Calibration procedures for linear regression models with multiplicative distortion measurement errors

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Abstract. This paper considers linear regression models when neither the response variable nor the covariates can be directly observed, but are measured with multiplicative distortion measurement errors. To eliminate the effect caused by the distortion, we propose two calibration procedures: the conditional absolute mean calibration and the conditional variance calibration. Both calibration procedures avoid using the nonzero expectation conditions imposed on the variables in the literature. Utilizing these calibrated variables, the least squares estimators are obtained, associated with their asymptotic results. The asymptotic normal confidence intervals and empirical likelihood confidence intervals are also proposed. Simulation studies are conducted to compare the proposed method.

1 Introduction

In many applications of regression analysis, variables of interest may be observed with measurement errors. A multiplicative distortion errors-in-variables linear regression model is written as

$$Y = \alpha_0 + X^{\mathrm{T}} \boldsymbol{\beta}_0 + \epsilon, \qquad \widetilde{Y} = \phi(U)Y, \qquad \widetilde{X} = \boldsymbol{\psi}(U)X, \qquad (1.1)$$

where Y is an unobservable response variable, $X = (X_1, X_2, ..., X_p)^T$ is an unobservable continuous covariate vector (the superscript "T" denotes the transpose operator throughout this paper), \tilde{Y} and \tilde{X} are the observed response and covariate vector. The parameter β_0 is unknown $p \times 1$ parameter vector on a compact parameter space $\Theta_{\beta} \subset \mathbb{R}^p$. The model error ϵ satisfies $E(\epsilon|X) = 0$ and $E(\epsilon^2|X) < \infty$. The confounding variable $U \in \mathbb{R}^1$ is observable and independent of (X, Y). The multiplicative distortion function $\psi(\cdot)$ is a $p \times p$ -diagonal matrix: diag $(\psi_1(\cdot), \ldots, \psi_p(\cdot))$. Moreover, we assume that $(\phi(\cdot), \psi_r(\cdot)), r = 1, \ldots, p$, are unknown continuous distortion functions. It is noted that $\phi(\cdot)$ and $\psi(\cdot)$ distort unobserved Y and X in a multiplicative relation.

Multiplicative distortion measurement data usually occur in biomedical research and health related studies. The collected data are often needed to adjust for some measures like body mass index, body surfaces area, height or weight. Kaysen et al. (2002) studied the relation between fibrinogen level and serum transferrin level among haemodialysis patients, and realized that BMI plays the role of confounding variable U that may contaminate the fibrinogen level and the serum transferrin level simultaneously. To eliminate the potential bias, Kaysen et al. (2002) numerically normalized the observed data with the confounding variable-BMI, and this procedure in analyzing the collected dataset implies that there may exist a multiplicative fashion between the primary unobserved variables and the confounding variable. As claimed in Şentürk and Müller (2005, 2006), because the exact relations between the confounding variable and primary observed variables are usually unknown, the

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way of simply dividing the confounding variable on the primary observed variables may lead to an inconsistent estimator of the parameter in the model. As a remedy, Şentürk and Müller (2005, 2006) introduced a flexible multiplicative adjustment by adopting some unknown smooth distortion functions $\phi(u)$ and $\psi_r(u)$ on the confounding variable. Recently, a number of authors have studied the multiplicative distortion measurement errors models in various parametric or semi-parametric settings: Cui et al. (2009), Nguyen, Şentürk and Carroll (2008), Li, Lin and Cui (2010), Nguyen and Şentürk (2007, 2008), Şentürk and Nguyen (2009), Li, Zhang and Feng (2016). These above literature focus on the nonzero expectation assumptions of the underlying unobserved variables. Delaigle, Hall and Zhou (2016), Zhao and Xie (2018), Li et al. (2018), Lu, Li and Feng (2019), Zhang, Lin and Li (2019), Feng, Gai and Zhang (2019), Xie and Zhu (2019) estimated the distortion functions by using the kernel smoothing methods with absolute value of distorting variables and the confounding variable, and these techniques need not impose the nonzero expectation conditions on the underlying unobserved variables.

In this paper, we first use the conditional absolute mean calibration to obtain the calibrated variables. With these calibrated variables, a least squares estimator is obtained. We study the asymptotic normality of this estimator and further estimate the asymptotic covariance matrix to construct asymptotic confidence intervals. Next, we propose another calibration method by using of conditional variance function between the distorted variables and confounding variable. Using the conditional variance calibration, the least squares estimator is proposed and compared its asymptotic covariance matrix with the one obtained from the conditional absolute mean calibration, and we further discuss the efficiency between these estimators. After estimating the asymptotic covariance matrix obtained from the conditional variance calibration, asymptotic confidence intervals are constructed. Lastly, we made use the empirical likelihood statistic asymptotically follows a centered chi-squared distribution. Monte Carlo simulation experiments are conducted to examine the performance of the proposed calibration procedures.

The reminder of the paper is organized as follows: In Section 2, we introduce two calibration procedures and present some asymptotic results. In Section 3, we provide to use the asymptotic normality and asymptotic empirical likelihood to construct the confidence intervals. In Section 4, we conduct Monte Carlo simulation experiments to compare the performance of estimators and confidence intervals obtained from two calibration procedures. An analysis of the air quality dataset will be reported on Section 5. All the technical proofs of the asymptotic results are given in "online-supplementary materials" (Zhang and Zhou, 2020).

2 Estimation methods and asymptotic results

2.1 Conditional absolute mean calibration

In this subsection, we first calibrate unobserved Y and X by using the observed i.i.d. sample $\{\widetilde{Y}_i, \widetilde{X}_i, U_i\}_{i=1}^n$. To ensure identifiability, it is assumed that

$$E[\phi(U)] = 1, \qquad E[\psi_r(U)] = 1, \quad r = 1, \dots, p.$$
 (2.1)

These identifiability conditions (2.1) are introduced by Şentürk and Müller (2006, 2005), and it is analogous to the classical additive measurement errors: E(e) = 0 for W = X + e, where W is error-prone and X is error-free (Li, Zhang and Feng, 2016, Tomaya and de Castro, 2018, Yang, Tong and Li, 2019).

Under the independence condition between U and (Y, X), the identifiability conditions (2.1) entailed that

$$E[\widetilde{Y}|U=u] = \phi(u)E(Y),$$

$$E[\widetilde{X}_r|U=u] = \psi_r(u)E(X_r), \quad r = 1, \dots, p.$$
(2.2)

It is seen that we need the condition that U is independent of (Y, X), and this is the usual assumptions of the distortion measurement errors model. If the independence condition fails, then $E(\tilde{Y}|U=u) = \phi(u)E(Y|U=u)$ and $E(\tilde{X}_r|U=u) = \psi_r(u)E(X_r|U=u)$, r = 1, ..., p. In fact, both $\phi(u)$ and E(Y|U=u) are unknown, and also Y is unobservable. So, we could not get an estimate of $\phi(u)$ and E(Y|U=u) because of the model unidentifiability. The independence condition between U and (Y, X) makes the model identifiability, and this is also a usual assumption of the distortion measurement errors models literature.

From (2.2), Cui et al. (2009) assumed the conditions $E(Y) \neq 0$, $E(X_r) \neq 0$ and obtained that $\phi(u) = \frac{E[\tilde{Y}|U=u]}{E(Y)}$, $\psi_r(u) = \frac{E[\tilde{X}_r|U=u]}{E(X_r)}$. In the population level, Cui et al. (2009) proposed to use the calibrated variables $Y = \frac{\tilde{Y}}{\phi(U)}$ and $X_r = \frac{\tilde{X}_r}{\psi_r(U)}$ for further statistical analysis. Such conditions of nonzero expectations of variables $(E(Y) \neq 0, E(X_r) \neq 0)$ are slightly strict. According to the assumptions of distortion functions in condition (C1), we have $E(|\tilde{Y}||U) = \phi(U)E(|Y|)$ and $E(|\tilde{X}_r||U) = \psi_r(U)E(|X_r|)$, r = 1, ..., p. Thus, Delaigle, Hall and Zhou (2016), Zhao and Xie (2018), Feng, Gai and Zhang (2019), Xie and Zhu (2019) used the following conditional absolute mean relations:

$$m_{|\widetilde{Y}|}(u) \stackrel{\text{def}}{=} E[|\widetilde{Y}||U=u] = \phi(u)E(|Y|), \qquad (2.3)$$

$$m_{|\widetilde{X}_r|}(u) \stackrel{\text{def}}{=} E\big[|\widetilde{X}_r||U=u\big] = \psi_r(u)E\big(|X_r|\big), \quad r = 1, \dots, p.$$
(2.4)

