbf{E}\bigg[\sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{\xi_n^E(a)^2}{m_n^4}\bigg] = \mathbf{E}\bigg[\sum_{a\in\mathbb{N}_0} \Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}\big(S^{i;x_i^n}\big)\frac{\mathbf{E}[\xi_n^E(a)^2]}{m_n^4}\bigg] = O\big(m_n^{-1}\big);$$

hence we have convergence to zero in probability.

By combining the convergence of the leading terms (7.8), our analysis of the error terms, and (2.10) and (2.18), we conclude that (7.5) holds.

7.1.2. *Convergence of the expected value*

Next, we prove that

$$\lim_{n\to\infty} \mathbf{E}\bigg[\prod_{i=1}^k F_{n,t_i}\big(S^{i;x_i^n}\big)m_n \int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y\bigg]$$
$$= \mathbf{E}\bigg[\prod_{i=1}^k \mathbf{1}_{\{\tau_0(B^{i;x_i})>t_i\}} \mathrm{e}^{-\langle L_{t_i}(B^{i;x_i}), Q'\rangle} g_i\big(B^{i;x_i}(t_i)\big)\bigg] \tag{7.9}$$

pointwise in $x_1, \dots, x_k \geq 0$. Given (7.5), we must prove that the sequence of variables inside the expected value on the left-hand side of (7.9) are uniformly integrable. For this, we prove that

$$\sup_{n\geq N} \mathbf{E}\Bigg[\prod_{i=1}^k \bigg(F_{n,t_i}\big(S^{i;x_i^n}\big) m_n \int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y\bigg)^2\Bigg] \leq \sup_{n\geq N}\prod_{i=1}^k \mathbf{E}\Bigg[\bigg(F_{n,t_i}\big(S^{i;x_i^n}\big) m_n \int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y\bigg)^{2k}\Bigg]^{1/k} < \infty$$

for large enough N, where the first upper bound is due to Hölder's inequality.

Since the g_i's are bounded,

$$m_n \int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y \leq \|g_i\|_\infty < \infty,$$

uniformly in n, and thus we need only prove that

$$\sup_{n\geq N} \mathbf{E}\big[\big|F_{n,t_i}\big(S^{i;x_i^n}\big)\big|^{2k}\big] < \infty, \quad 1 \leq i \leq k. \tag{7.10}$$

Since indicator functions are bounded by 1, their contribution to (7.10) may be ignored. For the other terms, we note that for $E \in \{D, U, L\}$ we can write

$$1 - \frac{E_n(a)}{m_n^2} = \frac{m_n^2 - V_n^E(a) - \xi_n^E(a)}{m_n^2} = \bigg(1 - \frac{V_n^E(a)}{m_n^2}\bigg)\bigg(1 - \frac{\xi_n^E(a)}{m_n^2 - V_n^E(a)}\bigg). \tag{7.11}$$

By (2.11), for large n we have $|1 - V_n^E(a)/m_n^2| \leq 1$, hence by applying Hölder's inequality in (7.10), we need only prove that

$$\sup_{n\geq N} \mathbf{E}\Bigg[\prod_{a\in\mathbb{N}_0}\bigg|1 - \frac{\xi_n^E(a)}{m_n^2 - V_n^E(a)}\bigg|^{6k\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(S^{i;x_i^n})}\Bigg] < \infty, \quad E \in \{D, U, L\}. \tag{7.12}$$

Let us fix $E \in \{D, U, L\}$ and define

$$\zeta_n(a) := \frac{\xi_n^E(a)}{m_n^{1/2}} \quad \text{and} \quad r_n(a) := \frac{m_n^{1/2}}{m_n^2 - V_n^E(a)}.$$

By (2.16), we know that there exists $C > 0$ and $0 < \gamma < 2/3$ such that $\mathbf{E}[|\zeta_n(a)|^q] \leq C^q q^{\gamma q}$ for every $q \in \mathbb{N}$ and n large enough. Thus, since the variables $\xi_n^E(0), \dots, \xi_n^E(n)$ are independent, it follows from the upper bound [20, (4.25)] that there exists $C' > 0$ and $2 < \gamma' < 3$ both independent of n such that (7.12) is bounded above by

$$\mathbf{E}\Bigg[\exp\Bigg(C'\bigg(\sum_{a\in\mathbb{N}_0}\big|r_n(a)\big|\Lambda_{\vartheta_i}^a\big(S^{i;x_i^n}\big)\big|\mathbf{E}\big[\zeta_n(a)\big]\big| + \sum_{a\in\mathbb{N}_0} r_n(a)^2\Lambda_{\vartheta_i}^a\big(S^{i;x_i^n}\big)^2 + \sum_{a\in\mathbb{N}_0}\big|r_n(a)\big|^{\gamma'}\Lambda_{\vartheta_i}^a\big(S^{i;x_i^n}\big)^{\gamma'}\bigg)\Bigg)\Bigg], \tag{7.13}$$

where we use the trivial bound $\Lambda_\vartheta^{(a,b)}, \Lambda_\vartheta^{(b,a)} \leq \Lambda_\vartheta^a$ for all a, b.

For any fixed x_i, we know that $S^{i;x_i^n}(u) = O(m_n^2)$ uniformly in $0 \leq u \leq \vartheta_i$ because $\vartheta_i = O(m_n^2)$. Thus, the only values of a for which $\Lambda_{\vartheta_i}^a$ is possibly nonzero are at most of order $O(m_n^2) = o(n)$. For any such values of a, the assumption (2.12) implies that $V_n^E(a) = o(m_n^2)$, hence $r_n(a) = O(m_n^{-3/2})$. By combining all of these estimates with (2.15), (7.12) is then a consequence of the following proposition.

Proposition 7.3. *Let $\vartheta = \vartheta(n,t) := \lfloor m_n^2 t\rfloor$ and $x^n := \lfloor m_n x\rfloor$ for some $t > 0$ and $x \geq 0$. For every $C > 0$ and $1 \leq q < 3$,*

$$\sup_{n\in\mathbb{N}, x\geq 0} \mathbf{E}\bigg[\exp\bigg(\frac{C}{m_n}\sum_{a\in\mathbb{Z}}\frac{\Lambda_\vartheta^a(S^{x^n})^q}{m_n^q}\bigg)\bigg] < \infty.$$

Since the proof of Proposition 7.3 is rather long and technical, we provide it later in Section 7.3 so as to not interrupt the flow of the present argument.

7.1.3. *Convergence of the integral*

We now complete the proof that (7.2) converges to (7.3). With (7.9) established, it only remains to justify passing the limit inside the integral in $\mathrm{d}x_1 \cdots \mathrm{d}x_k$. In order to prove this, we aim to use the Vitali convergence theorem (e.g., [12, Theorem 2.24]). For this, we need a more refined version of the uniform integrability estimate used in Section 7.1.2. By Hölder's inequality,

$$\Bigg(\prod_{i=1}^k f_i(x_i)\Bigg)\mathbf{E}\Bigg[\prod_{i=1}^k F_{n,t_i}\big(S^{i;x_i^n}\big)m_n \int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y\Bigg] \\ \leq \prod_{i=1}^k \|f_i\|_\infty \|g_i\|_\infty \mathbf{E}\big[\big|F_{n,t_i}\big(S^{i;x_i^n}\big)\big|^k\big]^{1/k}. \tag{7.14}$$

Our aim is to find a suitable upper bounds for the functions

$$x \mapsto \mathbf{E}\big[\big|F_{n,t_i}\big(S^{i;x^n}\big)\big|^k\big]^{1/k}, \quad 1 \leq i \leq k.$$

In order to achieve this, we fix a small $\varepsilon > 0$ (precisely how small will be determined in the following paragraphs), and we consider separately the two cases $x \in [0, n^{1-\varepsilon}/m_n)$ and $x \in [n^{1-\varepsilon}/m_n, (n+1)/m_n)$.

Let us first consider the case $x \in [0, n^{1-\varepsilon}/m_n)$. Note that for any $E \in \{D, U, L\}$,

$$\mathbf{1}_{\{\tau^{(n)}(S)>\vartheta\}} \prod_{a\in\mathbb{N}_0} \left|1 - \frac{E_n(a)}{m_n^2}\right|^{\Lambda_\vartheta^{(a_E,\bar{a}_E)}(S)} \leq \prod_{a\in\mathbb{Z}} \left|1 - \frac{E_n(|a|)}{m_n^2}\right|^{\Lambda_\vartheta^{(a_E,\bar{a}_E)}(S)}.$$

Then, by combining Hölder's inequality with a rearrangement similar to (7.11), $\mathbf{E}[|F_{n,t_i}(S^{i;x^n})|^k]^{1/k}$ is bounded above by the product of the two terms

$$\prod_{E\in\{D,U,L\}} \mathbf{E}\Bigg[\prod_{a\in\mathbb{Z}} \left|1 - \frac{\xi_n^E(|a|)}{m_n^2 - V_n^E(|a|)}\right|^{6k\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(S^{i;x^n})}\Bigg]^{1/6k}, \tag{7.15}$$

$$\prod_{E\in\{D,U,L\}} \mathbf{E}\Bigg[\prod_{a\in\mathbb{Z}} \left|1 - \frac{V_n^E(|a|)}{m_n^2}\right|^{6k\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(S^{i;x^n})}\Bigg]^{1/6k}. \tag{7.16}$$

