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# LATTICE APPROXIMATION TO THE DYNAMICAL $\Phi_3^4$ MODEL[1]

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We study the lattice approximations to the dynamical $\Phi_3^4$ model by paracontrolled distributions proposed in [*Forum Math. Pi* **3** (2015) e6]. We prove that the solutions to the lattice systems converge to the solution to the dynamical $\Phi_3^4$ model in probability, locally uniformly in time. Since the dynamical $\Phi_3^4$ model is not well defined in the classical sense and renormalisation has to be performed in order to define the nonlinear term, a corresponding suitable drift term is added in the stochastic equations for the lattice systems.

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**1. Introduction.** Recall that the usual continuum Euclidean $\Phi^4_d$-quantum field is heuristically described by the following probability measure:

$$N^{-1} \prod_{x\in\mathbb{T}^d} d\phi(x) \exp\biggl(-\int_{\mathbb{T}^d} \bigl(|\nabla\phi(x)|^2 + \phi^2(x) + \phi^4(x)\bigr)\,dx\biggr), \tag{1.1}$$

where $N$ is a normalization constant and $\phi$ is the real-valued field and $\mathbb{T}^d$ is the $d$-dimensional torus. There have been many approaches to the problem of giving a meaning to the above heuristic measure for $d = 2$ and $d = 3$ (see [10, 14] and references therein). In [31], Parisi and Wu proposed a program for Euclidean quantum field theory of getting Gibbs states of classical statistical mechanics as limiting distributions of stochastic processes, especially as solutions to nonlinear stochastic differential equations. Then one can use the stochastic differential equations to study the properties of the Gibbs states. This procedure is called stochastic field quantization (see [25]). The $\Phi^4_d$ model is the simplest nontrivial Euclidean quantum field (see [10] and the reference therein). The issue of the stochastic quantization of the $\Phi^4_d$ model is to solve the following equation:

$$d\Phi = \bigl(\Delta\Phi - \Phi^3\bigr)\,dt + dW(t), \qquad \Phi(0) = \Phi_0, \tag{1.2}$$

where $W$ is a cylindrical Wiener process on $L^2(\mathbb{T}^d)$. The solution $\Phi$ is also called dynamical $\Phi^4_d$ model. (1.2) is ill-posed in both two and three dimensions.

In two spatial dimensions, the dynamical $\Phi^4_2$ model was previously treated in [2, 9] and [29]. In three spatial dimensions, this equation (1.2) is ill-posed and the main difficulty in this case is that W, and hence the solutions are so singular that the nonlinear term is not well defined in the classical sense. It was a long-standing open problem to give a meaning to equation (1.2) in the three-dimensional case. A breakthrough result was achieved recently by Martin Hairer in [18], where he introduced a theory of regularity structures and gave a meaning to equation (1.2) successfully. He also proved existence and uniqueness of a local (in time) solution. By using the paracontrolled distributions proposed by Gubinelli, Imkeller and Perkowski in [12], existence and uniqueness of local solutions to (1.2) has also been obtained in [7]. Recently, these two approaches have been successful in giving a meaning to several other ill-posed stochastic PDEs like the Kardar–Parisi–Zhang (KPZ) equation [4, 17, 26], the Navier–Stokes equation driven by space–time white noise [37, 38], the dynamical sine-Gordon equation [23] and so on (see [22] for further interesting examples). From a philosophical perspective, the theory of regularity structures and the paracontrolled distributions are inspired by the theory of controlled rough paths [11, 28]. The main difference is that the regularity structure theory considers the problem locally, while the paracontrolled distribution method is a global approach using Fourier analysis. In [27], the author also uses renormalization group techniques to make sense of the dynamical $\Phi^4_3$ model.

The lattice approximation is an important tool in constructing and studying the continuum $\Phi^4_3$ field (see [1, 32, 33]). It also preserves Osterwalder–Schrader positivity and also the ferromagnetic nature of the measure (see [10] and the references therein). Let us set $\Lambda_\varepsilon := \{\varepsilon x \in \mathbb{T}^3, x \in \mathbb{Z}^3\}$. Heuristically, the quantities $\int |\nabla \phi(x)|^2 \, dx$, $\int \phi^2(x) \, dx$, and $\int \phi^4(x) \, dx$ can be approximated by $\varepsilon \sum_{|x-y|=\varepsilon, x,y\in\Lambda_\varepsilon} (\phi(x) - \phi(y))^2$, $\varepsilon^3 \sum_{x\in\Lambda_\varepsilon} \phi(x)^2$ and $\varepsilon^3 \sum_{x\in\Lambda_\varepsilon} \phi(x)^4$, respectively, as $\varepsilon$ tends to zero. Thus heuristically (1.1) can be approximated by the following probability measure $\mu_\varepsilon$:

$$
\begin{aligned}
& N_\varepsilon^{-1} \prod_{x\in\Lambda_\varepsilon} d\phi_x \exp\Biggl(2\varepsilon \sum_{|x-y|=\varepsilon, x,y\in\Lambda_\varepsilon} \phi(x)\phi(y) \\
(1.3) \qquad & \qquad - \bigl(\varepsilon^3 + 12\varepsilon\bigr) \sum_{x\in\Lambda_\varepsilon} \phi^2(x) - \varepsilon^3 \sum_{x\in\Lambda_\varepsilon} \phi^4(x)\Biggr),
\end{aligned}
$$

where $N_\varepsilon$ is a normalization constant. (1.3) is still just a heuristic expression, but one can give a rigorous meaning to it since it is a finite dimensional Gaussian measure with a density (see [10] and the references therein). We call this the lattice $\Phi^4_3$-field measure. From $\mu_\varepsilon$ by deriving suitable bounds on its moments and choosing subsequences if necessary, one gets limit measures by weak convergence. These are then the continuum $\Phi^4_3$-field measures.

The following stochastic PDE on $\Lambda_\varepsilon$, $\varepsilon > 0$, is the stochastic quantization associated to the lattice $\Phi^4_3$-field measure:

$$
\begin{aligned}
& d\Phi^\varepsilon(t,x) = \bigl(\Delta_\varepsilon \Phi^\varepsilon(t,x) - \bigl(\Phi^\varepsilon\bigr)^3(t,x) + \bigl(3C_0^\varepsilon - 9C_1^\varepsilon\bigr)\Phi^\varepsilon(t,x)\bigr)\, dt \\
(1.4) \qquad & \qquad + \varepsilon^{-3/2} \, dW_\varepsilon(t,x), \\
& \Phi^\varepsilon(0) = \Phi^\varepsilon_0.
\end{aligned}
$$

Here, $W_\varepsilon(t) = \{W(t,x)\}_{x\in\Lambda_\varepsilon}$ is a family of independent Brownian motions, $\Phi^\varepsilon_0$ and $W_\varepsilon$ are independent and $C_0^\varepsilon$ and $C_1^\varepsilon$ are constants defined in (6.3) and (1.10) below. For $x \in \Lambda_\varepsilon$, define

$$
\Delta_\varepsilon f(x) := \varepsilon^{-2} \sum_{y\in\Lambda_\varepsilon, y\sim x} \bigl(f(y) - f(x)\bigr),
$$

and the nearest neighbor relation $x \sim y$ is to be understood with periodic boundary conditions on $\Lambda_\varepsilon$.

The aim of this paper is to prove that as $\varepsilon \to 0$ the dynamical lattice approximation, that is, the solution to (1.4), converges to the dynamical $\Phi^4_3$ model. This problem is also related to the convergence of a rescaled discrete spin system to the solution of the dynamical $\Phi^4_3$ model (see [30] for the dynamical $\Phi^4_2$ model). We emphasize that to make sense of (1.2) we need to renormalise some ill-defined terms in (1.2). This is done by adding the renormalisation terms $C_0^\varepsilon \Phi^\varepsilon$ and $C_1^\varepsilon \Phi^\varepsilon$ in the approximating equation (1.4).

In the one-dimensional case, approximations to general stochastic partial differential equations driven by space–time white noise have been very well studied (see [8, 15, 16, 19, 20, 24] and the reference therein). In [13], the authors study the Sasamoto–Spohn-type discretizations of the conservative stochastic Burgers equation. In the three-dimensional case, we have also studied the discrete approximations to stochastic Navier–Stokes equations driven by space–time white noise (see [37]).

In this paper, we use the paracontrolled distribution method to prove that the solutions to the lattice approximation equation converge to the dynamical $\Phi_3^4$ model. The theory of paracontrolled distributions combines the idea of Gubinelli's controlled rough path [11] and Bony's paraproduct [5], which is defined as follows: Let $\Delta_j f$ be the $j$th Littlewood–Paley block of a (Scharwtz) distribution $f$. For its definition, we refer to Section 2. Define for distributions $f$ and $g$

$$\pi_<(f,g)=\pi_>(g,f)=\sum_{j\geq -1}\sum_{i<j-1}\Delta_i f\Delta_j g,\qquad \pi_0(f,g)=\sum_{|i-j|\leq 1}\Delta_i f\Delta_j g.$$

Formally, $fg=\pi_<(f,g)+\pi_0(f,g)+\pi_>(f,g)$. Observing that, if $f$ is regular, $\pi_<(f,g)$ behaves like $g$ and is the only term in the Bony's paraproduct not improving the regularities, the authors in [12] consider a paracontrolled ansatz of the type

$$u=\pi_<(u',g)+u^\sharp,$$

where $\pi_<(u',g)$ represents the "bad-part" of the solution, $u'$ is some suitable function and $g$ is some functional of the Gaussian field and $u^\sharp$ is regular enough to define the multiplication required. Then to make sense of the product of $uf$ we only need to define $gf$.

Using the paracontrolled distribution method, to perform the lattice approximation of the dynamical $\Phi_3^4$ model we shall meet the projection operators $P_N$, which do not commute with the paraproduct. Here, we use a random operator technique from [13] to handle these operators. However, for the dynamical $\Phi_3^4$ model this technique is not enough and we have to estimate an additional error term $D_N$ by stochastic calculations in Section 6.4 (see Remark 4.4).

*Framework and main result.* For $N\geq 1$, let $\Lambda^N=\{-N,-(N-1),\dots,N\}^3$. Set $\varepsilon=\frac{2}{2N+1}$. Every point $k\in\Lambda^N$ can be identified with $x=\varepsilon k\in\Lambda_\varepsilon=\{x=(x^1,x^2,x^3)\in\varepsilon\mathbb{Z}^3:-1<x^1,x^2,x^3<1\}$. We view $\Lambda_\varepsilon$ as a discretisation of the continuous three-dimensional torus $\mathbb{T}^3$ identified with $[-1,1]^3$. In the following for simplicity, we fix a cylindrical Wiener process in (1.2) on $L^2(\mathbb{T}^3)$ given by $2^{-\frac{3}{2}}\sum_k\beta_k e^{\iota\pi k\cdot x}$ for $x\in\mathbb{T}^3$ and restrict it to $L^2(\Lambda_\varepsilon)$ as $W_N(x)=2^{-\frac{3}{2}}\sum_{|k|_\infty\leq N}\beta_k e^{\iota\pi k\cdot x}$ for $x\in\Lambda_\varepsilon$, which is also a cylindrical Wiener process on $L^2(\Lambda_\varepsilon)$. Here, $\{\beta_k\}$ is a family of complex-valued Brownian motions with $\bar\beta_{-k}(t)=\beta_k(t)$ and $E[\beta_{k_1}(t_1)\beta_{k_2}(t_2)]=1_{\{k_1+k_2=0\}}t_1\wedge t_2$ and $|k|_\infty=$

$\max(|k^1|, |k^2|, |k^3|)$. For fixed $N$, (1.4) is a finite dimensional SDE and we can easily obtain existence and uniqueness of solutions to (1.4) by [34], which implies that the solution to (1.4) has the same distribution as the solution to the following equation:

$$\begin{aligned} d\Phi^\varepsilon(t,x) &= \big(\Delta_\varepsilon \Phi^\varepsilon(t,x) - \big(\Phi^\varepsilon\big)^3(t,x) + \big(3C_0^\varepsilon - 9C_1^\varepsilon\big)\Phi^\varepsilon(t,x)\big)\,dt \\ &\quad + dW_N(t,x), \\ \Phi^\varepsilon(0) &= \Phi_0^\varepsilon. \end{aligned} \tag{1.5}$$

Following [30], we discuss a suitable extension of functions defined on $\Lambda_\varepsilon$ onto all of the torus $\mathbb{T}^3$ (which we identify with the interval $[-1,1]^3$). For any function $Y : \Lambda_\varepsilon \to \mathbb{R}$, we define the discrete Fourier transform $\hat{Y}$ through

$$\hat{Y}(k) = \begin{cases} \sum_{x\in\Lambda_\varepsilon} \varepsilon^3 Y(x) e^{-\imath\pi k\cdot x} & \text{if } k \in \{-N,\dots,N\},^3 \\ 0 & \text{if } k \in \mathbb{Z}^3 \setminus \{-N,\dots,N\}^3. \end{cases}$$

In this context, Fourier inversion states

$$Y(x) = \frac{1}{8}\sum_{k\in\mathbb{Z}^3} \hat{Y}(k) e^{\imath\pi k\cdot x} \qquad \text{for all } x \in \Lambda_\varepsilon. \tag{1.6}$$

It is thus natural to extend $Y$ to all of $\mathbb{T}^3$ by taking (1.6) as a definition of $Y(x)$ for $x \in \mathbb{T}^3 \setminus \Lambda_\varepsilon$. More explicitly, for $Y : \Lambda_\varepsilon \to \mathbb{R}$ we define $(\operatorname{Ext} Y) : \mathbb{T}^3 \to \mathbb{R}$ as

$$\operatorname{Ext} Y(x) = \frac{1}{2^3}\sum_{k\in\{-N,\dots,N\}^3}\sum_{y\in\Lambda_\varepsilon} \varepsilon^3 e^{\imath\pi k\cdot(x-y)} Y(y).$$

By the definition of the operators $\Delta_\varepsilon$, we have

$$\widehat{e^{t\Delta_\varepsilon} v}(k) = \begin{cases} e^{-|k|^2 f(\varepsilon k)}\hat{v}(k) & \text{if } k \in \{-N,\dots,N\}^3, \\ 0 & \text{if } k \in \mathbb{Z}^3 \setminus \{-N,\dots,N\}^3. \end{cases}$$

Here, for $x = (x^1, x^2, x^3)$,

$$f(x) = \frac{4}{|x|^2}\bigg(\sin^2\frac{x^1\pi}{2} + \sin^2\frac{x^2\pi}{2} + \sin^2\frac{x^3\pi}{2}\bigg).$$

Now we extend the solutions to all of $\mathbb{T}^3$. It is easy to see that

$$\begin{aligned} \operatorname{Ext}\Phi^\varepsilon(t) &= P_t^\varepsilon \operatorname{Ext}\Phi_0^\varepsilon - \int_0^t P_{t-s}^\varepsilon Q_N\big[\big(\operatorname{Ext}\Phi^\varepsilon\big)^3 - \big(3C_0^\varepsilon - 9C_1^\varepsilon\big)\operatorname{Ext}\Phi^\varepsilon\big]\,ds \\ &\quad + \int_0^t P_{t-s}^\varepsilon \operatorname{Ext} dW_N. \end{aligned} \tag{1.7}$$

Here, $P_t^\varepsilon = \operatorname{Ext} e^{t\Delta_\varepsilon}$ and $Q_N u(x) = P_N u(x) + \Pi_N u(x)$ with

$$P_N = \mathcal{F}^{-1} 1_{|k|_\infty \le N}\mathcal{F}, \tag{1.8}$$

and $\Pi_N$ is defined for $u$ satisfying $\operatorname{supp}\mathcal{F}u \subset \{k : |k|_\infty \le 3N\}$ as follows:

$$\begin{aligned}\Pi_N u(x) &= \sum_{i_1,i_2,i_3\in\{-1,0,1\},\sum_{j=1}^3 i_j^2\neq 0} e_N^{i_1i_2i_3}(x)\mathcal{F}^{-1}1_{k\in P^{i_1i_2i_3}}\mathcal{F}u(x)\\ &= \sum_{i_1,i_2,i_3\in\{-1,0,1\},\sum_{j=1}^3 i_j^2\neq 0} P_N\big[e_N^{i_1i_2i_3}u\big](x),\end{aligned} \tag{1.9}$$

where $P^{i_1i_2i_3} = \{k = (k^1, k^2, k^3) : k^j i_j > N \text{ if } i_j = -1, 1; |k^j| \le N, \text{ if } i_j = 0, j = 1, 2, 3\}$ is a rectangular division of $\mathbb{Z}^3 \setminus \{k \in \mathbb{Z}^3, |k|_\infty \le N\}$, $e_N^{i_1i_2i_3}(x) = \prod_{j=1}^3 e^{-\iota\pi(2N+1)i_j x^j}$ and $|k|_\infty = \max(|k^1|, |k^2|, |k^3|)$. Here and in the following, the Fourier transform and the inverse Fourier transform are denoted by $\mathcal{F}$ and $\mathcal{F}^{-1}$, respectively.

REMARK 1.1. When we use (1.6) to write $f$ and $g$ in terms of discrete Fourier transform and take the product of $f$ and $g$, it is easy to see where the $\Pi_N$ part comes from. When $\operatorname{supp}\mathcal{F}(\operatorname{Ext} f \operatorname{Ext} g) \nsubseteq \{k \in \mathbb{Z}^3 : |k|_\infty \le N\}$, we should multiply $e_N^{i_1i_2i_3}$ to make $\operatorname{supp}\mathcal{F}(\operatorname{Ext} f \operatorname{Ext} g e_N^{i_1i_2i_3})$ belong to the set $\{k \in \mathbb{Z}^3 : |k|_\infty \le N\}$.

Now choose $C_0^\varepsilon$ as in (6.3) and

$$C_1^\varepsilon = C_{11}^\varepsilon + \sum_{i_1,i_2,i_3\in\{-1,0,1\},\sum_{j=1}^3 i_j^2\neq 0} C_{12}^{\varepsilon,i_1i_2i_3}, \tag{1.10}$$

with $C_{11}^\varepsilon$, $C_{12}^{\varepsilon,i_1i_2i_3}$ defined in (6.4) and (6.5), respectively. In the following, we omit the summation with respect to $i_1, i_2, i_3$ if there is no confusion.

The main result of this paper is the following theorem.

THEOREM 1.2. *Let $z \in (1/2, 2/3)$ and $\Phi_0 \in \mathcal{C}^{-z}$. Let $(\Phi, \tau)$ be the unique (maximal in time) solution to (1.2) and let for $\varepsilon \in (0, 1)$ the function $\Phi^\varepsilon$ be the unique solution to (1.5) on $[0, \infty)$. If the initial data satisfies $\operatorname{Ext}\Phi_0^\varepsilon - \Phi_0 \to 0$ in $\mathcal{C}^{-z}$, then there exists a sequence of random times $\tau_L$ such that $\lim_{L\to\infty}\tau_L = \tau$ and*

$$\sup_{t\in[0,\tau_L]} \|\operatorname{Ext}\Phi^\varepsilon - \Phi\|_{-z} \to 0 \qquad \textit{in probability, as } \varepsilon \to 0.$$

REMARK 1.3. (i) Existence and uniqueness of $(\Phi, \tau)$ has been obtained in [7, 18]. For the definition of $\mathcal{C}^{-z}$ and the norm $\|\cdot\|_{-z}$, see Section 2 below.

(ii) The constant $C_1^\varepsilon$ is the corresponding renormalization constant of order $-\log\varepsilon$ and is divided into two parts: $C_{11}^\varepsilon$ and $C_{12}^\varepsilon$ which come from terms with $P_N$ and $\Pi_N$ defined in (1.8) and (1.9), respectively. Moreover,

$$C_0^\varepsilon \backsimeq \frac{1}{\varepsilon}, \qquad C_{11}^\varepsilon \backsimeq -\log\varepsilon, \qquad C_{12}^{\varepsilon,i_1i_2i_3} \backsimeq 1.$$

(iii) After our original paper was published on arXiv, Hairer and Matetski in [21] also obtained similar results by using the theory of regularity structure. Moreover, by using the results in [6] they obtained the existence of a global solution to the dynamical $\Phi^4_3$ model starting from almost every point and the $\Phi^4_3$ field is an invariant measure of the solution to (1.2) when the coupling constant is small in their Corollary 1.2. By a similar argument as in the proof of Corollary 1.2 in [21], we can also obtain these results. Compared to the piecewise constant extension in [21], Corollary 1.2, our extension is smooth and based on discrete Fourier transform and does not change the inner product from $L^2(\Lambda_\varepsilon)$ to $L^2(\mathbb{T}^3)$, which coincides with the extension considered in [10]. Moreover, we use the lattice approximation and this extension to study the Dirichlet form associated with the $\Phi^4_3$ field in our forthcoming paper.

The structure of the paper is organized as follows. In Section 2, we recall some basic notions and results for the paracontrolled distribution method. In Section 3, we prove some estimates for the approximating operators. In Section 4, we use the paracontrolled distribution method to prove uniform bounds for the lattice approximation equations. In Section 5, we give the proof of our main result Theorem 1.2. In Section 6, convergence of several stochastic terms is proved.

**2. Besov spaces and paraproduct.** In the following, we recall the definitions and some properties of Besov spaces and paraproducts. For a general introduction, we refer to [3, 12]. First, we introduce the following notation. Throughout the paper, we use the notation $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$, and we write $a \backsimeq b$ if $a \lesssim b$ and $b \lesssim a$. Given a Banach space $E$ with norm $\|\cdot\|_E$ and $T > 0$, we write $C_T E = C([0, T]; E)$ for the space of continuous functions from $[0, T]$ to $E$, equipped with the supremum norm $\|\cdot\|_{C_T E}$. For $\alpha \in (0, 1)$, we also define $C^\alpha_T E$ as the space of $\alpha$-Hölder continuous functions from $[0, T]$ to $E$, endowed with the seminorm $\|f\|_{C^\alpha_T E} = \sup_{s,t\in[0,T],s\neq t} \frac{\|f(s)-f(t)\|_E}{|t-s|^\alpha}$.

The space of real valued infinitely differentiable functions of compact support is denoted by $\mathcal{D}(\mathbb{R}^d)$ or $\mathcal{D}$. The space of Schwartz functions is denoted by $\mathcal{S}(\mathbb{R}^d)$. Its dual, the space of tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^d)$.

Let $\chi, \theta \in \mathcal{D}$ be nonnegative radial functions on $\mathbb{R}^d$, such that:

i. the support of $\chi$ is contained in a ball and the support of $\theta$ is contained in an annulus;

ii. $\chi(z) + \sum_{j\geq 0} \theta(2^{-j}z) = 1$ for all $z \in \mathbb{R}^d$.

iii. $\operatorname{supp}(\chi) \cap \operatorname{supp}(\theta(2^{-j}\cdot)) = \varnothing$ for $j \geq 1$ and $\operatorname{supp}(\theta(2^{-i}\cdot)) \cap \operatorname{supp}(\theta(2^{-j}\cdot)) = \varnothing$ for $|i - j| > 1$.

We call the pair $(\chi, \theta)$ a dyadic partition of unity, and refer to [3], Proposition 2.10, for its existence. The Littlewood–Paley blocks are now defined as

$$\Delta_{-1}u = \mathcal{F}^{-1}(\chi \mathcal{F}u), \qquad \Delta_j u = \mathcal{F}^{-1}(\theta(2^{-j}\cdot)\mathcal{F}u).$$

For $\alpha \in \mathbb{R}$, the Hölder–Besov space $\mathcal{C}^\alpha$ is given by $\mathcal{C}^\alpha = B^\alpha_{\infty,\infty}(\mathbb{R}^d)$, where for $p, q \in [1, \infty]$ we define

$$B^\alpha_{p,q}(\mathbb{R}^d) = \left\{u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{B^\alpha_{p,q}} = \left(\sum_{j\geq -1} (2^{j\alpha}\|\Delta_j u\|_{L^p})^q\right)^{1/q} < \infty\right\},$$

with the usual interpretation as $l^\infty$ norm in case $q = \infty$. For $\alpha \in \mathbb{R}$, we write $\|\cdot\|_\alpha$ instead of $\|\cdot\|_{B^\alpha_{\infty,\infty}}$ in the following for simplicity.

We point out that everything above and everything that follows can be applied to distributions on the torus (see [35, 36]). More precisely, let $\mathcal{S}'(\mathbb{T}^d)$ be the space of distributions on $\mathbb{T}^d$. Therefore, Besov spaces on the torus with general indices $p, q \in [1, \infty]$ are defined as

$$B^\alpha_{p,q}(\mathbb{T}^d) = \left\{u \in \mathcal{S}'(\mathbb{T}^d) : \|u\|_{B^\alpha_{p,q}} = \left(\sum_{j\geq -1} (2^{j\alpha}\|\Delta_j u\|_{L^p(\mathbb{T}^d)})^q\right)^{1/q} < \infty\right\}.$$

We will need the following Besov embedding theorem on the torus (cf. [12], Lemma 41).

LEMMA 2.1. *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$, and let $\alpha \in \mathbb{R}$. Then $B^\alpha_{p_1,q_1}(\mathbb{T}^d)$ is continuously embedded in $B^{\alpha - d(\frac{1}{p_1} - \frac{1}{p_2})}_{p_2,q_2}(\mathbb{T}^d)$.*

Now we recall the following paraproduct introduced by Bony (see [5]). In general, the product $fg$ of two distributions $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$ is well defined if and only if $\alpha + \beta > 0$. In terms of Littlewood–Paley blocks, the product $fg$ of two distributions $f$ and $g$ can be formally decomposed as

$$fg = \sum_{j\geq -1}\sum_{i\geq -1} \Delta_i f \Delta_j g = \pi_<(f, g) + \pi_0(f, g) + \pi_>(f, g),$$

with

$$\pi_<(f, g) = \pi_>(g, f) = \sum_{j\geq -1}\sum_{i<j-1} \Delta_i f \Delta_j g, \qquad \pi_0(f, g) = \sum_{|i-j|\leq 1} \Delta_i f \Delta_j g.$$

For $j \geq 0$, we also use the notation

$$S_j f = \sum_{i\leq j-1} \Delta_i f,$$

and for $k_1, k_2 \in \mathbb{Z}^3$

$$\psi_<(k_1, k_2) = \sum_{j\geq -1}\sum_{i<j-1} \theta_i(k_1)\theta_j(k_2), \qquad \psi_0(k_1, k_2) = \sum_{|i-j|\leq 1} \theta_i(k_1)\theta_j(k_2),$$

with $\theta_i = \theta(2^{-i}\cdot)$ for $i \geq 0$ and $\theta_{-1} = \chi$. We will use without comment that $\|\cdot\|_\alpha \leq \|\cdot\|_\beta$ for $\alpha \leq \beta$, that $\|\cdot\|_{L^\infty} \lesssim \|\cdot\|_\alpha$ for $\alpha > 0$, and that $\|\cdot\|_\alpha \lesssim \|\cdot\|_{L^\infty}$ for $\alpha \leq 0$.