From (2.3)–(2.4), the conditions E(|Y|) = 0 and $E(|X_r|) = 0$ are equivalent to P(Y = 0) = 1, $P(X_r = 0) = 1$, respectively. Thus, we can avoid the nonzero expectation conditions (Cui et al., 2009) and use (2.3)–(2.4) to obtain estimators of $\phi(u) = \frac{m_{|\tilde{Y}|}(u)}{E(|Y|)} = \frac{m_{|\tilde{Y}|}(u)}{E(|\tilde{Y}|)}$ and $\psi_r(u) = \frac{m_{|\tilde{X}_r|}(u)}{E(|X_r|)} = \frac{m_{|\tilde{X}_r|}(u)}{E(|X_r|)} = \frac{m_{|\tilde{X}_r|}(u)}{E(|X_r|)}$, r = 1, ..., p. The Nadaraya–Watson estimators (Nadaraya, 1964, Watson, 1964) are used to estimate them, and they are defined as

$$\hat{\phi}_M(u) = \frac{1}{n\hat{f}_U(u)|\widetilde{Y}|} \sum_{i=1}^n K_h(U_i - u)|\widetilde{Y}_i|, \qquad (2.5)$$

$$\hat{\psi}_{M,r}(u) = \frac{1}{n\hat{f}_U(u)|\tilde{X}_r|} \sum_{i=1}^n K_h(U_i - u)|\tilde{X}_{ri}|, \qquad (2.6)$$

$$\hat{f}_U(u) = \frac{1}{n} \sum_{i=1}^n K_h(U_i - u), \qquad \overline{|\tilde{Y}|} = \frac{1}{n} \sum_{i=1}^n |\tilde{Y}_i|, \qquad \overline{|\tilde{X}_r|} = \frac{1}{n} \sum_{i=1}^n |\tilde{X}_{ri}|,$$

where $K_h(\cdot) = h^{-1}K(\cdot/h)$, $K(\cdot)$ denotes a density function, h is a positive-valued bandwidth. Using (2.5) and (2.6), we obtain the conditional absolute mean calibrated variables $\{\hat{Y}_{M,i}, \hat{X}_{M,ri}\}_{i=1}^{n}$ as

$$\hat{Y}_{M,i} = \frac{\widetilde{Y}_i}{\hat{\phi}_M(U_i)}, \qquad \hat{X}_{M,ri} = \frac{\widetilde{X}_{ri}}{\hat{\psi}_{M,r}(U_i)}, \quad r = 1, \dots, p.$$
 (2.7)

When some values of the estimated distortion functions are closed to zero, a useful remedy is to use their estimators by adding a small constant, say, 1/n in practice. It is a commonly used

remedy in nonparametric regression estimation. This remedy will not damage their asymptotic properties of the root-*n* convergent estimators.

In the following, we define $A^{\otimes 2} = AA^{T}$ for any matrix or vector A. The least squares estimator of (α_0, β_0) is obtained as

$$(\hat{\alpha}_{M}, \hat{\boldsymbol{\beta}}_{M}^{\mathrm{T}})^{\mathrm{T}} = \left\{ \frac{1}{n} \sum_{i=1}^{n} [(1, \hat{\boldsymbol{X}}_{M,i}^{\mathrm{T}})^{\mathrm{T}}]^{\otimes 2} \right\}^{-1} \frac{1}{n} \sum_{i=1}^{n} (1, \hat{\boldsymbol{X}}_{M,i}^{\mathrm{T}})^{\mathrm{T}} \hat{\boldsymbol{Y}}_{M,i},$$
(2.8)

where $\hat{X}_{M,i} = (\hat{X}_{M,1i}, \dots, \hat{X}_{M,pi})^{\mathrm{T}}$. In Theorem 2.1, we show that the least squares estimator (2.8) is asymptotically normally distributed.

We now list the assumptions needed in the following theorems:

(C1) The distortion functions are such that $\phi(u) > 0$ and $\psi_r(u) > 0$, r = 1, ..., p, for all $u \in [\mathcal{U}_L, \mathcal{U}_R]$, where $[\mathcal{U}_L, \mathcal{U}_R]$ denotes the compact support of U. Moreover, the distortion functions $\phi(u)$, $\psi_r(u)$'s, have three continuous derivatives. The density function $f_U(u)$ of the random variable U is bounded away from 0 and satisfies the Lipschitz condition of order 1 on $[\mathcal{U}_L, \mathcal{U}_R]$.

(C2) For some $s \ge 4$, $E(|Y|^s) < \infty$, $E(|X_r|^s) < \infty$, r = 1, ..., p. The matrix Σ defined in Theorem 2.1 is a positive-definite matrix.

(C3) The kernel function $K(\cdot)$ is a bounded symmetric density function supported on [-A, A] satisfying a Lipschitz condition. $K(\cdot)$ also has second-order continuous bounded derivatives, satisfying $K^{(j)}(\pm A) = 0$ with $K^{(j)}(s) = \frac{d^j K(s)}{ds^j}$, and also $\int_{-A}^{A} s^2 K(s) ds \neq 0$.

(C4) As $n \to \infty$, the bandwidth h satisfies $h^3 \log n \to 0$, $\frac{\log^2 n}{nh^2} \to 0$ and $nh^4 \to 0$.

These are mild conditions that are satisfied with most practical situations. Condition (C1) is recently used in the study of distortion measurement error models, see Zhao and Xie (2018), Delaigle, Hall and Zhou (2016), Zhang, Lin and Li (2019), Feng, Gai and Zhang (2019), Xie and Zhu (2019). The bounded assumption of the support of U entails no loss of generality as the variable can always be carried out by monotone transformations, even if the support before transformation is unbounded. For practical computations, it suffices to transform the empirical support to [-1, 1]. Condition (C2) is used in calibration procedures and the proof of theorems. Condition (C3) is the usual condition for the kernel function $K(\cdot)$. The Epanechnikov kernel satisfies this condition. Condition (C4) is the condition for the bandwidth h in the nonparametric kernel smoothing.

Let I_{p+1} be an identity matrix of size p + 1. We define the following notations:

$$\boldsymbol{\Sigma} = E\{[(1, \boldsymbol{X}^{\mathrm{T}})^{\mathrm{T}}]^{\otimes 2}\}, \qquad \boldsymbol{\Sigma}_{\epsilon} = E\{\epsilon^{2}[(1, \boldsymbol{X}^{\mathrm{T}})^{\mathrm{T}}]^{\otimes 2}\},\$$
$$\boldsymbol{G}(\boldsymbol{X}, \boldsymbol{\psi}(\boldsymbol{U})) = \operatorname{diag}\left(0, \frac{(\psi_{1}(\boldsymbol{U}) - 1)|\boldsymbol{X}_{1}|}{E(|\boldsymbol{X}_{1}|)}, \dots, \frac{(\psi_{p}(\boldsymbol{U}) - 1)|\boldsymbol{X}_{p}|}{E(|\boldsymbol{X}_{p}|)}\right),\$$
$$\boldsymbol{\Sigma}_{\phi, \boldsymbol{\psi}} = E\left\{\left[\left[\frac{(\phi(\boldsymbol{U}) - 1)|\boldsymbol{Y}|}{E(|\boldsymbol{Y}|)}\boldsymbol{I}_{p+1} - \boldsymbol{G}(\boldsymbol{X}, \boldsymbol{\psi}(\boldsymbol{U}))\right]\binom{\alpha_{0}}{\boldsymbol{\beta}_{0}}\right]^{\otimes 2}\right\}$$

Theorem 2.1. Suppose $E(|Y|) \prod_{r=1}^{p} E(|X_r|) > 0$, and the conditions (C1)–(C4) hold, we have

$$\sqrt{n} \left\{ \begin{pmatrix} \hat{\alpha}_M \\ \hat{\beta}_M \end{pmatrix} - \begin{pmatrix} \alpha_0 \\ \beta_0 \end{pmatrix} \right\} \stackrel{L}{\longrightarrow} N(\mathbf{0}_{p+1}, \mathbf{\Sigma}^{-1} \mathbf{\Sigma}_{\epsilon} \mathbf{\Sigma}^{-1} + \mathbf{\Sigma}_{\phi, \psi}).$$

Remark. The first term $\Sigma^{-1}\Sigma_{\epsilon}\Sigma^{-1}$ is the usual asymptotic covariance matrix for the least squares estimator when the data are exactly observed, i.e., $\phi(U) \equiv 1$, $\psi_r(U) \equiv 1$,

r = 1, ..., p. If the model error ϵ is further independent of X, this term reduces to $E(\epsilon^2)\Sigma^{-1}$. The second term $\Sigma_{\phi, \psi}$ is dedicated to the multiplicative distortion measurement errors involved in the response variable and covariates. It is also seen that the conditional absolute mean calibration can eliminate the effect of distortion function $\phi(U)$ and $\psi_r(U)$'s for estimating β_{0r} when $\beta_{0r} = 0$, that is, $\operatorname{Avar}(\hat{\beta}_{M,r}) = e_{r+1}^{\mathrm{T}}\Sigma^{-1}\Sigma_{\epsilon}\Sigma^{-1}e_{r+1}$, where $\hat{\beta}_{M,r}$ is the *r*-th component of $\hat{\beta}_M$, e_{r+1} is a (p+1)-dimensional vector with 1 in the (r+1)-th position and 0's elsewhere, r = 1, ..., p, and $\operatorname{Avar}(\hat{\beta}_{M,r})$ stands for the asymptotic variance of $\hat{\beta}_{M,r}$ obtained in Theorem 2.1.