Since $m_n x = O(n^{1-\varepsilon}) = o(n)$, the random walk $S^{i;x^n}$ can only attain values of order $o(n)$ in $\vartheta_i = O(m_n^2) = o(n)$ steps. Thus, for $E \in \{D, U, L\}$, it follows from (2.12) that $V_n^E(a) = o(m_n^2)$ for any value attained by the walk when $x \in [0, n^{1-\varepsilon}/m_n)$. By using the same argument as for (7.12) (namely, the inequality [20, (4.25)] followed by Proposition 7.3), we conclude that (7.15) is bounded by a constant for large n. For (7.16), let us assume without loss of generality that V_n^D is the sequence (or at least one of the sequenes) that satisfies (2.13). According to (2.11), we have

$$\prod_{E\in\{U,L\}} \mathbf{E}\Bigg[\prod_{a\in\mathbb{Z}} \left|1 - \frac{V_n^E(|a|)}{m_n^2}\right|^{6k\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(S^{i;x^n})}\Bigg]^{1/6k} \leq 1$$

for large enough n. For the terms involving V_n^D, since $|1-y| \leq \mathrm{e}^{-y}$ for any $y \in [0,1]$, it follows from (2.13) that, up to a constant C independent of n (depending on θ through $c = c(\theta)$ in (2.13)), we have the upper bound

$$\mathbf{E}\Bigg[\prod_{a\in\mathbb{Z}} \left|1 - \frac{V_n^D(|a|)}{m_n^2}\right|^{6k\Lambda_{\vartheta_i}^{(a,a)}(S^{i;x^n})}\Bigg]^{1/6k} \leq C\mathbf{E}\Bigg[\exp\Bigg(-\frac{6k\theta}{m_n^2}\sum_{a\in\mathbb{Z}} \log\big(1+|a|/m_n\big)\Lambda_{\vartheta_i}^{(a,a)}\big(S^{i;x^n}\big)\Bigg)\Bigg]^{1/6k} \tag{7.17}$$

for large enough n. If we define $S^{i;x^n} := x^n + S^0$ for all $x \geq 0$, then $\Lambda_{\vartheta_i}^{(a,a)}(S^{i;x^n}) = \Lambda_{\vartheta_i}^{(a-x^n,a-x^n)}(S^0)$. By combining this change of variables with the inequality

$$\log\big(1+|z+\bar{z}|\big) \geq \log\big(1+|z|\big) - \log\big(1+|\bar{z}|\big) \geq \log\big(1+|z|\big) - |\bar{z}|,$$

which is valid for all $z, \bar{z} \in \mathbb{R}$, we obtain that, up to a multiplicative constant independent of n, (7.17) is bounded by

$$\mathbf{E}\bigg[\exp\bigg(-\frac{6k\theta}{m_n^2}\sum_{a\in\mathbb{Z}}\big(\log(1+x)-|a/m_n|\big)\Lambda_{\vartheta_i}^{(a,a)}\big(S^0\big)\bigg)\bigg]^{1/6k}.$$

Noting that $\Lambda_{\vartheta_i}^{(a,a)} \le \Lambda_{\vartheta_i}^{a}$ for every $a \in \mathbb{Z}$ and that the vertex-occupation measures satisfy (5.5), an application of Hölder's inequality then implies that (7.17) is bounded above by the product of the two terms

$$\mathbf{E}\bigg[\exp\bigg(-\frac{12k\theta\log(1+x)}{m_n^2}\sum_{a\in\mathbb{Z}}\Lambda_{\vartheta_i}^{(a,a)}\big(S^0\big)\bigg)\bigg]^{1/12k}, \tag{7.18}$$

$$\mathbf{E}\bigg[\exp\bigg(\frac{12k\theta}{m_n^2}\sum_{0\le u\le\vartheta_i}\frac{|S^0(u)|}{m_n}\bigg)\bigg]^{1/12k}. \tag{7.19}$$

Recall the definition of the range $\mathcal{R}_{\vartheta_i}(S^0)$ in (5.10). Since

$$\mathcal{R}_{\vartheta_i}\big(S^0\big) \ge \max_{0\le u\le\vartheta_i}\big|S^0(u)\big|,$$

we conclude that there exists $C > 0$ independent of n such that (7.19) is bounded by the exponential moment $\mathbf{E}[\mathrm{e}^{C\mathcal{R}_{\vartheta_i}(S^0)/m_n}]^{1/12k}$. Thus, by (5.12), we see that (7.19) is bounded by a constant independent of n. It now remains to control (7.18). To this end, we note that $\sum_{a\in\mathbb{Z}}\Lambda_{\vartheta_i}^{(a,a)}(S^0)$, which represents the total number of visits on the self-edges of $\mathbb{Z}$ by S^0 before the ϑ_i^{th} step, is a Binomial random variable with ϑ_i trials and probability $1/3$. Thus, for small enough $\nu > 0$, it follows from Hoeffding's inequality that

$$\mathbf{P}\bigg[\sum_{a\in\mathbb{Z}}\Lambda_{\vartheta_i}^{(a,a)}\big(S^0\big) < \nu m_n^2\bigg] \le \mathrm{e}^{-cm_n^2} \tag{7.20}$$

for some $c > 0$ independent of n. By separating the expectation in (7.18) with respect to whether or not the walk has taken less than νm_n^2 steps on self-edges, we may bound it above by

$$\big(\mathrm{e}^{-12k\nu\theta\log(1+x)} + \mathrm{e}^{-cm_n^2}\big)^{1/12k} \le (1+x)^{-\nu\theta} + \mathrm{e}^{-(c/12k)m_n^2}.$$

Combining all of these bounds together with the fact that m_n is of order $n^{\mathfrak{d}}$ by (2.1), we finally conclude that for every $1 \le i \le k$, there exists constants $c_1, c_2, c_3 > 0$ independent of n such that, for large enough n,

$$\mathbf{E}\big[\big|F_{n,t_i}\big(S^{i;x^n}\big)\big|^k\big]^{1/k} \le c_1\big((1+x)^{-c_2\theta} + \mathrm{e}^{-c_3n^{2\mathfrak{d}}}\big), \quad x \in \big[0, n^{1-\varepsilon}/m_n\big). \tag{7.21}$$

Remark 7.4. We emphasize that c_2 does not depend on θ, and thus the assumption (2.13) implies that we can make $c_2\theta$ arbitrarily large by taking a large enough θ. In particular, if we take $\theta > 1/c_2$, then $(1+x)^{-c_2\theta}$ is integrable on $[0,\infty)$.

We now turn to the estimate in the case where $x \in [n^{1-\varepsilon}/m_n, (n+1)/m_n)$. By taking $\varepsilon > 0$ small enough (more specifically, such that $1-\varepsilon > 2\mathfrak{d}$, with $\mathfrak{d}$ as in (2.1)), we can ensure that $m_nx \ge n^{1-\varepsilon}$ implies that, for any constant $0 < C < 1$, we have $S^{i;x^n}(u) \ge Cn^{1-\varepsilon}$ for all $0 \le u \le \vartheta_i$ and n large enough. Let us assume without loss of generality that V_n^D satisfies (2.14). Provided $\varepsilon > 0$ is small enough (namely, at least as small as the ε in (2.14)), for any $a \in \mathbb{N}_0$ that can be visited by the random walk, we have that $V_n^D(a) \ge \kappa(Cn^{1-\varepsilon}/m_n)^\alpha$; hence

$$\begin{aligned}\bigg|1-\frac{D_n(a)}{m_n^2}\bigg| &\le \frac{m_n^2-V_n^D(a)}{m_n^2} + \frac{|\xi_n^D(a)|}{m_n^2} \le \frac{m_n^2-\kappa(Cn^{1-\varepsilon}/m_n)^\alpha}{m_n^2} + \frac{|\xi_n^D(a)|}{m_n^2} \\ &= \bigg(1-\frac{\kappa(Cn^{1-\varepsilon}/m_n)^\alpha}{m_n^2}\bigg)\bigg(1+\frac{|\xi_n(a)|}{m_n^2-\kappa(Cn^{1-\varepsilon}/m_n)^\alpha}\bigg).\end{aligned} \tag{7.22}$$

According to (2.1), we know that $(n^{1-\varepsilon}/m_n)^\alpha \asymp n^{\alpha(1-\mathfrak{d})-\alpha\varepsilon}$. Since α is chosen such that $\mathfrak{d}/2 < \alpha(1-\mathfrak{d}) \le 2\mathfrak{d}$ in Assumption 2.12, we can always choose $\varepsilon > 0$ small enough so as to guarantee that

$$n^{\mathfrak{d}/2} = o\big(n^{\alpha(1-\mathfrak{d})-\alpha\varepsilon}\big) \quad\text{and}\quad \big(n^{1-\varepsilon}/m_n\big)^\alpha = o\big(n^{2\mathfrak{d}}\big) = o\big(m_n^2\big). \tag{7.23}$$

As a consequence of the second equation in (7.23), for n large enough (7.22) yields

$$\left|1-\frac{D_n(a)}{m_n^2}\right| \le \left(1-\frac{\kappa(Cn^{1-\varepsilon}/m_n)^\alpha}{m_n^2}\right)\left(1+\frac{2|\xi_n(a)|}{m_n^2}\right).$$

As for $E \in \{U, L\}$, we have from (2.11) that

$$\left|1-\frac{E_n(a)}{m_n^2}\right| \le \frac{|m_n^2 - V_n^E(a)|}{m_n^2} + \frac{|\xi_n^E(a)|}{m_n^2} \le 1 + \frac{|\xi_n^E(a)|}{m_n^2}.$$

Thus, for any $x \in [n^{1-\varepsilon}/m_n, (n+1)/m_n)$ and large enough n, it follows from Hölder's inequality that the expectation $\mathbf{E}[|F_{n,t_i}(S^{i;x^n})|^k]^{1/k}$ is bounded above by the product of the following three terms:

$$\mathbf{E}\left[\prod_{a\in\mathbb{Z}}\left(1-\frac{\kappa(Cn^{1-\varepsilon}/m_n)^\alpha}{m_n^2}\right)^{4k\Lambda_{\vartheta_i}^{(a,a)}(S^{i;x^n})}\right]^{1/4k}, \tag{7.24}$$

$$\mathbf{E}\left[\prod_{a\in\mathbb{Z}}\left(1+\frac{2|\xi_n^D(|a|)|}{m_n^2}\right)^{4k\Lambda_{\vartheta_i}^{(a,a)}(S^{i;x^n})}\right]^{1/4k}, \tag{7.25}$$

$$\prod_{E\in\{U,L\}}\mathbf{E}\left[\prod_{a\in\mathbb{Z}}\left(1+\frac{|\xi_n^E(|a|)|}{m_n^2}\right)^{4k\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(S^{i;x^n})}\right]^{1/4k}. \tag{7.26}$$