We will also use that $\|S_j u\|_{L^\infty} \lesssim 2^{-j\alpha}\|u\|_\alpha$ for $\alpha < 0$, $j \geq 0$ and $u \in \mathcal{C}^\alpha$, where $\|\cdot\|_\alpha$ denotes the norm in $\mathcal{C}^\alpha$, $\alpha \in \mathbb{R}$.

The basic results about these bilinear operations are given by the following estimates: From these estimates, we know that $\pi_<(f,g)$ part is the only term in the paraproduct not improving the regularity even if $f$ is regular. $\pi_0(f,g)$ part is the only term in the paraproduct not well defined for arbitrary distributions $f, g$.

LEMMA 2.2 (Paraproduct estimates, [5], [12], Lemma 2). *For any $\beta \in \mathbb{R}$ we have*

$$\|\pi_<(f,g)\|_\beta \lesssim \|f\|_{L^\infty}\|g\|_\beta, \qquad f \in L^\infty, g \in \mathcal{C}^\beta,$$

*and for $\alpha < 0$ furthermore*

$$\|\pi_<(f,g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha\|g\|_\beta, \qquad f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta.$$

*For $\alpha + \beta > 0$, we have*

$$\|\pi_0(f,g)\|_{\alpha+\beta} \lesssim \|f\|_\alpha\|g\|_\beta, \qquad f \in \mathcal{C}^\alpha, g \in \mathcal{C}^\beta.$$

The following basic commutator lemma is important for our use.

LEMMA 2.3 ([12], Lemma 5). *Assume that $\alpha \in (0,1)$ and $\beta, \gamma \in \mathbb{R}$ are such that $\alpha + \beta + \gamma > 0$ and $\beta + \gamma < 0$. Then for smooth $f, g, h$, the trilinear operator*

$$C(f,g,h) = \pi_0\big(\pi_<(f,g),h\big) - f\pi_0(g,h)$$

*satisfies the bound*

$$\|C(f,g,h)\|_{\alpha+\beta+\gamma} \lesssim \|f\|_\alpha\|g\|_\beta\|h\|_\gamma.$$

*Thus, $C$ can be uniquely extended to a bounded trilinear operator from $\mathcal{C}^\alpha \times \mathcal{C}^\beta \times \mathcal{C}^\gamma$ to $\mathcal{C}^{\alpha+\beta+\gamma}$.*

Now we recall the following properties of the heat semigroup $P_t := e^{t\Delta}$, which corresponds to the smoothing effect of the heat semigrop.

LEMMA 2.4 ([12], Lemma 47). *Let $u \in \mathcal{C}^\alpha$ for some $\alpha \in \mathbb{R}$. Then for every $\delta \geq 0$*

$$\|P_t u\|_{\alpha+\delta} \lesssim t^{-\frac{\delta}{2}}\|u\|_\alpha.$$

LEMMA 2.5 ([7], Lemma A.1). *Let $u \in \mathcal{C}^\alpha$ for some $\alpha < 1$ and $v \in \mathcal{C}^\beta$ for some $\beta \in \mathbb{R}$. Then for $\delta \geq \alpha + \beta$*

$$\|P_t\pi_<(u,v) - \pi_<(u,P_t v)\|_\delta \lesssim t^{\frac{\alpha+\beta-\delta}{2}}\|u\|_\alpha\|v\|_\beta.$$

LEMMA 2.6 ([7], Lemma 2.5). *Let $u \in \mathcal{C}^{\alpha+\delta}$ for some $\alpha \in \mathbb{R}$, $\delta > 0$. Then for every $t \geq 0$,*

$$\|(P_t - I)u\|_\alpha \lesssim t^{\frac{\delta}{2}} \|u\|_{\alpha+\delta}.$$

We also have the following result, which will be used later.

LEMMA 2.7 (Bernstein-type lemma). *Let $u \in \mathcal{C}^\alpha$ for some $\alpha \in \mathbb{R}$.*

(1) *If* $\operatorname{supp} \mathcal{F}u \subset \{k : |k| \leq CN\}$ *for some $C > 0$, then for $\beta > \alpha$*

$$\|u\|_\beta \lesssim N^{\beta-\alpha} \|u\|_\alpha.$$

(2) *If* $\operatorname{supp} \mathcal{F}u \subset \{k : |k| > CN\}$ *for some $C > 0$, then for $\beta < \alpha$*

$$\|u\|_\beta \lesssim N^{\beta-\alpha} \|u\|_\alpha.$$

*Here, all the constants we omit are independent of $N$.*

PROOF. We have

$$\|u\|_\beta = \sup_j 2^{j\beta} \|\Delta_j u\|_{L^\infty} = \sup_j 2^{j(\beta-\alpha)} 2^{j\alpha} \|\Delta_j u\|_{L^\infty}.$$

For the first case, we have that $\Delta_j u \neq 0$ iff $2^j \lesssim N$, which implies the first result. If $\operatorname{supp} \mathcal{F}u \subset \{k : |k| > CN\}$, we have that $\Delta_j u \neq 0$ iff $2^j \gtrsim N$, which implies the second result. $\square$

**3. Estimates for the approximating operators.** In this section, we prove the estimates for the approximating operators on $\mathbb{T}^3$, which will be used to prove the main result. First, we prove estimates for $P_N$ and $\Pi_N$ defined in (1.8) and (1.9). Compared to the estimates proved in the one dimensional case in [13], we prove them here in the three-dimensional case. Moreover, we prove a commutator estimate for $P_t^\varepsilon$ whereas in [13] a commutator estimate for $\Delta_N$ was proved.

LEMMA 3.1. *Let $u \in \mathcal{C}^\alpha$ for some $\alpha \in \mathbb{R}$. Then for any $\kappa > 0$ small enough we have the following estimates*:

(1) (*Estimates for $P_N$*)

$$\|P_N u\|_{\alpha-\kappa} \lesssim \|u\|_\alpha, \qquad \|(I - P_N)u\|_{\alpha-\kappa} \lesssim N^{-\frac{\kappa}{2}} \|u\|_\alpha.$$

(2) (*Estimates for $\Pi_N$*) *If $\alpha > \frac{5\kappa}{4}$, then for $u$ satisfying* $\operatorname{supp} \mathcal{F}u \subset \{k : |k|_\infty \leq 3N\}$

$$\|\Pi_N u\|_{\alpha-\kappa} \lesssim N^{-\frac{\kappa}{2}} \|u\|_\alpha.$$

*If $\alpha < 0$ and* $\operatorname{supp} \mathcal{F}u \subset \{k : |k|_\infty \leq N\}$, *then*

$$\|e_N^{i_1 i_2 i_3} u\|_{\alpha-\kappa} \lesssim N^{-\frac{\kappa}{2}} \|u\|_\alpha.$$

*Here, all the constants we omit are independent of $N$.*

PROOF. We have for $p > 1$ large enough

$$\|P_N u\|_{\alpha-\kappa} \lesssim \|P_N u\|_{B^\alpha_{p,\infty}} \lesssim \|u\|_{B^\alpha_{p,\infty}} \lesssim \|u\|_\alpha,$$

where in the first inequality we used Lemma 2.1 and in the second inequality we used that $\mathcal{F}^{-1}1_{|k|_\infty \le N}\mathcal{F}$ is an $L^p$-multiplier. Similarly,

$$\|(I-P_N)u\|_{\alpha-\kappa} \lesssim N^{-\frac{\kappa}{2}}\|(I-P_N)u\|_{\alpha-\frac{\kappa}{2}} \lesssim N^{-\frac{\kappa}{2}}\|u\|_\alpha,$$

where in the first inequality we used Lemma 2.7 and in the second inequality we used the result for $P_N$. For (2), we have that for $\alpha > \frac{5\kappa}{4}$

$$\begin{aligned}\|\Pi_N u\|_{\alpha-\kappa} &\lesssim N^{\alpha-\frac{5\kappa}{4}}\|\Pi_N u\|_{\frac{\kappa}{4}} \lesssim N^{\alpha-\kappa}\|\mathcal{F}^{-1}1_{k\in P^{i_1i_2i_3}}\mathcal{F}u\|_{\frac{\kappa}{4}} \\ &\lesssim N^{-\frac{\kappa}{2}}\|\mathcal{F}^{-1}1_{k\in P^{i_1i_2i_3}}\mathcal{F}u\|_{\alpha-\frac{\kappa}{4}} \lesssim N^{-\frac{\kappa}{2}}\|u\|_\alpha.\end{aligned}$$

Here, in the first and third inequalities we used Lemma 2.7, in the second inequality we used that $\|e_N^{i_1i_2i_3}\|_{\frac{\kappa}{4}} \lesssim N^{\frac{\kappa}{4}}$ and in the last inequality we used a similar argument for $P_N$ since $\mathcal{F}^{-1}1_{k\in P^{i_1i_2i_3}}\mathcal{F}$ is an $L^p$-multiplier. Similarly, for $\alpha < 0$

$$\|e_N^{i_1i_2i_3}u\|_{\alpha-\kappa} \lesssim N^{\alpha-3\frac{\kappa}{2}}\|e_N^{i_1i_2i_3}u\|_{\frac{\kappa}{2}} \lesssim N^{\alpha-\kappa}\|u\|_{\frac{\kappa}{2}} \lesssim N^{-\frac{\kappa}{2}}\|u\|_\alpha.$$

Here, we used $\operatorname{supp}\mathcal{F}(e_N^{i_1i_2i_3}u) \subset \{k : |k| > N\}$ and Lemma 2.7 in the first inequality as well as Lemma 2.7 in the last inequality. Thus the results in (2) follows. $\square$

Now we prove several properties for the approximating semigroup $P_t^\varepsilon = \operatorname{Ext} e^{t\Delta_\varepsilon}$ such as smoothing effect, commutator estimate, which are parallel to the properties of the heat semigroup in Lemmas 2.4–2.6. In fact,

$$P_t^\varepsilon = \mathcal{F}^{-1}1_{|k|_\infty\le N}e^{-t|k|^2 f(\varepsilon k)}\mathcal{F} = \mathcal{F}^{-1}1_{|k|_\infty\le N}e^{-t|k|^2 f(\varepsilon k)}\varphi(\varepsilon k)\mathcal{F} = P_N\tilde{P}_t^\varepsilon,$$

with

$$\tilde{P}_t^\varepsilon := \mathcal{F}^{-1}e^{-t|k|^2 f(\varepsilon k)}\varphi(\varepsilon k)\mathcal{F},$$

where $\varphi$ is a smooth function and equals 1 on $\{|x|_\infty \le 1\}$ with $\operatorname{supp}\varphi \subset \{|x| \le 1.8\}$. Here, we introduce $\tilde{P}_t^\varepsilon$ for the following technique calculations. Then by similar arguments as in [12], Lemma 47, we have the following results.

LEMMA 3.2. *Let $u \in \mathcal{C}^\alpha$ for some $\alpha \in \mathbb{R}$. Then for every $\delta \ge 0$, $\kappa > 0$, $t > 0$,*

$$\|P_t^\varepsilon u\|_{\alpha+\delta-\kappa} \lesssim t^{-\frac{\delta}{2}}\|u\|_\alpha,$$

$$\|(P_t^\varepsilon - P_t)u\|_{\alpha+\delta-\kappa} \lesssim \varepsilon^{\frac{\kappa}{2}}t^{-\frac{\delta}{2}}\|u\|_\alpha.$$

*Here, the constants we omit are independent of $N$.*

PROOF. To obtain the first result, by Lemma 3.1 it suffices to prove that for every $\delta \geq 0$

$$\|\tilde{P}_t^\varepsilon u\|_{\alpha+\delta} \lesssim t^{-\frac{\delta}{2}}\|u\|_\alpha. \tag{3.1}$$

In the following, we prove (3.1) and have that for $j \geq 0$

$$\begin{aligned}\|\Delta_j \tilde{P}_t^\varepsilon u\|_{L^\infty} &= \|\mathcal{F}^{-1}\theta_j \phi^\varepsilon \mathcal{F}u\|_{L^\infty} = \|\mathcal{F}^{-1}\theta_j\tilde{\theta}(2^{-j}\cdot)\phi^\varepsilon \mathcal{F}u\|_{L^\infty} \\ &\leq \|\mathcal{F}^{-1}(\phi^\varepsilon\tilde{\theta}(2^{-j}\cdot))\|_{L^1}\|\Delta_j u\|_{L^\infty}.\end{aligned}$$

Here and in the following

$$\phi^\varepsilon(\xi) = e^{-t|\xi|^2 f(\varepsilon\xi)}\varphi(\varepsilon\xi), \qquad \phi(\xi) = e^{-t|\xi|^2}, \tag{3.2}$$

and $\tilde{\theta}$ is a smooth function supported in an annulus such that $\tilde{\theta}\theta = \theta$. Then we get that for $\delta \geq 0$,

$$\begin{aligned}\|\mathcal{F}^{-1}(\phi^\varepsilon\tilde{\theta}(2^{-j}\cdot))\|_{L^1} &= \|\mathcal{F}^{-1}(\phi^\varepsilon(2^j\cdot)\tilde{\theta})\|_{L^1} \lesssim \|(1-\Delta)^2(\phi^\varepsilon(2^j\cdot)\tilde{\theta})\|_{L^1} \\ &\lesssim \sum_{0\leq|m|\leq 4} 2^{j|m|}\|(D_m\phi^\varepsilon)(2^j\cdot)|_{\cdot\in\operatorname{supp}\tilde{\theta}}\|_{L^\infty} \\ &\lesssim \sum_{0\leq|m|\leq 4} 2^{j|m|}\frac{1}{2^{j|m|}(2^j\sqrt{t})^\delta} \lesssim (2^j\sqrt{t})^{-\delta}.\end{aligned}$$

Here, in the third inequality we used that $f(\varepsilon\xi) \geq c > 0$ and $|\varepsilon\xi| \lesssim 1$ on the support of $\phi^\varepsilon$, which implies that for any multiindex $m$ satisfying $|m| \leq 4$ and every $\delta \geq 0$ we have $|D_m\phi^\varepsilon(\xi)| \lesssim \frac{1}{|\xi|^{|m|+\delta}t^{\delta/2}}$. For $j = -1$, we can use Bernstein's lemma to obtain the estimate. Thus (3.1) follows.

For the second result, we have

$$P_t^\varepsilon - P_t = P_N(\tilde{P}_t^\varepsilon - P_t) + (I - P_N)P_t.$$

By Lemmas 2.4 and 3.1, it is sufficient to consider $\tilde{P}_t^\varepsilon - P_t$. Since $\phi^\varepsilon(\xi) - \phi(\xi) = \varphi(\varepsilon\xi)(e^{-t|\xi|^2 f(\varepsilon\xi)} - e^{-t|\xi|^2}) + (\varphi(\varepsilon\xi) - 1)e^{-t|\xi|^2}$ and $|\varphi(\varepsilon\xi) - 1| \lesssim |\varepsilon\xi|^\eta$, $|f(\varepsilon\xi) - \pi^2| \lesssim |\varepsilon\xi|^\eta$ for every $0 < \eta < 1$, we obtain that for any multiindex $m$ satisfying $|m| \leq 4$ and every $\delta \geq 0$, $0 < \eta < 1$, we have $|D_m(\phi^\varepsilon - \phi)(\xi)| \leq \frac{(\varepsilon|\xi|)^\eta}{|\xi|^{|m|+\delta}t^{\frac{\delta}{2}}}$. Thus the second result follows by a similar argument as in the proof of (3.1). □

In the following, we prove a commutator estimate for $P_t^\varepsilon$. However, $P_N$ does not commute with paraproduct and we can only obtain the following.

LEMMA 3.3. *Let $u \in \mathcal{C}^\alpha$ for some $\alpha < 1$ and $v \in \mathcal{C}^\beta$ for some $\beta \in \mathbb{R}$. Then for $\delta \geq \alpha + \beta$ and any $\kappa > 0$,*

$$\|P_t^\varepsilon \pi_<(u, v) - P_N\pi_<(u, \tilde{P}_t^\varepsilon v)\|_{\delta-\kappa} \lesssim t^{\frac{\alpha+\beta-\delta}{2}}\|u\|_\alpha\|v\|_\beta, \tag{3.3}$$

$$
\begin{aligned}
&\|(P_t^\varepsilon - P_t)\pi_<(u,v) - P_N\pi_<(u,\tilde P_t^\varepsilon v) + \pi_<(u,P_t v)\|_{\delta-\kappa} \\
&\quad\lesssim \varepsilon^{\frac{\kappa}{2}} t^{\frac{\alpha+\beta-\delta}{2}}\|u\|_\alpha\|v\|_\beta.
\end{aligned}
\tag{3.4}
$$

*Here, the constants we omit are independent of $N$.*

PROOF. We have

$$P_t^\varepsilon\pi_<(u,v) - P_N\pi_<(u,\tilde P_t^\varepsilon v) = P_N(\tilde P_t^\varepsilon\pi_<(u,v) - \pi_<(u,\tilde P_t^\varepsilon v)).$$

By Lemma 3.1, it suffices to prove that

$$\|\tilde P_t^\varepsilon\pi_<(u,v) - \pi_<(u,\tilde P_t^\varepsilon v)\|_\delta \lesssim t^{\frac{\alpha+\beta-\delta}{2}}\|u\|_\alpha\|v\|_\beta. \tag{3.5}$$

In fact, we have that

$$\tilde P_t^\varepsilon\pi_<(u,v) - \pi_<(u,\tilde P_t^\varepsilon v) = \sum_{j=-1}^\infty (\tilde P_t^\varepsilon(S_{j-1}u\Delta_j v) - S_{j-1}u\tilde P_t^\varepsilon\Delta_j v),$$

and that the Fourier transform of $\tilde P_t^\varepsilon(S_{j-1}u\Delta_j v) - S_{j-1}u\tilde P_t^\varepsilon\Delta_j v$ has its support in a suitable annulus $2^j\mathcal{A}$. Let $\psi\in\mathcal{D}(\mathbb{R}^3)$ with support in an annulus $\tilde{\mathcal{A}}$ be such that $\psi = 1$ on $\mathcal{A}$.

Thus by the same argument as in the proof of [7], Lemma A.1, we obtain that

$$
\begin{aligned}
&\|\tilde P_t^\varepsilon(S_{j-1}u\Delta_j v) - S_{j-1}u\tilde P_t^\varepsilon\Delta_j v\|_{L^\infty} \\
&\quad\lesssim \sum_{\eta\in\mathbb{N}^d,|\eta|=1}\|x^\eta\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\phi^\varepsilon)\|_{L^1}\|\partial^\eta S_{j-1}u\|_{L^\infty}\|\Delta_j v\|_{L^\infty},
\end{aligned}
$$

where $\phi^\varepsilon$ is introduced in (3.2). Now we have that

$$
\begin{aligned}
&\|x^\eta\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\phi^\varepsilon)\|_{L^1} \\
&\quad\le 2^{-j}\|\mathcal{F}^{-1}((\partial^\eta\psi)(2^{-j}\cdot)\phi^\varepsilon)\|_{L^1} + \|\mathcal{F}^{-1}(\psi(2^{-j}\cdot)\partial^\eta\phi^\varepsilon)\|_{L^1} \\
&\quad= 2^{-j}\|\mathcal{F}^{-1}(\partial^\eta\psi(\cdot)\phi^\varepsilon(2^j\cdot))\|_{L^1} + \|\mathcal{F}^{-1}(\psi(\cdot)\partial^\eta\phi^\varepsilon(2^j\cdot))\|_{L^1} \\
&\quad\lesssim 2^{-j}\|(1+|\cdot|^2)^2\mathcal{F}^{-1}(\partial^\eta\psi(\cdot)\phi^\varepsilon(2^j\cdot))\|_{L^\infty} \\
&\qquad + \|(1+|\cdot|^2)^2\mathcal{F}^{-1}(\psi(\cdot)\partial^\eta\phi^\varepsilon(2^j\cdot))\|_{L^\infty} \\
&\quad= 2^{-j}\|\mathcal{F}^{-1}((1-\Delta)^2(\partial^\eta\psi(\cdot)\phi^\varepsilon(2^j\cdot)))\|_{L^\infty} \\
&\qquad + \|\mathcal{F}^{-1}((1-\Delta)^2(\psi(\cdot)\partial^\eta\phi^\varepsilon(2^j\cdot)))\|_{L^\infty} \\
&\quad\lesssim 2^{-j}\|(1-\Delta)^2(\partial^\eta\psi(\cdot)\phi^\varepsilon(2^j\cdot))\|_{L^1} + \|(1-\Delta)^2(\psi(\cdot)\partial^\eta\phi^\varepsilon(2^j\cdot))\|_{L^1} \\
&\quad\lesssim 2^{-j}\sum_{0\le|m|\le4}(2^j)^{|m|}\frac{t^{-\mu}2^{-2j\mu}}{(2^j)^{|m|}} + \sum_{|m|\le5}(2^j)^{|m|}\frac{t^{-\mu}2^{-2j\mu}}{(2^j)^{|m|+1}} \\
&\quad\lesssim 2^{-j}t^{-\mu}2^{-2j\mu},
\end{aligned}
$$

where in the fourth inequality we used that $|D^m\phi^\varepsilon(\xi)| \lesssim |\xi|^{-|m|}t^{-\mu}|\xi|^{-2\mu}$, $\mu \geq 0$, for any multiindex $m$ satisfying $|m| \leq 5$. Hence we get that

$$\|\tilde{P}_t^\varepsilon(S_{j-1}u\Delta_j v) - S_{j-1}u\tilde{P}_t^\varepsilon\Delta_j v\|_{L^\infty} \lesssim t^{\frac{\alpha+\beta-\delta}{2}}2^{j(\alpha+\beta-\delta)}2^{-j(\alpha+\beta)}\|u\|_\alpha\|v\|_\beta,$$

which yields (3.5) by [3], Lemma 2.69.

Moreover, we have

$$\begin{aligned}&(P_t^\varepsilon - P_t)\pi_<(u,v) - P_N\pi_<(u,\tilde{P}_t^\varepsilon v) + \pi_<(u,P_t v)\\&\quad = P_N[(\tilde{P}_t^\varepsilon - P_t)\pi_<(u,v) - \pi_<(u,(\tilde{P}_t^\varepsilon - P_t)v)]\\&\qquad - (I-P_N)(P_t\pi_<(u,v) - \pi_<(u,P_t v)).\end{aligned}$$

The estimate for the second term can be obtained by Lemmas 2.5 and 3.1. By a similar argument as the proof of Lemma 3.2, we obtain that for any multiindex $m$ satisfying $|m| \leq 5$ and every $\delta \geq 0, 0 < \eta < 1$; we have $|D_m(\phi^\varepsilon - \phi)(\xi)| \leq \frac{(\varepsilon|\xi|)^\eta}{|\xi|^{|m|+\delta}t^{\frac{\delta}{2}}}$. Thus (3.4) follows by a similar argument as in the proof of (3.5). Here, $\phi^\varepsilon$ and $\phi$ are introduced in (3.2). □

The continuity result for $P_t^\varepsilon$ takes as follows.

LEMMA 3.4. *Let $u \in \mathcal{C}^{\alpha+\delta}$ for some $\alpha \in \mathbb{R}, 0 < \delta < 1$. Then for every $\varepsilon \in (0,1), \kappa > 0, t > s > 0$.*

$$\|(P_t^\varepsilon - P_s^\varepsilon)u\|_{\alpha-\kappa} \lesssim (t-s)^{\frac{\delta}{2}}\|u\|_{\alpha+\delta}.$$

*Here, the constants are independent of $N$.*

PROOF. We have $(P_t^\varepsilon - P_s^\varepsilon)u = P_N(\tilde{P}_t^\varepsilon - \tilde{P}_s^\varepsilon)u$. By Lemma 3.1, it suffices to prove that

$$\|(\tilde{P}_t^\varepsilon - \tilde{P}_s^\varepsilon)u\|_\alpha \lesssim (t-s)^{\frac{\delta}{2}}\|u\|_{\alpha+\delta}.$$

Since $|1 - e^{-(t-s)f(\varepsilon\xi)|\xi|^2}| \leq (t-s)^{\frac{\delta}{2}}|\xi|^\delta$, we obtain that for any multiindex $m$ satisfying $|m| \leq 4$ and any $\delta \geq 0$, we have $|D_m(\phi_t^\varepsilon - \phi_s^\varepsilon)(\xi)| \lesssim \frac{(t-s)^{\frac{\delta}{2}}|\xi|^\delta}{|\xi|^{|m|}}$, where $\phi^\varepsilon$ is introduced in (3.2). Thus by a similar argument as in the proof of Lemma 3.2 the result follows. □

## 4. Paracontrolled analysis for the approximating equations.

Now for simplicity let $u^\varepsilon = \operatorname{Ext}\Phi^\varepsilon$. Then we have the following equation:

$$\begin{aligned}u^\varepsilon(t) &= P_t^\varepsilon\operatorname{Ext}\Phi_0^\varepsilon - \int_0^t P_{t-s}^\varepsilon Q_N[(u^\varepsilon)^3 - (3C_0^\varepsilon - 9C_1^\varepsilon)u^\varepsilon]\,ds\\&\quad + \int_0^t P_{t-s}^\varepsilon P_N\,dW.\end{aligned}\tag{4.1}$$

Therefore, it suffices to prove the convergence result for solutions to (4.1). In this section, we give a uniform estimate for solutions to (4.1) by using paracontrolled analysis.

In this section, we fix $\kappa, \gamma > 0$ satisfying

$$z - \frac{1}{2} > 2\kappa, \qquad 6\kappa < \gamma, \qquad 10\kappa + 3\gamma < 2 - 3z.$$

Here, we recall that $\Phi_0 \in \mathcal{C}^{-z}$ and $z \in (\frac{1}{2}, \frac{2}{3})$. Parameters $\kappa, \gamma$ satisfying the above conditions can always be found. Indeed, we first choose $\gamma < \frac{2-3z}{3}$ then the conditions are satisfied if we choose $\kappa > 0$ small enough satisfying $\kappa < \frac{\gamma}{6} \wedge \frac{2z-1}{4} \wedge \frac{2-3z-3\gamma}{10}$.