2.2 Conditional variance calibration

In the following, we define that
$$\sigma_Y = \sqrt{\operatorname{Var}(Y)}, \ \sigma_{X_r} = \sqrt{\operatorname{Var}(X_r)}, \ r = 1, \dots, p$$
, and
 $m_{\widetilde{Y}}(u) \stackrel{\text{def}}{=} E(\widetilde{Y}|U=u), \qquad \sigma_{\widetilde{Y}|U}(u) \stackrel{\text{def}}{=} \sqrt{\operatorname{Var}(\widetilde{Y}|U=u)},$
 $m_{\widetilde{X}_r}(u) \stackrel{\text{def}}{=} E(\widetilde{X}_r|U=u), \qquad \sigma_{\widetilde{X}_r|U}(u) \stackrel{\text{def}}{=} \sqrt{\operatorname{Var}(\widetilde{X}_r|U=u)}.$

Under the independence condition between U and (Y, X), the identifiability condition (2.1) and condition (C1) entail that

$$\sigma_{\widetilde{Y}|U}(u) = \phi(u)\sigma_{Y}, \qquad E(\sigma_{\widetilde{Y}|U}(U)) = \sigma_{Y}, \tag{2.9}$$

$$\sigma_{\widetilde{X}_r|U}(u) = \psi_r(u)\sigma_{X_r}, \qquad E\left(\sigma_{\widetilde{X}_r|U}(U)\right) = \sigma_{X_r}, \quad r = 1, \dots, p.$$
(2.10)

Suppose that $\sigma_Y \prod_{r=1}^p \sigma_{X_r} > 0$, these equations (2.9)–(2.10) entail that

$$\phi(u) = \frac{\sigma_{\widetilde{Y}|U}(u)}{E(\sigma_{\widetilde{Y}|U}(U))}, \qquad \psi_r(u) = \frac{\sigma_{\widetilde{X}_r|U}(u)}{E(\sigma_{\widetilde{X}_r|U}(U))}.$$
(2.11)

Together with (2.11), we have

$$\begin{cases} Y = \frac{\widetilde{Y}}{\phi(U)} = \frac{E(\sigma_{\widetilde{Y}|U}(U))}{\sigma_{\widetilde{Y}|U}(U)} \widetilde{Y}, \\ X_r = \frac{\widetilde{X}_r}{\psi_{M,r}(U)} = \frac{E(\sigma_{\widetilde{X}_r|U}(U))}{\sigma_{\widetilde{X}_r|U}(U)} \widetilde{X}_r. \end{cases}$$
(2.12)

Thus, unobserved variables *Y* and X_r , r = 1, ..., p, can be obtained through (2.12) at the population level. It is seen that the conditional variance calibration involved in (2.9)–(2.11) does not need the nonzero expectation condition $(E(Y) \neq 0, E(X_r) \neq 0)$ imposed in Cui et al. (2009). Also, $\sigma_Y = 0$ and $\sigma_{X_r} = 0$ are equivalent to P(Y = E(Y)) = 1 and $P(X_r = E(X_r)) = 1$, the conditions P(Y = E(Y)) < 1 and $P(X_r = E(X_r)) < 1$ are much less strict than $E(Y) \neq 0$ and $E(X_r) \neq 0$.

In the following, we summarize the conditional variance calibration estimation procedure.

• The Nadaraya–Watson estimators are used to estimate $\phi(u)$ and $\psi_r(u)$. Define that

$$\hat{m}_{\widetilde{Y}}(u) = \frac{1}{nh\hat{f}_{U}(u)} \sum_{i=1}^{n} K_{h}(U_{i} - u)\widetilde{Y}_{i},$$

$$\hat{\sigma}_{\widetilde{Y}|U}^{2}(u) = \frac{1}{nh\hat{f}_{U}(u)} \sum_{i=1}^{n} K_{h}(U_{i} - u) [\widetilde{Y}_{i} - \hat{m}_{\widetilde{Y}}(U_{i})]^{2},$$

$$\hat{m}_{\widetilde{X}_{r}}(u) = \frac{1}{nh\hat{f}_{U}(u)} \sum_{i=1}^{n} K_{h}(U_{i} - u)\widetilde{X}_{ri},$$

$$\hat{\sigma}_{\widetilde{X}_{r}|U}^{2}(u) = \frac{1}{nh\hat{f}_{U}(u)} \sum_{i=1}^{n} K_{h}(U_{i} - u) [\widetilde{X}_{ri} - \hat{m}_{\widetilde{X}_{r}}(U_{i})]^{2}.$$

We obtain $\hat{\sigma}_{\widetilde{Y}|U}(u) = \sqrt{\hat{\sigma}_{\widetilde{Y}|U}^2(u)}, \hat{\sigma}_{\widetilde{X}_r|U}(u) = \sqrt{\hat{\sigma}_{\widetilde{X}_r|U}^2(u)}, \text{ and}$ $\hat{E}(\sigma_{\widetilde{Y}|U}(U)) = \hat{\sigma}_Y = \frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{\widetilde{Y}|U}(U_i),$ $\hat{E}(\sigma_{\widetilde{X}_r|U}(U)) = \hat{\sigma}_{X_r} = \frac{1}{n} \sum_{i=1}^n \hat{\sigma}_{\widetilde{X}_r|U}(U_i).$

Then, we have

$$\hat{\phi}_{V}(u) = \frac{\hat{\sigma}_{\widetilde{Y}|U}(u)}{\hat{E}(\sigma_{\widetilde{Y}|U}(U))}, \qquad \hat{\psi}_{V,r}(u) = \frac{\hat{\sigma}_{\widetilde{X}_{r}|U}(u)}{\hat{E}(\sigma_{\widetilde{X}_{r}|U}(U))}.$$
(2.13)

• Using (2.13), the conditional variance calibrated variables for $\{Y_i, X_{ri}, r = 1, ..., p\}_{i=1}^n$ are defined as

$$\hat{Y}_{V,i} = \frac{\widetilde{Y}_i}{\hat{\phi}_V(U_i)}, \qquad \hat{X}_{V,ri} = \frac{\widetilde{X}_{ri}}{\hat{\psi}_{V,r}(U_i)}.$$
(2.14)

• The least squares estimator of (α_0, β_0) is obtained as

$$(\hat{\alpha}_{V}, \hat{\beta}_{V}^{\mathrm{T}})^{\mathrm{T}} = \left\{ \frac{1}{n} \sum_{i=1}^{n} [(1, \hat{X}_{V,i}^{\mathrm{T}})^{\mathrm{T}}]^{\otimes 2} \right\}^{-1} \frac{1}{n} \sum_{i=1}^{n} (1, \hat{X}_{V,i}^{\mathrm{T}})^{\mathrm{T}} \hat{Y}_{V,i},$$

$$(2.15)$$

where $\hat{X}_{V,i} = (\hat{X}_{V,1i}, \dots, \hat{X}_{V,pi})^{\mathrm{T}}$.

In the following theorems, we introduce the asymptotic results of the conditional variance calibration procedure.

Theorem 2.2. Let $W = (Y, X^T)^T$ and M(W) be a function of W, satisfying $E\{[M(W)]^2\} < \infty$. Suppose that P(Y = E(Y)) < 1, $P(X_r = E(X_r)) < 1$, r = 1, ..., p, and also the conditions (C1)–(C4) hold, we have

$$\frac{1}{n} \sum_{i=1}^{n} (\hat{Y}_{V,i} - Y_i) M(W_i)$$

= $\frac{1}{n} \sum_{i=1}^{n} \left[\frac{(Y_i - E(Y))^2}{2\sigma_Y^2} + \frac{1}{2} \right] [\phi(U_i) - 1] E[YM(W)] + o_P(n^{-1/2}).$

For $r = 1, \ldots, p$, we have

$$\frac{1}{n} \sum_{i=1}^{n} (\hat{X}_{V,ri} - X_{ri}) M(\mathbf{W}_{i})$$

= $\frac{1}{n} \sum_{i=1}^{n} \left[\frac{(X_{ri} - E(X_{r}))^{2}}{2\sigma_{X_{r}}^{2}} + \frac{1}{2} \right] [\psi_{r}(U_{i}) - 1] E[X_{r}M(\mathbf{W})] + o_{P}(n^{-1/2}).$

Remark. Theorem 2.2 gives a fundamental theory to construct asymptotic results of estimators with the conditional variance calibration estimation procedure. Theorem 2.2 and Lemma 1.1 in the on-line supplementary material of Zhao and Xie (2018) provides different calibration procedures for multiplicative distortion measurement errors in the light of the circumstances. Separate calibration procedures will result in different calibrated variables, and finally lead to different asymptotic properties of estimators. This is far from trivial, and should be analyzed case by case.