By repeating the bound (7.20) and the argument thereafter, we conclude that there exists $c_4, c_5 > 0$ independent of n such that (7.24) is bounded by $\mathrm{e}^{-c_4 n^{\alpha(1-\mathfrak{d})-\alpha\varepsilon}} + \mathrm{e}^{-c_5 n^{2\mathfrak{d}}}$. For (7.25), let us define $\zeta_n(a) := |\xi_n^D(a)|/m_n^{1/2}$. By applying [20, (4.25)] in similar fashion to (7.13), we see that (7.25) is bounded above by

$$\mathbf{E}\left[\exp\left(C'\left(\frac{1}{m_n^{1/2}}\sum_{a\in\mathbb{Z}}\frac{\Lambda_{\vartheta_i}^a(S^{i;x_i^n})}{m_n}\mathbf{E}\big[\big|\zeta_n\big(|a|\big)\big|\big] + \frac{1}{m_n}\sum_{a\in\mathbb{Z}}\frac{\Lambda_{\vartheta_i}^a(S^{i;x_i^n})^2}{m_n^2} + \frac{1}{m_n^{\gamma'/2}}\sum_{a\in\mathbb{Z}}\frac{\Lambda_{\vartheta_i}^a(S^{i;x_i^n})^{\gamma'}}{m_n^{\gamma'}}\right)\right)\right] \tag{7.27}$$

for some $C' > 0$ and $2 < \gamma' < 3$ independent of n. By (2.16), the moments $\mathbf{E}[|\zeta_n(a)|]$ are uniformly bounded in n, and thus

$$\frac{1}{m_n^{1/2}}\sum_{a\in\mathbb{Z}}\frac{\Lambda_{\vartheta_i}^a(S^{i;x_i^n})}{m_n}\mathbf{E}\big[\big|\zeta_n\big(|a|\big)\big|\big] = O\big(m_n^{1/2}\big) = O\big(n^{\mathfrak{d}/2}\big).$$

By applying the uniform exponential moment bounds of Proposition 7.3 to the remaining terms in (7.27), we conclude that there exists a constant $c_6 > 0$ independent of n such that (7.25) is bounded by $\mathrm{e}^{c_6 n^{\mathfrak{d}/2}}$. A similar bound applies to (7.26). Then, by using the first equality in (7.23) and combining the inequalities for (7.24)–(7.26), we see that there exists $\bar{c}_4, \bar{c}_5 > 0$ independent of n such that

$$\mathbf{E}\big[\big|F_{n,t_i}\big(S^{i;x^n}\big)\big|^k\big]^{1/k} \le \mathrm{e}^{-\bar{c}_4 n^{\alpha(1-\mathfrak{d})-\alpha\varepsilon}} + \mathrm{e}^{-\bar{c}_5 n^{2\mathfrak{d}}}, \quad x \in \big[n^{1-\varepsilon}/m_n, (n+1)/m_n\big) \tag{7.28}$$

By combining (7.21) and (7.28), we conclude that, for large n, the integral of the absolute value of (7.14) on the set $[0, (n+1)/m_n)^k$ is bounded above by

$$\left(\prod_{i=1}^k \|f_i\|_\infty \|g_i\|_\infty\right)\left(c_1\int_0^{n^{1-\varepsilon}/m_n}(1+x)^{-c_2\theta} + \mathrm{e}^{-c_3 n^{2\mathfrak{d}}}\,\mathrm{d}x + \int_{n^{1-\varepsilon}/m_n}^{(n+1)/m_n}\mathrm{e}^{-\bar{c}_4 n^{\alpha(1-\mathfrak{d})-\alpha\varepsilon}} + \mathrm{e}^{-\bar{c}_5 n^{2\mathfrak{d}}}\,\mathrm{d}x\right)^k$$

for some $c_1, c_2, c_3, \bar{c}_4, \bar{c}_5 > 0$ independent of n. If we take $\theta > 0$ large enough so that $(1+x)^{-c_2\theta}$ is integrable, then the sequence of functions

$$\left(\prod_{i=1}^k \mathbf{1}_{[0,(n+1)/m_n)}(x_i) f_i(x_i)\right)\mathbf{E}\left[\prod_{i=1}^k F_{n,t_i}\big(S^{i;x_i^n}\big) m_n \int_{S^{i;x_i^n}(\vartheta_i)/m_n}^{(S^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y\right]$$

is uniformly integrable in the sense of [12, Theorem 2.24-(ii),(iii)], concluding the proof of the convergence of moments in Theorem 2.20-(1).

7.2. *Step* 2: *Convergence in distribution*

Up to writing each f_i and g_i as the difference of their positive and negative parts, there is no loss of generality in assuming that $f_i, g_i \geq 0$. The convergence in joint distribution follows from the convergence in moments proved in Section 7.1. The argument we use to prove this is essentially the same as [20, Lemma 4.4]:

For any $\underline{R} \in [-\infty, 0]$ and $\bar{R} \in [0, \infty]$, let us define

$$\hat{K}_n^{\underline{R},\bar{R}}(t)g(x) := \mathbf{E}^{\lfloor m_n x\rfloor}\left[\left(\underline{R} \vee F_{n,t}(S) \wedge \bar{R}\right)m_n \int_{S(\vartheta)/m_n}^{S(\vartheta)+1)/m_n} g(y)\,\mathrm{d}y\right]$$

and

$$\hat{K}^{\underline{R},\bar{R}}(t)g(x) := \mathbf{E}^x\left[\left(\underline{R} \vee \mathbf{1}_{\{\tau_0(B)>t\}}\mathrm{e}^{-\langle L_t(B),Q'\rangle} \wedge \bar{R}\right)g\big(B(t)\big)\right],$$

where we use the convention $\underline{R} \vee y \wedge \bar{R} := \max\{\underline{R}, \min\{y, \bar{R}\}\}$ for any $y \in \mathbb{R}$. We note a few elementary properties of these truncated operators:

1. $\hat{K}_n^{-\infty,\infty}(t) = \hat{K}_n(t)$, and $\hat{K}^{-\underline{R},\infty}(t) = \hat{K}(t)$ for all $\underline{R} \leq 0$.
2. Arguing as in Section 7.1, for every $\underline{R} \in [-\infty, 0]$ and $\bar{R} \in [0, \infty]$,

$$\lim_{n\to\infty}\langle f_i, \hat{K}_n^{\underline{R},\bar{R}}(t_i)g_i\rangle = \langle f_i, \hat{K}^{\underline{R},\bar{R}}(t_i)g_i\rangle, \quad 1 \leq i \leq k \tag{7.29}$$

   in joint moments.
3. If $|\underline{R}|, \bar{R} < \infty$, then the $\langle f_i, \hat{K}_n^{\underline{R},\bar{R}}(t_i)g_i\rangle$ are bounded uniformly in n; hence the moment convergence of (7.29) implies convergence in joint distribution.

Let $\underline{R} > -\infty$ be fixed. Since $\langle f_i, \hat{K}_n^{\underline{R},\infty}(t_i)g_i\rangle \to \langle f_i, \hat{K}^{\underline{R},\infty}(t_i)g_i\rangle$ in joint moments, the sequences in question are tight (e.g., [4, Problem 25.17]). Therefore, it suffices to prove that every subsequence that converges in joint distribution has $\langle f_i, \hat{K}^{\underline{R},\infty}(t_i)g_i\rangle$ as a limit (e.g., [4, Theorem–Corollary 25.10]). Let $\mathcal{A}_1^{\underline{R}}, \ldots, \mathcal{A}_k^{\underline{R}}$ be limit points of $\langle f_1, \hat{K}^{\underline{R},\infty}(t_1)g_1\rangle, \ldots, \langle f_k, \hat{K}^{\underline{R},\infty}(t_k)g_k\rangle$. Since $f_i, g_i \geq 0$, the variables $\langle f_i, \hat{K}_n^{\underline{R},\bar{R}}(t_i)g_i\rangle$ and $\langle f_i, \hat{K}^{\underline{R},\bar{R}}(t_i)g_i\rangle$ are increasing in $\bar{R}$. Therefore, for every $\bar{R} < \infty$, we have

$$\left(\mathcal{A}_1^{\underline{R}}, \ldots, \mathcal{A}_k^{\underline{R}}\right) \geq \left(\langle f_1, \hat{K}^{\underline{R},\bar{R}}(t_1)g_1\rangle, \ldots, \langle f_k, \hat{K}^{\underline{R},\bar{R}}(t_k)g_k\rangle\right) \tag{7.30}$$

in the sense of stochastic dominance in the space $\mathbb{R}^k$ with the componentwise order (e.g. [22, Theorem 1 and Proposition 3]). By the monotone convergence theorem,

$$\lim_{\bar{R}\to\infty}\langle f_i, \hat{K}^{\underline{R},\bar{R}}(t_i)g_i\rangle = \langle f_i, \hat{K}^{\underline{R},\infty}(t_i)g_i\rangle, \quad 1 \leq i \leq k$$

almost surely; hence the stochastic dominance (7.30) also holds for $\bar{R} = \infty$. Since $\mathcal{A}_i^{\underline{R}}$ and $\langle f_i, \hat{K}^{\underline{R},\infty}(t_i)g_i\rangle$ have the same mixed moments, we thus infer that their joint distributions coincide. In conclusion, for any finite $\underline{R}$, we have that

$$\lim_{n\to\infty}\langle f_i, \hat{K}_n^{\underline{R},\infty}(t_i)g_i\rangle = \langle f_i, \hat{K}_n^{\underline{R},\infty}(t_i)g_i\rangle$$

in joint distribution. In order to get the result for $\underline{R} = -\infty$, we use the same stochastic domination argument by sending $\underline{R} \to -\infty$.

7.3. *Proof of Proposition* 7.3

If we prove that

$$\sup_{n\in\mathbb{N}} \mathbf{E}\left[\exp\left(\frac{C}{m_n}\sum_{a\in\mathbb{Z}}\frac{\Lambda_\vartheta^a(S^0)^q}{m_n^q}\right)\right] < \infty,$$

then we get the desired result by a simple change of variables. Similarly to [20, Proposition 4.3], a crucial tool for proving this consists of combinatorial identities involving the quantile transform for random walks derived in [1]. However, such results only apply to the simple symmetric random walk.