*Paracontrolled analysis of solutions to* (4.1). Now we split (4.1) into the following three equations. We also use the graph notation similar as in [18]: Here, the symbol $\cdot$ corresponds to the white noise and $\<1>$ corresponds to convolution with the kernel associated with $P_t^\varepsilon$. Moreover, $\<0>$ corresponds to the operator $\int_0^t P_{t-s}^\varepsilon Q_N \cdot ds$ and $\<0'>$ corresponds to convolution with the kernel associated with $\tilde{P}_t^\varepsilon$:

$$u_1^\varepsilon(t) = \int_{-\infty}^t P_{t-s}^\varepsilon P_N \, dW = \<1>,$$

$$u_2^\varepsilon(t) = -\int_0^t P_{t-s}^\varepsilon Q_N\big[(u_1^\varepsilon)^{\diamond,3}\big] \, ds = -\<30>,$$

and

$$\begin{aligned} u_3^\varepsilon(t) = {} & P_t^\varepsilon\big(\mathrm{Ext}\,\Phi_0^\varepsilon - u_1^\varepsilon(0)\big) \\ & - \int_0^t P_{t-s}^\varepsilon\big[Q_N\big[-6\<31> u_3^\varepsilon + 3\<1>(u_3^\varepsilon)^2 + 3\<1>(\<30>)^2 + (-\<30> + u_3^\varepsilon)^3\big] \\ & + P_N\big[3(-\<32> + \<2> \diamond u_3^\varepsilon) + 3(-\<32o> + \<2o> \diamond u_3^\varepsilon) - 9\varphi^\varepsilon u^\varepsilon\big]\big] \, ds. \end{aligned} \tag{4.2}$$

Here,

$$\<2> := \<1>^2 - C_0^\varepsilon,$$

$$\<3> := \<1>^3 - 3C_0^\varepsilon \<1>,$$

$$\<31> := \<1>\<30>,$$

$$\<32> := \<2>\<30> - 3(C_{11}^\varepsilon + \varphi_1^\varepsilon)\<1>,$$

$$\<2> \diamond u_3^\varepsilon := u_3^\varepsilon \<2> + 3(C_{11}^\varepsilon + \varphi_1^\varepsilon)(-\<30> + u_3^\varepsilon),$$

$$:= e_N^{i_1 i_2 i_3} ,$$

$$:= \quad - 3\big(C_{12}^{\varepsilon, i_1 i_2 i_3} + \varphi_2^{\varepsilon, i_1 i_2 i_3}\big) ,$$

$$\diamond u_3^\varepsilon := u_3^\varepsilon + 3\big(C_{12}^{\varepsilon, i_1 i_2 i_3} + \varphi_2^{\varepsilon, i_1 i_2 i_3}\big)\big(- \quad + u_3^\varepsilon\big),$$

$$\varphi^\varepsilon := \varphi_1^\varepsilon + \varphi_2^\varepsilon = \varphi_1^\varepsilon + \sum \varphi_2^{\varepsilon, i_1 i_2 i_3},$$

where $C_0^\varepsilon \in \mathbb{R}, C_{1i}^\varepsilon \in \mathbb{R}, \varphi_i^\varepsilon \in C((0,T];\mathbb{R})$ are defined as in Section 6 below and there exists $\varphi_1 \in C((0,T];\mathbb{R})$ such that for every $\rho > 0$ small enough $\sup_{t\in[0,T]} t^\rho |\varphi_1^\varepsilon - \varphi_1| \to 0$ and $\sup_{t\in[0,T]} t^\rho |\varphi_2^\varepsilon| \to 0$ as $\varepsilon \to 0$. In fact, , , and denote $(u_1^\varepsilon)^{\diamond,2}$, $(u_1^\varepsilon)^{\diamond,3}$, $-(u_1^\varepsilon)^{\diamond,2} \diamond u_2^\varepsilon$ and $-e_N^{i_1 i_2 i_3}(u_1^\varepsilon)^{\diamond,2} \diamond u_2^\varepsilon$, respectively. Furthermore,

$$\pi_{0,\diamond}( \ , \ ) := \pi_0( \ , \ ) - 3\big(C_{11}^\varepsilon + \varphi_1^\varepsilon\big) ,$$

$$\pi_{0,\diamond}\big(u_3^\varepsilon, \ \big) := \pi_0\big(u_3^\varepsilon, \ \big) + 3\big(C_{11}^\varepsilon + \varphi_1^\varepsilon\big)\big(- \quad + u_3^\varepsilon\big),$$

$$\pi_{0,\diamond}( \ , \ ) := \pi_0( \ , \ ) - 3\big(C_{12}^{\varepsilon, i_1 i_2 i_3} + \varphi_2^{\varepsilon, i_1 i_2 i_3}\big) ,$$

$$\pi_{0,\diamond}\big(u_3^\varepsilon, \ \big) := \pi_0\big(u_3^\varepsilon, \ \big) + 3\big(C_{12}^{\varepsilon, i_1 i_2 i_3} + \varphi_2^{\varepsilon, i_1 i_2 i_3}\big)\big(- \quad + u_3^\varepsilon\big).$$

In (4.2), the most difficult term to be handled is $u_3 \diamond$ , which requires us to use paracontrolled ansatz and the commutator estimates. For this, we introduce the following notation:

$$K^\varepsilon(t) := \int_0^t P_{t-s}^\varepsilon \ ds := \ , \qquad \tilde{K}^\varepsilon(t) := \int_0^t \tilde{P}_{t-s}^\varepsilon \ ds := \ ,$$

and

$$K_1^\varepsilon(t) := \int_0^t P_{t-s}^\varepsilon \ ds := \ , \qquad \tilde{K}_1^\varepsilon(t) := \int_0^t \tilde{P}_{t-s}^\varepsilon \ ds := \ .$$

Also define

$$\pi_{0,\diamond}( \ , \ ) := \pi_0( \ , \ ) - C_{11}^\varepsilon - \varphi_1^\varepsilon,$$

and

$$\pi_{0,\diamond}( \ , \ ) := \pi_0( \ , \ ) - C_{12}^{\varepsilon, i_1 i_2 i_3} - \varphi_2^{\varepsilon, i_1 i_2 i_3}.$$

Here, we introduce and since Lemma 3.3 about the commutator estimates only holds for $\tilde{P}^\varepsilon_t$, not for $P^\varepsilon_t$. Now we introduce the following notation: for $T > 0$,

$$C^\varepsilon_W(T) := \sup_{t\in[0,T]} [\| \|_{-\frac{1}{2}-2\kappa} + \| \|_{-1-2\kappa} + \| \|_{\frac{1}{2}-2\kappa} + \|\pi_0( , )\|_{-2\kappa} + \|\pi_{0,\diamond}( , )\|_{-\frac{1}{2}-2\kappa} + \|\pi_{0,\diamond}( , )\|_{-2\kappa}] + \| \|_{C_T^{\frac{1}{8}}\mathcal{C}^{\frac{1}{4}-2\kappa}},$$

and

$$E^\varepsilon_W(T) := \sup_{t\in[0,T]} [\| \|_{-1-2\kappa} + \|\pi_0( , e_N^{i_1 i_2 i_3} )\|_{-2\kappa} + \|\pi_{0,\diamond}( , )\|_{-\frac{1}{2}-2\kappa} + \|\pi_0( , )\|_{-2\kappa} + \|\pi_0( , )\|_{-2\kappa} + \|\pi_{0,\diamond}( , )\|_{-2\kappa}].$$

In the following, we write $C^\varepsilon_W$ and $E^\varepsilon_W$ for simplicity if there is no confusion. Here, $E^\varepsilon_W$ appears as an error term for the lattice approximations, which goes to 0 in probability (see Section 6.2). Lemma 3.2 and (3.1) imply that for $t \in [0, T]$

$$\| (t)\|_{1-3\kappa} + \| (t)\|_{1-3\kappa} \lesssim C^\varepsilon_W \tag{4.3}$$

and

$$\| (t)\|_{1-3\kappa} + \| (t)\|_{1-3\kappa} \lesssim E^\varepsilon_W. \tag{4.4}$$

Now we write the paracontrolled ansatz as follows:

$$u^\varepsilon_3 = -3P_N[\pi_<(- + u^\varepsilon_3, + )] + u^{\varepsilon,\sharp}$$

with $u^{\varepsilon,\sharp}(t) \in \mathcal{C}^{1+3\kappa}$ for $t > 0$. Then Lemma 2.2 yields that, for $t > 0$,

$$\|u^\varepsilon_3(t)\|_{1-3\kappa} \lesssim \|- (t) + u^\varepsilon_3(t)\|_\gamma (C^\varepsilon_W + E^\varepsilon_W) + \|u^{\varepsilon,\sharp}(t)\|_{1-3\kappa}. \tag{4.5}$$

Furthermore, $u_3^\varepsilon$ solves (4.2) if and only if $u^{\varepsilon,\sharp}$ solves the following equation:

$$\begin{aligned}
u^{\varepsilon,\sharp}(t) &= P_t^\varepsilon\big(\mathrm{Ext}\,\Phi_0^\varepsilon - u_1^\varepsilon(0)\big) \\
&\quad - \int_0^t P_{t-s}^\varepsilon\big[Q_N\big[-6\; u_3^\varepsilon + 3\,(u_3^\varepsilon)^2 + 3\,(\;)^2 + (-\; + u_3^\varepsilon)^3\big] \\
&\quad + 3P_N\big[(\pi_> + \pi_{0,\diamond})(-\; + u_3^\varepsilon, \; + \;)\big] - 9\varphi^\varepsilon u^\varepsilon\big]\,ds \\
&\quad - 3\int_0^t P_{t-s}^\varepsilon P_N\big[\pi_<(-\; + u_3^\varepsilon, \; + \;)\big]\,ds \qquad (4.6) \\
&\quad + 3P_N\big[\pi_<(-\; + u_3^\varepsilon, \; + \;)\big] \\
&:= P_t^\varepsilon\big(\mathrm{Ext}\,\Phi_0^\varepsilon - u_1^\varepsilon(0)\big) \\
&\quad + \int_0^t P_{t-s}^\varepsilon\big[Q_N\phi_1^{\varepsilon,\sharp} + P_N\phi_2^{\varepsilon,\sharp} + 9\varphi^\varepsilon u^\varepsilon\big]\,ds + F^\varepsilon,
\end{aligned}$$

where $F^\varepsilon$ represents the last two terms.

In the following, we give estimates of terms on the right-hand side of (4.6).

*Estimates of $\phi_i^{\varepsilon,\sharp}$.* First, we prove an estimate for $\phi_1^{\varepsilon,\sharp}$.

PROPOSITION 4.1. *For $\phi_1^{\varepsilon,\sharp}$ defined in (4.6), the following estimate holds:*

$$\|Q_N\phi_1^{\varepsilon,\sharp}\|_{-\frac{1}{2}-4\kappa} \lesssim C(C_W^\varepsilon, E_W^\varepsilon)\big(1 + \|u_3^\varepsilon\|_{\frac{1}{2}+4\kappa}(\|u_3^\varepsilon\|_\gamma + 1) + \|u_3^\varepsilon\|_\gamma^3\big).$$

*Here, the constant we omit is independent of $N$.*

PROOF. Since

$$\Pi_N[u_3^\varepsilon\;] = P_N[u_3^\varepsilon e_N^{i_1i_2i_3}\;],$$

we have

$$\begin{aligned}
&\|\Pi_N[u_3^\varepsilon\;]\|_{-\frac{1}{2}-4\kappa} \\
&\quad\lesssim \|u_3^\varepsilon\; e_N^{i_1i_2i_3}\|_{-\frac{1}{2}-3\kappa} \\
&\quad\lesssim \big(\|e_N^{i_1i_2i_3}\;\|_{-\frac{1}{2}-3\kappa}\|\;\|_{\frac{1}{2}-2\kappa} + \|\pi_0(\;, e_N^{i_1i_2i_3}\;)\|_{-2\kappa}\big)\|u_3^\varepsilon\|_{\frac{1}{2}+4\kappa} \\
&\quad\lesssim \big(N^{-\kappa/2}\|\;\|_{\frac{1}{2}-2\kappa}\|\;\|_{-\frac{1}{2}-2\kappa} + \|\pi_0(\;, e_N^{i_1i_2i_3}\;)\|_{-2\kappa}\big)\|u_3^\varepsilon\|_{\frac{1}{2}+4\kappa},
\end{aligned}$$

where we used Lemma 3.1 in the first and last inequalities as well as Lemma 2.2 in the second inequality.

Using the paraproduct, one has

$$\begin{aligned}\Pi_N[\vdots(u_3^\varepsilon)^2] &= P_N[\vdots e_N^{i_1i_2i_3}(u_3^\varepsilon)^2] \\ &= P_N[\pi_<((u_3^\varepsilon)^2, e_N^{i_1i_2i_3}\vdots) + \pi_0((u_3^\varepsilon)^2, e_N^{i_1i_2i_3}\vdots) + \pi_>((u_3^\varepsilon)^2, e_N^{i_1i_2i_3}\vdots)] \\ &= P_N[\pi_<((u_3^\varepsilon)^2, e_N^{i_1i_2i_3}\vdots) + \pi_0(\pi_0(u_3^\varepsilon, u_3^\varepsilon), e_N^{i_1i_2i_3}\vdots) \\ &\quad + \pi_>((u_3^\varepsilon)^2, e_N^{i_1i_2i_3}\vdots) + 2C(u_3^\varepsilon, u_3^\varepsilon, e_N^{i_1i_2i_3}\vdots) + 2u_3^\varepsilon\pi_0(u_3^\varepsilon, e_N^{i_1i_2i_3}\vdots)].\end{aligned}$$

Here, $C(u_3^\varepsilon, u_3^\varepsilon, e_N^{i_1i_2i_3}\vdots)$ is the trilinear operator as defined in Lemma 2.3. Then by using Lemmas 2.2, 2.3 and 3.1 we obtain

$$\|\Pi_N[(u_3^\varepsilon)^2\vdots]\|_{-\frac{1}{2}-4\kappa} \lesssim N^{-\frac{\kappa}{2}}\|u_3^\varepsilon\|_{\frac{1}{2}+4\kappa}\|u_3^\varepsilon\|_\gamma\|\vdots\|_{-\frac{1}{2}-2\kappa}. \tag{4.7}$$

Moreover, by a similar argument as for (4.7) we have

$$\begin{aligned}&\|\Pi_N[(\Psi)^2\vdots]\|_{-\frac{1}{2}-4\kappa} \\ &\quad \lesssim N^{-\frac{\kappa}{2}}\|\Psi\|^2_{\frac{1}{2}-2\kappa}\|\vdots\|_{-\frac{1}{2}-2\kappa} + \|\Psi\|_{\frac{1}{2}-2\kappa}\|\pi_0(\Psi, \vdots e_N^{i_1i_2i_3})\|_{-2\kappa}.\end{aligned}$$

Furthermore, Lemma 3.1 implies that

$$\|Q_N[(-\Psi + u_3^\varepsilon)^3]\|_{\gamma-\kappa} \lesssim \|-\Psi + u_3^\varepsilon\|^3_\gamma.$$

Estimates for the terms containing $P_N$ can be obtained similarly. Hence the result follows from the above estimates. $\square$

Now we consider $\phi_2^{\varepsilon,\sharp}$. To prove an estimate for $\pi_{0,\diamond}(u_3^\varepsilon, \vee + \mathring{\vee})$, we have to use the paracontrolled ansatz. However, the Fourier cutoff operator $P_N$ does not commute with the paraproduct. Here, we follow the random operator technique from [13], Lemma 8.16, and prove the following result.

LEMMA 4.2. *Let $\alpha + \beta + \gamma > 0$, $\beta + \gamma < 0$, assume that $\alpha \in (0, 1)$ and let $f \in \mathcal{C}^\alpha$, $g \in \mathcal{C}^\beta$, $h \in \mathcal{C}^\gamma$. Define the operators*

$$A_N^1(g, h)(f) := -\pi_0((I - P_N)\pi_<(f, P_N g), h)$$

*and*

$$A_N^2(g, h)(f) := \pi_0(P_N\pi_<(f, (P_{3N} - P_N)g), h).$$

*Then for all $\eta < 0$*

$$\begin{aligned}&\|\pi_0(P_N\pi_<(f, P_{3N}g), h) - f\pi_0(P_N g, h)\|_\eta \\ &\quad \lesssim \|f\|_\alpha\|P_N g\|_\beta\|h\|_\gamma + \|A_N^1(g, h) + A_N^2(g, h)\|_{L(\mathcal{C}^\alpha, \mathcal{C}^\eta)}\|f\|_\alpha.\end{aligned}$$

*Here, the constant we omit is independent of $N$ and $L(\mathcal{C}^\alpha, \mathcal{C}^\eta)$ denotes the space of bounded operators between $\mathcal{C}^\alpha$ and $\mathcal{C}^\eta$, equipped with the operator norm.*

PROOF. We have that

$$\pi_0\big(P_N\pi_<(f, P_{3N}g), h\big) = A_N^2(g,h)(f) + \pi_0\big(\pi_<(f, P_N g), h\big) + A_N^1(g,h)(f).$$

Thus the result follows from Lemma 2.3. $\square$

By using Lemma 4.2, we have the following estimate for $\phi_2^{\varepsilon,\sharp}$.

PROPOSITION 4.3. *For $\phi_2^{\varepsilon,\sharp}$ defined in (4.6), the following estimate holds*:

$$\|P_N\phi_2^{\varepsilon,\sharp}\|_{-\frac{1}{2}-6\kappa} \lesssim C(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N)(1 + \|u_3^\varepsilon\|_{\frac{1}{2}+4\kappa} + \|u^{\varepsilon,\sharp}\|_{1+3\kappa})$$

*with*

$$A_N(T) := \|(A_N^1 + A_N^2)( \quad + \quad , \quad + \quad )\|_{C_T L(\mathcal{C}^{1-3\kappa}, \mathcal{C}^{-\frac{1}{2}-5\kappa})}$$

*and*

$$D_N(T) := \sup_{t\in[0,T]} \big(\|\pi_0\big((I-P_N)\pi_<( \quad , \quad + \quad ), \quad + \quad \big)$$

$$- \pi_0\big(P_N\pi_<( \quad , (P_{3N}-P_N)( \quad + \quad )), \quad + \quad \big)\|_{-\kappa}\big).$$

PROOF. First, we consider $\pi_0(u_3^\varepsilon, \quad + \quad )$. By the paracontrolled ansatz, we obtain

$$\begin{aligned}
&\pi_0(u_3^\varepsilon, \quad + \quad )\\
&\quad = -3\pi_0\big(P_N[\pi_<(- \quad + u_3^\varepsilon, P_{3N}( \quad + \quad ))], \quad + \quad \big)\\
&\qquad + \pi_0(u^{\varepsilon,\sharp}, \quad + \quad ).
\end{aligned}$$

Here, in the equality we used that $P_{3N}( \quad + \quad ) = \quad + \quad$. Then by using Lemma 4.2 and that $P_N( \quad + \quad ) = \quad + \quad$, we obtain that

$$\begin{aligned}
&\|\pi_{0,\diamond}(u_3^\varepsilon, \quad + \quad )\|_{-\frac{1}{2}-5\kappa}\\
&\quad \lesssim (\| \quad \|_{\frac{1}{2}-2\kappa} + \|u_3^\varepsilon\|_{\frac{1}{2}+4\kappa})[\| \quad + \quad \|_{1-3\kappa}\| \quad + \quad \|_{-1-2\kappa}\\
&\qquad + \|\pi_{0,\diamond}( \quad + \quad , \quad + \quad )\|_{-2\kappa}]\\
&\qquad + A_N\|u_3^\varepsilon\|_{1-3\kappa} + D_N + \|u^{\varepsilon,\sharp}\|_{1+3\kappa}\| \quad + \quad \|_{-1-2\kappa}.
\end{aligned}$$

The estimate for $\pi_>(- \quad + u_3^\varepsilon, \quad + \quad )$ can be obtained by Lemma 2.2. Thus the result follows from (4.3), (4.4) and (4.5). $\square$

REMARK 4.4. (i) In this paper, we split equation (4.2) by using $Q_N(uv) = P_N(uv + uve_N^{i_1,i_2,i_3})$ and introduce a new random operator which is different from [13]. In fact, by using $Q_N((u_2^\varepsilon + u_3^\varepsilon)(u_1^\varepsilon)^{\diamond,2}) = Q_N((u_2^\varepsilon + u_3^\varepsilon)Q_N[(u_1^\varepsilon)^{\diamond,2}])$, we can use the paracontrolled ansatz and use a random operator similar as in [13] to deduce the result. We would like to thank the referee for pointing out this to us. However, the idea of these two operators is the same and the calculations for these two operators are essentially similar.

(ii) By the calculations in Section 6.3, we know that in order to get $\|A_N^1 + A_N^2\|_{L(\mathcal{C}^\alpha,\mathcal{C}^\eta)} \to 0$ we need $\alpha > \eta + 3/2$. Also the regularity of $u^{\varepsilon,\sharp}$ requires that $\eta > -1 + 3\kappa$, which implies that $\alpha > 1/2 + 3\kappa$. However, the best regularity we can obtain for $u_2^\varepsilon$ is $\mathcal{C}^{1/2-}$. Thus, for the error terms including $u_2^\varepsilon$ we have to bound them directly by stochastic calculations, which corresponds to $D_N$ (see Section 6.4).

*Estimates of $F^\varepsilon$.* We now turn to $F^\varepsilon$: We divide $F^\varepsilon$ into two parts:

$$
\begin{aligned}
&\|F^\varepsilon(t)\|_{1+3\kappa} \\
&\quad\lesssim \Big\| \int_0^t P^\varepsilon_{t-s}\pi_<\big(-\Psi(s) + u_3^\varepsilon(s) - (-\Psi(t) + u_3^\varepsilon(t)), \\
&\qquad \vee(s) + \vee_\circ(s)\big)\,ds \Big\|_{1+3\kappa} \\
&\qquad + \Big\| \int_0^t P^\varepsilon_{t-s}\pi_<\big(-\Psi(t) + u_3^\varepsilon(t), \vee(s) + \vee_\circ(s)\big)\,ds \\
&\qquad - P_N\pi_<\big(-\Psi(t) + u_3^\varepsilon(t), Y(t) + Y_\circ(t)\big) \Big\|_{1+3\kappa} \\
&\quad := I_1 + I_2.
\end{aligned}
$$

Estimate of $I_2$ can be obtained by Lemma 3.3:

$$
I_2 \lesssim t^{\frac{\gamma-6\kappa}{2}} \|-\Psi(t) + u_3^\varepsilon(t)\|_\gamma (C_W^\varepsilon + E_W^\varepsilon), \tag{4.8}
$$

where by the condition on $\gamma$ we have $\gamma > 6\kappa$.

For $I_1$, we will use the regularity of $u_2^\varepsilon + u_3^\varepsilon$ with respect to time to control it. Lemmas 2.2 and 3.2 yield that

$$
\begin{aligned}
I_1 &\lesssim \int_0^t (t-s)^{-1-3\kappa} \|\vee(s) + \vee_\circ(s)\|_{-1-2\kappa} \\
&\quad \times \|-\Psi(t) + u_3^\varepsilon(t) + \Psi(s) - u_3^\varepsilon(s)\|_{\frac{\kappa}{2}}\,ds \\
&\lesssim (C_W^\varepsilon + E_W^\varepsilon)\Big(C_W^\varepsilon + \int_0^t (t-s)^{-1-3\kappa} \|u_3^\varepsilon(t) - u_3^\varepsilon(s)\|_{\frac{\kappa}{2}}\,ds\Big),
\end{aligned}
$$

and we note that by (4.2), Lemmas 3.2 and 3.4 that, for $t > s > 0$,

$$
\begin{aligned}
&\|u_3^\varepsilon(t) - u_3^\varepsilon(s)\|_{\frac{\kappa}{2}} \\
&\quad\lesssim \|(P^\varepsilon_{\frac{t}{2}} - P^\varepsilon_{\frac{s}{2}})(P^\varepsilon_{\frac{t}{2}} + P^\varepsilon_{\frac{s}{2}})(\operatorname{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0))\|_{\frac{\kappa}{2}} \\
&\qquad + \left\|\int_0^s (P^\varepsilon_{t-r} - P^\varepsilon_{s-r})G^\varepsilon(r)\,dr\right\|_{\frac{\kappa}{2}} + \left\|\int_s^t P^\varepsilon_{t-r}G^\varepsilon(r)\,dr\right\|_{\frac{\kappa}{2}} \\
&\quad\lesssim (t-s)^{b_0} s^{-\frac{z+2\kappa+2b_0}{2}} \|\operatorname{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} \\
&\qquad + (t-s)^b \int_0^s (s-r)^{-\frac{1+4\kappa+2b}{2}} \|G^\varepsilon(r)\|_{-1-3\kappa}\,dr \\
&\qquad + (t-s)^{b_1}\left(\int_s^t (t-r)^{-\frac{1+4\kappa}{2(1-b_1)}} \|G^\varepsilon(r)\|_{-1-3\kappa}^{\frac{1}{1-b_1}}\,dr\right)^{1-b_1},
\end{aligned}
$$

where in the last inequality for the third term we used Hölder's inequality. Here, $6\kappa < 2b_0 < 2 - z - 2\kappa$, $6\kappa < 2b < 1 - 4\kappa$, $3\kappa < b_1 < 1 - \frac{3(\gamma+z+\kappa)}{2} < \frac{1}{2}(1-4\kappa)$ and

$$
\begin{aligned}
G^\varepsilon = {}& Q_N[3(\ )^2 - 6\ u_3^\varepsilon + 3(u_3^\varepsilon)^2 + (-\ + u_3^\varepsilon)^3] \\
&+ P_N[3(-\ + \ \diamond u_3^\varepsilon) + 3(-\ + \ \diamond u_3^\varepsilon)] - 9\varphi^\varepsilon u^\varepsilon.
\end{aligned}
$$

Moreover, by Propositions 4.1 and 4.3 and Lemma 2.2 one has the following estimate:

$$
\|G^\varepsilon(t)\|_{-1-3\kappa} \lesssim C(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N)U_0^\varepsilon(t) + t^{-\rho}(C_W^\varepsilon + \|u_3^\varepsilon(t)\|_\gamma). \tag{4.9}
$$

Here and in the following

$$
U_0^\varepsilon(t) = 1 + \|u_3^\varepsilon(t)\|_{\frac{1}{2}+4\kappa}(\|u_3^\varepsilon(t)\|_\gamma + 1) + \|u_3^\varepsilon(t)\|_\gamma^3 + \|u^{\varepsilon,\sharp}(t)\|_{1+3\kappa}.
$$

Thus we obtain that

$$
\begin{aligned}
I_1 \lesssim {}& (C_W^\varepsilon + E_W^\varepsilon)\Bigg(C_W^\varepsilon + t^{-\frac{z}{2}-4\kappa}\|\operatorname{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} \\
&+ \int_0^t \int_r^t (t-s)^{-1-3\kappa+b}(s-r)^{-\frac{1+4\kappa+2b}{2}}\,ds\,\|G^\varepsilon(r)\|_{-1-3\kappa}\,dr \\
&+ \left(\int_0^t (t-s)^{-1-3\kappa+b_1}\,ds\right)^{b_1}\Bigg(\int_0^t \int_0^r (t-s)^{-1-3\kappa+b_1}(t-r)^{-\frac{1+4\kappa}{2(1-b_1)}} \\
&\times \|G^\varepsilon(r)\|_{-1-3\kappa}^{\frac{1}{1-b_1}}\,ds\,dr\Bigg)^{1-b_1}\Bigg),
\end{aligned}
$$

where for the last term we used Hölder's inequality. Then by changing variable $s = r + (t-r)\sigma$ for the third term and using (4.9) we have

$$
\begin{aligned}
I_1 \lesssim\ & (C^\varepsilon_W + E^\varepsilon_W) t^{-\frac{z}{2}-4\kappa} \|\mathrm{Ext}\,\Phi^\varepsilon_0 - u^\varepsilon_1(0)\|_{-z} + C(C^\varepsilon_W, E^\varepsilon_W, A_N, D_N) \\
& + C(C^\varepsilon_W, E^\varepsilon_W, A_N, D_N) \int_0^t (t-r)^{-\frac{1}{2}-5\kappa} \big(U^\varepsilon_0(r) + r^{-\rho} \|u^\varepsilon_3\|_\gamma\big)\, dr \\
& + C(C^\varepsilon_W, E^\varepsilon_W, A_N, D_N) \\
& \times \bigg[ \int_0^t (t-r)^{-\frac{1+4\kappa}{2(1-b_1)}} \big(U^\varepsilon_0(r) + r^{-\rho}\|u^\varepsilon_3\|_\gamma\big)^{\frac{1}{1-b_1}}\, dr \bigg]^{1-b_1}.
\end{aligned}
\tag{4.10}
$$

Combining (4.8) and (4.10), we could control $\|F^\varepsilon\|_{1+3\kappa}$ by the right-hand side of (4.8) and (4.10).