In the following, we define

$$Q(X, \psi(U)) = \left(0, \left(\psi_1(U_i) - 1\right) \left[\frac{(X_1 - E(X_1))^2}{2\sigma_{X_1}^2} + \frac{1}{2}\right], \dots, \\ \left(\psi_p(U) - 1\right) \left[\frac{(X_p - E(X_p))^2}{2\sigma_{X_p}^2} + \frac{1}{2}\right]\right), \\ \Xi_{\phi, \psi} = E\left\{\left[\left[\left(\frac{(Y - E(Y))^2}{2\sigma_Y^2} + \frac{1}{2}\right) [\phi(U) - 1] I_{p+1} - Q(X, \psi(U))\right] {\alpha_0 \choose \beta_0}\right]^{\otimes 2}\right\}.$$

Theorem 2.3. Suppose conditions in Theorem 2.2 hold, we have

$$\sqrt{n} \left\{ \begin{pmatrix} \hat{\alpha}_V \\ \hat{\boldsymbol{\beta}}_V \end{pmatrix} - \begin{pmatrix} \alpha_0 \\ \boldsymbol{\beta}_0 \end{pmatrix} \right\} \stackrel{L}{\longrightarrow} N(\boldsymbol{0}_{p+1}, \boldsymbol{\Sigma}^{-1}\boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}\boldsymbol{\Sigma}^{-1} + \boldsymbol{\Xi}_{\boldsymbol{\phi},\boldsymbol{\psi}}).$$

Remark. Compared with Theorem 2.1, it is seen that the estimator $(\hat{\alpha}_V, \hat{\beta}_V)$ is asymptotically more efficient than $(\hat{\alpha}_M, \hat{\beta}_M)$ when $\sum_{\phi, \psi} - \Xi_{\phi, \psi}$ is a positive-definite matrix, and vice versa. In detail, we denote the asymptotic variance of $\hat{\alpha}_M$ as Avar $(\hat{\alpha}_M)$, and we further denote the asymptotic variance of $\hat{\alpha}_V$ as Avar $(\hat{\alpha}_V)$. It is easily seen that

Avar
$$(\hat{\alpha}_M)$$
 – Avar $(\hat{\alpha}_V)$ = Var $(\phi(U))$ $\left\{ \frac{E(Y^2)}{[E(|Y|)]^2} - \frac{E[(Y - E(Y))^4]}{4\sigma_Y^4} - \frac{3}{4} \right\}$

Similarly, we denote the asymptotic variance of $\hat{\beta}_{V,r}$ (the *r*-th component of $\hat{\beta}_V$) as Avar $(\hat{\beta}_{V,r})$. We have

$$\begin{aligned} \operatorname{Avar}(\beta_{M,r}) &- \operatorname{Avar}(\beta_{V,r}) \\ &= \beta_{0r}^{2} \operatorname{Var}(\phi(U)) \left\{ \frac{E(Y^{2})}{[E(|Y|)]^{2}} - \frac{E[(Y - E(Y))^{4}]}{4\sigma_{Y}^{4}} - \frac{3}{4} \right\} \\ &+ \beta_{0r}^{2} \operatorname{Var}(\psi_{r}(U)) \left\{ \frac{E(X_{r}^{2})}{[E(|X_{r}|)]^{2}} - \frac{E[(X_{r} - E(X_{r}))^{4}]}{4\sigma_{X_{r}}^{4}} - \frac{3}{4} \right\} \\ &+ 2\beta_{0r}^{2} \operatorname{Cov}(\phi(U), \psi_{r}(U)) \left\{ \frac{E[(Y - E(Y))^{2}(X_{r} - E(X_{r}))^{2}]}{4\sigma_{Y}^{2}\sigma_{X_{r}}^{2}} + \frac{3}{4} - \frac{E|YX_{r}|}{E(|Y|)E(|X_{r}|)} \right\}. \end{aligned}$$

If the response variable Y is exactly observed, that is, $\phi(U) \equiv 1$, so $Var(\phi(U)) = 0$ and $Cov(\phi(U), \psi_r(U)) = 0$, and the difference between the asymptotic variances $Avar(\hat{\beta}_{M,r})$ and $Avar(\hat{\beta}_{V,r})$ reduces to

$$\beta_{0r}^2 \operatorname{Var}(\psi_r(U)) \bigg\{ \frac{E(X_r^2)}{[E(|X_r|)]^2} - \frac{E[(X_r - E(X_r))^4]}{4\sigma_{X_r}^4} - \frac{3}{4} \bigg\}.$$

Next, it is also seen that if the true parameter $\beta_{0r} = 0$, we have

$$\operatorname{Avar}(\hat{\beta}_{M,r}) = \operatorname{Avar}(\hat{\beta}_{V,r}) = e_{r+1}^{\mathrm{T}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\Sigma}_{\epsilon} \boldsymbol{\Sigma}^{-1} e_{r+1}.$$

When $\beta_{0r} = 0$, both the conditional absolute mean calibration and conditional variance calibration result in asymptotic efficiency estimators, that is, the calibration estimation procedures eliminate the effect caused by the distorting multiplicative functions $\phi(U)$ and $\psi_r(U)$'s.

3 Confidence intervals

3.1 Asymptotic normal approximation

According to Theorem 2.1 and Theorem 2.3, the $(1 - \alpha) \times 100\%$ ($0 < \alpha < 1$) confidence interval for β_{0r} can be obtained by estimating the asymptotic variances.

3.1.1 Conditional absolute mean calibration. According to Theorem 2.1, let $\hat{\epsilon}_{M,i} = \hat{Y}_{M,i} - \hat{\alpha}_M - \hat{\beta}_M^T \hat{X}_{M,i}$, i = 1, ..., n, we define

$$\hat{\boldsymbol{\Sigma}}_{M} = \frac{1}{n} \sum_{i=1}^{n} [(1, \hat{\boldsymbol{X}}_{M,i}^{\mathrm{T}})^{\mathrm{T}}]^{\otimes 2}, \qquad \hat{\boldsymbol{\Sigma}}_{M,\epsilon} = \frac{1}{n} \sum_{i=1}^{n} \hat{\epsilon}_{M,i}^{2} [(1, \hat{\boldsymbol{X}}_{M,i}^{\mathrm{T}})^{\mathrm{T}}]^{\otimes 2},$$

and

$$\hat{\boldsymbol{G}}(\hat{\boldsymbol{X}}_{M,i}, \hat{\boldsymbol{\psi}}_{M}(U_{i})) = \operatorname{diag}\left(0, \frac{(|\tilde{\boldsymbol{X}}_{1i}| - |\hat{\boldsymbol{X}}_{M,1i}|)}{|\tilde{\boldsymbol{X}}_{1}|}, \dots, \frac{(|\tilde{\boldsymbol{X}}_{pi}| - |\hat{\boldsymbol{X}}_{M,pi}|)}{|\tilde{\boldsymbol{X}}_{p}|}\right),$$
$$\hat{\boldsymbol{\Sigma}}_{\phi, \boldsymbol{\psi}} = \frac{1}{n} \sum_{i=1}^{n} \left[\left[\frac{(|\tilde{Y}_{i}| - |\hat{Y}_{M,i}|)}{|\tilde{Y}|} \boldsymbol{I}_{p+1} - \hat{\boldsymbol{G}}(\hat{\boldsymbol{X}}_{M,i}, \hat{\boldsymbol{\psi}}_{M}(U_{i})) \right] \begin{pmatrix} \hat{\boldsymbol{\alpha}}_{M} \\ \hat{\boldsymbol{\beta}}_{M} \end{pmatrix} \right]^{\otimes 2},$$

and also

$$\hat{\sigma}_{M,r}^2 = e_{r+1}^{\mathrm{T}} \hat{\boldsymbol{\Sigma}}_M^{-1} \hat{\boldsymbol{\Sigma}}_{M,\epsilon} \hat{\boldsymbol{\Sigma}}_M^{-1} e_{r+1} + e_{r+1}^{\mathrm{T}} \hat{\boldsymbol{\Sigma}}_{\phi,\psi} e_{r+1}$$

Based on the estimator $\hat{\sigma}_{M,r}^2$, the $(1 - \alpha) \times 100\%$ (0 < α < 1) confidence interval for β_{0r} is

$$\left(\hat{\beta}_{M,r}-\sqrt{\frac{\hat{\sigma}_{M,r}^2}{n}}z_{\alpha/2},\hat{\beta}_{M,r}+\sqrt{\frac{\hat{\sigma}_{M,r}^2}{n}}z_{\alpha/2}\right),$$

where $\hat{\beta}_{M,r}$ is the *r*-th component of $\hat{\beta}_M$, and $z_{\alpha/2}$ is the quantile satisfying $P(N(0, 1) \ge z_{\alpha/2}) = \alpha/2$.

3.1.2 Conditional variance calibration. According to Theorem 2.3, let $\hat{\epsilon}_{V,i} = \hat{Y}_{V,i} - \hat{\alpha}_V - \hat{\beta}_V^T \hat{X}_{V,i}$, i = 1, ..., n, we define

$$\hat{\boldsymbol{\Sigma}}_{V} = \frac{1}{n} \sum_{i=1}^{n} [(1, \hat{\boldsymbol{X}}_{V,i}^{\mathrm{T}})^{\mathrm{T}}]^{\otimes 2}, \qquad \hat{\boldsymbol{\Sigma}}_{V,\epsilon} = \frac{1}{n} \sum_{i=1}^{n} \hat{\epsilon}_{V,i}^{2} [(1, \hat{\boldsymbol{X}}_{V,i}^{\mathrm{T}})^{\mathrm{T}}]^{\otimes 2},$$

and

$$\hat{\boldsymbol{Q}}(\boldsymbol{X}_{V,i}, \hat{\boldsymbol{\psi}}_{V}(U_{i})) = \left(0, \left(\hat{\psi}_{V,1}(U_{i}) - 1\right) \left[\frac{(\hat{\boldsymbol{X}}_{V,1i} - \overline{\tilde{\boldsymbol{X}}_{1}})^{2}}{2\hat{\sigma}_{X_{1}}^{2}} + \frac{1}{2}\right], \dots, \\ \left(\hat{\psi}_{V,p}(U_{i}) - 1\right) \left[\frac{(\hat{\boldsymbol{X}}_{V,pi} - \overline{\tilde{\boldsymbol{X}}_{p}})^{2}}{2\hat{\sigma}_{X_{p}}^{2}} + \frac{1}{2}\right]\right),$$

$$\hat{\boldsymbol{\Xi}}_{\phi,\boldsymbol{\psi}} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \left[\left(\frac{(\hat{Y}_{V,i} - \overline{\tilde{Y}})^2}{2\hat{\sigma}_Y^2} + \frac{1}{2} \right) [\hat{\phi}_V(U_i) - 1] \boldsymbol{I}_{p+1} - \hat{\boldsymbol{Q}}(\hat{\boldsymbol{X}}_{V,i}, \hat{\boldsymbol{\psi}}_V(U_i)) \right] \begin{pmatrix} \hat{\alpha}_V \\ \hat{\boldsymbol{\beta}}_V \end{pmatrix} \right\}^{\otimes 2}.$$