In order to get around this requirement, we decompose the vertex-occupation measures in terms of the edge-occupations measures as follows: By combining

$$\Lambda^a_\vartheta(S^0) = \Lambda^{(a,a-1)}_\vartheta(S^0) + \Lambda^{(a,a)}_\vartheta(S^0) + \Lambda^{(a,a+1)}_\vartheta(S^0) + \mathbf{1}_{\{S^0(\vartheta)=a\}}, \quad a \in \mathbb{Z}$$

with the inequality $(z+\bar{z})^q \leq 2^{q-1}(z^q+\bar{z}^q)$ (for $z, \bar{z} \geq 0$ and $q \geq 1$), it suffices by an application of Hölder's inequality to prove that the exponential moments of

$$\frac{1}{m_n}\sum_{a\in\mathbb{Z}} \frac{(\Lambda^{(a,a-1)}_\vartheta(S^0) + \Lambda^{(a,a+1)}_\vartheta(S^0))^q}{m_n^q} \tag{7.31}$$

and

$$\frac{1}{m_n}\sum_{a\in\mathbb{Z}} \frac{\Lambda^{(a,a)}_\vartheta(S^0)^q}{m_n^q} \tag{7.32}$$

are uniformly bounded in n.

7.3.1. *Non-self-edges*
Let us begin with (7.31).

Definition 7.5. Let $\mathfrak{S}$ be a simple symmetric random walk on $\mathbb{Z}$, that is, the increments $\mathfrak{S}(u) - \mathfrak{S}(u-1)$ are i.i.d. uniform on $\{-1, 1\}$. For any $a, b \in \mathbb{Z}$ and $u \in \mathbb{N}_0$, we denote $\mathfrak{S}^a := (\mathfrak{S}|\mathfrak{S}(0) = a)$ and $\mathfrak{S}^{a,b}_u := (\mathfrak{S}|\mathfrak{S}(0) = a \text{ and } \mathfrak{S}(u) = b)$ (note that the latter only makes sense if $|b-a|$ and u have the same parity).

For every $u \in \mathbb{N}_0$, let

$$\mathcal{H}_u(S^0) := \sum_{a\in\mathbb{Z}} \Lambda^{(a,a)}_u(S^0), \tag{7.33}$$

i.e., the number of times S^0 visits self-edges by the u^{th} step. Then, it is easy to see that we can couple S^0 and $\mathfrak{S}^0$ in such a way that

$$S^0(u) = \mathfrak{S}^0(u - \mathcal{H}_u(S^0)), \quad u \in \mathbb{N},$$

i.e., $\mathfrak{S}^0$ is the same path as S^0 with the visits to self-edges removed. If we define the edge-occupation measures for $\mathfrak{S}^0$ in the same way as (5.4), then it is clear that the coupling of S and $\mathfrak{S}$ satisfies

$$\Lambda^{(a,a-1)}_\vartheta(S) + \Lambda^{(a,a+1)}_\vartheta(S) \leq \Lambda^a_\vartheta(\mathfrak{S}).$$

Thus, for (7.31) we need only prove that the exponential moments of

$$\frac{1}{m_n}\sum_{a\in\mathbb{Z}} \frac{\Lambda^a_\vartheta(\mathfrak{S}^0)^q}{m_n^q} \tag{7.34}$$

are uniformly bounded in n.

By the total probability rule, we note that

$$\mathbf{E}\left[\exp\left(\frac{C}{m_n}\sum_{a\in\mathbb{Z}} \frac{\Lambda^a_\vartheta(\mathfrak{S}^0)^q}{m_n^q}\right)\right] = \sum_{b\in\mathbb{Z}} \mathbf{E}\left[\exp\left(\frac{C}{m_n}\sum_{a\in\mathbb{Z}} \frac{\Lambda^a_\vartheta(\mathfrak{S}^{0,b}_\vartheta)^q}{m_n^q}\right)\right]\mathbf{P}[\mathfrak{S}^0(\vartheta) = b].$$

According to the proof of [20, Proposition 4.3] (more specifically, [20, (4.19)] and the following paragraph, explaining the distribution of the quantity denoted $M(N, \tilde{T})$ in [20, (4.19)]), there exists a constant $\bar{C} > 0$ that only depends on C, q, and the number t in $\vartheta = \lfloor m_n^2 t \rfloor$ such that

$$\frac{1}{m_n}\sum_{a\in\mathbb{Z}} \frac{\Lambda_\vartheta^a(\mathfrak{S}_\vartheta^{0,b})^q}{m_n^q} \le \bar{C}\big((\mathfrak{R}_\vartheta^{0,b}/m_n)^{q-1} + \big((|b|+2)/m_n\big)^{q-1}\big), \tag{7.35}$$

where $\mathfrak{R}_\vartheta^{0,b}$ is equal in distribution to the range of $\mathfrak{S}_\vartheta^{0,b}$, that is,

$$\mathfrak{R}_\vartheta^{0,b} \overset{\mathrm{d}}{=} \mathcal{R}_\vartheta\big(\mathfrak{S}_\vartheta^{0,b}\big) := \max_{0\le u\le\vartheta} \mathfrak{S}_\vartheta^{0,b}(u) - \min_{0\le u\le\vartheta} \mathfrak{S}_\vartheta^{0,b}(u).$$

Hence, if $\mathcal{R}_\vartheta(\mathfrak{S}^0)$ denotes the range of the unconditioned random walk $\mathfrak{S}^0$, then

$$\mathbf{E}\bigg[\exp\bigg(\frac{\bar{C}}{m_n}\sum_{a\in\mathbb{Z}}\frac{\Lambda_\vartheta^a(\mathfrak{S}^0)^q}{m_n^q}\bigg)\bigg] \le \mathbf{E}\bigg[\exp\bigg(C\bigg(\frac{(\mathcal{R}_\vartheta(\mathfrak{S}^0))^{q-1}}{m_n^{q-1}} + \frac{(|\mathfrak{S}^0(\vartheta)|+2)^{q-1}}{m_n^{q-1}}\bigg)\bigg)\bigg].$$

Since $q-1<2$, the result then follows from the same moment estimate leading up to (5.12), but by applying [9, (6.2.3)] to the random walk $\mathfrak{S}^0$ instead of S^0.

7.3.2. *Self-edges*
We now control the exponential moments of (7.32). By referring to the uniform boundedness of the exponential moments of (7.31) that we have just proved, we know that for any $b\in\{-1,1\}$, the exponential moments of

$$\frac{1}{m_n}\sum_{a\in\mathbb{Z}}\frac{\Lambda_\vartheta^{(a,a+b)}(S^0)^q}{m_n^q} \quad \text{and} \quad \frac{1}{m_n}\sum_{a\in\mathbb{Z}}\frac{\Lambda_\vartheta^{(a+b,a)}(S^0)^q}{m_n^q}$$

are uniformly bounded in n. Thus, by applying $(x+y)^q \le 2^q(|x|^q+|y|^q)$, the exponential moments of

$$\frac{1}{m_n}\sum_{a\in\mathbb{Z}}\frac{(\Lambda_\vartheta^{(a+1,a)}(S^0)+\Lambda_\vartheta^{(a-1,a)}(S^0)+\Lambda_\vartheta^{(a,a+1)}(S^0)+\Lambda_\vartheta^{(a,a-1)}(S^0))^q}{m_n^q}$$

are uniformly bounded in n. Consequently, it suffices to prove that there exists $c,\bar{c}>0$ such that for every $n\in\mathbb{N}$ and y large enough (independently of n),

$$\begin{aligned}&\mathbf{P}\bigg[\sum_{a\in\mathbb{Z}}\Lambda_\vartheta^{(a,a)}\big(S^0\big)^q > y\bigg]\\ &\quad\le \mathbf{P}\bigg[\sum_{a\in\mathbb{Z}}\big(\Lambda_\vartheta^{(a+1,a)}\big(S^0\big)+\Lambda_\vartheta^{(a-1,a)}\big(S^0\big)+\Lambda_\vartheta^{(a,a+1)}\big(S^0\big)+\Lambda_\vartheta^{(a,a-1)}\big(S^0\big)\big)^q > cy-\bar{c}\bigg].\end{aligned} \tag{7.36}$$

We now prove (7.36).

Definition 7.6. If ϑ is even, let $\mathcal{S}_0,\mathcal{S}_1,\ldots,\mathcal{S}_{\vartheta/2-1}$ be defined as the path segments

$$\mathcal{S}_u = \big(S^0(2u), S^0(2u+1), S^0(2u+2)\big), \quad 0\le u\le \vartheta/2-1.$$

If ϑ is odd, then we similarly define $\mathcal{S}_0,\mathcal{S}_1,\ldots,\mathcal{S}_{(\vartheta-1)/2-1},\mathcal{S}_{(\vartheta-1)/2}$ as

$$\mathcal{S}_u = \begin{cases}(S^0(2u), S^0(2u+1), S^0(2u+2)) & \text{if } 0\le u\le(\vartheta-1)/2-1,\\ (S^0(2u), S^0(2u+1)) & \text{if } u=(\vartheta-1)/2.\end{cases}$$

In words, we partition the path formed by the first ϑ steps of S^0 into successive segments of two steps, with the exception that the very last segment may contain only one step if ϑ is odd (see Figure 2 for an illustration of this partition).

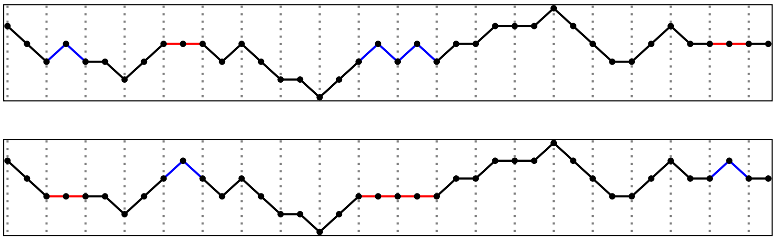

Fig. 2. The partition into two-step segments is represented by dashed gray lines. Type 2 segments are red, and type 3 segments are blue. The two paths represent S^0 and $\hat{S}^0$, as related to each other by the permutation of type 2 and 3 segments.