In the following, we will bound $\|u^\varepsilon_3\|_{\frac{1}{2}+4\kappa}$ and $\|u^\varepsilon_3\|_\gamma$. Estimates for these two terms are much easier. We do not need to use Lemma 3.3 and can obtain the following estimates for $\int_0^t P^\varepsilon_{t-s} P_N[\pi_<(- \; + u^\varepsilon_3, \; + \;)]\, ds$ by Lemmas 2.2 and 3.2 directly:

$$
\begin{aligned}
& \bigg\| \int_0^t P^\varepsilon_{t-s} P_N[\pi_<(- \; + u^\varepsilon_3, \; + \;)]\, ds \bigg\|_{\frac{1}{2}+4\kappa} \\
& \quad \lesssim (C^\varepsilon_W + E^\varepsilon_W) \int_0^t (t-s)^{-\frac{3}{4}-\frac{7\kappa}{2}} \|u^\varepsilon_3\|_\gamma\, ds + C(C^\varepsilon_W, E^\varepsilon_W)
\end{aligned}
\tag{4.11}
$$

and

$$
\begin{aligned}
& \bigg\| \int_0^t P^\varepsilon_{t-s} P_N[\pi_<(- \; + u^\varepsilon_3, \; + \;)]\, ds \bigg\|_{\gamma} \\
& \quad \lesssim (C^\varepsilon_W + E^\varepsilon_W) \int_0^t (t-r)^{-\frac{1+3\kappa+\gamma}{2}} \|u^\varepsilon_3\|_\gamma\, dr + C(C^\varepsilon_W, E^\varepsilon_W).
\end{aligned}
\tag{4.12}
$$

*Uniform estimates of the solutions to* (4.2). Now we introduce the following random times: Define for any $L \geq 1$

$$
\begin{aligned}
\tau^\varepsilon_L &:= \inf\{t \geq 0 : \|u^\varepsilon(t)\|_{-z} \geq L\} \wedge L, \\
\rho^\varepsilon_L &:= \inf\{t \geq 0 : C^\varepsilon_W(t) + E^\varepsilon_W(t) + A_N(t) + D_N(t) \geq L\}.
\end{aligned}
$$

PROPOSITION 4.5. *For any $L, L_1 \geq 1$, we have*

$$
\begin{aligned}
\sup_{t \in [0, \tau^\varepsilon_L \wedge \rho^\varepsilon_{L_1}]} & \big( t^{\frac{3(\gamma+z+\kappa)}{2}} \|u^{\varepsilon,\sharp}(t)\|_{1+3\kappa} + t^{\frac{\frac{1}{2}+z+5\kappa}{2}} \|u^\varepsilon_3(t)\|_{\frac{1}{2}+4\kappa} + t^{\frac{\gamma+z+\kappa}{2}} \|u^\varepsilon_3(t)\|_\gamma \big) \\
& \lesssim C(L, L_1).
\end{aligned}
$$

*Moreover, before $\tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon$ one has that $u_3^\varepsilon(t)$ depends in a Lipschitz continuous way on the data* $\operatorname{Ext}\Phi_0^\varepsilon$ *and terms in $(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N)$. Here, we consider $u_3^\varepsilon(t)$ with respect to $\|\cdot\|_{-z}$ norm and the Lipschitz constant can be chosen uniformly over $t \in [0, \tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon]$.*

PROOF. It follows from Propositions 4.1, 4.3 and (4.8), (4.10) that for $\frac{3(\gamma+z+\kappa)}{2} < 1$ and $t \in [0, \tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon]$,

$$
\begin{aligned}
&t^{\frac{3(\gamma+z+\kappa)}{2}}\|u^{\varepsilon,\sharp}(t)\|_{1+3\kappa} \\
&\quad\lesssim C\|\operatorname{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} \\
&\qquad + t^{\frac{3(\gamma+z+\kappa)}{2}} C\int_0^t (t-r)^{-\frac{3}{4}-5\kappa}\big(r^{-\frac{3(\gamma+z+\kappa)}{2}}U^\varepsilon(r) + r^{-\rho}\|u_3^\varepsilon(r)\|_\gamma\big)\,dr + C \\
&\qquad + Ct^{\frac{3(\gamma+z+\kappa)}{2}}\int_0^t (t-r)^{-\frac{1}{2}-5\kappa}\big(r^{-\frac{3(\gamma+z+\kappa)}{2}}U^\varepsilon(r) + r^{-\rho}\|u_3^\varepsilon(r)\|_\gamma\big)\,dr \\
&\qquad + t^{\frac{3(\gamma+z+\kappa)}{2(1-b_1)}}\int_0^t (t-r)^{-\frac{1+4\kappa}{2(1-b_1)}}\big(r^{-\frac{3(\gamma+z+\kappa)}{2}}U^\varepsilon(r) + r^{-\rho}\|u_3^\varepsilon(r)\|_\gamma\big)^{\frac{1}{1-b_1}}\,dr \\
&\qquad + t^{\frac{\gamma+z+\kappa}{2}}\|u_3^\varepsilon(t)\|_\gamma.
\end{aligned}
\tag{4.13}
$$

Here and in the following, $C = C(L_1)$ and $U^\varepsilon(r) = r^{\frac{3(\gamma+z+\kappa)}{2}}U_0^\varepsilon(r)$. A similar argument is that for (4.13) and using (4.11), (4.12) one also has that, for $t \in [0, \tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon]$ and $0 < 9\kappa < \frac{3}{2} - 2z - 3\gamma$,

$$
\begin{aligned}
&t^{\frac{\frac{1}{2}+z+5\kappa}{2}}\|u_3^\varepsilon(t)\|_{\frac{1}{2}+4\kappa} \\
&\quad\lesssim \|\operatorname{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} \\
&\qquad + t^{\frac{\frac{1}{2}+z+5\kappa}{2}} C\int_0^t (t-r)^{-\frac{1+11\kappa}{2}}\big(r^{-\frac{3(\gamma+z+\kappa)}{2}}U^\varepsilon(r) + r^{-\rho}\|u_3^\varepsilon(r)\|_\gamma\big)\,dr \\
&\qquad + C + Ct^{\frac{\frac{1}{2}+z+5\kappa}{2}}\int_0^t (t-r)^{-\frac{3}{4}-\frac{7\kappa}{2}} r^{-\frac{\gamma+z+\kappa}{2}} r^{\frac{\gamma+z+\kappa}{2}}\|u_3^\varepsilon(r)\|_\gamma\,dr
\end{aligned}
\tag{4.14}
$$

and

$$
\begin{aligned}
&t^{\frac{\gamma+z+\kappa}{2}}\|u_3^\varepsilon(t)\|_\gamma \\
&\quad\lesssim \|\operatorname{Ext}\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} \\
&\qquad + t^{\frac{\gamma+z+\kappa}{2}} C\int_0^t (t-r)^{-\frac{1}{4}-\frac{\gamma+7\kappa}{2}}\big(r^{-\frac{3(\gamma+z+\kappa)}{2}}U^\varepsilon(r) + r^{-\rho}\|u_3^\varepsilon(r)\|_\gamma\big)\,dr \\
&\qquad + C + Ct^{\frac{\gamma+z+\kappa}{2}}\int_0^t (t-r)^{-\frac{1+\gamma+3\kappa}{2}} r^{-\frac{(\gamma+z+\kappa)}{2}} r^{\frac{\gamma+z+\kappa}{2}}\|u_3^\varepsilon(r)\|_\gamma\,dr.
\end{aligned}
\tag{4.15}
$$

Since $\frac{\frac{1}{2}+5\kappa+z}{2} \leq \gamma+z+\kappa$, combining with (4.13)–(4.15), we get that by Hölder's inequality and Bihari's inequality there exists some $T_0$ (depending on $L_1$) such that

$$\sup_{t\in[0,T_0]} \big(t^{\frac{3(\gamma+z+\kappa)}{2}} \|u^{\varepsilon,\sharp}(t)\|_{1+3\kappa} + t^{\frac{\frac{1}{2}+z+5\kappa}{2}} \|u_3^\varepsilon(t)\|_{\frac{1}{2}+4\kappa} + t^{\frac{\gamma+z+\kappa}{2}} \|u_3^\varepsilon(t)\|_\gamma\big) \\ \lesssim C(L, L_1),$$

which combined with Propositions 4.1 and 4.3 implies that

$$\sup_{t\in[0,T_0]} t^{\frac{3(\gamma+z+\kappa)}{2}} \|Q_N\phi_1^{\varepsilon,\sharp} + P_N\phi_2^{\varepsilon,\sharp}\|_{-\frac{1}{2}-6\kappa} \lesssim C(L, L_1). \tag{4.16}$$

Moreover, by (4.2) and Lemma 2.2 we obtain that for $t \in [0, T_0]$ and $10\kappa + 3\gamma < \frac{3}{2} - 2z$

$$\begin{aligned}\|u_3^\varepsilon(t)\|_{-z} \lesssim{}& C + \|\mathrm{Ext}\,\Phi_0^\varepsilon - u_1^\varepsilon(0)\|_{-z} + \int_0^t s^{-\rho}\|u_3^\varepsilon(s)\|_\gamma\,ds \\ &+ \int_0^t \big[(t-s)^{-\frac{\frac{1}{2}+7\kappa-z}{2}} \vee 1\big] s^{-\frac{3(\gamma+z+\kappa)}{2}} s^{\frac{3(\gamma+z+\kappa)}{2}} \\ &\times \|Q_N\phi_1^{\varepsilon,\sharp} + P_N\phi_2^{\varepsilon,\sharp}\|_{-\frac{1}{2}-6\kappa}\,ds \\ &+ \int_0^t (t-s)^{-\frac{1+3\kappa-z}{2}} s^{-\frac{\gamma+\kappa+z}{2}}\,ds \sup_{s\in[0,t]} s^{\frac{\gamma+\kappa+z}{2}} \|u_2^\varepsilon + u_3^\varepsilon\|_\gamma \\ \lesssim{}& C(L, L_1).\end{aligned}$$

Here in the last inequality we used (4.16). Moreover, similar arguments as above imply that $u_3^\varepsilon(t)$ before $T_0$ depends in a Lipschitz continuous way on the data $\mathrm{Ext}\,\Phi_0^\varepsilon$ and terms in $(C_W^\varepsilon, E_W^\varepsilon, A_N, D_N)$. The Lipschitz constant can be chosen uniformly for $t \in [0, T_0]$. Furthermore, we can extend the time from $T_0$ to $\tau_L^\varepsilon \wedge \rho_{L_1}^\varepsilon$ as we did in [37]. □

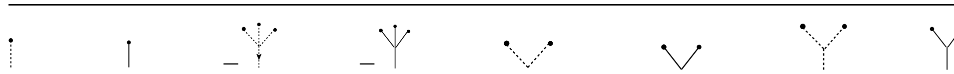
**5. Proof of main result.** In [7], it is proved that the solution to (1.1) can be obtained as a limit of solutions $\bar{\Phi}^\varepsilon$ to the following equation:

$$\begin{aligned} d\bar{\Phi}^\varepsilon &= \Delta\bar{\Phi}^\varepsilon\,dt + P_N\,dW - (\bar{\Phi}^\varepsilon)^3\,dt + (3\bar{C}_0^\varepsilon - 9\bar{C}_1^\varepsilon)\bar{\Phi}^\varepsilon\,dt, \\ \bar{\Phi}^\varepsilon(0) &= \Phi_0.\end{aligned}$$

Here, $\bar{C}_0^\varepsilon$ and $\bar{C}_1^\varepsilon$ are defined in Section 6.1 below. For this equation, we can also divide it into three equations and define $\bar{u}_1^\varepsilon, \bar{u}_2^\varepsilon, \bar{u}_3^\varepsilon, \bar{K}^\varepsilon$ and other terms similarly as $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, K^\varepsilon$ and the associated terms, respectively. For $L \geq 0$, define $\tau_L := \inf\{t \geq 0 : \|\Phi(t)\|_{-z} \geq L\} \wedge L$. Then $\tau_L$ increases to the explosion

TABLE 1

| $u_1^\varepsilon$ | $\bar{u}_1^\varepsilon$ | $u_2^\varepsilon$ | $\bar{u}_2^\varepsilon$ | $(u_1^\varepsilon)^{\diamond,2}$ | $(\bar{u}_1^\varepsilon)^{\diamond,2}$ | $K^\varepsilon$ | $\bar{K}^\varepsilon$ |
|---|---|---|---|---|---|---|---|

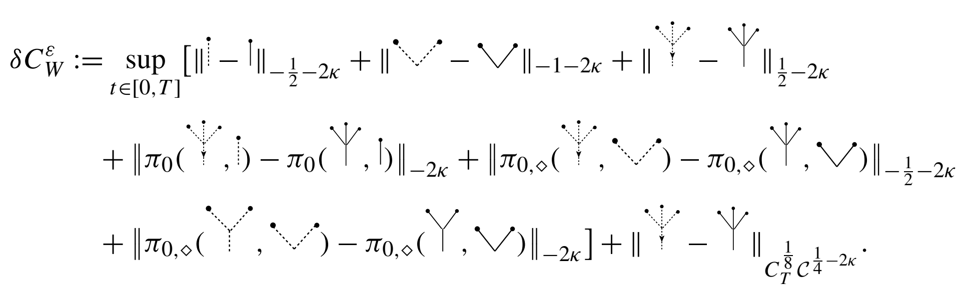


time $\tau$ as $L \to \infty$. Moreover, define $\bar{\tau}_L^\varepsilon := \inf\{t \geq 0 : \|\bar{\Phi}^\varepsilon(t)\|_{-z} \geq L\} \wedge L$ and $\bar{\rho}_L^\varepsilon := \inf\{t \geq 0 : \bar{C}_W^\varepsilon(t) \geq L\}$. Here, $\bar{C}_W^\varepsilon$ defined similarly as $C_W^\varepsilon$ with $u_i^\varepsilon$ replaced by corresponding $\bar{u}_i^\varepsilon$. A similar argument as in the proof in [7] implies that for $L, L_3, L_4 \geq 1$

$$\sup_{t \in [0, \tau_L \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|\bar{\Phi}^\varepsilon(t) - \Phi(t)\|_{-z} \to^P 0 \qquad \text{as } \varepsilon \to 0. \tag{5.1}$$

Here, $\Phi$ is the solution to (1.2). To make our paper more readable, we also introduce the graph notation similarly as in [18] for $\bar{u}_i^\varepsilon$ and we also recall the graph for $u_i^\varepsilon$ in Table 1.

Define



In Section 6, we will prove that $\delta C_W^\varepsilon \to^P 0$, $E_W^\varepsilon \to^P 0$, $A_N \to^P 0$ and $D_N \to^P 0$ as $\varepsilon \to 0$. Then using the estimates for $P_t^\varepsilon - P_t$ obtained in Section 3 and by similar arguments as in Section 4 we have that for $L, L_i \geq 1$ with $i = 1, 2, 3, 4$ that

$$\sup_{t \in [0, \tau_L \wedge \tau_{L_1}^\varepsilon \wedge \rho_{L_2}^\varepsilon \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|u^\varepsilon(t) - \bar{\Phi}^\varepsilon(t)\|_{-z} \to^P 0, \qquad \varepsilon \to 0. \tag{5.2}$$

Here, $E_W^\varepsilon, A_N, D_N$ appear as error terms for the lattice approximations. Then (5.1) and (5.2) imply that

$$\sup_{t \in [0, \tau_L \wedge \tau_{L_1}^\varepsilon \wedge \rho_{L_2}^\varepsilon \wedge \bar{\rho}_{L_3}^\varepsilon \wedge \bar{\tau}_{L_4}^\varepsilon]} \|u^\varepsilon(t) - \Phi(t)\|_{-z} \to^P 0, \qquad \varepsilon \to 0. \tag{5.3}$$

Moreover, we have the following estimate: for each $\epsilon > 0$,

$$
\begin{aligned}
&P\Big(\sup_{t\in[0,\tau_L]} \|u^\varepsilon - \Phi\|_{-z} > \epsilon\Big)\\
&\quad \le P\Big(\sup_{t\in[0,\tau_L\wedge\tau^\varepsilon_{L_1}\wedge\rho^\varepsilon_{L_2}\wedge\bar\rho^\varepsilon_{L_3}\wedge\bar\tau^\varepsilon_{L_4}]} \|u^\varepsilon - \Phi\|_{-z} > \epsilon\Big)\\
&\qquad + P\big(\tau_L\wedge\rho^\varepsilon_{L_2}\wedge\bar\rho^\varepsilon_{L_3}\wedge\bar\tau^\varepsilon_{L_4} > \tau^\varepsilon_{L_1}\big)\\
&\qquad + P\big(\tau_L\wedge\bar\rho^\varepsilon_{L_3} > \bar\tau^\varepsilon_{L_4}\big) + P\big(\tau_L > \rho^\varepsilon_{L_2}\big) + P\big(\tau_L > \bar\rho^\varepsilon_{L_3}\big).
\end{aligned}
\tag{5.4}
$$

The first term goes to zero as $\varepsilon \to 0$ by (5.3). Also for $L_1 > L + \epsilon$,

$$
P\big(\tau_L\wedge\rho^\varepsilon_{L_2}\wedge\bar\rho^\varepsilon_{L_3}\wedge\bar\tau^\varepsilon_{L_4} > \tau^\varepsilon_{L_1}\big) \le P\Big(\sup_{t\in[0,\tau_L\wedge\tau^\varepsilon_{L_1}\wedge\rho^\varepsilon_{L_2}\wedge\bar\rho^\varepsilon_{L_3}\wedge\bar\tau^\varepsilon_{L_4}]} \|u^\varepsilon - \Phi\|_{-z} > \epsilon\Big),
$$

which goes to zero as $\varepsilon \to 0$ by (5.3). Furthermore, for $L_4 > L + \epsilon$ we have

$$
P\big(\tau_L\wedge\bar\rho^\varepsilon_{L_3} > \bar\tau^\varepsilon_{L_4}\big) \le P\Big(\sup_{t\in[0,\tau_L\wedge\bar\rho^\varepsilon_{L_3}\wedge\bar\tau^\varepsilon_{L_4}]} \|\bar\Phi^\varepsilon - \Phi\|_{-z} > \epsilon\Big)
$$

which goes to zero by (5.1) as $\varepsilon \to 0$. The last two terms on the right-hand side of (5.4) go to zero uniformly over $\varepsilon \in (0,1)$ as $L_2, L_3$ go to $\infty$. Thus the result follows.

**6. Stochastic convergence.** In this section, we will prove that $\delta C^\varepsilon_W \to 0$, $E^\varepsilon_W \to 0$, $A_N \to 0$, $D_N \to 0$ in probability as $\varepsilon \to 0$.

To simplify the arguments below, we assume that $\mathcal{F}W(0) = 0$ and restrict ourselves to the flow of $\int_{\mathbb{T}^3} u(x)\,dx = 0$. We follow the notation from [13], Section 9. We represent the white noise in terms of its spatial Fourier transform. More precisely, let $E = \mathbb{Z}^3 \setminus \{0\}$ and let $W(s,k) = \langle W(s), e_k\rangle$ for $e_k(x) = 2^{-\frac{3}{2}} e^{\iota\pi x\cdot k}$, $x \in \mathbb{T}^3$, and we view $W(s,k)$ as a Gaussian process on $\mathbb{R}\times E$ with covariance given by

$$
E\Big[\int_{\mathbb{R}\times E} f(\eta) W(d\eta) \int_{\mathbb{R}\times E} g(\eta') W(d\eta')\Big] = \int_{\mathbb{R}\times E} g(\eta_1) f(\eta_{-1})\,d\eta_1,
$$

where $\eta_a = (s_a, k_a)$, $s_{-a} = s_a$, $k_{-a} = -k_a$ and the measure $d\eta_a = ds_a\,dk_a$ is the product of the Lebesgue measure $ds_a$ on $\mathbb{R}$ and of the counting measure $dk_a$ on $E$. Then

$$
u^\varepsilon_1(t,x) = \int_{\mathbb{R}\times E} e_k(x) P^\varepsilon_{t-s}(k) W(d\eta), \qquad \bar u^\varepsilon_1(t,x) = \int_{\mathbb{R}\times E} e_k(x) \bar P^\varepsilon_{t-s}(k) W(d\eta),
$$

where $p^\varepsilon_t(k) = e^{-|k|^2 f(\varepsilon k) t} 1_{\{t\ge 0\}}$, $P^\varepsilon_t(k) = p^\varepsilon_t(k) 1_{\{|k|_\infty \le N\}}$, $p_t(k) = e^{-|k|^2\pi^2 t} 1_{\{t\ge 0\}}$, and $\bar P^\varepsilon_t(k) = p_t(k) 1_{\{|k|_\infty\le N\}}$. Moreover,

$$
\int P^\varepsilon_{t-s}(k) P^\varepsilon_{\sigma-s}(k)\,ds = \frac{e^{-|k|^2 f(\varepsilon k)|t-\sigma|} 1_{\{|k|_\infty \le N\}}}{2|k|^2 f(\varepsilon k)} := V^\varepsilon_{t-\sigma}(k) \tag{6.1}
$$

and

$$\int \bar{P}^{\varepsilon}_{t-s}(k)\bar{P}^{\varepsilon}_{\sigma-s}(k)\,ds = \frac{e^{-|k|^2\pi^2|t-\sigma|}1_{\{|k|_\infty\le N\}}}{2\pi^2|k|^2} := \bar{V}^{\varepsilon}_{t-\sigma}(k). \tag{6.2}$$

Now we introduce the following notation: $k_{[1\dots n]} = \sum_{i=1}^n k_i$, $\eta_{1\dots n} = (\eta_1,\dots,\eta_n) \in (\mathbb{R}\times E)^n$, $d\eta_{1\dots n} = d\eta_1\cdots d\eta_n$, $dk_{1\dots n} = dk_1\cdots dk_n$, $\tilde{k}^{i_1i_2i_3} = (k^j - i_j(2N+1))_{j=1,2,3}$ for $i_j = 1,0,-1$ and $\sum_{j=1}^3 i_j^2 \neq 0$. In the following, we always omit the superscript of $\tilde{k}$ if there is no confusion. Denote by

$$\int_{(\mathbb{R}\times E)^n} f(\eta_{1\dots n})W(d\eta_{1\dots n})$$

a generic element of the $n$th chaos of $W$ on $\mathbb{R}\times E$. By [13], Section 9.2, we know that

$$E\left[\left|\int_{(\mathbb{R}\times E)^n} f(\eta_{1\dots n})W(d\eta_{1\dots n})\right|^2\right] \le (n!)\int_{(\mathbb{R}\times E)^n} \left|f(\eta_{1\dots n})\right|^2 d\eta_{1\dots n}.$$

Hence for bounding the variance of the chaos, it is enough to bound the $L^2$ norm of the unsymmetrized kernels. To obtain the results, we first recall the following lemma from [38] for our later use.

LEMMA 6.1 ([38], Lemma 3.10). *Let $0 < l, m < d, l + m - d > 0$. Then we have*

$$\sum_{k_1,k_2\in\mathbb{Z}^d\setminus\{0\},k_1+k_2=k} \frac{1}{|k_1|^l|k_2|^m} \lesssim \frac{1}{|k|^{l+m-d}}.$$

By similar arguments as in the proof of [38], Lemma 3.11, we have the following results.