Moreover, we define

$$\hat{\sigma}_{V,r}^2 = e_{r+1}^{\mathrm{T}} \hat{\boldsymbol{\Sigma}}_V^{-1} \hat{\boldsymbol{\Sigma}}_{V,\epsilon} \hat{\boldsymbol{\Sigma}}_V^{-1} e_{r+1} + e_{r+1}^{\mathrm{T}} \hat{\boldsymbol{\Xi}}_{\phi, \boldsymbol{\psi}} e_{r+1}.$$

Based on the estimator $\hat{\sigma}_{V,r}^2$, the $(1 - \alpha) \times 100\%$ (0 < α < 1) confidence interval for β_{0r} is

$$\left(\hat{\beta}_{V,r}-\sqrt{\frac{\hat{\sigma}_{V,r}^2}{n}}z_{\alpha/2},\hat{\beta}_{V,r}+\sqrt{\frac{\hat{\sigma}_{V,r}^2}{n}}z_{\alpha/2}\right),$$

where $\hat{\beta}_{V,r}$ is the *r*-th component of $\hat{\beta}_{V}$.

3.2 Empirical likelihood method

Another popular method to construct confidence intervals is the empirical likelihood (EL) method method proposed by Owen (2001). The EL method is an appealing nonparametric approach for constructing confidence intervals (regions) for the parameter of interest without estimating the covariance matrix. There has been much literature to discuss the EL method and its applications. For example, Lian (2012), Li, Lin and Zhu (2012), Cui et al. (2009), Liang et al. (2009), Yang, Xue and Cheng (2011, 2009), Guo et al. (2015).

Then, we make statistical inference based on the EL principle. Usually, the EL method needs an auxiliary vector $\wp_{n,i}(\boldsymbol{\beta}') = (\wp_{n,i}^{[1]}(\boldsymbol{\beta}'), \dots, \wp_{n,i}^{[p]}(\boldsymbol{\beta}'))^{\mathrm{T}}$ with the property of that $E \wp_{n,i}(\boldsymbol{\beta}') = 0$ when $\boldsymbol{\beta}' = \boldsymbol{\beta}_0$:

$$\wp_{n,i}(\boldsymbol{\beta}') = X_i(Y_i - \alpha_0 - X_i^{\mathrm{T}}\boldsymbol{\beta}').$$

Because of $\{Y_i, X_i\}_{i=1}^n$ are unavailable, we need the "calibrated" variables $\{\hat{Y}_{M,i}, \hat{X}_{M,i}\}_{i=1}^n$ or $\{\hat{Y}_{V,i}, \hat{X}_{V,i}\}_{i=1}^n$, respectively. We now define two calibrated EL principles by plugging in the estimated arguments $\{\hat{Y}_{M,i}, \hat{X}_{M,i}\}_{i=1}^n$ and $\{\hat{Y}_{V,i}, \hat{X}_{V,i}\}_{i=1}^n$ into $\wp_{n,i}(\boldsymbol{\beta}')$:

$$\hat{l}_{M,n}(\boldsymbol{\beta}') = -2 \max\left\{\sum_{i=1}^{n} \log(np_i) : p_i \ge 0, \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i \hat{\wp}_{M,n,i}(\boldsymbol{\beta}') = 0\right\},\$$
$$\hat{l}_{V,n}(\boldsymbol{\beta}') = -2 \max\left\{\sum_{i=1}^{n} \log(np_i) : p_i \ge 0, \sum_{i=1}^{n} p_i = 1, \sum_{i=1}^{n} p_i \hat{\wp}_{V,n,i}(\boldsymbol{\beta}') = 0\right\},\$$

where $\hat{\wp}_{M,n,i}(\boldsymbol{\beta}') = \hat{\boldsymbol{X}}_{M,i}(\hat{\boldsymbol{Y}}_{M,i} - \hat{\alpha}_M - \hat{\boldsymbol{X}}_{M,i}^{\mathrm{T}}\boldsymbol{\beta}'), \ \hat{\wp}_{V,n,i}(\boldsymbol{\beta}') = \hat{\boldsymbol{X}}_{V,i}(\hat{\boldsymbol{Y}}_{V,i} - \hat{\alpha}_V - \hat{\boldsymbol{X}}_{V,i}^{\mathrm{T}}\boldsymbol{\beta}'),$ $i = 1, \dots, n.$ By the Lagrange multiplier method, we have $\hat{l}_{M,n}(\boldsymbol{\beta}') = 2\sum_{i=1}^{n}\log\{1 + \hat{\lambda}_M^{\mathrm{T}}\hat{\wp}_{M,n,i}(\boldsymbol{\beta}')\},$ and also $\hat{l}_{V,n}(\boldsymbol{\beta}') = 2\sum_{i=1}^{n}\log\{1 + \hat{\lambda}_V^{\mathrm{T}}\hat{\wp}_{V,n,i}(\boldsymbol{\beta}')\},$ where $\hat{\lambda}_M$ and $\hat{\lambda}_V$ are determined by $\frac{1}{n}\sum_{i=1}^{n}\frac{\hat{\wp}_{M,n,i}(\boldsymbol{\beta}')}{1+\hat{\lambda}_M^{\mathrm{T}}\hat{\wp}_{M,n,i}(\boldsymbol{\beta}')} = 0, \ \frac{1}{n}\sum_{i=1}^{n}\frac{\hat{\wp}_{V,n,i}(\boldsymbol{\beta}')}{1+\hat{\lambda}_V^{\mathrm{T}}\hat{\wp}_{V,n,i}(\boldsymbol{\beta}')} = 0,$ respectively. **Theorem 3.1.** Suppose conditions in Theorem 2.2 hold, both $\hat{l}_{M,n}(\boldsymbol{\beta}_0)$ and $\hat{l}_{V,n}(\boldsymbol{\beta}_0)$ asymptotically converge in distribution to χ_p^2 , namely, a centered Chi-squared distribution with p degrees of freedom.

From Theorem 3.1, we can construct a confidence region of $\boldsymbol{\beta}_0$ by $I_{\alpha,M} = \{\boldsymbol{\beta}' : \hat{l}_{M,n}(\boldsymbol{\beta}') \le c_\alpha\}$ and $I_{\alpha,V} = \{\boldsymbol{\beta}' : \hat{l}_{V,n}(\boldsymbol{\beta}') \le c_\alpha\}$, where c_α denotes the α quantile of the χ_p^2 distribution. If we focus on the confidence intervals for the parameter β_{0r} , we can construct the EL statistics as

$$\begin{split} \hat{l}_{M,n}^{\{r\}}(\beta_{r}) &= -2 \max\left\{\sum_{i=1}^{n} \log(np_{i}) : p_{i} \geq 0, \sum_{i=1}^{n} p_{i} = 1, \sum_{i=1}^{n} p_{i} \hat{\wp}_{M,n,i}^{\{r\}}(\beta_{r}') = 0\right\}, \\ \hat{l}_{V,n}^{\{r\}}(\beta_{r}') &= -2 \max\left\{\sum_{i=1}^{n} \log(np_{i}) : p_{i} \geq 0, \sum_{i=1}^{n} p_{i} = 1, \sum_{i=1}^{n} p_{i} \hat{\wp}_{V,n,i}^{\{r\}}(\beta_{r}') = 0\right\}, \\ \frac{\{r\}}{M,n,i}(\beta_{r}') &= \hat{X}_{M,ri}(\hat{Y}_{M,i} - \hat{\alpha}_{M} - \sum_{s \neq r} \hat{X}_{M,si} \hat{\beta}_{M,s} - \hat{X}_{M,ri} \beta_{r}'), \text{ and } \hat{\wp}_{V,n,i}^{\{r\}}(\beta_{r}') \end{split}$$

where $\hat{\wp}_{M,n,i}^{\{r\}}(\beta_r') = X_{M,ri}(Y_{M,i} - \hat{\alpha}_M - \sum_{s \neq r} X_{M,si}\beta_{M,s} - X_{M,ri}\beta_r')$, and $\hat{\wp}_{V,n,i}^{\{r\}}(\beta_r') = \hat{X}_{V,ri}(\hat{Y}_{V,i} - \hat{\alpha}_V - \sum_{s \neq r} \hat{X}_{V,si}\hat{\beta}_{V,s} - \hat{X}_{V,ri}\beta_r')$, i = 1, ..., n. Similar to the proof of Theorem 3.1, we can construct confidence intervals of β_{0r} by $I_{\alpha,M}^{\{r\}} = \{\beta_r': \hat{l}_{M,n}^{\{r\}}(\beta_r') \le c_{\alpha}^*\}$ and $I_{\alpha,V}^{\{r\}} = \{\beta_r': \hat{l}_{V,n}^{\{r\}}(\beta_r') \le c_{\alpha}^*\}$, where c_{α}^* denotes the α quantile of the χ_1^2 distribution.