Definition 7.7. Let $\mathcal{S}_u$ be a path segment as in the previous definition. We say that $\mathcal{S}_u$ is a *type 1* segment if there exists some $a \in \mathbb{Z}$ and $b \in \{-1, 1\}$ such that

$$\mathcal{S}_u = \begin{cases} (a, a, a+b), \\ (a+b, a, a), \text{ or} \\ (a, a), \end{cases}$$

we say that $\mathcal{S}_u$ is a *type 2* segment if there exists some $a \in \mathbb{Z}$ such that

$$\mathcal{S}_u = (a, a, a),$$

and we say that $\mathcal{S}_u$ is a *type 3* segment if there exists some $a \in \mathbb{Z}$ such that

$$\mathcal{S}_u = (a, a+1, a).$$

Given a realization of the first ϑ steps of the lazy random walk S^0, we define the transformed path $(\hat{S}^0(u))_{0 \le u \le \vartheta}$ by replacing every type 2 segment (a, a, a) in $(S^0(u))_{0 \le u \le \vartheta}$ by the corresponding type 3 segment $(a, a+1, a)$, and vice versa. (see Figure 2 for an illustration of this transformation). Given that this path transformation is a bijection on the set of all possible realizations of $(S^0(u))_{0 \le u \le \vartheta}$, $(\hat{S}^0(u))_{0 \le u \le \vartheta}$ is also a lazy random walk.

Every contribution of S^0 to $\sum_a \Lambda_\vartheta^{(a,a)}(S^0)$ comes from type 1 and 2 segments. Moreover, if a type 1 segment $\mathcal{S}_u$ is not at the end of the path and adds a contribution of one to $\Lambda_\vartheta^{(a,a)}(S^0)$ for some $a \in \mathbb{Z}$, then it must also add one to

$$\Lambda_\vartheta^{(a+1,a)}(S^0) + \Lambda_\vartheta^{(a-1,a)}(S^0) + \Lambda_\vartheta^{(a,a+1)}(S^0) + \Lambda_\vartheta^{(a,a-1)}(S^0). \tag{7.37}$$

Lastly, for every type 2 segment, a contribution of two to $\Lambda_\vartheta^{(a,a)}(S^0)$ for some $a \in \mathbb{Z}$ is turned into a contribution of two to (7.37) in $\hat{S}^0$. In short, we observe that there is at most one $a_0 \in \mathbb{Z}$ (i.e., the one level, if any, where a type 1 segment occurs at the very end of the path of $S^0(u)$, $u \le \vartheta$) such that

$$\Lambda_\vartheta^{(a,a)}(S^0) \le \Lambda_\vartheta^{(a+1,a)}(\hat{S}^0) + \Lambda_\vartheta^{(a-1,a)}(\hat{S}^0) + \Lambda_\vartheta^{(a,a+1)}(\hat{S}^0) + \Lambda_\vartheta^{(a,a-1)}(\hat{S}^0)$$

for every $a \in \mathbb{Z} \setminus \{a_0\}$, and

$$\Lambda_\vartheta^{(a_0,a_0)}(S^0) \le \Lambda_\vartheta^{(a_0+1,a_0)}(\hat{S}^0) + \Lambda_\vartheta^{(a_0-1,a_0)}(\hat{S}^0) + \Lambda_\vartheta^{(a_0,a_0+1)}(\hat{S}^0) + \Lambda_\vartheta^{(a_0,a_0-1)}(\hat{S}^0) + 1$$

Given that $(z+1)^q \le 2^{q-1} z^q + 2^{q-1}$ for every $z, q \ge 1$, we obtain (7.36).

8. Proof of Theorem 2.20-(2)

This proof is very similar to that of Theorem 2.20-(1), except that we deal with random walks and Brownian motions conditioned on their endpoint.

8.1. *Step* 1: *Convergence of moments*

We begin with a generic mixed moment of traces, which we can always write in the form

$$\mathbf{E}\left[\prod_{i=1}^{k}\mathrm{Tr}\big[\hat{K}_n(t_i)\big]\right].$$

By Fubini's theorem, this is equal to

$$\int_{[0,(n+1)/m_n]^k}\mathbf{E}\left[\prod_{i=1}^{k}m_n\mathbf{P}\big[S^0(\vartheta_i)=0\big]F_{n,t_i}\big(S^{i;x_i^n,x_i^n}_{\vartheta_i}\big)\right]\mathrm{d}x_1\cdots\mathrm{d}x_k, \tag{8.1}$$

and by the trace formula in Remark 2.8 the corresponding continuum limit is

$$\mathbf{E}\left[\prod_{i=1}^{k}\mathrm{Tr}\big[\hat{K}(t_i)\big]\right]=\int_{\mathbb{R}_+}\mathbf{E}\left[\prod_{i=1}^{k}\frac{1}{\sqrt{2\pi t_i}}\mathbf{1}_{\{\tau_0(B^{i;x_i,x_i}_{t_i})>t_i\}}\mathrm{e}^{-\langle L_t(B^{i;x_i,x_i}_{t_i}),Q'\rangle}\right]\mathrm{d}x_1\cdots\mathrm{d}x_k,$$

where ϑ_i and x_i^n are as in Section 7.1, and

1. $S^{1;x_1^n,x_1^n}_{\vartheta_1},\dots,S^{k;x_k^n,x_k^n}_{\vartheta_k}$ are independent copies of random walk bridges $S^{x,x}_{\vartheta}$ with $x=x_i^n$ and $\vartheta=\vartheta_i$;
2. $B^{1;x_1,x_1}_{t_1},\dots,B^{k;x_k,x_k}_{t_k}$ are independent copies of standard Brownian bridges $B^{x,x}_t$ with $x=x_i$ and $t=t_i$.

Also, $S^{i;x_i^n,x_i^n}_{\vartheta_i}$ are independent of Q_n, and $B^{i;x_i,x_i}_{t_i}$ are independent of Q.

According to the local central limit theorem,

$$\lim_{n\to\infty}m_n\mathbf{P}\big[S^0(\vartheta_i)=0\big]=\frac{1}{\sqrt{2\pi t_i}},\quad 1\le i\le k.$$

Moreover, we have the following analog of Proposition 7.1:

Proposition 8.1. *The conclusion of Proposition* 7.1 *holds with every instance of* $S^{i;x_i^n}$ *replaced by* $S^{i;x_i^n,x_i^n}_{\vartheta_i}$, *and every instance of* $B^{i;x_i}$ *replaced by* $B^{i;x_i,x_i}_{t_i}$.

Proof. Arguing as in the proof of Proposition 7.1, this follows from coupling $S^{i;x_i^n,x_i^n}$ with a Brownian bridge $\tilde{B}^{i;x_i,x_i}_{3t_i/2}$ with variance $2/3$ using Theorem 5.2, and then defining $B^{i;x_i,x_i}_{t_i}(s):=\tilde{B}^{i;x_i,x_i}_{3t_i/2}(3s/2)$. □

With these results in hand, by repeating the arguments in Section 7.1.1, for any $x_1,\dots,x_k\ge 0$, we can find a coupling such that

$$\lim_{n\to\infty}m_n\mathbf{P}\big[S^0(\vartheta_i)=0\big]F_{n,t_i}\big(S^{i;x_i^n,x_i^n}_{\vartheta_i}\big)=\frac{1}{\sqrt{2\pi t_i}}\mathbf{1}_{\{\tau_0(B^{i;x_i,x_i}_{t_i})>t_i\}}\mathrm{e}^{-\langle L_t(B^{i;x_i,x_i}_{t_i}),Q'\rangle}$$

in probability for $1\le i\le k$. Then, by arguing as in Section 7.1.2 (more specifically, the estimate for (7.10)), we get the convergence

$$\lim_{n\to\infty}\mathbf{E}\left[\prod_{i=1}^{k}m_n\mathbf{P}\big[S^0(\vartheta_i)=0\big]F_{n,t_i}\big(S^{i;x_i^n,x_i^n}_{\vartheta_i}\big)\right]=\mathbf{E}\left[\prod_{i=1}^{k}\frac{1}{\sqrt{2\pi t_i}}\mathbf{1}_{\{\tau_0(B^{i;x_i,x_i}_{t_i})>t_i\}}\mathrm{e}^{-\langle L_t(B^{i;x_i,x_i}_{t_i}),Q'\rangle}\right]$$

pointwise in $x_1,\dots,x_k$ thanks to the following proposition, which we prove at the end of this section.

Proposition 8.2. *Let* $\vartheta=\vartheta(n,t):=\lfloor m_n^2t\rfloor$ *and* $x^n:=\lfloor m_nx\rfloor$ *for some* $t>0$ *and* $x\ge 0$. *For every* $C>0$ *and* $1\le q<3$,

$$\sup_{n\in\mathbb{N},x\ge 0}\mathbf{E}\left[\exp\left(\frac{C}{m_n}\sum_{a\in\mathbb{Z}}\frac{\Lambda^a_\vartheta(S^{x^n,x^n}_\vartheta)^q}{m_n^q}\right)\right]<\infty.$$

It only remains to prove that we can pass the limit outside the integral (8.1). We once again use [12, Theorem 2.24]. For this, it is enough to prove that, for n large enough, there exists constants $c_1, c_2, c_3, \bar{c}_4, \bar{c}_5 > 0$ such that

$$\int_{[0,(n+1)/m_n]^k} \left| \mathbf{E}\left[\prod_{i=1}^{k} F_{n,t_i}\left(S_{\vartheta_i}^{i;x_i^n,x_i^n}\right) \right] \right| \mathrm{d}x_1 \cdots \mathrm{d}x_k$$
$$\le \prod_{i=1}^{k} \int_0^{(n+1)/m_n} \mathbf{E}\left[\left| F_{n,t_i}\left(S_{\vartheta_i}^{i;x^n,x^n}\right) \right|^k \right]^{1/k} \mathrm{d}x$$
$$\le \left(c_1 \int_0^{n^{1-\varepsilon}/m_n} \left((1+x)^{-c_2\theta} + \mathrm{e}^{-c_3 n^{2\mathfrak{d}}} \right) \mathrm{d}x + \int_{n^{1-\varepsilon}/m_n}^{(n+1)/m_n} \left(\mathrm{e}^{-\bar{c}_4 n^{\alpha(1-\mathfrak{d})-\alpha\varepsilon}} + \mathrm{e}^{-2\bar{c}_5 n^{2\mathfrak{d}}} \right) \mathrm{d}x \right)^k, \tag{8.2}$$

where θ is taken large enough so that $(1+x)^{-c_2\theta}$ is integrable. To this end, for every $\vartheta \in \mathbb{N}$, let us define $\mathcal{R}_\vartheta(S_\vartheta^{0,0})$ as the range of $S_\vartheta^{0,0}$. By replicating the estimates in Section 7.1.3, we see that (8.2) is the consequence of the following two propositions, concluding the proof of the convergence of moments.