LEMMA 6.2. *For every $0 < \kappa < 1$, $i \ge 0$, $t \ge 0$, $k_1, k_2 \in E$ we have*

$$\left|e^{-|k_{[12]}|^2\pi^2 t}\theta(2^{-i}k_{[12]}) - e^{-|k_2|^2\pi^2 t}\theta(2^{-i}k_2)\right| \lesssim |k_1|^{\kappa}2^{-i\kappa}.$$

LEMMA 6.3. *For every $0 < \kappa < 1$, $i \ge 0$, $t \ge 0$, we have that for $k_1, k_2 \in E$ with $|k_{[12]}|_\infty \le N$, $|k_2|_\infty \le N$:*

$$\left|e^{-|k_{12}|^2 t f(\varepsilon k_{[12]})}\theta(2^{-i}k_{[12]}) - e^{-|k_2|^2 t f(\varepsilon k_2)}\theta(2^{-i}k_2)\right| \lesssim |k_1|^{\kappa}2^{-i\kappa}.$$

Now we prove the following estimate for the approximating operators.

LEMMA 6.4. *For any $0 < \kappa < 1$ and $t > 0$, $k \in E$, $\varepsilon > 0$:*

(i)

$$\left|p^{\varepsilon}_t(k) - p_t(k)\right| \lesssim e^{-|k|^2\bar{c}_f t}|\varepsilon k|^{\kappa}, \qquad \left|P^{\varepsilon}_t(k) - p_t(k)\right| \lesssim e^{-|k|^2\bar{c}_f t}|\varepsilon k|^{\kappa};$$

(ii)

$$\big|P_t^\varepsilon(k) - \bar{P}_t^\varepsilon(k)\big| \lesssim e^{-|k|^2\bar{c}_f t}|\varepsilon k|^\kappa, \qquad \big|V_t^\varepsilon(k) - \bar{V}_t^\varepsilon(k)\big| \lesssim \frac{e^{-|k|^2\bar{c}_f t}|\varepsilon k|^\kappa}{|k|^2}.$$

*Here*, $\bar{c}_f = c_f \wedge \pi^2 > 0$, $c_f = \min\{f(x) : |x| \le 1.8\}$.

PROOF. The results follow from $|f(\varepsilon k) - \pi^2| \lesssim |\varepsilon k|^\kappa$ and

$$\begin{aligned} \big|e^{-|k|^2 t f(\varepsilon k)} - e^{-|k|^2\pi^2 t}\big| &\lesssim e^{-|k|^2\bar{c}_f t}\big[1 \wedge \big(t^\kappa \big|f(\varepsilon k) - \pi^2\big|^\kappa |k|^{2\kappa}\big)\big] \\ &\lesssim e^{-|k|^2\bar{c}_f t}|\varepsilon k|^\kappa. \end{aligned}$$

□

We prove the following two lemmas for the convergence of the error terms.

LEMMA 6.5. *For every* $q \ge 0, 0 < r < 3$,

$$\int_E \theta\big(2^{-q}\tilde{k}\big)^2 \frac{1}{|k|^r}\, dk \lesssim 2^{(3-r)q} \quad \textit{and} \quad \int_E \theta\big(2^{-q}\tilde{\tilde{k}}\big)^2 \frac{1}{|k|^r}\, dk \lesssim 2^{(3-r)q}.$$

PROOF. We only treat the first, the second can be obtained by a similar argument. We have

$$\begin{aligned} \int \theta\big(2^{-q}\tilde{k}\big)^2 \frac{1}{|k|^r}\, dk &\lesssim \int 1_{|k|\le 2^q}\theta\big(2^{-q}\tilde{k}\big)^2 \frac{1}{|k|^r}\, dk + \int 1_{|k|>2^q}\theta\big(2^{-q}\tilde{k}\big)^2 \frac{1}{|k|^r}\, dk \\ &\lesssim 2^{(3-r)q} \end{aligned}$$

Here, in the last inequality we used that the cardinality of $k$ with $\theta(2^{-q}\tilde{k}) \ne 0$ is of order $2^{3q}$. □

LEMMA 6.6. *For every* $q \ge 0, 0 < r < 3$,

$$\int \theta\big(2^{-q}\tilde{k}\big)^2 \frac{1}{|k|^r}\, dk \lesssim \varepsilon^\kappa 2^{(3-r+\kappa)q}.$$

*Here*, $\kappa > 0$ *is small enough.*

PROOF. We have

$$\begin{aligned} \int \theta\big(2^{-q}\tilde{k}\big)^2 \frac{1}{|k|^r}\, dk &\lesssim \int 1_{|k|\le N}\theta\big(2^{-q}\tilde{k}\big)^2 \frac{1}{|k|^r}\, dk \\ &\quad + \varepsilon^\kappa \int 1_{|k|\ge N}\theta\big(2^{-q}\tilde{k}\big)^2 \frac{1}{|k|^{r-\kappa}}\, dk \\ &\lesssim \varepsilon^\kappa 2^{(3-r+\kappa)q}, \end{aligned}$$

where in the last inequality we used that $|k| \le N \backsimeq |\tilde{k}| \backsimeq 2^q$ and Lemma 6.5. □

6.1. *Convergence of renormalization terms*. In this subsection, we prove $\delta C_W^\varepsilon \to 0$ in probability as $\varepsilon \to 0$. In the following, we use the graph notation similar as in [18] to make this paper more readable. We use $\eta \dashrightarrow \sigma$ or $s \overset{k}{\dashrightarrow} \sigma$ to represent a factor $P^\varepsilon_{\sigma-s}(k)$ for $\eta = (s,k)$ and use $\sigma \overset{k}{\rightsquigarrow} t$ to represent $V^\varepsilon_{t-\sigma}(k)$ or $\bar{V}^\varepsilon_{t-\sigma}(k)$ for simplicity. We use $\eta \longrightarrow \sigma$ to represent a factor $\bar{P}^\varepsilon_{\sigma-s}(k)$ for $\eta = (s,k)$, and use $\sigma \xrightarrow{k} t$ to represent $\bar{P}^\varepsilon_{t-\sigma}(k)$ or $p_{t-\sigma}(k)$ if there is no confusion. We also use the convention that if a vertex is drawn in grey, then the corresponding variable is integrated out. Here, we use two different graphs to denote $P^\varepsilon_{\sigma-s}(k)$ and $\bar{P}^\varepsilon_{\sigma-s}(k)$. The second one is to emphasize the appearance of $k$.

For the terms containing $u_2^\varepsilon$, there are error terms ($J_t^i$ in the following) appears. For these terms, we use $|k_i| \simeq N$ or Lemma 6.6 to produce $\varepsilon^\kappa$.

*Convergence of* $\dot{\vdots} - \dot{|}$

In this part, we prove the convergence of $\dot{\vdots} - \dot{|}$. We have

$$
\begin{aligned}
&E|\Delta_q[\dot{\vdots}(t) - \dot{|}(t)]|^2 \\
&\quad\lesssim \int_{\mathbb{R}\times E} \theta(2^{-q}k)^2 \big|e_k\big(P^\varepsilon_{t-s}(k) - \bar{P}^\varepsilon_{t-s}(k)\big)\big|^2\, d\eta \\
&\quad\lesssim \int \theta(2^{-q}k)^2 (\varepsilon|k|)^\kappa |k|^{-2}\, dk \lesssim \varepsilon^\kappa 2^{q(\kappa+1)}.
\end{aligned}
$$

Here, $\kappa > 0$ is small enough and in the second inequality we used Lemma 6.4. Similarly, by using

$$
\big|1 - e^{-|t_2-t_1| f(\varepsilon k)|k|^2}\big| \lesssim |t_1 - t_2|^\kappa |k|^{2\kappa},
$$

we get the desired estimates for $E|\Delta_q[(\dot{\vdots}(t_2) - \dot{|}(t_2)) - (\dot{\vdots}(t_1) - \dot{|}(t_1))]|^2$ with $t_1, t_2 \in [0,T]$, which combined with Gaussian hypercontractivity implies that for $p > 1$, $\epsilon > 0$ small enough,

$$
\begin{aligned}
&E\big[\big\|(\dot{\vdots}(t_2) - \dot{|}(t_2)) - (\dot{\vdots}(t_1) - \dot{|}(t_1))\big\|^p_{B_{p,p}^{-\frac{1}{2}-\kappa-\epsilon}}\big] \\
&\quad\lesssim \varepsilon^{p\kappa/2} |t_2 - t_1|^{\kappa p/4}.
\end{aligned}
$$

Then by Lemma 2.1, we obtain that for every $\delta > 0$, $p > 1$, $\dot{\vdots} - \dot{|} \to 0$ in $L^p(\Omega; C_T\mathcal{C}^{-1/2-\delta})$ as $\varepsilon \to 0$.

*Convergence of* $\dot{\vee} - \vee$

In this part, we prove the convergence of $\dot{\vee}$. Recall that $\dot{\vee} = \dot{\vdots}^2 - C_0^\varepsilon$ and $\vee = \dot{|} - \bar{C}_0^\varepsilon$. Now take

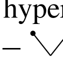
$$
C_0^\varepsilon = 2^{-3} \int_E \frac{1_{\{|k|_\infty \le N\}}}{2|k|^2 f(\varepsilon k)}\, dk, \qquad \bar{C}_0^\varepsilon = 2^{-3} \int \frac{1_{\{|k|_\infty \le N\}}}{2|k|^2 \pi^2}\, dk. \tag{6.3}
$$

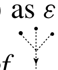
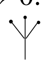
Then we have

$$E\big|\Delta_q\big[$$  $$(t) -$$  $$(t)\big]\big|^2$$

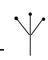
$$\lesssim \int_{(\mathbb{R}\times E)^2} \theta\big(2^{-q}k_{[12]}\big)^2 \big|\big(P^\varepsilon_{t-s_1}(k_1)P^\varepsilon_{t-s_2}(k_2) - \bar{P}^\varepsilon_{t-s_1}(k_1)\bar{P}^\varepsilon_{t-s_2}(k_2)\big)\big|^2 \, d\eta_{12}$$

$$\lesssim \varepsilon^\kappa \int \theta\big(2^{-q}k_{[12]}\big)^2 \frac{|k_1|^\kappa + |k_2|^\kappa}{|k_1|^2|k_2|^2} \, dk_{12} \lesssim \varepsilon^\kappa 2^{(\kappa+2)q}.$$

Here, $\kappa > 0$ is small enough and in the second inequality we used Lemma 6.4 and in the last inequality we used Lemma 6.1. Then by Gaussian hypercontractivity and Lemma 2.1, we obtain that for every $\delta > 0$, $p > 1$,  $-$  $\to 0$ in $L^p(\Omega; C_T\mathcal{C}^{-1-\delta})$ as $\varepsilon \to 0$.

*Convergence of*  $-$ 

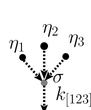
In this part, we consider the convergence of $u^\varepsilon_2$. Recall that

 $$(t) -$$  $$(t) = I^3_t - \bar{I}^3_t + J^3_t.$$

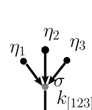
Here,

$$I^3_t = 2^{-3} \int_{(\mathbb{R}\times E)^3} e_{k_{[123]}} \int_0^t P^\varepsilon_{t-\sigma}(k_{[123]}) P^\varepsilon_{\sigma-s_1}(k_1) \times P^\varepsilon_{\sigma-s_2}(k_2) P^\varepsilon_{\sigma-s_3}(k_3) \, d\sigma \, W(d\eta_{123}),$$

and $\bar{I}^3_t$ is defined similarly as $I^3_t$ with $P^\varepsilon_{t-\sigma}(k_{[123]})$ replaced by $p_{t-\sigma}(k_{[123]})$ and with other $P^\varepsilon$ replaced by $\bar{P}^\varepsilon$ and $J^3_t$ is defined similarly as $I^3_t$ with $e_{k_{[123]}}, k_{[123]}$ replaced by $e_{\tilde{k}_{[123]}}, \tilde{k}_{[123]}$, respectively. We use a graph notation to indicate the main part in $I^3_t$ and $\bar{I}^3_t$:



to indicate $\displaystyle\int_0^t P^\varepsilon_{t-\sigma}(k_{[123]}) \cdots d\sigma$,



for the corresponding term in $\bar{I}^3_t$.

The graph for $J_t^3$ is the same as that for $I_t^3$ with $k_{[123]}$ replaced by $\tilde{k}_{[123]}$. By Lemma 6.4 and a straightforward calculation, we obtain that

$$
\begin{aligned}
&E|\Delta_q(I_t^3 - \bar{I}_t^3)|^2 \\
&\quad \lesssim \int_{(\mathbb{R}\times E)^2} \theta(2^{-q}k_{[123]})^2 \bigg| \int_0^t \bigg( P^{\varepsilon}_{t-\sigma}(k_{[123]}) \prod_{i=1}^3 P^{\varepsilon}_{\sigma-s_i}(k_i) \\
&\qquad - p_{t-\sigma}(k_{[123]}) \prod_{i=1}^3 \bar{P}^{\varepsilon}_{\sigma-s_i}(k_i) \bigg) d\sigma \bigg|^2 d\eta_{123} \\
&\quad \lesssim \int \theta(2^{-q}k_{[123]}) \frac{\varepsilon^{\kappa}\sum_{i=1}^3 |k_i|^{\kappa} + |k_{[123]}|^{\kappa}\varepsilon^{\kappa}}{|k_1|^2|k_2|^2|k_3|^2[|k_1|^2+|k_2|^2+|k_3|^2]|k_{[123]}|^2}\, dk_{123} \\
&\quad \lesssim \int_E \theta(2^{-q}k) \frac{\varepsilon^{\kappa}}{|k|^{4-\kappa}}\, dk \lesssim \varepsilon^{\kappa} 2^{q(-1+\kappa)},
\end{aligned}
$$

where we used Lemma 6.1 in the third inequality and we used the graph notation in the second inequality. From the graph, we can use Lemma 6.4 to control each $-$, and $\varepsilon^{\kappa}|k_i|^{\kappa}$ is produced. Then for each integral w.r.t. time $s_i$, $|k_i|^{-2}e^{-|\sigma-\bar{\sigma}||k_i|^2}$ is produced and taking integrals w.r.t. $\sigma$ and $\bar{\sigma}$, we obtain the second inequality. Similar calculations also imply that

$$
\begin{aligned}
E|\Delta_q J_t^3|^2 &\lesssim \int \theta(2^{-q}\tilde{k}_{[123]}) \frac{1_{\{|k_{123}|>N, 2^q \lesssim N\}}}{|k_1|^2|k_2|^2|k_3|^2[|k_1|^2+|k_2|^2+|k_3|^2]|\tilde{k}_{[123]}|^2}\, dk_{123} \\
&\lesssim \int_E \theta(2^{-q}\tilde{k}) \frac{1_{\{|k|>N, 2^q\lesssim N\}}}{|k|^2|\tilde{k}|^2}\, dk \lesssim \varepsilon^{\kappa} 2^{q(-1+\kappa)}.
\end{aligned}
$$

Here, we use $|k| > N$ and $\frac{1}{|k|^2}$ to produce $\varepsilon^{\kappa}$. By a similar argument as above, we also obtain that for every $\delta > 0$, $p > 1$, $- \to 0$ in $L^p(\Omega; C_T\mathcal{C}^{\frac{1}{2}-\delta})$. Similarly, we obtain that $- \to 0$ in $L^p(\Omega; C_T^{\frac{1}{8}}\mathcal{C}^{\frac{1}{4}-\delta})$.

*Convergence of* $\pi_{0,\diamond}( , ) - \pi_{0,\diamond}( , )$

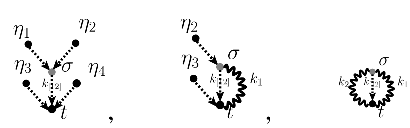
In this part, we focus on $\pi_0( , )$ and prove that $\pi_{0,\diamond}( , ) - \pi_{0,\diamond}( , ) \to 0$ in $C_T\mathcal{C}^{-\delta}$ for every $\delta > 0$. Now we have the following identity for $t \in [0, T]$:

$$
\pi_0( , )(t) - \pi_0( , )(t) = I_t^1 + 4I_t^2 + 2I_t^3 - [\bar{I}_t^1 + 4\bar{I}_t^2 + 2\bar{I}_t^3],
$$

where

$$
\begin{aligned}
I_t^1 &= 2^{-\frac{9}{2}} \int e_{k_{[1234]}} \psi_0(k_{[12]}, k_{[34]}) \int_0^t d\sigma\, P^\varepsilon_{t-\sigma}(k_{[12]}) P^\varepsilon_{\sigma-s_1}(k_1) \\
&\quad \times P^\varepsilon_{\sigma-s_2}(k_2) P^\varepsilon_{t-s_3}(k_3) P^\varepsilon_{t-s_4}(k_4) W(d\eta_{1234}), \\
I_t^2 &= 2^{-\frac{9}{2}} \int \int e_{k_{[23]}} \psi_0(k_{[12]}, k_3 - k_1) \int_0^t d\sigma\, P^\varepsilon_{t-\sigma}(k_{[12]}) P^\varepsilon_{\sigma-s_2}(k_2) \\
&\quad \times P^\varepsilon_{t-s_3}(k_3) V^\varepsilon_{t-\sigma}(k_1)\, dk_1\, W(d\eta_{23}), \\
I_t^3 &= 2^{-6} \int_{E^2} \int_0^t d\sigma\, V^\varepsilon_{t-\sigma}(k_1) V^\varepsilon_{t-\sigma}(k_2) P^\varepsilon_{t-\sigma}(k_{[12]})\, dk_{12},
\end{aligned}
$$

and for $i = 1, 2, 3$, $\bar{I}_t^i$ is defined similarly with $P^\varepsilon_{t-\sigma}(k_{[12]})$ replaced by $p_{t-\sigma}(k_{[12]})$ and other $P^\varepsilon, V^\varepsilon$ replaced by $\bar{P}^\varepsilon, \bar{V}^\varepsilon$, respectively. We use a graph notation to indicate the main part in $I_t^1$ and $I_t^2$, $I_t^3$:



The graphs for $\bar{I}_t^1$, $\bar{I}_t^2$ and $\bar{I}_t^3$ should be the same with $\bullet\!\dashrightarrow\!\bullet$ replaced by $\bullet\!\longrightarrow\!\bullet$.

In fact, choose

$$
C_{11}^\varepsilon = 2^{-5} \int \int_{-\infty}^t d\sigma\, V^\varepsilon_{t-\sigma}(k_1) V^\varepsilon_{t-\sigma}(k_2) P^\varepsilon_{t-\sigma}(k_{[12]})\, dk_{12} \tag{6.4}
$$

and $\bar{C}_1^\varepsilon$ is defined with each $P^\varepsilon, V^\varepsilon$ replaced by $p, \bar{V}^\varepsilon$, respectively. Choose $\varphi_1^\varepsilon(t) = 2I_t^3 - C_{11}^\varepsilon$ and $\bar{\varphi}_1^\varepsilon(t) = 2\bar{I}_t^3 - \bar{C}_1^\varepsilon$ and

$$
\varphi_1(t) = -2^{-7} \int \frac{e^{-t\pi^2(|k_1|^2 + |k_2|^2 + |k_{[12]}|^2)}}{|k_1|^2 |k_2|^2 (|k_1|^2 + |k_2|^2 + |k_{[12]}|^2)\pi^6}\, dk_{12}.
$$

Then we easily obtain that

$$
\sup_{t \in [0,T]} t^\rho |\varphi_1^\varepsilon - \varphi_1| \lesssim \varepsilon^\kappa, \qquad \sup_{t \in [0,T]} t^\rho |\bar{\varphi}_1^\varepsilon - \varphi_1| \lesssim \varepsilon^\kappa,
$$

for every $\rho > 0$, $0 < \kappa < 2\rho$.

*Terms in the second chaos*: Now we consider $I_t^2$ and by graph notation and (6.1), (6.2) we have

$$
\begin{aligned}
&E\big|\Delta_q(I_t^2-\bar I_t^2)\big|^2\\
&\quad\lesssim\int\psi_0(k_{[12]},k_3-k_1)\psi_0(k_{[24]},k_3-k_4)\theta(2^{-q}k_{[23]})^2\\
&\qquad\times\frac{|\varepsilon k_{[12]}|^{\kappa/2}|\varepsilon k_{[24]}|^{\kappa/2}+|\varepsilon k_1|^{\kappa/2}|\varepsilon k_4|^{\kappa/2}+|\varepsilon k_2|^{\kappa}+|\varepsilon k_3|^{\kappa}}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2(|k_1|^2+|k_{[12]}|^2)(|k_4|^2+|k_{[24]}|^2)}\,dk_{1234}\\
&\quad\lesssim\varepsilon^{\kappa}\int\theta(2^{-q}k_{[23]})^2\frac{2^{-2q+2\kappa}}{|k_2|^{2-\kappa}|k_3|^2}\,dk_{23}\\
&\quad\lesssim\varepsilon^{\kappa}2^{q3\kappa},
\end{aligned}
$$

with $\kappa>0$ small enough. Here, we used that $|k_{[i2]}|\gtrsim 2^q$ on the support of $\psi_0(k_{[i2]},k_3-k_i)\theta(2^{-q}k_{[23]})$ for $i=1,4$ in the second inequality and Lemma 6.1 in the last inequality.

*Terms in the fourth chaos*: Now for $I_t^1$ by (6.1), (6.2) and graph notation we have

$$
\begin{aligned}
&E\big[\big|\Delta_q(I_t^1-\bar I_t^1)\big|^2\big]\\
&\quad\lesssim\varepsilon^{\kappa}\int\theta(2^{-q}k_{[1234]})^2\frac{\psi_0(k_{[12]},k_{[34]})}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2|k_{[12]}|^4}\\
&\qquad\times\left(|k_{[12]}|^{\kappa}+\sum_{i=1}^4|k_i|^{\kappa}\right)dk_{1234}\\
&\quad\lesssim\int\theta(2^{-q}k_{[1234]})^2\psi_0(k_{[12]},k_{[34]})\\
&\qquad\times\left(\frac{\varepsilon^{\kappa}}{|k_{[34]}||k_{[12]}|^{5-\kappa}}+\frac{\varepsilon^{\kappa}}{|k_{[34]}|^{1-\kappa}|k_{[12]}|^5}\right)dk_{[12][34]}\\
&\quad\lesssim\int\theta(2^{-q}k)^2 2^{-q(2+\kappa)}\frac{\varepsilon^{\kappa}}{|k|^{1-2\kappa}}\,dk\lesssim\varepsilon^{\kappa}2^{q\kappa},
\end{aligned}
$$

where we used Lemma 6.1 in the second inequality and that $|k_{[12]}|\gtrsim 2^q$ on the support of $\theta(2^{-q}k_{[1234]})\ \psi_0(k_{[12]},k_{[34]})$ in the third inequality. Now we have that for $\kappa>0$ small enough

$$
E\big[\big|\Delta_q(I_t^1-\bar I_t^1)\big|^2\big]\lesssim 2^{q\kappa}\varepsilon^{\kappa}.
$$

By a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta>0$, $p>1$,

$$
\pi_{0,\diamond}(\ ,\ )-\pi_{0,\diamond}(\ ,\ )\to 0\qquad\text{in } L^p(\Omega;C_T\mathcal{C}^{-\delta}).
$$

*Convergence of* $\pi_0(\dot\Psi, \dot{\mathbf{|}}) - \pi_0(\Psi, \mathbf{|})$

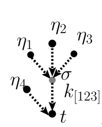
In this part, we focus on $\pi_0(\dot\Psi, \dot{\mathbf{|}})$ and prove that $\pi_0(\dot\Psi, \dot{\mathbf{|}}) - \pi_0(\Psi, \mathbf{|}) \to 0$ in $C_T\mathcal{C}^{-\delta}$. We have the following identity for $t \in [0, T]$:

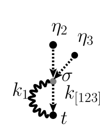
$$-\pi_0(\Psi, \mathbf{|})(t) + \pi_0(\dot\Psi, \dot{\mathbf{|}})(t) = I_t^1 + 3I_t^2 - [\bar{I}_t^1 + 3\bar{I}_t^2] + J_t^1 + 3J_t^2,$$

where

$$\begin{aligned} I_t^1 &= 2^{-\frac{9}{2}} \int e_{k_{[1234]}} \psi_0(k_{[123]}, k_4) \int_0^t d\sigma\, P^\varepsilon_{t-\sigma}(k_{[123]}) P^\varepsilon_{\sigma-s_1}(k_1) \\ &\quad \times P^\varepsilon_{\sigma-s_2}(k_2) P^\varepsilon_{\sigma-s_3}(k_3) P^\varepsilon_{t-s_4}(k_4) W(d\eta_{1234}), \\ I_t^2 &= 2^{-\frac{9}{2}} \int\int e_{k_{[23]}} \psi_0(k_{[123]}, k_1) \int_0^t d\sigma\, P^\varepsilon_{t-\sigma}(k_{[123]}) P^\varepsilon_{\sigma-s_2}(k_2) \\ &\quad \times P^\varepsilon_{\sigma-s_3}(k_3) V^\varepsilon_{t-\sigma}(k_1)\, dk_1 W(d\eta_{23}), \end{aligned}$$

and for $i = 1, 2$, $\bar{I}_t^i$ is defined with $P^\varepsilon_{t-\sigma}(k_{[123]})$ replaced by $p_{t-\sigma}(k_{[123]})$ and other $P^\varepsilon$, $V^\varepsilon$ replaced by $\bar{P}^\varepsilon$, $\bar{V}^\varepsilon$, respectively, and for $i = 1, 2$, $J_t^i$ is defined similarly as $I_t^i$ with $k_{[123]}, e_{k_{[1234]}}, e_{k_{[23]}}$ replaced by $\tilde{k}_{[123]}, e_{\tilde{k}_{[1234]}}, e_{\tilde{k}_{[23]}}$, respectively. We also use graph notation to indicate the main part in $I_t^1$ and $I_t^2$:



,


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The graphs for $\bar{I}_t^1, \bar{I}_t^2$ should be the same with $\bullet\cdots\!\!\rightarrow\!\bullet$ replaced by $\bullet\!\!\longrightarrow\!\bullet$ and the graphs for $J_t^1, J_t^2$ should be the same as above only with $k_{[123]}$ replaced by $\tilde{k}_{[123]}$.