4 Simulation studies

Simulation studies are made in this section to show the performance of our proposed method. The Epanechnikov kernel $K(t) = 0.75(1 - t^2)I\{|t| \le 1\}$ is used here. The bandwidth h should be chosen according to condition (C4), and the optimal bandwidth for h cannot be obtained because under-smoothing $(nh^4 \rightarrow 0)$ for the non-parametric estimates is necessary. The consequence of under-smoothing is that the biases of the non-parametric estimate are kept small and preclude the optimal bandwidth for h. The asymptotic covariance matrices in Theorem 2.1 and Theorem 2.3 depend on neither the bandwidth h nor the kernel function K(t). Hence, we can use the rule of thumb: $h = \hat{\sigma}_U n^{-1/3}$, and $\hat{\sigma}_U$ is the sample standard deviation of U. This method is fairly effective and easy to apply in practice. Our numerical experience suggests that the numerical results were stable when we shifted several values around this data-driven bandwidth.

Example. We consider the model

$$Y = \alpha_0 + \beta_{01}X_1 + \beta_{02}X_2 + \beta_{03}X_3 + \epsilon.$$
(4.1)

2000 realizations are generated and sample size are n = 300, n = 500 and n = 1000, respectively. In this example, $\alpha_0 = 1$, $\boldsymbol{\beta}_0 = (\beta_{01}, \beta_{02}, \beta_{03})^{\mathrm{T}} = (2, -0.5, 0)^{\mathrm{T}}$, $X \sim N_3(\mu_X, \boldsymbol{\Sigma}_X)$ with $\mu_X = \mathbf{0}_{3\times 1}$ and $\boldsymbol{\Sigma}_X = (\sigma_{ij})_{1 \le i, j \le 3}$, $\sigma_{ij} = 0.75^{|i-j|}$. The model error ϵ is independent of X and generated from $N(0, 0.25^2)$. The variable U follows an uniform distribution U[0, 1], and the distortion functions are chosen as the following two cases:

Case 1
$$\phi(u) = \frac{12((u-0.5)^2+1)}{13}$$
, $\psi_1(u) = 1 + 0.2\sin(2\pi u)$, $\psi_2(u) = 1 - \frac{u^2-1/3}{5}$, and $\psi_3(u) = \frac{2(u+3)}{7}$.
Case 2 $\phi(u) = u + \frac{1}{2}$, $\psi_1(u) = 1.5 - u$, $\psi_2(u) = u^2 + \frac{2}{3}$, and $\psi_3(u) = 1.25 - u^3$.

		n = 300				n = 500			n = 1000		
_		М	SD	MSE	М	SD	MSE	М	SD	MSE	
Case 1	$\hat{\alpha}_T$	1.0001	0.0143	0.2058	1.0004	0.0112	0.1276	1.0001	0.0080	0.0641	
	$\hat{\alpha}_M$	0.9961	0.0184	0.3540	0.9981	0.0138	0.1938	0.9989	0.0093	0.0894	
	\hat{lpha}_V	1.0053	0.0164	0.3004	1.0036	0.0124	0.1688	1.0018	0.0085	0.0765	
	\hat{lpha}_N	1.0004	0.0221	0.4912	1.0007	0.0172	0.2984	1.0001	0.0122	0.1489	
	$\hat{\beta}_{T,1}$	2.0002	0.0222	0.4935	1.9996	0.0167	0.2804	1.9999	0.0116	0.1364	
	$\hat{\beta}_{M,1}$	1.9894	0.0348	1.3226	1.9940	0.0263	0.7496	1.9983	0.0173	0.3064	
	$\hat{\beta}_{V,1}$	1.9946	0.0342	1.2009	1.9975	0.0270	0.7368	2.0002	0.0167	0.2810	
	$\hat{\beta}_{N,1}$	1.9317	0.0411	6.3469	1.9320	0.0325	5.6810	1.9326	0.0221	5.0212	
	$\hat{\beta}_{T,2}$	-0.5004	0.0276	0.7660	-0.4998	0.0212	0.4507	-0.5001	0.0148	0.2199	
	$\hat{\beta}_{M,2}$	-0.4917	0.0320	1.0922	-0.4942	0.0237	0.5969	-0.4973	0.0158	0.2565	
	$\hat{\beta}_{V,2}$	-0.4955	0.0306	0.9561	-0.4969	0.0232	0.5500	-0.4987	0.0152	0.2348	
	$\hat{\beta}_{N,2}$	-0.4822	0.0406	1.9659	-0.4816	0.0325	1.3927	-0.4818	0.0225	0.8371	
	$\hat{\beta}_{T,3}$	0.0004	0.0220	0.4869	-0.0003	0.0172	0.2985	0.0000	0.0117	0.1384	
	$\hat{\beta}_{M,3}$	0.0005	0.0249	0.6220	-0.0002	0.0187	0.3511	-0.0001	0.0122	0.1506	
	$\hat{\beta}_{V,3}$	0.0002	0.0235	0.5529	-0.0003	0.0181	0.3295	-0.0001	0.0119	0.1434	
	$\hat{\beta}_{N,3}$	0.0336	0.0332	2.2364	0.0326	0.0264	1.7659	0.0324	0.0183	1.3899	

Table 1 Simulation results of Mean (M), Standard Error (SD) and Mean Squared Error (MSE) for true estimator $(\hat{\alpha}_T, \hat{\beta}_T)$, the proposed estimators $(\hat{\alpha}_M, \hat{\beta}_M)$ and $(\hat{\alpha}_V, \hat{\beta}_V)$, and the naive estimator $(\hat{\alpha}_N, \hat{\beta}_N)$ for case 1

Note: MSE is in the scale of $\times 10^{-3}$.

Table 2 Simulation results of Mean (M), Standard Error (SD) and Mean Squared Error (MSE) for true estimator $(\hat{\alpha}_T, \hat{\beta}_T)$, the proposed estimators $(\hat{\alpha}_M, \hat{\beta}_M)$ and $(\hat{\alpha}_V, \hat{\beta}_V)$, and the naive estimator $(\hat{\alpha}_N, \hat{\beta}_N)$ for case 2

		n = 300				n = 500			n = 1000		
		М	SD	MSE	М	SD	MSE	М	SD	MSE	
Case 2	$\hat{\alpha}_T$	1.0002	0.0143	0.2049	0.9997	0.0110	0.1219	0.9998	0.0080	0.0624	
	\hat{lpha}_M	0.9979	0.0272	0.7475	0.9975	0.0205	0.4292	0.9992	0.0143	0.2049	
	\hat{lpha}_V	1.0090	0.0292	0.9379	1.0044	0.0207	0.4250	1.0022	0.0138	0.1966	
	\hat{lpha}_N	1.0028	0.0576	3.3347	0.9985	0.0430	1.8564	1.0012	0.0314	0.9902	
	$\hat{\beta}_{T,1}$	1.9996	0.0221	0.4883	2.0002	0.0173	0.3024	1.9995	0.0120	0.1450	
	$\hat{\beta}_{M,1}$	1.9919	0.0859	7.4498	1.9940	0.0664	4.4445	1.9972	0.0460	2.1245	
	$\hat{\beta}_{V,1}$	1.9907	0.1119	12.6208	1.9972	0.0722	5.2293	1.9989	0.0466	2.1798	
	$\hat{\beta}_{N,1}$	1.3368	0.0954	448.8398	1.3337	0.0735	446.2293	1.3305	0.0528	451.0061	
	$\hat{\beta}_{T,2}$	-0.5001	0.0269	0.7234	-0.5005	0.0216	0.4676	-0.4998	0.0149	0.2237	
	$\hat{\beta}_{M,2}$	-0.4912	0.0332	1.1175	-0.4952	0.0243	0.6169	-0.4971	0.0163	0.2756	
	$\hat{\beta}_{V,2}$	-0.4881	0.0573	3.4271	-0.4950	0.0286	0.8479	-0.4973	0.0163	0.2734	
	$\hat{\beta}_{N,2}$	0.3632	0.1041	756.0647	0.3647	0.0817	754.4667	0.3678	0.0580	756.4651	
	$\hat{\beta}_{T,3}$	0.0007	0.0216	0.4707	0.0003	0.0170	0.2898	-0.0004	0.0117	0.1376	
	$\hat{\beta}_{M,3}$	0.0000	0.0247	0.6131	0.0005	0.0188	0.3538	-0.0005	0.0124	0.1545	
	$\hat{\beta}_{V,3}$	-0.0016	0.0290	0.8435	-0.0009	0.0182	0.3330	-0.0012	0.0119	0.1432	
	$\hat{\beta}_{N,3}$	-0.3344	0.0694	116.7067	-0.3352	0.0551	115.4227	-0.3374	0.0399	115.4609	

Note: MSE is in the scale of $\times 10^{-3}$.

In Table 1 and Table 2, we report the mean, standard errors and mean squared errors for the true estimator $(\hat{\alpha}_T, \hat{\beta}_T)$ (the least squares estimator (MSE) by using the simulated data

set $\{Y_i, X_i\}_{i=1}^n$, the proposed estimators $(\hat{\alpha}_M, \hat{\beta}_M)$ and $(\hat{\alpha}_V, \hat{\beta}_V)$, and the naive estimator

 $(\hat{\alpha}_N, \hat{\beta}_N)$ (the least squares estimator by using the data set $\{\widetilde{Y}_i, \widetilde{X}_i\}_{i=1}^n$). Here, MSE for each estimator is defined as $MSE = \frac{1}{2000} \sum_{s=1}^{2000} (\hat{\theta}_s - \theta_0)^2$, θ_0 is the true value, and $\hat{\theta}_s$ is the estimator of θ_0 for the *s*-th simulated data, s = 1, ..., 2000. For example, the MSE of $\hat{\alpha}_T$ is defined as $MSE = \frac{1}{2000} \sum_{s=1}^{2000} (\hat{\alpha}_{T,s} - \alpha_0)^2$. The MSE for other estimators are defined similarly.