Proposition 8.3. *Let $\vartheta = \vartheta(n,t) := \lfloor m_n^2 t \rfloor$ for some $t > 0$. For every $C > 0$,*

$$\sup_{n\in\mathbb{N}} \mathbf{E}\left[\mathrm{e}^{C\mathcal{R}_\vartheta(S_\vartheta^{0,0})/m_n} \right] < \infty.$$

Proposition 8.4. *Let $\vartheta = \vartheta(n,t) := \lfloor m_n^2 t \rfloor$ for some $t > 0$. For small enough $\nu > 0$, there exists some $c > 0$ independent of n such that*

$$\mathbf{P}\left[\sum_{a\in\mathbb{Z}} \Lambda_\vartheta^{(a,a+b)}\left(S_\vartheta^{0,0}\right) < \nu m_n^2 \right] \le \mathrm{e}^{-cm_n^2}, \quad b \in \{-1,0,1\}.$$

Proof of Proposition 8.3. Let us define

$$\mathcal{M}\left(S_\vartheta^{0,0}\right) := \max_{0\le u\le\vartheta} \left| S_\vartheta^{0,0}(u) \right|.$$

It is easy to see that $\mathcal{R}_\vartheta(S_\vartheta^{0,0}) \le 2\mathcal{M}(S_\vartheta^{0,0})$, and thus it suffices to prove that the exponential moments of $\mathcal{M}(S_\vartheta^{0,0})/m_n$ are uniformly bounded in n.

Let $\mathfrak{S}$ be as in Definition 7.5, and define

$$\mathcal{M}\left(\mathfrak{S}_v^{0,0}\right) := \max_{0\le u\le v} \left| \mathfrak{S}_u^{0,0} \right|, \quad v \in 2\mathbb{N}_0.$$

According to [20, (4.7)] (up to normalization, the quantity denoted $\tilde{M}(N,\tilde{T})$ in [20, (4.7)] is essentially the same as what we denote by $\mathcal{M}(\mathfrak{S}_\vartheta^{0,0})$; see the definition of the former on [20, Page 2302]) we know that for every $0 < q < 2$ and $C > 0$,

$$\sup_{u\in\mathbb{N}} \mathbf{E}\left[\mathrm{e}^{C(\mathcal{M}(\mathfrak{S}_u^{0,0})/\sqrt{u})^q} \right] < \infty. \tag{8.3}$$

Let us define

$$\mathcal{H}\left(S_u^{0,0}\right) := \sum_{a\in\mathbb{Z}} \Lambda_\vartheta^{(a,a)}\left(S_u^{0,0}\right), \quad u \in 2\mathbb{N}_0. \tag{8.4}$$

For any $h \in \mathbb{N}_0$, we can couple the bridges of S and $\mathfrak{S}$ in such a way that

$$\left(S_\vartheta^{0,0}(u) | \mathcal{H}\left(S_\vartheta^{0,0}\right) = h\right) = \mathfrak{S}_{\vartheta-h}^{0,0}\left(u - \mathcal{H}\left(S_u^{0,0}\right)\right).$$

In words, we obtain $\mathfrak{S}_{\vartheta-\cdot}^{0,0}$ from $S_\vartheta^{0,0}(u)$ by removing all segments that visit self-edges. Since visits to self-edges do not contribute to the magnitude of $S_\vartheta^{0,0}$,

$$\left(\mathcal{M}\left(S_\vartheta^{0,0}\right) | \mathcal{H}\left(S_\vartheta^{0,0}\right) = h\right) = \mathcal{M}\left(\mathfrak{S}_{\vartheta-h}^{0,0}\right).$$

Thus, (8.3) for $q = 1$ implies that

$$\sup_{n\in\mathbb{N}} \mathbf{E}\big[\mathrm{e}^{C\mathcal{M}(S_\vartheta^{0,0})/m_n}\big] = \sup_{n\in\mathbb{N}} \sum_{h\in\mathbb{N}_0} \mathbf{E}\big[\mathrm{e}^{C\mathcal{M}(S_\vartheta^{0,0})/m_n} | \mathcal{H}\big(S_\vartheta^{0,0}\big) = h\big]\mathbf{P}\big[\mathcal{H}\big(S_\vartheta^{0,0}\big) = h\big]$$

$$\leq \sup_{n\in\mathbb{N}} \sup_{1\leq u\leq\vartheta} \mathbf{E}\big[\mathrm{e}^{(\sqrt{u}/m_n)C\mathcal{M}(\mathfrak{S}_u^{0,0})/\sqrt{u}}\big] < \infty \tag{8.5}$$

for every $C > 0$, as desired. □

Proof of Proposition 8.4. Note that

$$\mathbf{P}\Bigg[\sum_{a\in\mathbb{N}_0} \Lambda_\vartheta^{(a,a+b)}\big(S_\vartheta^{0,0}\big) < \nu m_n^2\Bigg] \leq \mathbf{P}\Bigg[\sum_{a\in\mathbb{N}_0} \Lambda_\vartheta^{(a,a+b)}\big(S^0\big) < \nu m_n^2\Bigg]\mathbf{P}\big[S^0(\vartheta) = 0\big]^{-1}.$$

By the local central limit theorem, $\mathbf{P}[S^0(\vartheta) = 0]^{-1} = O(m_n)$, and thus the result follows from the same binomial concentration argument used for (7.20). □

Proof of Proposition 8.2. In similar fashion to the proof of Proposition 7.3, it suffices to prove that the exponential moments of

$$\frac{1}{m_n}\sum_{a\in\mathbb{Z}} \frac{(\Lambda_\vartheta^{(a,a-1)}(S_\vartheta^{0,0}) + \Lambda_\vartheta^{(a,a+1)}(S_\vartheta^{0,0}))^q}{m_n^q} \quad \text{and} \quad \frac{1}{m_n}\sum_{a\in\mathbb{Z}} \frac{\Lambda_\vartheta^{(a,a)}(S_\vartheta^{0,0})^q}{m_n^q} \tag{8.6}$$

are uniformly bounded in n. We start with the first term in (8.6). Under the coupling in the proof of Proposition 8.3,

$$\bigg(\sum_{a\in\mathbb{Z}} \big(\Lambda_\vartheta^{(a,a-1)}\big(S_\vartheta^{0,0}\big) + \Lambda_\vartheta^{(a,a+1)}\big(S_\vartheta^{0,0}\big)\big)^q | \mathcal{H}\big(S_\vartheta^{0,0}\big) = h\bigg) \leq \sum_{a\in\mathbb{Z}} \Lambda_{\vartheta-h}^a\big(\mathfrak{S}_{\vartheta-h}^{0,0}\big)^q$$

for every $h \in \mathbb{N}_0$. By conditioning on $\mathcal{H}(S_\vartheta^{0,0})$ as in (8.5), we need only prove that

$$\sup_{n\in\mathbb{N}} \mathbf{E}\Bigg[\exp\Bigg(\frac{C}{m_n}\sum_{a\in\mathbb{Z}} \frac{\Lambda_\vartheta^a(\mathfrak{S}_\vartheta^{0,0})^q}{m_n^q}\Bigg)\Bigg] < \infty.$$

By using (7.35) in the case $b = 0$ (i.e., [20, (4.19)]), this follows from (8.3). With this established, the exponential moments of the second term in (8.6) can be controlled by using the same argument in Section 7.3.2 (the path transformation used therein does not change the endpoint of the path that is being modified; hence the transformed version of $S_\vartheta^{0,0}$ is a random walk bridge). □

8.2. *Step* 2: *Convergence in distribution*

The convergence in distribution follows from the convergence of mixed moments by using the same truncation/stochastic domination argument as in Section 7.2.

9. Proof of Theorem 2.21

This follows roughly the same steps as the proof of Theorem 2.20-(1).

9.1. *Step* 1: *Convergence of moments*

9.1.1. *Expression for mixed moments and convergence result*
By Fubini's theorem, any mixed moment $\mathbf{E}[\prod_{i=1}^k \langle f_i, \hat{K}_n^w(t_i)g_i\rangle]$ can be written as

$$\int_{[0,(n+1)/m_n)^k} \Bigg(\prod_{i=1}^k f_i(x_i)\Bigg)\mathbf{E}\Bigg[\prod_{i=1}^k F_{n,t_i}\big(T^{i;x_i^n}\big)m_n \int_{T^{i;x_i^n}(\vartheta_i)/m_n}^{(T^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y\Bigg]\mathrm{d}x_1\cdots\mathrm{d}x_k, \tag{9.1}$$

and the corresponding continuum limit is

$$\mathbf{E}\Bigg[\prod_{i=1}^k \langle f_i, \hat{K}(t_i) g_i\rangle\Bigg] = \int_{\mathbb{R}_+^k}\Bigg(\prod_{i=1}^k f_i(x_i)\Bigg)\mathbf{E}\Bigg[\prod_{i=1}^k \mathrm{e}^{-\langle L_{t_i}(X^{i;x_i}),Q'\rangle - w\mathfrak{L}^0_{t_i}(X^{i;x_i})} g_i\big(X^{i;x_i}(t_i)\big)\Bigg]\,\mathrm{d}x_1\cdots\mathrm{d}x_k, \tag{9.2}$$

where ϑ_i and x_i^n are as in Section 7.1,

1. $T^{1;x_1^n},\dots,T^{k;x_k^n}$ are independent copies of the Markov chain T with respective starting points $x_1^n,\dots,x_k^n$; and
2. $X^{1;x_1},\dots,X^{k;x_k}$ are independent copies of X with respective starting points $x_1,\dots,x_k$.

$T^{i;x_i^n}$ are independent of Q_n and $X^{i;x_i}$ are independent of Q.