*Terms in the second chaos*: First, we consider $I_t^2$ and have the following calculations:

$$\begin{aligned} &E\bigl|\Delta_q\bigl(I_t^2 - \bar{I}_t^2\bigr)\bigr|^2 \\ &\quad\lesssim \int \psi_0(k_{[123]}, k_1)\psi_0(k_{[234]}, k_4)\theta\bigl(2^{-q}k_{[23]}\bigr)^2 \\ &\qquad \times \frac{|\varepsilon k_{[123]}|^{\kappa/2}|\varepsilon k_{[234]}|^{\kappa/2} + |\varepsilon k_1|^{\kappa/2}|\varepsilon k_4|^{\kappa/2} + |\varepsilon k_2|^{\kappa} + |\varepsilon k_3|^{\kappa}}{|k_2|^2|k_3|^2|k_1|^2(|k_1|^2 + |k_{[123]}|^2)|k_4|^2(|k_4|^2 + |k_{[234]}|^2)}\, dk_{1234} \\ &\quad\lesssim \varepsilon^\kappa \int 2^{-q(2-2\kappa)}\theta\bigl(2^{-q}k_{[23]}\bigr)^2 \frac{1}{|k_2|^{2-\kappa}|k_3|^2}\, dk_{23} \lesssim \varepsilon^\kappa 2^{3q\kappa}, \end{aligned}$$

where $\kappa > 0$ is small enough. Here, we used (6.1), (6.2) and graph notation in the first inequality and that $|k_{[123]}| \gtrsim 2^q$, $k_{[234]} \gtrsim 2^q$ in the second inequality and we used Lemma 6.1 in the last inequality. By a similar calculation as above, we see that

$$E\big|\Delta_q J_t^2\big|^2 \lesssim \int 2^{-q(2-2\kappa)}\theta\big(2^{-q}\tilde{k}_{[23]}\big)^2 \frac{\varepsilon^\kappa}{|k_2|^{2-\kappa}|k_3|^2}\, dk_{23} \lesssim \varepsilon^\kappa 2^{3\kappa q}.$$

Here, $\kappa > 0$ is small enough and in the first inequality we used $|k_{[123]}| \backsimeq N$ to deduce that $|k_i| \backsimeq N$ for some $i \in \{1,2,3\}$, which produces $\varepsilon^\kappa$, and in the last inequality we used Lemmas 6.1 and 6.5.

*Terms in the fourth chaos*: Now for $I_t^1$ we have

$$\begin{aligned} &E\big[\big|\Delta_q\big(I_t^1 - \bar{I}_t^1\big)\big|^2\big] \\ &\quad\lesssim \varepsilon^\kappa \int \frac{\theta(2^{-q}k_{[1234]})^2\psi_0(k_{[123]},k_4)(|k_{[123]}|^\kappa + \sum_{i=1}^4 |k_i|^\kappa)}{|k_1|^2|k_2|^2|k_3|^2|k_4|^2[|k_1|^2+|k_2|^2+|k_3|^2]|k_{[123]}|^2}\, dk_{1234} \\ &\quad\lesssim \int 2^{-q(2-\kappa)}\theta\big(2^{-q}k\big)^2 \frac{\varepsilon^\kappa}{|k|}\, dk \lesssim \varepsilon^\kappa 2^{q\kappa}, \end{aligned}$$

where we used (6.1), (6.2) and graph notation in the first inequality, Lemma 6.1 and that $|k_{[123]}| \gtrsim 2^q$ in the second inequality. For $J_t^1$, using Lemma 6.5 and by a similar argument, we also obtain that

$$E\big|\Delta_q J_t^1\big|^2 \lesssim \varepsilon^\kappa 2^{\kappa q}.$$

Now by a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we have that for every $\delta > 0$, $p > 1$,

$$\pi_0( \;,\; ) - \pi_0( \;,\; ) \to 0 \qquad \text{in } L^p\big(\Omega; C_T\mathcal{C}^{-\delta}\big).$$

*Convergence of* $\pi_{0,\diamond}( \;,\; ) - \pi_{0,\diamond}( \;,\; )$

In this part, we focus on $\pi_{0,\diamond}( \;,\; )$ and prove that $\pi_{0,\diamond}( \;,\; ) - \pi_{0,\diamond}( \;,\; ) \to 0$ in $C_T\mathcal{C}^{-\frac{1}{2}-\delta}$. We have the following identity for $t \in [0,T]$:

$$\begin{aligned} &\pi_{0,\diamond}( \;,\; )(t) - \pi_{0,\diamond}( \;,\; )(t) \\ &\qquad = I_t^1 + 6I_t^2 + 6I_t^3 - \big[\bar{I}_t^1 + 6\bar{I}_t^2 + 6\bar{I}_t^3\big] + J_t^1 + 6J_t^2 + 6J_t^3, \end{aligned}$$

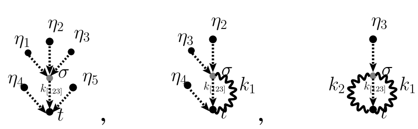
where

$$I_t^1 = 2^{-6} \int e_{k_{[12345]}} \psi_0(k_{[123]}, k_{[45]}) \int_0^t d\sigma\, P^\varepsilon_{t-\sigma}(k_{[123]}) \\ \times \prod_{i=1}^3 P^\varepsilon_{\sigma - s_i}(k_i) \prod_{i=4}^5 P^\varepsilon_{t-s_i}(k_i) W(d\eta_{12345}),$$

$$I_t^2 = 2^{-6} \int\int e_{k_{[234]}} \psi_0(k_{[123]}, k_4 - k_1) \int_0^t d\sigma\, P^\varepsilon_{t-\sigma}(k_{[123]}) \\ \times \prod_{i=2}^3 P^\varepsilon_{\sigma - s_i}(k_i) P^\varepsilon_{t-s_4}(k_4) V^\varepsilon_{t-\sigma}(k_1)\, dk_1\, W(d\eta_{234}),$$

$$I_t^3 = 2^{-6} \int\int e_{k_3} \psi_0(k_{[123]}, k_{[12]}) \int_0^t d\sigma\, P^\varepsilon_{\sigma - s_3}(k_3) \\ \times V^\varepsilon_{t-\sigma}(k_1) V^\varepsilon_{t-\sigma}(k_2) P^\varepsilon_{t-\sigma}(k_{[123]})\, dk_{12}\, W(d\eta_3),$$

and for $i = 1, 2, 3$, $\bar{I}^i_t$ is defined similarly with $P^\varepsilon_{t-\sigma}(k_{[123]})$ replaced by $p_{t-\sigma}(k_{[123]})$ and other $P^\varepsilon$, $V^\varepsilon$ replaced by $\bar{P}^\varepsilon$, $\bar{V}^\varepsilon$, respectively, and for $i = 1, 2, 3$, $J^i_t$ is defined similarly as $I^i_t$ with each $k_{[123]}$, $e_{k_{[12345]}}$, $e_{k_{[234]}}$, $e_{k_3}$ replaced by $\tilde{k}_{[123]}$, $e_{\tilde{k}_{[12345]}}$, $e_{\tilde{k}_{[234]}}$, $e_{\tilde{k}_3}$, respectively. We use graph notation to indicate the main parts in $I^1_t$ and $I^2_t$, $I^3_t$:

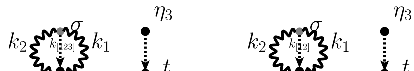



The graph for $\bar{I}^1_t$, $\bar{I}^2_t$, $\bar{I}^3_t$ should be the same with $\bullet\cdots\!\!\to\bullet$ replaced by $\bullet\!\longrightarrow\!\bullet$ and the graph for $J^1_t$, $J^2_t$, $J^3_t$ should be the same as above only with $k_{[123]}$ replaced by $\tilde{k}_{[123]}$.

We consider the following term first:

$$I^3_t - \bar{I}^3_t - [\tilde{I}^3_t - \bar{\tilde{I}}^3_t] + \tilde{I}^3_t - \bar{\tilde{I}}^3_t - C^\varepsilon(t)\,\vdots\,(t) + \bar{C}^\varepsilon(t)\,|\,(t),$$

where $\tilde{I}^3_t$, $\bar{\tilde{I}}^3_t$ are defined similarly as $I^3_t$, $\bar{I}^3_t$ with $P^\varepsilon_{\sigma-s_3}(k_3)$, $\bar{P}^\varepsilon_{\sigma-s_3}(k_3)$ replaced by $P^\varepsilon_{t-s_3}(k_3)$, $\bar{P}^\varepsilon_{t-s_3}(k_3)$, respectively, and $C^\varepsilon(t) = \frac{1}{2}[C^\varepsilon_{11} + \varphi^\varepsilon_1(t)]$, $\bar{C}^\varepsilon(t) = \frac{1}{2}[\bar{C}^\varepsilon_{11} + \bar{\varphi}^\varepsilon_1(t)]$. We also use graph notation to indicate the main parts in $\tilde{I}^3_t$ and $C^\varepsilon(t)\,\vdots\,(t)$:




Since for $\kappa > 0$ small enough, $\int |P^\varepsilon_{t-s_3}(k_3) - P^\varepsilon_{\sigma-s_3}(k_3)|^2\, ds_3 \lesssim \frac{(t-\sigma)^{\kappa/2}}{|k_3|^{2-\kappa}}$ and

$$\int \big|P^\varepsilon_{t-s_3}(k_3) - P^\varepsilon_{\sigma-s_3}(k_3) - [\bar{P}^\varepsilon_{t-s_3}(k_3) - \bar{P}^\varepsilon_{\sigma-s_3}(k_3)]\big|^2\, ds_3 \lesssim \frac{(t-\sigma)^{\kappa/2} \wedge \varepsilon^\kappa}{|k_3|^{2-\kappa}},$$

by (6.1), (6.2) and graph notation we obtain that for $\kappa > 0$ small enough

$$
\begin{aligned}
&E\big[\big|\Delta_q\big(I_t^3 - \bar{I}_t^3 - [\tilde{I}_t^3 - \bar{\tilde{I}}_t^3]\big)\big|^2\big] \\
&\quad \lesssim \int \theta\big(2^{-q}k_3\big)^2 \bigg[\frac{1}{|k_3|^{2-2\kappa}} \bigg(\int_0^t \int \varepsilon^{\kappa/2}\big(|k_{[123]}|^{\kappa/2} + |k_2|^{\kappa/2} + |k_1|^{\kappa/2}\big) \\
&\qquad \times \frac{e^{-(|k_{[123]}|^2+|k_1|^2+|k_2|^2)\bar{c}_f(t-\sigma)}}{|k_1|^2|k_2|^2}(t-\sigma)^{\kappa/2}\,dk_{12}\,d\sigma\bigg)^2 \\
&\qquad + \frac{\varepsilon^\kappa}{|k_3|^{2-2\kappa}}\bigg(\int_0^t \int \frac{e^{-(|k_{[123]}|^2+|k_1|^2+|k_2|^2)(t-\sigma)}}{|k_1|^2|k_2|^2}(t-\sigma)^{\kappa/4}\,dk_{12}\,d\sigma\bigg)^2\bigg]\,dk_3 \\
&\quad \lesssim \varepsilon^\kappa 2^{q(1+3\kappa)}.
\end{aligned}
$$

Here, in the last inequality we used that $\sup_{a\geq 0} a^r e^{-a} \leq C$ for $r \geq 0$ and Lemma 6.1. Moreover, by Lemmas 6.2 and 6.3 and graph notation we obtain that

$$
\begin{aligned}
&E\big[\big|\Delta_q\big(\tilde{I}_t^3 - \bar{\tilde{I}}_t^3 - \overset{\bullet}{\vdots}(t)C^\varepsilon(t) + \overset{\bullet}{|}(t)\bar{C}^\varepsilon(t)\big)\big|^2\big] \\
&\quad \lesssim \int \frac{1}{|k_3|^2}\theta\big(2^{-q}k_3\big)\bigg(\int\int_0^t |k_{[12]}|^{-\kappa}|k_3|^\kappa \\
&\qquad \times \big(\varepsilon^{\kappa/2}|k_2|^{\kappa/2} + \varepsilon^{\kappa/2}|k_1|^{\kappa/2} + \varepsilon^{\kappa/2}|k_3|^{\kappa/2}\big) \\
&\qquad \times \frac{e^{-|k_1|^2(t-\sigma)\bar{c}_f - |k_2|^2(t-\sigma)\bar{c}_f}}{|k_1|^2|k_2|^2}\,dk_{12}\,d\sigma\bigg)^2\,dk_3 \\
&\qquad + \int \frac{\varepsilon^\kappa|k_3|^\kappa}{|k|^2}\theta\big(2^{-q}k_3\big)^2 \\
&\qquad \times \bigg(\int\int_0^t \frac{e^{-|k_2|^2(t-\sigma)-|k_1|^2(t-\sigma)}}{|k_1|^2|k_2|^2}|k_3|^\kappa|k_{[12]}|^{-\kappa}\,dk_{12}\,d\sigma\bigg)^2\,dk_3 \\
&\quad \lesssim \varepsilon^\kappa \int \theta\big(2^{-q}k_3\big)\frac{1}{|k_3|^{2-3\kappa}}\,dk_3 \lesssim \varepsilon^\kappa 2^{q(1+3\kappa)}.
\end{aligned}
$$

For $J_t^3$, we have

$$
\begin{aligned}
E\big[\big|\Delta_q J_t^3\big|^2\big] &\lesssim \int \frac{1}{|k_3|^2}\theta\big(2^{-q}\tilde{k}_3\big) \\
&\qquad \times \bigg(\int \frac{1_{|k_1|\leq N, |k_2|\leq N, |k_3|\leq N}}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|\tilde{k}_{[123]}|^2)}\,dk_{12}\bigg)^2\,dk_3 \\
&\lesssim \varepsilon^\kappa 2^{q(1+3\kappa)}.
\end{aligned}
$$

Here, we used that $2^q \backsimeq N \backsimeq |\tilde{k}_3|$ and Lemma 6.6 in the last inequality.

*Terms in the third chaos*: Now we focus on the bounds for $I_t^2$. We obtain the following inequalities:

$$\begin{aligned}
&E\big|\Delta_q\big(I_t^2-\bar{I}_t^2\big)\big|^2 \\
&\quad\lesssim \int \theta\big(2^{-q}k_{[234]}\big)\psi_0(k_{[123]},k_4-k_1)\psi_0(k_{[235]},k_4-k_5) \\
&\qquad\times\prod_{i=1}^{5}\frac{1}{|k_i|^2}\frac{|k_{[123]}|^{\kappa/2}|k_{[235]}|^{\kappa/2}\varepsilon^{\kappa}+\sum_{i=1}^{4}(\varepsilon|k_i|)^{\kappa}}{(|k_1|^2+|k_{[123]}|^2+|k_2|^2)(|k_5|^2+|k_{[235]}|^2)}\,dk_{12345} \\
&\quad\lesssim \int 2^{-q(1-\kappa)}\frac{\varepsilon^{\kappa}\theta(2^{-q}k_{[234]})}{|k_2|^{3-2\kappa}|k_3|^2|k_4|^2}\,dk_{234}\lesssim \varepsilon^{\kappa}2^{q(1+3\kappa)},
\end{aligned}$$

where we used graph notation in the first inequality and Lemma 6.1 in the last inequality. For $J_t^2$ by a similar calculation as above, we know that

$$E\big|\Delta_q J_t^2\big|^2\lesssim\int 2^{-q(1-\kappa)}\theta\big(2^{-q}\tilde{k}_{[234]}\big)^2\frac{1}{|k_2|^{3-\kappa}|k_3|^2|k_4|^2}\,dk_{234}\lesssim\varepsilon^{\kappa}2^{(1+3\kappa)q}.$$

Here, $\kappa>0$ is small enough and in the last inequality we used Lemmas 6.1 and 6.6.

*Terms in the fifth chaos*: Now we focus on the bounds for $I_t^1$. By graph notation, we obtain the following inequalities:

$$\begin{aligned}
&E\big|\Delta_q\big(I_t^1-\bar{I}_t^1\big)\big|^2 \\
&\quad\lesssim \int \theta\big(2^{-q}k_{[12345]}\big)^2\psi_0(k_{[123]},k_{[45]})^2 \\
&\qquad\times\prod_{i=1}^{5}\frac{1}{|k_i|^2}\frac{(\sum_{i=1}^{5}|\varepsilon k_i|^{\kappa}+|\varepsilon k_{[123]}|^{\kappa})}{|k_{[123]}|^2(|k_1|^2+|k_2|^2+|k_{[123]}|^2)}\,dk_{12345} \\
&\quad\lesssim \varepsilon^{\kappa}2^{q(1+2\kappa)}.
\end{aligned}$$

For $J_t^1$ by similar calculations as for $I_t^1$ and using the fact that $|k_{[123]}|\backsimeq N\gtrsim|\tilde{k}_{[123]}|$, we obtain that

$$E\big|\Delta_q J_t^1\big|^2\lesssim\varepsilon^{\kappa}2^{q(1+2\kappa)}.$$

By a similar calculation as above, we also obtain that there exist $\kappa,\epsilon,\gamma>0$ small enough such that for any $t_1,t_2\in[0,T]$

$$\begin{aligned}
&E\big[\big|\Delta_q\big(\pi_{0,\diamond}(\ ,\ )(t_1)-\pi_{0,\diamond}(\ ,\ )(t_2) \\
&\qquad-\pi_{0,\diamond}(\ ,\ )(t_1)+\pi_{0,\diamond}(\ ,\ )(t_2)\big)\big|^2\big] \\
&\quad\lesssim\varepsilon^{\gamma}|t_1-t_2|^{\kappa}2^{q(1+\epsilon)},
\end{aligned}$$

which by Gaussian hypercontractivity and Lemma 2.1 implies that for every $\delta > 0$, $p > 1$, $\pi_{0,\diamond}( \; , \; ) - \pi_{0,\diamond}( \; , \; ) \to 0$ in $L^p(\Omega; C_T\mathcal{C}^{-\frac{1}{2}-\delta})$.

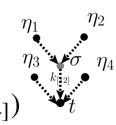
6.2. *Convergence of error terms*. In this subsection, we prove that $E_W^\varepsilon \to^P 0$ as $\varepsilon \to 0$. For the estimate, we use Lemma 6.6 or the fact that there exists some $|k_i| \simeq N$ to produce $\varepsilon^\kappa$. Due to this reason most error terms converge to zero. However, for $\pi_{0,\diamond}( \; , \; )$ and $\pi_{0,\diamond}( \; , \; )$ we still need to do renormalization such that it converges to zero. This is where $C_{12}^\varepsilon$ comes from.

*Convergence of* $\pi_0( \; , \; )$

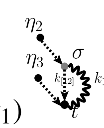
We have the following identity for $t \in [0, T]$:

$$\pi_0( \; , \; )(t) = I_t^1 + 4I_t^2 + 2I_t^3,$$

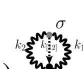
where

$$I_t^1 = 2^{-\frac{9}{2}} \int e_{\tilde{k}_{[1234]}} \psi_0(k_{[12]}, \tilde{k}_{[34]})$$ 

$$W(d\eta_{1234}),$$

$$I_t^2 = 2^{-\frac{9}{2}} \int \int e_{\tilde{k}_{[23]}} \psi_0(k_{[12]}, \tilde{k}_3 - k_1)$$ 

$$dk_1 W(d\eta_{23})$$

$$I_t^3 = 2^{-6} \int e_N^{i_1 i_2 i_3} \psi_0(k_{[12]}, \widetilde{-k_{[12]}})$$ 

$$dk_{12}.$$

*Term in the* 0*th chaos*: We have

$$E\big[|\Delta_q I_t^3|^2\big] \lesssim \left( \int \frac{1_{|k_{[12]}| \simeq N \simeq 2^q} \psi_0(k_{[12]}, \tilde{k}_{[12]})}{|k_{[12]}|^3} \, dk_{[12]} \right)^2 \lesssim \varepsilon^\kappa 2^{q(3\kappa)}.$$

*Term in the second chaos*: Now we consider $I_t^2$. We have

$$\begin{aligned} E\big|\Delta_q I_t^2\big|^2 &\lesssim \int \psi_0(k_{[12]}, \tilde{k}_3 - k_1)\psi_0(k_{[24]}, \tilde{k}_3 - k_4)\theta\big(2^{-q}\tilde{k}_{[23]}\big)^2 \\ &\quad \times \frac{1}{|k_2|^2|k_3|^2|k_1|^2(|k_1|^2 + |k_{12}|^2)|k_4|^2(|k_4|^2 + |k_{[24]}|^2)} \, dk_{1234} \\ &\lesssim \int 2^{(-2+\kappa)q}\theta\big(2^{-q}\tilde{k}_{[23]}\big)^2 \frac{1}{|k_2|^2|k_3|^2} \, dk_{23} \lesssim \varepsilon^\kappa 2^{q2\kappa}, \end{aligned}$$

where $\kappa > 0$ is small enough. Here, we used that $|k_{[i2]}| \gtrsim 2^q$ for $i = 1, 4$ in the second inequality and used Lemmas 6.1 and 6.6 in the third inequality.

*Term in the fourth chaos*: Now for $I_t^1$ we have

$$E\big[|\Delta_q I_t^1|^2\big] \lesssim \int \theta\big(2^{-q}\tilde{k}_{[1234]}\big)^2 \psi_0(k_{[12]}, \tilde{k}_{[34]}) \frac{1}{|k_{[34]}||k_{[12]}|^{5-\kappa}}\, dk_{[12][34]}$$

$$\lesssim 2^{-2q} \int \theta\big(2^{-q}\tilde{k}_{[1234]}\big)^2 \frac{1}{|k_{[34]}||k_{[12]}|^{3-\kappa}}\, dk_{[12][34]} \lesssim \varepsilon^{\kappa} 2^{q\kappa},$$

where we used Lemmas 6.1 and 6.6 in the last inequality. By a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0$, $p > 1$,

$$\pi_0( \;,\; ) \to 0 \qquad \text{in } L^p\big(\Omega; C_T\mathcal{C}^{-\delta}\big).$$

*Convergence of* $\pi_0( \;,\; )$

Now we have the following identity for $t \in [0, T]$:

$$\pi_0( \;,\; )(t) = I_t^1 + 4I_t^2 + 2I_t^3,$$

where

$$I_t^1 = 2^{-\frac{9}{2}} \int e_{\tilde{k}_{[1234]}} \psi_0(\tilde{k}_{[12]}, k_{[34]}) \qquad W(d\eta_{1234}),$$

$$I_t^2 = 2^{-\frac{9}{2}} \int\int e_{\tilde{k}_{[23]}} \psi_0(\tilde{k}_{[12]}, k_3 - k_1) \qquad dk_1 W(d\eta_{23}),$$

$$I_t^3 = 2^{-6} \int e_N^{i_1 i_2 i_3} \psi_0(\tilde{k}_{[12]}, -k_{[12]}) \qquad dk_{12}.$$

$I_t^3, I_t^2$ can be estimated similarly as for the case of $\pi_0( \;,\; )$ and we only consider *Terms in the fourth chaos*: Now for $I_t^1$ we have

$$E|\Delta_q I_t^1|^2 \lesssim \int \psi_0(\tilde{k}_{[12]}, k_{[34]})\theta\big(2^{-q}\tilde{k}_{[1234]}\big)^2$$

$$\times \frac{1}{|k_2|^2|k_3|^2|k_1|^2(|k_1|^2+|k_2|^2)|k_4|^2|\tilde{k}_{[12]}|^2}\, dk_{1234}$$

$$\lesssim \int 2^{-2q}\theta\big(2^{-q}\tilde{k}_{[1234]}\big)^2 \frac{1}{|k_{[12]}|^{3-\kappa}|k_{[34]}|}\, dk_{[12][34]} \lesssim \varepsilon^{\kappa} 2^{q2\kappa},$$

where we used Lemmas 6.1 and 6.6 in the last inequality. By a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for $\delta > 0$,

$p>1$

$$\pi_0( \ , \ ) \to 0 \qquad \text{in } L^p(\Omega; C_T\mathcal{C}^{-\delta}).$$

*Convergence of* $\pi_{0,\diamond}( \ , \ )$

We have

$$\pi_0( \ , \ )(t) = I_t^1 + 4I_t^2 + 2I_t^3.$$

Here, $I_t^i, i=1,2$ is defined similarly as for the case of $\pi_0( \ , \ )$ with $k_{[12]}$, $e_{\tilde{k}_{[1234]}}$ and $e_{\tilde{k}_{[23]}}$ replaced by $\tilde{k}_{[12]}$, $e_{\tilde{\tilde{k}}_{[1234]}}$ and $e_{\tilde{\tilde{k}}_{[23]}}$, respectively, and

$$I_t^3 = 2^{-6}\int e_N^{i_1i_2i_3} e_N^{i_1'i_2'i_3'} \psi_0\big(\tilde{k}_{[12]}^{i_1'i_2'i_3'}, \widetilde{-k_{[12]}}^{i_1i_2i_3}\big)\int_0^t d\sigma\, P_{t-\sigma}^{\varepsilon}\big(\tilde{k}_{[12]}^{i_1'i_2'i_3'}\big) \\ \times V_{t-\sigma}^{\varepsilon}(k_1)V_{t-\sigma}^{\varepsilon}(k_2)\, dk_{12},$$

for $i_j, i_j' \in \{-1,0,1\}$ for $j=1,2,3$ with $\sum_j i_j^2 \neq 0$, $\sum_j (i_j')^2 \neq 0$. Choosing

$$C_{12}^{\varepsilon,i_1i_2i_3} = 2^{-5}\int\int_{-\infty}^t d\sigma\, P_{t-\sigma}^{\varepsilon}\big(\widetilde{-k_{[12]}}^{i_1i_2i_3}\big)V_{t-\sigma}^{\varepsilon}(k_1)V_{t-\sigma}^{\varepsilon}(k_2)\, dk_{12}, \tag{6.5}$$

and $\varphi_2^{\varepsilon,i_1i_2i_3}(t) = -2^{-5}\int\int_{-\infty}^0 d\sigma\, P_{t-\sigma}^{\varepsilon}(\widetilde{-k_{[12]}}^{i_1i_2i_3})V_{t-\sigma}^{\varepsilon}(k_1)V_{t-\sigma}^{\varepsilon}(k_2)\, dk_{12}$, we easily obtain that

$$\big|C_{12}^{\varepsilon,i_1i_2i_3}\big| \backsimeq 1, \qquad \sup_{t\in[0,T]} t^{\rho}\big|\varphi_2^{\varepsilon,i_1i_2i_3}(t)\big| \lesssim \varepsilon^{\kappa},$$

for every $\rho > \kappa/2 > 0$. For the terms in $2I_t^3 - C_{12}^{\varepsilon,i_1i_2i_3} - \varphi_2^{\varepsilon,i_1i_2i_3}$, we know that $e_N^{i_1i_2i_3}e_N^{i_1'i_2'i_3'} \neq 1$ and we easily obtain that

$$E\big[\big|\Delta_q\big(2I_t^3 - C_{12}^{\varepsilon} - \varphi_2^{\varepsilon}\big)\big|^2\big] \lesssim \varepsilon^{\kappa}2^{q(3\kappa)}.$$

*Term in the second chaos*: Now we consider $I_t^2$. We have

$$E|\Delta_q I_t^2|^2 \lesssim \int \psi_0(\tilde{k}_{[12]}, \tilde{k}_3 - k_1)\psi_0(\tilde{k}_{[24]}, \tilde{k}_3 - k_4)\theta\big(2^{-q}\tilde{\tilde{k}}_{[23]}\big)^2 1_{|k_{[12]}|>N, |k_{[24]}|>N} \\ \times \frac{1}{|k_2|^2|k_3|^2|k_1|^2(|k_1|^2 + |\tilde{k}_{12}|^2)|k_4|^2(|k_4|^2 + |\tilde{k}_{[24]}|^2)}\, dk_{1234} \\ \lesssim \varepsilon^{\kappa}\int 2^{(-2+2\kappa)q}\theta\big(2^{-q}\tilde{\tilde{k}}_{[23]}\big)^2\frac{1}{|k_2|^{2-\kappa}|k_3|^2}\, dk_{23} \lesssim \varepsilon^{\kappa}2^{3q\kappa},$$

where $\kappa>0$ is small enough and we used that $|k_i| \backsimeq N$ for some $i\in\{1,2,4\}$ in the second inequality and Lemmas 6.1 and 6.5 in the third inequality.