Comparison between true estimator $(\hat{\alpha}_T, \hat{\beta}_T)$ and the proposed estimators $(\hat{\alpha}_M, \hat{\beta}_M)$ and $(\hat{\alpha}_V, \hat{\beta}_V)$, it is not surprised that the true estimator performs better than the proposed estimators, because the values of MSE for $(\hat{\alpha}_T, \hat{\beta}_T)$ are all smaller than the proposed estimators $(\hat{\alpha}_M, \hat{\beta}_M)$ and $(\hat{\alpha}_V, \hat{\beta}_V)$. For the proposed estimators, all the mean values are close to the true value $(1, 2, -0.5, 0)^{T}$, and the values of MSE decrease as the sample size *n* increases. In Table 1, the conditional variance calibrated estimator $(\hat{\alpha}_V, \hat{\beta}_V)$ performs better than the conditional absolute mean calibrated estimator $(\hat{\alpha}_M, \hat{\beta}_M)$ for case 1, and $(\hat{\alpha}_M, \hat{\beta}_M)$ performs better than $(\hat{\alpha}_V, \hat{\beta}_V)$ for case 2 in Table 2. Also, it is also seen that the naive estimator $(\hat{\alpha}_N, \hat{\beta}_N)$ have large bias especially for β_{01} in Table 1, and all the values of MSE for the naive estimators are larger than the true estimator and proposed estimators in this table. In Table 2, the bias of naive estimator $(\hat{\alpha}_N, \hat{\beta}_N)$ is much larger especially for estimating β_{01} , and the values of MSE are much larger than the others. This indicates that ignoring multiplicative distortion functions $\phi(U)$ and $\psi_r(U)$'s increases the bias and result in an inconsistent estimator even the sample size *n* is large.

Next, we report the 95% normal approximation (NA) confidence intervals and empirical likelihood (EL) confidence intervals for $\beta_{0,s}$, s = 1, 2, 3. The simulation results are reported in Table 3 and Table 4. In Table 3, when the sample size n gets larger, we see that both the EL confidence intervals show satisfactory performances for case 1 both in terms of average

Table 3 Simulation results of confidence intervals for case 1. "NA" stands for the normal approximation and "EL" stands for the empirical likelihood. "Lower" stands for the average of lower bounds, "Upper" stands for the average of upper bounds, "AL" stands for average of lengths of confidence intervals, "CP" stands for the coverage probabilities

			n = 300			n = 500			n = 1000	1
		β_1	β_2	β_3	β_1	β_2	β_3	β_1	β_2	β_3
				C	onditional	absolute me	ean calibrati	ion		
NA	Lower	1.9177	-0.5532	-0.0490	1.9423	-0.5406	-0.0363	1.9625	-0.5283	-0.0244
	Upper	2.0601	-0.4279	0.0494	2.0466	-0.4485	0.0360	2.0328	-0.4661	0.0247
	AL	0.1424	0.1253	0.0984	0.1043	0.0920	0.0723	0.0702	0.0621	0.0491
	CP	94.9%	94.0%	95.2%	94.8%	94.9%	94.5%	94.6%	94.0%	94.9%
EL	Lower	1.9088	-0.5556	-0.0532	1.9335	-0.5439	-0.0376	1.9621	-0.5296	-0.0243
	Upper	2.0589	-0.4305	0.0484	2.0455	-0.4505	0.0340	2.0337	-0.4654	0.0242
	AL	0.1500	0.1250	0.1016	0.1120	0.0933	0.0717	0.0716	0.0642	0.0486
	СР	94.8%	94.3%	95.2%	95.3%	95.7%	94.0%	95.2%	94.7%	94.6%
					Condition	nal variance	calibration			
NA	Lower	1.9294	-0.5536	-0.0457	1.9497	-0.5411	-0.0349	1.9661	-0.5288	-0.0239
	Upper	2.0585	-0.4356	0.0463	2.0464	-0.4530	0.0341	2.0330	-0.4683	0.0239
	AL	0.1290	0.1180	0.0921	0.0967	0.0881	0.0691	0.0668	0.0605	0.0478
	CP	94.0%	94.2%	94.3%	94.1%	94.9%	94.3%	94.7%	94.1%	94.8%
EL	Lower	1.9233	-0.5598	-0.0478	1.9488	-0.5461	-0.0345	1.9649	-0.5307	-0.0239
	Upper	2.0593	-0.4395	0.0463	2.0491	-0.4560	0.0348	2.0333	-0.4689	0.0242
	AL	0.1360	0.1203	0.0942	0.1002	0.0900	0.0694	0.0684	0.0618	0.0482
	CP	94.6%	94.8%	94.6%	94.8%	95.3%	94.6%	95.4%	94.7%	95.1%

Table 4 Simulation results of confidence intervals for case 2. "NA" stands for the normal approximation and "EL" stands for the empirical likelihood. "Lower" stands for the average of lower bounds, "Upper" stands for the average of upper bounds, "AL" stands for the average of lengths of confidence intervals, "CP" stands for the coverage probabilities

		n = 300			n = 500			n = 1000		
		β_1	β_2	β_3	β_1	β_2	β_3	β_1	β_2	β_3
				Co	onditional	absolute me	ean calibrati	ion		
NA	Lower	1.8220	-0.5543	-0.0507	1.8650	-0.5418	-0.0368	1.9061	-0.5293	-0.0244
	Upper	2.1553	-0.4250	0.0481	2.1211	-0.4468	0.0358	2.0861	-0.4651	0.0246
	AL	0.3332	0.1293	0.0988	0.2560	0.0950	0.0727	0.1800	0.0642	0.0491
	CP	94.0%	93.8%	94.8%	94.4%	94.2%	95.1%	95.1%	94.2%	94.8%
EL	Lower	1.8136	-0.5607	-0.0489	1.8631	-0.5438	-0.0390	1.9037	-0.5303	-0.0250
	Upper	2.1606	-0.4256	0.0500	2.1201	-0.4501	0.0343	2.0886	-0.4642	0.0252
	AL	0.3470	0.1350	0.0990	0.2570	0.0937	0.0734	0.1849	0.0661	0.0503
	СР	95.2%	95.5%	94.7%	94.6%	94.6%	95.1%	95.7%	94.9%	95.3%
					Condition	nal variance	calibration			
NA	Lower	1.7633	-0.6464	-0.1134	1.8648	-0.5400	-0.0360	1.9069	-0.5281	-0.0235
	Upper	2.2165	-0.3283	0.1055	2.1255	-0.4479	0.0320	2.0887	-0.4664	0.0219
	AL	0.4531	0.3181	0.2190	0.2606	0.0920	0.0680	0.1817	0.0617	0.0454
	CP	93.5%	92.9%	94.0%	93.9%	93.6%	94.3%	94.9%	93.9%	93.9%
EL	Lower	1.8030	-0.5653	-0.0584	1.8570	-0.5430	-0.0393	1.9047	-0.5300	-0.0249
	Upper	2.1747	-0.4141	0.0487	2.1323	-0.4468	0.0336	2.0901	-0.4645	0.0234
	AL	0.3717	0.1511	0.1072	0.2753	0.0962	0.0729	0.1854	0.0655	0.0483
	СР	94.7%	95.7%	95.4%	95.3%	94.9%	95.2%	95.5%	95.5%	95.3%

lengths of the confidence intervals and the coverage probabilities, while NA confidence intervals have lower coverage probabilities than the EL confidence intervals. It is also seen that the conditional variance calibration performs better than the conditional absolute mean calibration, because the average lengths of conditional variance calibration are much shorter than the conditional absolute mean calibration. This coincides with the simulation results obtained in Table 1. The simulation results for case 2 are reported in Table 4. In this table, the conditional absolute mean calibration performs better than the conditional variance calibration. This meets our expectation because the parameter estimates obtained in Table 2 reveal that conditional absolute mean calibration works well, and it is not surprised that conditional absolute mean calibration works well for confidence intervals in case 2. It is worth noting that the EL method does not need to estimate the asymptotic covariance matrices of estimators, while the normal approximation methods need to estimate the asymptotic covariance matrices. When then sample size is 300, NA confidence intervals have much lower coverage probabilities especially for the conditional variance calibration in case 2. Generally, the NA asymptotic intervals and the EL method are both recommended for constructing asymptotic intervals when the sample size is large in practice.

From Theorem 2.1 and Theorem 2.3, we know that both the conditional absolute mean calibration and conditional variance calibration can eliminate the effect of distortion function $\phi(U)$ and $\psi_r(U)$'s for estimating β_{03} when $\beta_{03} = 0$. In Table 1 and Table 2, we find that the values of MSE for $\hat{\beta}_{M,3}$ and $\hat{\beta}_{V,3}$ are close to the true estimator $\hat{\beta}_{T,3}$ when the sample size *n* is 500 and 1000. The confidence intervals of β_{03} reported in Table 3 and Table 4 also reveal a similar phenomenon. NA confidence intervals of $\hat{\beta}_{V,3}$ for n = 300 performs not as good as $\hat{\beta}_{M,3}$ in case 2. This is because the asymptotic variance of $\hat{\beta}_{V,r}$ is more complex than $\hat{\beta}_{M,r}$, the finite-sample behaviors of the estimator $\hat{\sigma}_V^2$ usually had poor performance when the sample size is small.