Proposition 9.1. *Let $x_1,\dots,x_n\ge 0$ be fixed. The following limits hold jointly in distribution over $1\le i\le k$:*

1. $\lim_{n\to\infty}\sup_{0\le s\le t_i}\big|\frac{T^{i;x_i^n}(\lfloor m_n^2(3s/2)\rfloor)}{m_n} - X^{i;x_i}(s)\big| = 0.$
2. $\lim_{n\to\infty}\sup_{y>0}\big|\frac{\Lambda^{(y_n,\bar{y}_n)}_{\vartheta_i}(T^{i;x_i^n})}{m_n}(1-\frac{1}{2}\mathbf{1}_{\{(y_n,\bar{y}_n)=(0,0)\}}) - \frac{1}{2}L^y_{t_i}(X^{i;x_i})\big| = 0$, *jointly in $(y_n,\bar{y}_n)_{n\in\mathbb{N}}$ as in (5.1).*
3. $\lim_{n\to\infty}\big|\frac{\Lambda^{(0,0)}_{\vartheta_i}(T^{i;x_i^n})}{m_n} - 2\mathfrak{L}^0_{t_i}(X^{i;x_i})\big| = 0.$
4. $\lim_{n\to\infty} m_n\int_{T^{i;x_i^n}(\vartheta_i)/m_n}^{(T^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y = g_i(X^{i;x_i}(t)).$
5. *The convergences in (2.17).*
6. $\lim_{n\to\infty}\sum_{a\in\mathbb{N}_0}\frac{\Lambda^{(a_E,\bar{a}_E)}_{\vartheta_i}(X^{i;x_i^n})}{m_n}\frac{\xi_n^E(a)}{m_n} = \frac{1}{2}\int_{\mathbb{R}_+} L^y_{t_i}(T^{i;x_i})\,\mathrm{d}W^E(y)$ *for $E\in\{D,U,L\}$, where, for every $a\in\mathbb{N}_0$, $(a_E,\bar{a}_E)$ are as in (7.4).*

Proof. Arguing as in Proposition 7.1, the result follows by using Theorem 6.2 to couple the $T^{i;x_i^n}$ with reflected Brownian motions with variance $2/3$, $\tilde{X}^{i;x_i^n}$, and then defining $X^{i;x_i^n}(s) := \tilde{X}^{i;x_i^n}(3s/2)$, which yields a standard reflected Brownian motion such that $L^y_{3t_i/2}(\tilde{X}^{i;x_i}) = \frac{3}{2}L^y_{t_i}(X^{i;x_i})$ and $\mathfrak{L}^0_{3t_i/2}(\tilde{X}^{i;x_i}) = \frac{3}{2}\mathfrak{L}^0_{t_i}(X^{i;x_i})$. □

9.1.2. *Convergence inside the expected value*

We begin with the proof that for every $x_1,\dots,x_k\ge 0$, there is a coupling such that

$$\lim_{n\to\infty}\prod_{i=1}^k F_{n,t_i}\big(T^{i;x_i^n}\big) m_n\int_{T^{i;x_i^n}(\vartheta_i)/m_n}^{(T^{i;x_i^n}(\vartheta_i)+1)/m_n} g_i(y)\,\mathrm{d}y = \prod_{i=1}^k \mathrm{e}^{-\langle L_{t_i}(X^{i;x_i}),Q'\rangle - w\mathfrak{L}^0_{t_i}(X^{i;x_i})} g_i\big(X^{i;x_i}(t_i)\big) \tag{9.3}$$

in probability. Proposition 9.1 provides a coupling such that

$$\prod_{i=1}^k \mathbf{1}_{\{\tau^{(n)}(T^{i;x_i^n})>\vartheta_i\}}\Bigg(\prod_{a\in\mathbb{N}}\bigg(1-\frac{D_n(a)}{m_n^2}\bigg)^{\Lambda^{(a,a)}_{\vartheta_i}(T^{i;x_i^n})}\Bigg)$$
$$\cdot\Bigg(\prod_{a\in\mathbb{N}_0}\bigg(1-\frac{U_n(a)}{m_n^2}\bigg)^{\Lambda^{(a,a+1)}_{\vartheta_i}(T^{i;x_i^n})}\bigg(1-\frac{L_n(a)}{m_n^2}\bigg)^{\Lambda^{(a+1,a)}_{\vartheta_i}(T^{i;x_i^n})}\Bigg)$$

converges in probability to $\prod_{i=1}^k \mathrm{e}^{-\langle L_{t_i}(X^{i;x_i}),Q'\rangle}$. Combining this with Proposition 9.1-(4), it only remains to show that

$$\lim_{n\to\infty}\prod_{i=1}^k\bigg(1-\frac{(1-w_n)}{2}-\frac{D_n(0)}{2m_n^2}\bigg)^{\Lambda^{(0,0)}_{\vartheta_i}(T^{i;x_i^n})} = \prod_{i=1}^k \mathrm{e}^{-w\mathfrak{L}^0_{t_i}(X^{i;x_i})}.$$

To this effect, the Taylor expansion $\log(1+z) = z + O(z^2)$ yields

$$\bigg(1-\frac{(1-w_n)}{2}-\frac{D_n(0)}{2m_n^2}\bigg)^{\Lambda^{(0,0)}_{\vartheta_i}(T^{i;x_i^n})}$$
$$=\exp\bigg(-\Lambda^{(0,0)}_{\vartheta_i}\big(T^{i;x_i^n}\big)\bigg(\frac{(1-w_n)}{2}+\frac{D_n(0)}{2m_n^2}+O\bigg(\frac{(1-w_n)^2}{4}+\frac{D_n(0)^2}{2m_n^4}\bigg)\bigg)\bigg).$$

By Proposition 9.1-(3) and Assumption 2.2,

$$\lim_{n\to\infty}\Lambda^{(0,0)}_{\vartheta_i}\big(T^{i;x_i^n}\big)\left(\frac{(1-w_n)}{2}+\frac{D_n(0)}{2m_n^2}\right)=w\mathfrak{L}^0_{t_i}\big(X^{i;x_i}\big)$$

and

$$\lim_{n\to\infty}\Lambda^{(0,0)}_{\vartheta_i}\big(T^{i;x_i^n}\big)\left(\frac{(1-w_n)^2}{4}+\frac{D_n(0)^2}{2m_n^4}\right)=0$$

almost surely, as desired.

9.1.3. *Convergence of the expected value*
Next we prove

$$\begin{aligned}\lim_{n\to\infty}\mathbf{E}\left[\prod_{i=1}^k F^w_{n,t_i}\big(T^{i;x_i^n}\big)m_n\int_{T^{i;x_i^n}(\vartheta_i)/m_n}^{(T^{i;x_i^n}(\vartheta_i)+1)/m_n}g_i(y)\,\mathrm{d}y\right]\\ =\mathbf{E}\left[\prod_{i=1}^k \mathrm{e}^{-\langle L_{t_i}(X^{i;x_i}),Q'\rangle-w\mathfrak{L}^0_{t_i}(X^{i;x_i})}g_i\big(X^{i;x_i}(t_i)\big)\right]\end{aligned}\tag{9.4}$$

pointwise in $x_1,\ldots,x_k\geq 0$. Similarly to Section 7.1.2, this is done by combining (9.3) with the uniform integrability estimate

$$\sup_{n\geq N}\mathbf{E}\big[\big|F^w_{n,t_i}\big(T^{i;x_i^n}\big)\big|^{2k}\big]<\infty,\quad 1\leq i\leq k\tag{9.5}$$

for large enough N. To achieve this we combine Proposition 6.8 and the following:

Proposition 9.2. *Let $\vartheta=\vartheta(n,t)=\lfloor m_n^2t\rfloor$ for some $t>0$. For every $C>0$ and $1\leq q<3$,*

$$\sup_{n\in\mathbb{N},x\geq 0}\mathbf{E}\left[\exp\left(\frac{C}{m_n}\sum_{a\in\mathbb{N}}\frac{\Lambda^a_\vartheta(T^{x^n})^q}{m_n^q}\right)\right]<\infty.$$

Proof. If we couple X and S as in Definition 6.9, then we see that

$$\begin{aligned}\mathbf{E}\left[\exp\left(\frac{C}{m_n}\sum_{a\in\mathbb{N}}\frac{\Lambda^a_\vartheta(T^{x^n})^q}{m_n^q}\right)\right]&\leq\mathbf{E}\left[\exp\left(\frac{2^{q-1}C}{m_n}\sum_{a\in\mathbb{Z}\setminus\{0\}}\frac{\Lambda^a_{\varrho^{x^n}(\vartheta)}(S^0)^q}{m_n^q}\right)\right]\\ &\leq\mathbf{E}\left[\exp\left(\frac{2^{q-1}C}{m_n}\sum_{a\in\mathbb{Z}}\frac{\Lambda^a_\vartheta(S^0)^q}{m_n^q}\right)\right].\end{aligned}$$

Thus Proposition 9.2 follows directly from Proposition 7.3. □

Indeed, the arguments of Section 7.1.2 show that the contribution of the terms of the form (4.14) and (4.15) to (9.5) can be controlled by Proposition 9.2. Thus, it suffices to prove that for every $C>0$, there is some $N\in\mathbb{N}$ large enough so that

$$\sup_{n\geq N,x\geq 0}\mathbf{E}\left[\left|1-\frac{(1-w_n)}{2}-\frac{D_n(0)}{2m_n^2}\right|^{C\Lambda^{(0,0)}_{\vartheta_i}(T^{i;x_i^n})}\right]<\infty.\tag{9.6}$$

By using the bound $|1-z|\leq\mathrm{e}^{|z|}$, it suffices to control the exponential moments of

$$\Lambda^{(0,0)}_\vartheta\big(T^{x^n}\big)|1-w_n|\quad\text{and}\quad\frac{\Lambda^{(0,0)}_\vartheta(T^{x^n})|D_n(0)|}{m_n^2}.\tag{9.7}$$

We begin with the first term in (9.7). According to Proposition 6.8, for every $C>0$,

$$\sup_{n\in\mathbb{N},x\geq 0}\mathrm{E}\big[\mathrm{e}^{C\Lambda_\vartheta^{(0,0)}(T^{x^n})/m_n}\big]<\infty.$$

Thus, given that $|1-w_n|=O(m_n^{-1})$ by Assumption 2.2, we conclude that

$$\sup_{n\in\mathbb{N},x\geq 0}\mathrm{E}\big[\mathrm{e}^{C\Lambda_\vartheta^{(0,0)}(T^{x^n})|1-w_n|}\big]<\infty.$$

Let us now consider the second term in (9.7). By the tower property and Assumption 2.17, there exists $\bar{C},\bar{c}>0$ independent of n such that

$$\mathbf{E}\big[\mathrm{e}^{C(\Lambda_\vartheta^{(0,0)}(T^{x^n})/m_n^{3/2})(|D_n(0)|/m_n^{1/2})}\big]\leq\bar{C}\mathbf{E}\big[\mathrm{e}^{\bar{c}(C^2/m_n)(\Lambda_\vartheta^{(0,0)}(T^{x^n})/m_n)^2}\big].$$

Since $\bar{c}(C^2/m_n)\to 0$, it follows from Proposition 6.8 that

$$\sup_{n\geq N,x\geq 0}\mathbf{E}\big[\mathrm{e}^{\bar{c}(C^2/m_n)(\Lambda_\vartheta^{(0,0)}(T^{x^n})/m_n)^2}\big]<\infty$$

for large enough N, concluding the proof of (9.6).