*Term in the fourth chaos*: Now for $I_t^1$ we have

$$E\big[|\Delta_q I_t^1|^2\big] \lesssim \varepsilon^\kappa \int \theta(2^{-q}\tilde{\tilde{k}}_{[1234]})^2 \psi_0(\tilde{k}_{[12]}, \tilde{k}_{[34]}) \frac{1}{|k_{[34]}||\tilde{k}_{[12]}|^{5-\kappa}}\, dk_{[12][34]}$$

$$\lesssim \varepsilon^\kappa 2^{-2q} \int \theta(2^{-q}\tilde{\tilde{k}}_{[1234]})^2 \frac{1}{|k_{[34]}||\tilde{k}_{[12]}|^{3-\kappa}}\, dk_{[12][34]} \lesssim \varepsilon^\kappa 2^{q\kappa},$$

where we used that $|k_{[12]}| \backsimeq N \gtrsim |\tilde{k}_{[12]}|$ in the first inequality as well as Lemmas 6.1 and 6.5 in the last inequality. By a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0$, $p > 1$,

$\pi_{0,\diamond}($  ,  $) \to 0$ in $L^p(\Omega; C_T\mathcal{C}^{-\delta})$.

*Convergence of* 

By a similar calculation as that for  in Section 6.1, we know that

$E|\Delta_q[$  $]|^2 \lesssim \int \theta(2^{-q}\tilde{k}_{[12]})^2 \frac{1}{|k_1|^2|k_2|^2}\, dk_{12} \lesssim \varepsilon^\kappa 2^{(\kappa+2)q}.$

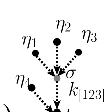
Here, $\kappa > 0$ is small enough and in the last inequality we used Lemmas 6.1, 6.6. Then by Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0$, $p > 1$,  $\to 0$ in $L^p(\Omega; C_T\mathcal{C}^{-1-\delta})$.

*Convergence of* $\pi_0($  $, e_N^{i_1 i_2 i_3}$  $)$

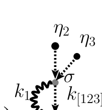
Now we have the following identity for $t \in [0, T]$:

$\pi_0($  $, e_N^{i_1 i_2 i_3}$  $)(t) = I_t^1 + 3I_t^2 + J_t^1 + 3J_t^2,$

where

$I_t^1 = 2^{-\frac{9}{2}} \int e_{\tilde{k}_{[1234]}} \psi_0(k_{[123]}, \tilde{k}_4)$ 

$W(d\eta_{1234}),$

$I_t^2 = 2^{-\frac{9}{2}} \int\int e_{\tilde{k}_{[23]}} \psi_0(k_{[123]}, \tilde{k}_1)$ 

$dk_1 W(d\eta_{23}),$

and for $i = 1, 2$, $J_t^i$ is defined similarly as $I_t^i$ with each $k_{[123]}, e_{\tilde{k}_{[1234]}}, e_{\tilde{k}_{[23]}}$ replaced by $\tilde{k}_{[123]}, e_{\tilde{\tilde{k}}_{[1234]}}, e_{\tilde{\tilde{k}}_{[23]}}$, respectively.

*Terms in the second chaos*: First, we consider $I_t^2$ and by similar calculations as that for $\pi_0($  ,  $)$, we obtain

$$E|\Delta_q I_t^2|^2 \lesssim \varepsilon^\kappa \int 2^{-q(2-2\kappa)} \theta(2^{-q}\tilde{k}_{[23]})^2 \frac{1}{|k_2|^2|k_3|^2}\, dk_{23} \lesssim \varepsilon^\kappa 2^{2q\kappa},$$

where $\kappa > 0$ is small enough and we used that $|k_{[123]}| \backsimeq |\tilde{k}_1| \backsimeq N$ in the first inequality and we used Lemmas 6.1 and 6.5 in the last inequality. By a similar calculation as above, we see that

$$E\big|\Delta_q J_t^2\big|^2 \lesssim \varepsilon^\kappa \int 2^{-q(2-2\kappa)}\theta\big(2^{-q}\tilde{\tilde{k}}_{[23]}\big)^2 \frac{1}{|k_2|^2|k_3|^2}\,dk_{23} \lesssim \varepsilon^\kappa 2^{2\kappa q}.$$

Here, $\kappa > 0$ is small enough and we used that $|\tilde{k}_{[123]}| \backsimeq |\tilde{k}_1| \backsimeq N$ in the first inequality, we used Lemmas 6.1 and 6.5 in the last inequality.

*Terms in the fourth chaos*: Now for $I_t^1$, $J_t^1$ we similarly get that

$$E\big[\big|\Delta_q I_t^1\big|^2 + \big|\Delta_q J_t^1\big|^2\big] \lesssim \varepsilon^\kappa 2^{\kappa q}.$$

Here, for $I_t^1$ we used that $|k_{[123]}| \simeq |\tilde{k}_4| \backsimeq N$ and for $J_t^1$ we used that $|k_{[123]}| \simeq N \gtrsim |\tilde{k}_{[123]}|$. Now by a similar calculation as above, Gaussian hypercontractivity and Lemma 2.1 we obtain that for every $\delta > 0$, $p > 1$,

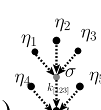
$$\pi_0(\ \ , e_N^{i_1 i_2 i_3}) \to 0 \qquad \text{in } L^p(\Omega; C_T\mathcal{C}^{-\delta}).$$

*Convergence of* $\pi_{0,\diamond}(\ \ ,\ \ )$

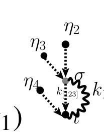
Now we have the following identity for $t \in [0,T]$:

$$\pi_0(\ \ ,\ \ )(t) = I_t^1 + 6I_t^2 + 6I_t^3 + J_t^1 + 6J_t^2 + 6J_t^3,$$

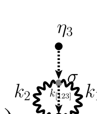
where

$$I_t^1 = 2^{-6}\int e_{\tilde{k}_{[12345]}}\psi_0(k_{[123]}, \tilde{k}_{[45]})$$


$$W(d\eta_{12345}),$$

$$I_t^2 = 2^{-6}\int\int e_{\tilde{k}_{[234]}}\psi_0(k_{[123]}, \tilde{k}_4 - k_1)$$


$$dk_1 W(d\eta_{234}),$$

$$I_t^3 = 2^{-6}\int\int e_{\tilde{k}_3}\psi_0(k_{[123]}, \tilde{k}_{[12]})$$


$$dk_{12} W(d\eta_3),$$

and for $i = 1, 2, 3$, $J_t^i$ is defined similarly as $I_t^i$ with each $k_{[123]}, e_{\tilde{k}_{[12345]}}, e_{\tilde{k}_{[234]}}, e_{\tilde{k}_3}$ replaced by $\tilde{k}_{[123]}, e_{\tilde{\tilde{k}}_{[12345]}}, e_{\tilde{\tilde{k}}_{[234]}}, e_{\tilde{\tilde{k}}_3}$, respectively.

*Terms in the first chaos*: We consider $J_t^3$. $I_t^3$ can be estimated similarly. We decompose $J_t^3 = J_t^{31} + J_t^{32}$, with $J_t^{31}, J_t^{32}$ associated with the terms that $\tilde{\tilde{k}}_3 \neq k_3$

and $\tilde{\tilde{k}}_3 = k_3$, respectively. For $J_t^{31}$, we have

$$
\begin{aligned}
&E\big[|\Delta_q J_t^{31}|^2\big] \\
&\quad\lesssim \int \frac{1_{|k_3|_\infty \le N}}{|k_3|^2}\theta\big(2^{-q}\tilde{\tilde{k}}_3\big)\bigg(\int \frac{1_{|k_1|\lesssim N, |k_2|\lesssim N}}{|k_1|^2|k_2|^2(|k_1|^2+|k_2|^2+|\tilde{k}_{[123]}|^2)}\, dk_{12}\bigg)^2 dk_3 \\
&\quad\lesssim \varepsilon^\kappa 2^{q(1+3\kappa)}.
\end{aligned}
$$

Here, we used that $|\tilde{\tilde{k}}_3| \backsimeq 2^q \backsimeq N$ in the last inequality. For $J_t^{32}$, we consider

$$
J_t^{32} - \tilde{J}_t^{32} + \tilde{J}_t^{32} - C_2^\varepsilon(t)\dot{|}(t),
$$

where $\tilde{J}_t^{32}$ is defined as $J_t^{32}$ with $P^\varepsilon_{\sigma-s_3}(k_3)$ replaced by $P^\varepsilon_{t-s_3}(k_3)$ and $C_2^\varepsilon(t) = \frac{1}{2}(C_{12}^\varepsilon + \varphi_2^\varepsilon(t))$.

Since $\int |P^\varepsilon_{t-s_3}(k_3) - P^\varepsilon_{\sigma-s_3}(k_3)|^2\, ds_3 \le C\frac{(t-\sigma)^{\kappa/2}}{|k_3|^{2-\kappa}}$, by a straightforward calculation we obtain that for $\kappa > 0$ small enough

$$
\begin{aligned}
&E\big[\big|\Delta_q\big(J_t^{32} - \tilde{J}_t^{32}\big)\big|^2\big] \\
&\quad\lesssim \int \theta\big(2^{-q}k_3\big)^2 \frac{1}{|k_3|^{2-4\kappa}} \\
&\qquad\times \bigg(\int_0^t \int \frac{e^{-(|\tilde{k}_{[123]}|^2+|k_1|^2+|k_2|^2)\bar{c}_f(t-\sigma)}}{|k_1|^2|k_2|^2}(t-\sigma)^\kappa 1_{|k_{[123]}|\backsimeq N}\, dk_{12}\, d\sigma\bigg)^2 dk_3 \\
&\quad\lesssim \varepsilon^\kappa 2^{q(1+5\kappa)}.
\end{aligned}
$$

Here, in the last inequality we used that $|k_{123}| \backsimeq N$ implies that $|k_i| \backsimeq N$ for some $i \in \{1,2,3\}$ and that $\sup_{a\ge 0} a^r e^{-a} \le C$ for $r \ge 0$ and Lemma 6.1. Moreover, by Lemmas 6.2 and 6.3 we obtain that

$$
\begin{aligned}
&E\big[\big|\Delta_q\big(\tilde{J}_t^{32} - \dot{|}(t)C_2^\varepsilon(t)\big)\big|^2\big] \\
&\quad\lesssim \int \frac{1}{|k_3|^2}\theta\big(2^{-q}k_3\big)\bigg(\int\int_0^t |\tilde{k}_{[12]}|^{-\kappa}|k_3|^\kappa \\
&\qquad\times \frac{e^{-|k_1|^2(t-\sigma)\bar{c}_f - |k_2|^2(t-\sigma)\bar{c}_f}}{|k_1|^2|k_2|^2}\, dk_{12}\, d\sigma\bigg)^2 dk_3 \\
&\quad\lesssim \int \theta\big(2^{-q}k_3\big)\frac{1}{|k_3|^{2-2\kappa}}\, dk_3 \bigg[\int_{|k_{[12]}|\le N} \frac{1}{|\tilde{k}_{[12]}|^\kappa |k_{[12]}|^3}\, dk_{[12]} \\
&\qquad + \varepsilon^{\kappa/2}\int_{|k_{[12]}|>N} \frac{1}{|\tilde{k}_{[12]}|^{3+\kappa/2}}\, dk_{[12]}\bigg]^2 \\
&\quad\lesssim \varepsilon^\kappa 2^{q(1+2\kappa)},
\end{aligned}
$$

where in the last inequality we used that if $|k_{[12]}| \leq N$, then $|\tilde{k}_{[12]}| \backsimeq N$.

Terms in the third and fifth chaos can be estimated similarly as done for the case of $\pi_{0,\diamond}( \ , \ )$ and we also obtain that there exist $\kappa, \epsilon, \gamma > 0$ small enough such that for any $t_1, t_2 \in [0, T]$,

$$E\big[\big|\Delta_q\big(\pi_{0,\diamond}( \ , \ )(t_1) - \pi_{0,\diamond}( \ , \ )(t_2)\big)\big|^2\big] \\ \lesssim \varepsilon^{\gamma}|t_1 - t_2|^{\kappa} 2^{q(1+\epsilon)},$$

which by Gaussian hypercontractivity and Lemma 2.1 implies that for every $\delta > 0$, $p > 1$, $\pi_{0,\diamond}( \ , \ ) \to 0$ in $L^p(\Omega; C_T\mathcal{C}^{-\frac{1}{2}-\delta})$.

6.3. *Convergence of random operators*. The purpose of this subsection is to prove that $A_N$ defined in Lemma 4.2 converges to zero in probability. Here, we follow essentially the same arguments as in [13], Section 10.2.

THEOREM 6.7. *For every $T \geq 0$, $0 < \eta < \kappa/2$, $r \geq 1$, we have*

$$E\big[(A_N)^r\big]^{1/r} \lesssim N^{-\frac{\kappa}{2}+\eta}.$$

*Here, $\kappa$ is fixed in Section 4.*

To prove Theorem 6.7, we use similar arguments as in [13], Section 10.2, and obtain the following two lemmas.

LEMMA 6.8. *We have*

$$\big(A_N^1 + A_N^2\big)( \ + \ , \ + \ )(f)(t,x) \\ = \sum_{p,q} \int_{\mathbb{T}^3} g_{p,q}^N(t,x,y)\Delta_p f(y)\,dy$$

*with*

$$\mathcal{F}g_{p,q}^N(t,x,\cdot)(k) = \sum_{k_1,k_2} \Gamma_{p,q}^N(x,k,k_1,k_2)\mathcal{F}( \ + \ )(t,k_1)\mathcal{F}( \ + \ )(t,k_2).$$

*Here,*

$$\Gamma_{p,q}^N(x,k,k_1,k_2) \\ = 2^{-9/2}e^{\imath(k_1+k_2-k)\pi x}\theta_q(k_1+k_2-k)\tilde{\theta}_p(k)\psi_<(k,k_1)\psi_0(k_1-k,k_2) \\ \times \big(-1_{|k_1-k|_\infty > N}1_{|k_1|_\infty \leq N} + 1_{|k_1-k|_\infty \leq N}1_{N<|k_1|_\infty \leq 3N}\big),$$

*with $\tilde{\theta}_p$ being a smooth function supported in an annulus $2^p\mathcal{A}$ such that $\tilde{\theta}_p\theta_p = \theta_p$.*

LEMMA 6.9. *For all* $r \geq 1, \kappa > 0$, *we have for* $A^1_N := A^1_N( \; + \; , \; + \; )$, $A^2_N := A^2_N( \; + \; , \; + \; )$:

$$E\big[\big\| A^1_N(t) + A^2_N(t) - \big(A^1_N(s) + A^2_N(s)\big)\big\|^r_{L(\mathcal{C}^{1-3\kappa}, B^{-\frac{1}{2}-4\kappa}_{r,r})}\big]$$

$$\lesssim \sum_{p,q} 2^{qr(-\frac{1}{2}-4\kappa)} 2^{-pr(1-3\kappa)}$$

$$\times \Big(\sup_{x\in\mathbb{T}^3} \sum_k E\big[\big|\big(\mathcal{F}g^N_{p,q}(t,x,\cdot) - \mathcal{F}g^N_{p,q}(s,x,\cdot)\big)(k)\big|^2\big]\Big)^{r/2}.$$

LEMMA 6.10. *For all* $p, q \geq -1$, *all* $0 \leq t_1 < t_2$, *and all* $\lambda, \kappa \in (0,1]$, *we have*

$$\sum_k E\big[\big|\big(\mathcal{F}g^N_{p,q}(t_2,x,\cdot) - \mathcal{F}g^N_{p,q}(t_1,x,\cdot)\big)(k)\big|^2\big]$$

$$\lesssim 1_{2^p,2^q \lesssim N}\big(2^{3p}2^{2q} + 2^{2p}2^{3q}\big) N^{-2+2\lambda+\kappa} |t_1 - t_2|^{\lambda}.$$

PROOF. We only prove the estimate for $\sum_k E[|\mathcal{F}g^N_{p,q}(t,x,\cdot)(k)|^2])$. The above assertion can be obtained by essentially the same arguments. First, we consider the term in $A_N$ containing and . We have the following chaos decomposition:

$$\mathcal{F}\,(t,l_1)\mathcal{F}\,(t,l_2) = I^1_t + 4I^2_t + 2I^3_t.$$

Here,

$$I^1_t = 2^{-3} \int 1_{k_{[12]}=l_1, k_{[34]}=l_2} \int_0^t d\sigma\, p^{\varepsilon}_{t-\sigma}(k_{[12]})\varphi(\varepsilon k_{[12]}) P^{\varepsilon}_{\sigma-s_1}(k_1)$$
$$\times P^{\varepsilon}_{\sigma-s_2}(k_2) P^{\varepsilon}_{t-s_3}(k_3) P^{\varepsilon}_{t-s_4}(k_4) W(d\eta_{1234}),$$
$$I^2_t = 2^{-3} \int 1_{k_{[12]}=l_1, k_3-k_1=l_2} \int_0^t d\sigma\, p^{\varepsilon}_{t-\sigma}(k_{[12]})\varphi(\varepsilon k_{[12]})$$
$$\times P^{\varepsilon}_{\sigma-s_2}(k_2) P^{\varepsilon}_{t-s_3}(k_3) V^{\varepsilon}_{t-\sigma}(k_1)\, dk_1 W(d\eta_{23}),$$
$$I^3_t = 2^{-3} \int \int_0^t d\sigma\, 1_{k_{[12]}=l_1, -k_{[12]}=l_2} V^{\varepsilon}_{t-\sigma}(k_1) V^{\varepsilon}_{t-\sigma}(k_2) p^{\varepsilon}_{t-\sigma}(k_{[12]})\varphi(\varepsilon k_{[12]})\, dk_{12}.$$

The graph for $I^i_t, i = 1,2,3$ is similar as that of $\pi_0( \; , \; )$ and we omit them here.

*Term in the chaos of order* 0: By a similar calculation as in Section 6.1, we have

$$\sum_k \Big|\sum_{k_1,k_2} \Gamma^N_{p,q}(x,k,k_1,k_2)1_{k_1+k_2=0}I^3_t\Big|^2$$
$$\lesssim \sum_k \Big|\sum_{k_1} \Gamma^N_{p,q}(x,k,k_1,-k_1)\frac{1}{|k_1|^3}\Big|^2$$
$$\lesssim \sum_k \tilde{\theta}_p(k)^2\theta_q(-k)^2\Big|\sum_{k_1}(1_{|k_1-k|_\infty>N}1_{|k_1|_\infty\le N}+1_{|k_1-k|_\infty\le N}1_{N<|k_1|_\infty\le 3N})$$
$$\times\,\psi_<(k,k_1)\psi_0(k_1-k,k_1)\frac{1}{|k_1|^3}\Big|^2.$$

In the first case without loss of generality, we may assume that $|k_1^i-k^i|>N$ for some $i$. Then there are at most $|k^i|$ values of $k_1^i$ with $|k_1^i|\le N$ and $|k_1^i-k^i|>N$. In the second case for $1_{|k_1-k|_\infty\le N}1_{N<|k_1|_\infty\le 3N}$ without loss of generality, we may assume that $|k_1^i|>N$ for some $i$. Then there are at most $|k^i|$ values of $k_1^i$ with $|k_1^i|>N$ and $|k_1^i-k^i|\le N$. Moreover, observe that $|k_1|\simeq N$ on the support of $(1_{|k_1-k|_\infty>N}1_{|k_1|_\infty\le N}+1_{|k_1-k|_\infty\le N}1_{|k_1|_\infty>N})\psi_0(k-k_1,k_1)$ and that $|k|\lesssim N$ whenever $1_{|k_1|_\infty\le 3N}\psi_<(k,k_1)\neq 0$, which implies that the above term is bounded by

$$\sum_k \tilde{\theta}_p(k)^2\theta_q(-k)^2|k|^2 1_{|k|\lesssim N}N^{-2}\lesssim 1_{2^p,2^q\lesssim N}2^{3p}2^{2q}N^{-2}.$$

*Term in the second chaos*: By a similar calculation as in Section 6.1, we have

$$\sum_k E\Big|\sum_{l_1,l_2}\Gamma^N_{p,q}(x,k,l_1,l_2)I^2_t\Big|^2$$
$$\lesssim \sum_k 1_{2^p,2^q\lesssim N}\tilde{\theta}_p(k)^2\int \theta_q(k_{[23]}-k)^2$$
$$\times\prod_{i=2}^3\frac{1}{|k_i|^2}\bigg[\int \psi_<(k,k_{[12]})\frac{1}{(|k_{[12]}|^2+|k_1|^2)|k_1|^2}$$
$$\times\,(1_{|k_{[12]}-k|_\infty>N,|k_{[12]}|_\infty\le N}+1_{|k_{[12]}-k|_\infty\le N,N<|k_{[12]}|_\infty\le 3N})\,dk_1\bigg]^2\,dk_{23}$$
$$\lesssim \sum_k 1_{2^p,2^q\lesssim N}\tilde{\theta}_p(k)^2\int \theta_q(k_{[23]}-k)^2\frac{1}{|k_{[23]}|}N^{-2+\kappa}\,dk_{[23]}$$
$$\lesssim \sum_k 1_{2^p,2^q\lesssim N}\tilde{\theta}_p(k)^2N^{-2+\kappa}2^{2q}\lesssim 1_{2^p,2^q\lesssim N}2^{3p}2^{2q}N^{-2+\kappa}.$$

Here, in the second inequality we used that $|k_{[12]}| \simeq N$ on the support of $1_{|k_{[12]}-k|_\infty > N, |k_{[12]}|_\infty \le N} \psi_<(k, k_{[12]})$ and in the third inequality we used Lemma 6.5.

*Term in the fourth chaos*: We have

$$\begin{aligned}
&\sum_k E\Big|\sum_{l_1,l_2} \Gamma^N_{p,q}(x,k,l_1,l_2) I^1_t\Big|^2 \\
&\quad\lesssim \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(k_{[1234]}-k)^2 \psi_<(k,k_{[12]})^2 \psi_0(k_{[12]}-k,k_{[34]})^2 \\
&\qquad\times \frac{1_{2^p,2^q \lesssim N}}{|k_2|^2|k_3|^2|k_1|^2|k_4|^2|k_{[12]}|^4} \\
&\qquad\times (1_{|k_{[12]}-k|_\infty > N, |k_{[12]}|_\infty \le N} + 1_{|k_{[12]}-k|_\infty \le N, N < |k_{[12]}|_\infty \le 3N})\, dk_{1234} \\
&\quad\lesssim \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(k_{[1234]}-k)^2 \frac{1_{2^p,2^q \lesssim N}}{|k_{[1234]}|^{1-\kappa}}\, dk_{[1234]} N^{-2-\kappa} \\
&\quad\lesssim 1_{2^p,2^q \lesssim N} \sum_k \tilde{\theta}_p(k)^2 N^{-2-\kappa} 2^{(2+\kappa)q} \\
&\quad\lesssim 1_{2^p,2^q \lesssim N} 2^{3p} 2^{2q} N^{-2},
\end{aligned}$$

where we used Lemma 6.1 and that $|k_{[12]}| \simeq N$ in the second inequality.

Moreover, we consider

$$\mathcal{F}(t,l_1)\mathcal{F}(t,l_2) = J^1_t + 4J^2_t + 2J^3_t.$$

Here, $J^i_t, i = 1, 2$, is defined similarly as $I^i_t, i = 1, 2$, with $k_{[12]}$ replaced by $\tilde{k}_{[12]}$ and

$$J^3_t = 2^{-3} \int \int_0^t d\sigma\, 1_{\tilde{k}_{[12]} = l_1, -k_{[12]} = l_2} V^\varepsilon_{t-\sigma}(k_1) V^\varepsilon_{t-\sigma}(k_2) p^\varepsilon_{t-\sigma}(\tilde{k}_{[12]}) \varphi(\varepsilon \tilde{k}_{[12]})\, dk_{12}.$$

*Terms in the chaos of order* 0: We have

$$\begin{aligned}
&\sum_k \Big|\sum_{k_1,k_2} \Gamma^N_{p,q}(x,k,\tilde{k}_1,k_2) 1_{k_1+k_2=0} J^3_t\Big|^2 \\
&\quad\lesssim \sum_k \Big|\sum_{k_1} \Gamma^N_{p,q}(x,k,\tilde{k}_1,-k_1) 1_{|k_1|_\infty \lesssim N} \frac{1}{|k_1||\tilde{k}_1|^2}\Big|^2 \\
&\quad\lesssim \sum_k \tilde{\theta}_p(k)^2 \theta_q(\tilde{k})^2 \Big|\sum_{k_1} (1_{|\tilde{k}_1-k|_\infty > N} 1_{|\tilde{k}_1|_\infty \le N} + 1_{|\tilde{k}_1-k|_\infty \le N} 1_{N < |\tilde{k}_1|_\infty \le 3N}) \\
&\qquad\times \psi_<(k,\tilde{k}_1) 1_{|k_1|_\infty \lesssim N} \frac{1}{|k_1||\tilde{k}_1|^2}\Big|^2.
\end{aligned}$$

Similarly, as above we obtain that there are at most $|k^i|$ values of $\tilde{k}_1^i$ with $|\tilde{k}_1^i| > N$ and $|\tilde{k}_1^i - k^i| \leq N$ or $|\tilde{k}_1^i| \leq N$ and $|\tilde{k}_1^i - k^i| > N$. Moreover, observe that $|\tilde{k}_1| \simeq N$ on the support of $1_{|\tilde{k}_1 - k|_\infty > N} 1_{|\tilde{k}_1|_\infty \leq N} \psi_<(k, \tilde{k}_1)$ and that $|k| \lesssim N$ whenever $1_{|k_1|_\infty < 3N} \psi_<(k, k_1) \neq 0$, which implies that the above term is bounded by

$$\sum_k \tilde{\theta}_p(k)^2 \theta_q(\tilde{k})^2 |k|^2 1_{|k| \lesssim N} N^{-2} \lesssim 1_{2^p, 2^q \lesssim N} 2^{2p} 2^{3q} N^{-2}.$$

For the terms in the second chaos by a similar calculation as above, we obtain the desired estimates.