5 Real data analysis

We analyze the air quality dataset from the machine learning repository (https://archive.ics.uci.edu/ml/machine-learning-databases/00360/). In this data set, it was recorded from March 2004 to February 2005 (one year) representing the longest freely available recordings of on field deployed air quality chemical sensor devices responses. The data set contains hourly averaged responses from an array of five metal oxide chemical sensors embedded in an Air Quality Chemical Multisensor Device. The device was located on the field in a significantly polluted area, at the road level, within an Italian city.

We use a linear regression model to analyze this data set. The sample size is 826 when we remove samples that contain missing values. The variables contained in this data set are \tilde{X}_1 -ground truth hourly averaged CO concentrations (COC), \tilde{X}_2 -PT08.S1 (tin oxide) hourly averaged sensor response (nominally CO targeted), \tilde{X}_3 -hourly averaged overall Non Metanic HydroCarbons concentration (NMHC) in microg /m³, \tilde{X}_4 -hourly averaged Benzene concentration (BC) in microg /m³, \tilde{X}_5 -PT08.S2 (titania) hourly averaged sensor response (nominally NMHC targeted), \tilde{X}_6 -hourly averaged NOx concentration (NOxC) in ppb, \tilde{X}_7 -PT08.S3 (tungsten oxide) hourly averaged sensor response (nominally NOx targeted), \tilde{X}_8 -hourly averaged NO2 concentration (NO2C) in microg /m³, \tilde{X}_9 -PT08.S4 (tungsten oxide) hourly averaged sensor response (nominally NO2 targeted), \tilde{X}_{10} -PT08.S5 (indium oxide) hourly averaged sensor response (nominally O3 targeted), \tilde{Y} -relative humidity, and the confounding variable U is the covariate-temperature (ŰC).

Directly using the least squares method on the observed $\{\tilde{Y}_i, \tilde{X}_i\}_{i=1}^n$, the mean squared residuals is obtained as 81.5. After using the conditional variance calibration, the mean squared residuals is obtained as $\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_{V,i}^2 = 192.9$. For the conditional absolute mean calibration, the mean squared residuals is obtained as $\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_{V,i}^2 = 192.9$. For the conditional absolute mean calibration, the mean squared residuals is obtained as $\frac{1}{n}\sum_{i=1}^{n}\hat{\epsilon}_{M,i}^2 = 20.0$. This shows that the conditional absolute mean calibration greatly reduces the mean squared residuals and fit the model more appropriately, and the conditional variance calibration performs not well for this data set. As such, we adopt the conditional absolute mean method here.

We present the patterns of $\hat{\psi}_{M,r}(u)$, r = 1, ..., 10 and $\hat{\phi}_M(u)$ in Figure 1 and Figure 2. The plots in Figure 1 and Figure 2 show that the distortion functions $\hat{\phi}_M(u)$, $\hat{\psi}_{M,r}(u)$'s are all not constant functions. In Figure 1 and Figure 2, we examine the form of distortions that



Figure 1 Estimated curves of distorting functions $\hat{\psi}_{M,r}(u)$, r = 1, ..., 10, against confounding variable-temperature ($\hat{A}^{\circ}C$), associated 95% pointwise confidence intervals (dotted lines). Five plots in the first row are $\hat{\psi}_{M,r}(u)$, r = 1, ..., 5 (left to right), and five plots in the second row are $\hat{\psi}_{M,r}(u)$, r = 6, ..., 10 (left to right).



Figure 2 Estimated curve of distorting functions $\hat{\phi}_M(u)$ against confounding variable-temperature ($\hat{A}^\circ C$), associated 95% pointwise confidence intervals (dotted lines).

	Estimate	Normal ap	proximation	Empirical likelihood		
		Lower	Upper	Lower	Upper	
$\hat{\beta}_{M,1}$	0.6675	0.6199	0.7150	0.5220	0.8097	
$\hat{\beta}_{M,2}$	0.0055	0.0002	0.0110	0.0052	0.0058	
$\hat{\beta}_{M,3}$	-0.0024	-0.0075	0.0026	-0.0038	-0.0011	
$\hat{\beta}_{M4}$	0.2673	-0.0253	0.5600	0.2368	0.2970	
$\hat{\beta}_{M5}$	-0.1243	-0.1344	-0.1142	-0.1247	-0.1240	
$\hat{\beta}_{M,6}$	-0.0173	-0.0341	-0.0005	-0.0196	-0.0150	
β _{M 7}	-0.0398	-0.0424	-0.0371	-0.0401	-0.0395	
$\hat{\beta}_{M,8}$	0.0206	-0.0080	0.0494	0.0175	0.0237	
$\hat{\beta}_M 9$	0.0852	0.0792	0.0913	0.0850	0.0855	
$\hat{\beta}_{M,10}$	-0.0028	-0.0056	-0.0002	-0.0032	-0.0025	

Table 5 Parameter estimate and 95% confidence intervals of $\hat{\beta}_{M,r}$ for the air quality dataset

the confounding variable temperature has effect on the observed variables \tilde{X}_r 's and \tilde{Y} . It is known that temperature-U affects the true humidity-Y, which in turn affects the potential for precipitation. The interaction of temperature and humidity also directly affects the health and well-being of humans. As air temperature increases, air can hold more water molecules, and its relative humidity increases. In Figure 2, the figure $\hat{\phi}(u)$ is revealed to show the downtrend in a nonparametric way between distorted \tilde{Y} and the confounding variable U. This shows that the distortion function $\phi(U)$ should be taken into account to reveal the relation between the true humidity-Y and temperature-U. After we obtain the calibrated humidity \hat{Y} , a simple linear regression models shows that $\hat{Y} = 49.1 + 0.002698U$. It is seen that the positive relation between the calibrated humidity \hat{Y} and temperature-U is more reasonable to meet the expectation. But, the p-value of this coefficient 0.002698 is 0.97. This shows that the underlying true variable Y is possibly independent of the confounding variable U in this dataset. This not spoils the independence condition between the unobserved variable Y and observed confounder U.

The estimator of α_0 is obtained as $\hat{\alpha}_M = 63.9$, and the estimator $\hat{\beta}_M$ and the associated 95% confidence intervals are presented in Table 5. In this table, all the 95% empirical likelihood intervals indicate that β_{0r} 's are all non-zeros at the 5% significant level, while the 95% normal approximation confidence intervals indicate that β_{03} , β_{04} and β_{08} should be zero at the 5% significant level. Moreover, the lengths of the empirical likelihood confidence intervals are shorter than the normal approximation confidence intervals. Together with the simulation

results obtained in Tables 3–4, we recommend to using the empirical likelihood confidence intervals for further analysis.

Compared with the least squares estimates by directly using the observed variables without calibration, it indicates that the parameters β_{01} , β_{02} and β_{03} and β_{010} should be excluded from the model at the 5% significant level, and β_{04} , β_{05} , β_{06} , β_{07} , β_{08} and β_{09} are all non-zeros at the 5% significant level, and the signs of the nonzero estimates are (-, -, -, -, +, +). If we delete those irrelative variables (X_1, X_2, X_3, X_{10}) from the model, the mean squared residuals is increased to 23.1, which is slightly larger than 22.9 obtained by the observed data set without calibration, but both of them are still larger than the calibrated one: $\frac{1}{n} \sum_{i=1}^{n} \hat{\epsilon}_{M,i}^2 = 20.0$.

In Table 5, it is seen that the signs of these estimates with the conditional absolute mean calibration are (+, -, -, -, +, +), and the sign of the estimate $\hat{\beta}_{M,4}$ is different from those obtained without calibration. This indicates that the larger values of benzene concentration may result in larger values of underlying relative humidity in the significantly polluted area. The existing studies showed that humidity depends on both the amount fractions of water vapour and benzene. If benzene concentrations are above the annual limit value, high values of benzene concentration will change the interference of environmental humidity and also the air quality situation in that area. The negative value of the estimate of β_{04} without calibration is an underestimation of benzene concentration, which may cause a mismanagement of air quality in this area. Moreover, the daily traffic-related pollutant concentrations (such as CO, NMHC and O3) also have impact on meteorological factor-relative humidity in the air pollution studies, and the variables (X_1, X_2, X_3, X_{10}) should be included in the model. Therefore, the confounding variable does have impact so that the significant predictors can really be revealed in the model and any further analysis can be more meaningful.

6 Discussions and further research

In this paper, we proposed two different calibration procedures for the parameter estimation and confidence intervals construction under multiplicative distortion measurement errors. This paper serves as a basis for studies on the multiplicative distortion measurement errors models in the light of the circumstances. Different calibration procedures will result in different calibrated variables, and finally lead to different asymptotic properties of estimators. In future work, the semi-parametric models such as partial linear models, single-index models and partial linear varying coefficient models, can be considered, especially the nonparametric part in these semi-parametric models are distorted. One can also consider the case of multivariate confounding variables, such as modeling the distortion functions as singleindex models or additive models with different calibration procedures. For the other directions of the multiplicative distortion measurement errors models, one can pursue to consider the comparison between different calibration procedures.

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Supplementary Material

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