9.1.4. *Convergence of the integral*

With (9.1.3) established, once more we aim to prove that (9.1) converges to (9.2) by using [12, Theorem 2.24]. Similarly to Section 7.1.3, for this we need upper bounds of the form

$$\mathbf{E}\big[\big|F_{n,t_i}^w\big(T^{i;x^n}\big)\big|^k\big]^{1/k}\leq c_1\big((1+x)^{-c_2\theta}+\mathrm{e}^{-c_3n^{2\mathfrak{d}}}\big),\quad x\in\big[0,n^{1-\varepsilon}/m_n\big)\tag{9.8}$$

and

$$\mathbf{E}\big[\big|F_{n,t_i}^w\big(T^{i;x^n}\big)\big|^k\big]^{1/k}\leq\mathrm{e}^{-\bar{c}_4n^{\alpha(1-\mathfrak{d})-\alpha\varepsilon}}+\mathrm{e}^{-\bar{c}_5n^{2\mathfrak{d}}},\quad x\in\big[n^{1-\varepsilon}/m_n,(n+1)/m_n\big),\tag{9.9}$$

where $\varepsilon,c_1,c_2,c_3,\bar{c}_4,\bar{c}_5>0$ are independent of n and $\theta>0$ is taken large enough so that $(1+x)^{-c_2\theta}$ is integrable.

We begin with $x\in[0,n^{1-\varepsilon}/m_n)$. Replicating the analysis leading up to (7.15) and (7.16) leads to bounding $\mathbf{E}[|F_{n,t_i}^w(T^{i;x^n})|^k]^{1/k}$ by the product of the following five terms:

$$\mathbf{E}\bigg[\bigg|1-\frac{(1-w_n)}{2}-\frac{D_n(0)}{2m_n^2}\bigg|^{7k\Lambda_{\vartheta_i}^{(0,0)}(T^{i;x_i^n})}\bigg]^{1/7k},\tag{9.10}$$

$$\prod_{E\in\{U,L\}}\mathbf{E}\bigg[\prod_{a\in\mathbb{N}_0}\bigg|1-\frac{\xi_n^E(a)}{m_n^2-V_n^E(a)}\bigg|^{7k\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(T^{i;x^n})}\bigg]^{1/7k},\tag{9.11}$$

$$\mathbf{E}\bigg[\prod_{a\in\mathbb{N}}\bigg|1-\frac{\xi_n^D(a)}{m_n^2-V_n^D(a)}\bigg|^{7k\Lambda_{\vartheta_i}^{(a,a)}(T^{i;x^n})}\bigg]^{1/7k},\tag{9.12}$$

$$\prod_{E\in\{U,L\}}\mathbf{E}\bigg[\prod_{a\in\mathbb{N}_0}\bigg|1-\frac{V_n^E(a)}{m_n^2}\bigg|^{7k\Lambda_{\vartheta_i}^{(a_E,\bar{a}_E)}(T^{i;x^n})}\bigg]^{1/7k},\tag{9.13}$$

$$\mathbf{E}\bigg[\prod_{a\in\mathbb{N}}\bigg|1-\frac{V_n^D(a)}{m_n^2}\bigg|^{7k\Lambda_{\vartheta_i}^{(a,a)}(T^{i;x^n})}\bigg]^{1/7k}.\tag{9.14}$$

Suppose without loss of generality that V_n^D satisfies (2.13). (9.10) can be controlled with (9.6); (9.11) and (9.12) can be controlled with Proposition 9.2; and (9.13) can be controlled with (2.11). For (9.14), up to a constant independent of n, we get from (2.13) the upper bound

$$\mathbf{E}\bigg[\exp\bigg(-\frac{7k\theta}{m_n^2}\sum_{a\in\mathbb{N}}\log\big(1+|a|/m_n\big)\Lambda_{\vartheta_i}^{(a,a)}\big(T^{i;x^n}\big)\bigg)\bigg]^{1/7k}.\tag{9.15}$$

Let us couple $T^{i;x^n}$ and $S^{x^n} = x^n + S^0$ as in Definition 6.9. The same argument used to control (7.17) implies that (9.15) is bounded above by the product of

$$\mathbf{E}\left[\exp\left(-\frac{14k\theta\log(1+x)}{m_n^2}\sum_{a\in\mathbb{Z}\setminus\{0\}}\Lambda^{(a,a)}_{\varrho^{x^n}(\vartheta_i)}\big(S^0\big)\right)\right]^{1/14k}, \tag{9.16}$$

$$\mathbf{E}\left[\exp\left(\frac{14k\theta}{m_n^2}\sum_{0\le u\le\varrho^{x^n}(\vartheta_i)}\frac{|S^0(u)|}{m_n}\right)\right]^{1/14k}. \tag{9.17}$$

Since $\varrho^{x^n}(\vartheta_i) \le \vartheta_i$, we can prove that (9.17) is bounded by a constant independent of n by using (5.12) directly. As for (9.16), we have the following proposition:

Proposition 9.3. *Let $\vartheta = \vartheta(n,t) := \lfloor m_n^2 t\rfloor$ for some $t > 0$. For every $x \ge 0$, let us couple T^{x^n} and $S^{x^n} := x^n + S^0$ as in Definition 6.9. For small enough $\nu > 0$, there exists $C, c > 0$ independent of x and n such that*

$$\sup_{x\ge0}\mathbf{P}\left[\sum_{a\in\mathbb{Z}\setminus\{0\}}\Lambda^{(a,a+b)}_{\varrho^{x^n}(\vartheta)}\big(S^0\big) < \nu m_n^2\right] \le C\mathrm{e}^{-cm_n^2}, \quad b\in\{-1,0,1\}.$$

Proof. By Proposition 6.8, for any $0<\delta<1$, we can find $\bar C, \bar c > 0$ such that

$$\sup_{x\ge0}\mathbf{P}\big[\Lambda^{(0,0)}_{\vartheta}\big(T^{x^n}\big)\ge\delta\vartheta\big] \le \bar C\mathrm{e}^{-\bar c m_n^2}.$$

Given that $\vartheta - \varrho^{x^n}(\vartheta) \le \Lambda^{(0,0)}_{\vartheta}(T^{x^n})$, it suffices to prove that

$$\sup_{x\ge0}\mathbf{P}\left[\sum_{a\in\mathbb{Z}\setminus\{0\}}\Lambda^{(a,a+b)}_{(1-\delta)\vartheta}\big(S^0\big) < \nu m_n^2\right] \le C\mathrm{e}^{-cm_n^2}, \quad b\in\{-1,0,1\}$$

for large enough N. This follows by Hoeffding's inequality. □

By arguing as in the passage following (7.20), Proposition 9.3 implies that (9.16) is bounded above by $c_1((1+x)^{-c_2\theta} + \mathrm{e}^{-c_3 n^{2\eth}})$ for $c_1, c_2, c_3 > 0$ independent of n (and c_2 independent of θ), hence (9.8) holds.

We now prove (9.9). Let $x\in[n^{1-\varepsilon}/m_n, (n+1)/m_n)$. Assuming without loss of generality that V_n^D satisfies (2.14), by arguing as in Section 7.1.3, we get that $\mathbf{E}[|F^w_{n,t_i}(T^{i;x^n})|^k]^{1/k}$ is bounded by the product of the four terms

$$\mathbf{E}\left[\left|1-\frac{(1-w_n)}{2}-\frac{D_n(0)}{2m_n^2}\right|^{5k\Lambda^{(0,0)}_{\vartheta_i}(T^{i;x^n_i})}\right]^{1/5k}\cdot\mathbf{E}\left[\prod_{a\in\mathbb{N}}\left(1-\frac{\kappa(Cn^{1-\varepsilon}/m_n)^\alpha}{m_n^2}\right)^{5k\Lambda^{(a,a)}_{\vartheta_i}(T^{i;x^n})}\right]^{1/5k}$$

$$\cdot\,\mathbf{E}\left[\prod_{a\in\mathbb{N}}\left(1+\frac{2|\xi_n^D(a)|}{m_n^2}\right)^{5k\Lambda^{(a,a)}_{\vartheta_i}(T^{i;x^n})}\right]^{1/5k}\cdot\prod_{E\in\{U,L\}}\mathbf{E}\left[\prod_{a\in\mathbb{N}_0}\left(1+\frac{|\xi_n^E(a)|}{m_n^2}\right)^{5k\Lambda^{(a_E,\bar a_E)}_{\vartheta_i}(T^{i;x^n})}\right]^{1/5k}.$$

By combining Propositions 9.2 and 9.3 with (9.6), the same arguments used in Section 7.1.3 yields (9.9), concluding the proof of the convergence of moments.

9.2. *Step* 2: *Convergence in distribution*

The convergence in joint distribution follows from the convergence of moments by using the same truncation/stochastic dominance argument Section 7.2, thus concluding the proof of Theorem 2.21.

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