*Terms in the fourth chaos*: We have

$$\begin{aligned}
&\sum_k E\Big|\sum_{l_1, l_2} \Gamma_{p,q}^N(x, k, l_1, l_2) J_t^1\Big|^2 \\
&\quad \lesssim \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(\tilde{k}_{[1234]} - k)^2 \psi_<(k, \tilde{k}_{[12]})^2 \psi_0(\tilde{k}_{[12]}, k_{[34]})^2 \\
&\qquad \times \frac{1_{2^p, 2^q \lesssim N}}{|k_2|^2 |k_3|^2 |k_1|^2 |k_4|^2 |\tilde{k}_{[12]}|^4} 1_{|k_{[12]}| \lesssim N, |k_{[34]}| \lesssim N} \, dk_{1234} \\
&\qquad \times (1_{|\tilde{k}_{12} - k|_\infty > N, |\tilde{k}_{12}|_\infty \leq N} + 1_{|\tilde{k}_{12} - k|_\infty \leq N, N < |\tilde{k}_{12}|_\infty \leq 3N}) \\
&\quad \lesssim 1_{2^p, 2^q \lesssim N} \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(\tilde{k}_{[1234]} - k)^2 \\
&\qquad \times \frac{1}{|k_1|^2 |k_2|^2 |k_3|^2 |k_4|^2} N^{-4} 1_{|k_{[12]}| \lesssim N, |k_{[34]}| \lesssim N} \, dk_{1234} \\
&\quad \lesssim 1_{2^p, 2^q \lesssim N} \sum_k \tilde{\theta}_p(k)^2 \int \theta_q(\tilde{k}_{[1234]} - k) \frac{1}{|k_{[12]}|^2 |k_{[34]}|^2} \, dk_{[12][34]} N^{-2} \\
&\quad \lesssim 1_{2^p, 2^q \lesssim N} 2^{3p} 2^{2q} N^{-2}.
\end{aligned}$$

Here, in the second inequality we used that $|\tilde{k}_{[12]}| \backsimeq N$ and in the third inequality we used that $N^{-1} \lesssim |k_{[12]}|^{-1}$, $N^{-1} \lesssim |k_{[34]}|^{-1}$ and in the last inequality we used Lemma 6.5.

Furthermore, for the terms associated with and we obtain the desired estimates by similar arguments. Thus the result follows. $\square$

PROOF OF THEOREM 6.7. For $t, s \geq 0, r > 0$ large enough, we have for $A^1_N := A^1_N( \; + \; , \; + \; )$, $A^2_N := A^2_N( \; + \; , \; + \; )$:

$$
\begin{aligned}
&E\big[\big\|(A^1_N(t) + A^2_N(t) - \big(A^1_N(s) + A^2_N(s)\big)\big\|^r_{L(\mathcal{C}^{1-3\kappa},\mathcal{C}^{-\frac{1}{2}-5\kappa})}\big] \\
&\quad \lesssim E\big[\big\|(A^1_N(t) + A^2_N(t) - \big(A^1_N(s) + A^2_N(s)\big)\big\|^r_{L(\mathcal{C}^{1-3\kappa},B^{-\frac{1}{2}-4\kappa}_{r,r})}\big] \\
&\quad \lesssim \sum_{p,q} 2^{qr(-\frac{1}{2}-4\kappa)} 2^{-pr(1-3\kappa)} \\
&\qquad \times 1_{2^p,2^q \lesssim N}\big[\big(2^{3p}2^{2q} + 2^{2p}2^{3q}\big)|t-s|^{\lambda} N^{-2+2\lambda+\kappa}\big]^{r/2} \\
&\quad \lesssim |t-s|^{r\lambda/2} N^{(-\kappa/2+\lambda+\delta)r}.
\end{aligned}
$$

Here, $\frac{\kappa}{2} > \delta + \lambda > 0$. Thus the result follows by using Kolmogorov's continuity criterion. □

6.4. *Convergence of $D^N$.* In this subsection, we prove that $D^N \to^P 0$ as $\varepsilon \to 0$. Here, we use the fact that there exists some $|k_j| \simeq N$ to produce $\varepsilon^\kappa$. We have the following identity for $t \in [0, T]$:

$$
\begin{aligned}
&\pi_0\big((I - P_N)\pi_<( \; , \; ), \; \big)(t) - \pi_0\big(P_N\pi_<\big( \; , (P_{3N} - P_N) \; \big), \; \big)(t) \\
&\quad = \sum_{i=1}^{4} (I^i_t + J^i_t),
\end{aligned}
$$

where

$$
\begin{aligned}
I^1_t &= 2^{-9} \int e_{k_{[1234567]}} \psi_0(k_{[12345]}, k_{[67]}) \psi_<(k_{[123]}, k_{[45]}) \\
&\qquad \times \big(1_{|k_{[12345]}|_\infty > N} 1_{|k_{[45]}|_\infty \leq N} - 1_{|k_{[12345]}|_\infty \leq N} 1_{N < |k_{[45]}|_\infty \leq 3N}\big) \\
&\qquad \times \int_0^t \int_0^t d\sigma \, d\bar{\sigma} \, P^\varepsilon_{t-\sigma}(k_{[123]}) \prod_{i=1}^{3} P^\varepsilon_{\sigma - s_i}(k_i) p^\varepsilon_{t-\bar{\sigma}}(k_{[45]}) \varphi(\varepsilon k_{[45]}) \\
&\qquad \times \prod_{i=4}^{5} P^\varepsilon_{\bar{\sigma}-s_i}(k_i) \prod_{i=6}^{7} P^\varepsilon_{t-s_i}(k_i) W(d\eta_{1234567}) \\
&\quad := \int G(t, x, \eta_{1234567}) W(d\eta_{1234567}), \\
I^2_t &= \sum_{i=1}^{3} I^{2i}_t, \qquad I^{21}_t = 6 \int \int G(t, x, \eta_{123(-3)567}) \, d\eta_3 \, W(d\eta_{12567}), \\
I^{22}_t &= 6 \int G(t, x, \eta_{12345(-3)7}) \, d\eta_3 \, W(d\eta_{12457}),
\end{aligned}
$$

$$I_t^{23} = 4\int G(t, x, \eta_{12345(-5)7})]\, d\eta_5 W(d\eta_{12347}),$$

$$I_t^3 = \sum_{i=1}^{6} I_t^{3i}, \qquad I_t^{31} = 6\int\int G(t, x, \eta_{123(-3)(-2)67})\, d\eta_{23} W(d\eta_{167}),$$

$$I_t^{32} = 24\int\int G(t, x, \eta_{123(-3)5(-2)7})\, d\eta_{23} W(d\eta_{157}),$$

$$I_t^{33} = 12\int\int G(t, x, \eta_{123(-3)5(-5)7})\, d\eta_{35} W(d\eta_{127})$$

$$I_t^{34} = 6\int\int G(t, x, \eta_{12345(-2)(-3)})\, d\eta_{23} W(d\eta_{145}),$$

$$I_t^{35} = 12\int\int G(t, x, \eta_{12345(-3)(-4)})\, d\eta_{34} W(d\eta_{125})$$

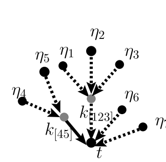
$$I_t^{36} = 2\int\int G(t, x, \eta_{12345(-4)(-5)})\, d\eta_{45} W(d\eta_{123}),$$

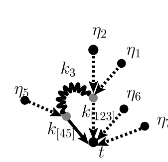
$$I_t^4 = \sum_{i=1}^{3} I_t^{4i}, \qquad I_t^{41} = 12\int\int G(t, x, \eta_{123(-1)(-2)(-3)7})\, d\eta_{123} W(d\eta_7),$$

$$I_t^{42} = 12\int\int G(t, x, \eta_{123(-3)5(-2)(-1)})\, d\eta_{123} W(d\eta_5),$$

$$I_t^{43} = 24\int\int G(t, x, \eta_{123(-3)5(-5)-2})\, d\eta_{235} W(d\eta_1),$$

and $J_t^1$ is defined similarly as $I_t^1$ with $k_{[123]}, k_{[12345]}, e_{k_{[1234567]}}$ replaced by $\tilde{k}_{[123]}, \tilde{k}_{[12345]}, e_{\tilde{k}_{[1234567]}}$, respectively, and $J_t^i, i = 2, 3, 4$ is defined similarly as $I_t^i$ with the $G$ replaced by that associated with $J^1$. For the reader's convenience, we use the graph notation to denote $G(t, x, \eta_{1234567})$ and the term for $I_t^{21}$. The graphs for other terms are similar as the graph for $I_t^{21}$ with the corresponding lines connected. Here, we omit $(I - P_N), P_N, \pi_0, \pi_<$ in the graph for simplicity:



,


.

*Terms in the seventh chaos*: We have

$$E|\Delta_q I_t^1|^2 \lesssim \int \theta(2^{-q} k_{[1234567]}) \psi_0(k_{[12345]}, k_{[67]}) \psi_<(k_{[123]}, k_{[45]})$$
$$\times (1_{|k_{[12345]}|_\infty > N, |k_{[45]}|_\infty \le N} + 1_{|k_{[12345]}|_\infty \le N, |k_{[45]}|_\infty > N})$$

$$\times 1_{|k_{[1234567]}|\lesssim N}\prod_{i=1}^{7}\frac{1}{|k_i|^2}\left(1\Big/\left(|k_{[123]}|^2|k_{[45]}|^2\left(|k_{[123]}|^2+\sum_{i=1}^{3}|k_i|^2\right)\right.\right.$$
$$\left.\left.\times\left(|k_{[45]}|^2+\sum_{i=4}^{5}|k_i|^2\right)\right)\right)dk_{1234567}.$$

Observe that $|k_{45}|_\infty \backsimeq N$ on the support of $\psi_<(k_{[123]},k_{[45]})1_{|k_{[12345]}|_\infty>N}$, which combined with Lemma 6.1 implies that the above term is bounded by

$$\int\theta(2^{-q}k_{[1234567]})1_{|k_{[45]}|_\infty\backsimeq N,2^q\lesssim N}\frac{1}{|k_{[123]}|^4|k_{[45]}|^5|k_{[67]}|}dk_{[123][45][67]}$$
$$\lesssim\int 1_{2^q\lesssim N}\theta(2^{-q}k_{[1234567]})\frac{N^{-2-\kappa}}{|k_{[12345]}|^{3-\kappa}}\frac{1}{|k_{[67]}|}dk_{[12345][67]}\lesssim\varepsilon^\kappa 2^{2q\kappa}.$$

*Terms in the fifth chaos*: Consider $I_t^{21}$ first: by the formula we know that $|k_5-k_3|\backsimeq N$, hence

$$E|\Delta_q I_t^{21}|^2$$
$$\lesssim 1_{2^q\lesssim N}\int\theta(2^{-q}k_{[12567]})\prod_{i=5}^{7}\frac{1}{|k_i|^2}\frac{1}{|k_1|^2|k_2|^2}$$
$$\times\left[\left(\int\frac{1}{|k_3|^2(|k_5-k_3|^2+|k_3|^2)(|k_{[123]}|^2+|k_5-k_3|^2)}dk_3\right)^2\right.$$
$$\left.+\left(\int\frac{1}{|k_3|^2(|k_5-k_3|^2+|k_{[123]}|^2)(|k_{[123]}|^2+|k_3|^2)}dk_3\right)^2\right]$$
$$\times 1_{\{|k_5-k_3|\backsimeq N,|k_5|\lesssim N,|k_{[12]}|\lesssim N\}}dk_{12567}$$
$$\lesssim\int\theta(2^{-q}k_{[12567]})1_{2^q\lesssim N}\frac{N^{-4+2\kappa}}{|k_{[12]}|^{3-\kappa}|k_5|^{2+2\kappa}|k_{[67]}|}dk_{[12]5[67]}\lesssim\varepsilon^\kappa 2^{2q\kappa}.$$

Here, in the first inequality we consider $\sigma>\bar\sigma$ and $\sigma\le\bar\sigma$ separately and we used that $|k_{[123]}|^2+|k_3|^2\gtrsim|k_{[12]}|^2$ in the second inequality.

Now we consider $I_t^{22}$ and in this case we have that $|k_{[45]}|\backsimeq N$, which implies that

$$E|\Delta_q I_t^{22}|^2\lesssim 1_{2^q\lesssim N}\int\theta(2^{-q}k_{[12457]})1_{|k_{45}|\backsimeq N}\frac{1}{|k_1|^2|k_2|^2|k_{[45]}|^5|k_7|^2}$$
$$\times\left(\int\frac{1}{(|k_{[123]}|^2+|k_3|^2)|k_3|^2}dk_3\right)^2dk_{12[45]7}$$
$$\lesssim 1_{2^q\lesssim N}\int\theta(2^{-q}k_{[12457]})\frac{N^{-2-\kappa}}{|k_{[12]}|^{3-\kappa}|k_{[45]}|^{3-\kappa}|k_7|^2}dk_{[12][45]7}$$
$$\lesssim\varepsilon^\kappa 2^{2q\kappa}.$$

Here, in the second inequality we used that $|k_{[123]}|^2+|k_3|^2\gtrsim|k_{[12]}|^2$.

For $I_t^{23}$ we have that $|k_{[45]}|\backsimeq N$, hence

$$\begin{aligned}E\big|\Delta_q I_t^{23}\big|^2\lesssim{}&1_{2^q\lesssim N}\int\theta\big(2^{-q}k_{[12347]}\big)1_{\{|k_{[45]}|\backsimeq N\}}\prod_{i=1}^4\frac{1}{|k_i|^2}\frac{1}{|k_7|^2}\\&\times\frac{1}{(|k_{[123]}|^2+\sum_{i=1}^3|k_i|^2)|k_{[123]}|^2}\\&\times\bigg(\int\frac{1}{(|k_{[45]}|^2+|k_5|^2)|k_5|^2}\,dk_5\bigg)^2\,dk_{12347}\\\lesssim{}&\int 1_{2^q\lesssim N}\theta\big(2^{-q}k_{[12347]}\big)\frac{1}{|k_{[1234]}|^{2-\kappa}}\frac{N^{-2+\kappa}}{|k_7|^2}\,dk_{[1234]7}\lesssim\varepsilon^\kappa 2^{2q\kappa}.\end{aligned}$$

*Terms in the third chaos*: For $I_t^{31}$, we have that

$$\begin{aligned}E\big|\Delta_q I_t^{31}\big|^2\lesssim{}&1_{2^q\lesssim N}\int\theta\big(2^{-q}k_{[167]}\big)\psi_0(k_1,k_{[67]})\psi_<(k_{[123]},k_{[23]})\\&\times(1_{|k_1|_\infty>N,|k_{[23]}|_\infty\le N}+1_{|k_1|_\infty\le N,N<|k_{[23]}|_\infty\le 3N})\\&\times\frac{1}{|k_1|^2}\frac{1}{|k_6|^2|k_7|^2}\bigg(\int\frac{1}{(|k_{[123]}|^2+|k_{[23]}|^2)|k_{[23]}|^2|k_2|^2|k_3|^2}\,dk_{23}\bigg)^2\\&\times 1_{|k_1|\lesssim N}\,dk_{167}\\\lesssim{}&1_{2^q\lesssim N}\int\theta\big(2^{-q}k_{[167]}\big)\frac{N^{-3-\kappa}}{|k_1|^{3-\kappa}|k_{[67]}|}\,dk_{1[67]}\lesssim\varepsilon^\kappa 2^{2q\kappa}.\end{aligned}$$

Here, we used that $|k_{[23]}|^2\lesssim|k_2|^2+|k_3|^2$ in the first inequality and in the second inequality we used that $|k_{[23]}|\backsimeq N$.

For $I_t^{32}$, we obtain that

$$\begin{aligned}E\big|\Delta_q I_t^{32}\big|^2\lesssim{}&1_{2^q\lesssim N}\int\theta\big(2^{-q}k_{[157]}\big)1_{\{|k_5-k_3|_\infty\backsimeq N\}}\frac{1}{|k_1|^2|k_5|^2|k_7|^2}1_{|k_5|\le N}\\&\times\bigg[\bigg(\int 1/\big(\big(|k_5-k_3|^2+|k_3|^2\big)\\&\times\big(|k_{[123]}|^2+|k_2|^2+|k_5-k_3|^2\big)|k_2|^2|k_3|^2\big)\,dk_{23}\bigg)^2\\&+\bigg(\int 1/\big(\big(|k_{[123]}|^2+|k_2|^2+|k_3|^2\big)\\&\times\big(|k_{[123]}|^2+|k_2|^2+|k_5-k_3|^2\big)|k_2|^2|k_3|^2\big)\,dk_{23}\bigg)^2\bigg]\,dk_{157}\end{aligned}$$

$$\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[157]}) N^{-3} \frac{1}{|k_1|^2 |k_5|^{3-\kappa} |k_7|^2} \, dk_{157} \lesssim \varepsilon^\kappa 2^{2q\kappa}.$$

Here, in the first inequality we consider $\sigma > \bar{\sigma}$ and $\sigma \le \bar{\sigma}$ separately. For $I_t^{33}$, we have that

$$\begin{aligned} E|\Delta_q I_t^{33}|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[127]}) 1_{|k_5 - k_3| \simeq N, |k_{[12]}| \lesssim N} \frac{1}{|k_1|^2 |k_2|^2 |k_7|^2} \\ &\quad \times \Big[ \Big( \int 1/((|k_3|^2 + |k_5 - k_3|^2 + |k_5|^2) \\ &\quad \times (|k_{[123]}|^2 + |k_5 - k_3|^2 + |k_5|^2) |k_3|^2 |k_5|^2) \, dk_{35} \Big)^2 \\ &\quad + \Big( \int 1/((|k_3|^2 + |k_{[123]}|^2) \\ &\quad \times (|k_{[123]}|^2 + |k_5 - k_3|^2 + |k_5|^2) |k_3|^2 |k_5|^2) \, dk_{35} \Big)^2 \Big] dk_{127} \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[127]}) \frac{N^{-2}}{|k_{[12]}|^{3-\kappa} |k_7|^2} \, dk_{[12]7} \lesssim \varepsilon^\kappa 2^{q\kappa}. \end{aligned}$$

Here, in the first inequality we consider $\sigma > \bar{\sigma}$ and $\sigma \le \bar{\sigma}$ separately. For $I_t^{34}$, we get that

$$\begin{aligned} E|\Delta_q I_t^{34}|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[145]}) 1_{|k_{[45]}| \simeq N} \frac{1}{|k_1|^2 |k_{[45]}|^5} \\ &\quad \times \Big( \int \frac{1_{|k_{[23]}| \lesssim N}}{(|k_{[123]}|^2 + \sum_{i=2}^3 |k_i|^2) |k_2|^2 |k_3|^2} \, dk_{23} \Big)^2 dk_{145} \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[145]}) 1_{|k_{[45]}| \simeq N} \frac{N^\kappa}{|k_{[45]}|^5 |k_1|^2} \, dk_{1[45]} \lesssim \varepsilon^\kappa 2^{q\kappa}. \end{aligned}$$

For $I_t^{35}$, we have

$$\begin{aligned} E|\Delta_q I_t^{35}|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[125]}) 1_{|k_{[45]}| \simeq N} \frac{1}{|k_1|^2 |k_2|^2 |k_5|^2} \\ &\quad \times \Big( \int \frac{1}{(|k_{[45]}|^2 + |k_4|^2)(|k_{[123]}|^2 + |k_3|^2) |k_3|^2 |k_4|^2} \, dk_{34} \Big)^2 dk_{125} \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q} k_{[125]}) \frac{N^{-2+\kappa}}{|k_5|^2 |k_{[12]}|^{3-\kappa}} \, dk_{[12]5} \lesssim \varepsilon^\kappa 2^{2q\kappa}. \end{aligned}$$

Here, in the second inequality we used $|k_{[123]}|^2 + |k_3|^2 \gtrsim |k_{[12]}|^2$. For $I_t^{36}$, we obtain that

$$E\big|\Delta_q I_t^{36}\big|^2 \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_{[123]}) \frac{1}{|k_{[123]}|^4} \times \bigg(\int (1_{|k_{[12345]}|_\infty > N} 1_{|k_{[45]}|_\infty \le N} + 1_{|k_{[12345]}|_\infty \le N} 1_{N<|k_{[45]}|_\infty \le 3N}) \Big/ \bigg(\bigg(|k_{[45]}|^2 + \sum_{i=4}^{5} |k_i|^2\bigg)|k_4|^2|k_5|^2\bigg) dk_{45}\bigg)^2 dk_{123}.$$

Now we use similar argument as in Section 6.3. For the case that $1_{|k_{[12345]}|_\infty > N} \times 1_{|k_{[45]}|_\infty \le N}$ without loss of generality we assume that $|k_{[123]}^i + k_{[45]}^i| > N$ for some $i$. Then there are at most $|k_{[123]}^i|$ values of $k_{[45]}^i$ with $|k_{[12345]}^i| > N$ and $|k_{[45]}^i| \le N$. For the case that $1_{|k_{[12345]}|_\infty \le N} 1_{N<|k_{[45]}|_\infty \le 3N}$, similarly we obtain that there are at most $|k_{[123]}^i|$ values of $k_{[45]}^i$ with $|k_{[45]}^i| > N$ and $|k_{[12345]}^i| \le N$. Thus we obtain

$$E\big|\Delta_q I_t^{36}\big|^2 \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_{[123]}) N^{-2+\kappa} \frac{1}{|k_{[123]}|^2} dk_{[123]} \lesssim \varepsilon^\kappa 2^{2q\kappa}.$$

*Terms in the first chaos*: For $I^{41}$, we obtain that

$$\begin{aligned} E\big|\Delta_q I_t^{41}\big|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_7) \frac{1}{|k_7|^2} \times \bigg[\int \frac{1_{|k_{[12]}| \simeq N}}{|k_2|^2|k_3|^2|k_1|^2(|k_{[123]}|^2 + |k_3|^2)|k_{[12]}|^2} dk_{123}\bigg]^2 dk_7 \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_7) \frac{N^{-2+\kappa}}{|k_7|^2} dk_7 \lesssim \varepsilon^\kappa 2^{2q\kappa}. \end{aligned}$$

For $I^{42}$, we have that

$$\begin{aligned} E\big|\Delta_q I_t^{42}\big|^2 &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_5) \\ &\quad \times \bigg(\int 1_{|k_5 - k_3| \simeq N} \prod_{i=1}^{3} \frac{1}{|k_i|^2} \frac{1_{\{|k_i|_\infty \le N, i=1,2,3\}}}{(|k_{[123]}|^2 + \sum_{i=1}^{2} |k_i|^2)|k_5 - k_3|^2} dk_{123}\bigg)^2 \\ &\quad \times \frac{1}{|k_5|^2} dk_5 \\ &\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_5) \frac{N^{-2+\kappa}}{|k_5|^2} dk_5 \lesssim \varepsilon^\kappa 2^{q\kappa}. \end{aligned}$$

For $I^{43}$, we get that

$$
\begin{aligned}
E|\Delta_q I_t^{43}|^2 \lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_1)\frac{1}{|k_1|^2} \\
\times \Big(\int 1_{|k_5-k_3|\simeq N} 1_{\{|k_i|_\infty \le N, i=2,3,5\}} \\
/\big(|k_2|^2|k_3|^2|k_5|^2\big(|k_{[123]}|^2+|k_2|^2+|k_3|^2\big) \\
\times \big(|k_5-k_3|^2+|k_5|^2\big)\big)\,dk_{235}\Big)^2 dk_1 \\
\lesssim 1_{2^q \lesssim N} \int \theta(2^{-q}k_1)\frac{N^{-2+\kappa}}{|k_1|^2}\,dk_1 \lesssim \varepsilon^\kappa 2^{q\kappa}.
\end{aligned}
$$

Similar arguments imply the same estimate for $J_t^i$. By a similar calculation as above, we also obtain that there exist $\kappa, \epsilon > 0$ small enough such that for any $t_1, t_2 \in [0, T]$,

$$
\begin{aligned}
&E\big[\big|\Delta_q\big(\pi_0\big((I-P_N)\pi_<( \;, \;), \big)(t_1) \\
&\quad - \pi_0\big(P_N\pi_<( \;, (P_{3N}-P_N) ), \big)(t_1) \\
&\quad - \pi_0\big((I-P_N)\pi_<( \;, \;), \big)(t_2) \\
&\quad + \pi_0\big(P_N\pi_<( \;, (P_{3N}-P_N) ), \big)(t_2)\big)\big|^2\big] \\
&\qquad \lesssim \varepsilon^\kappa |t_1-t_2|^\kappa 2^{q\epsilon}.
\end{aligned}
$$

Moreover, for the other terms in $D^N$ we can use similar calculations and Lemma 6.5 to obtain the same estimates. Then by using Gaussian hypercontractivity, Lemma 2.1 and Kolomogorov continuity criterion, we obtain that $D_N \to^P 0$ as $\varepsilon \to 0$.

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