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Fixed points and cycle structure of random permutations

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Abstract

Using the recently developed notion of permutation limits this paper derives the limiting distribution of the number of fixed points and cycle structure for any convergent sequence of random permutations, under mild regularity conditions. In particular this covers random permutations generated from Mallows Model with Kendall's Tau, μ random permutations introduced in [11], as well as a class of exponential families introduced in [15].

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1 Introduction

Study of random permutations is an area of classical interest in the intersection of Combinatorics and Probability theory. Permutation statistics of interest is indeed a long list which includes number of fixed points, cycle structure, length of longest increasing sub-sequence, number of descents, number of cycles, number of inversions, order of a permutation, etc. Most of this literature focuses on the case where the permutation π_n is chosen uniformly at random from S_n. For example it is well known that the number of fixed points of a uniformly random permutation converges to $Poi(1)$ in distribution. More generally, denoting the number of cycles of length l by $C_n(l)$, we have

$$\{C_n(1),\cdots,C_n(l)\}\stackrel{d}{\to}\{Poi(1),Poi(1/2),\cdots,Poi(1/l)\},$$

where the limiting Poisson variables are mutually independent. However, not much is known in this regard outside the realm of the uniform measure. Possibly the most widely studied non uniform probability measure on S_n is the Mallows model with Kendall's Tau, first introduced by Mallows in [13], which has a p.m.f. of the form

$$M_{n,q}(\pi)=\frac{1}{Z_{n,q}}q^{Inv(\pi_n)}. \tag{1.1}$$

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Here $Inv(\pi_n) := \sum_{1\le i<j\le n} 1\{(i-j)(\pi_n(i)-\pi_n(j)) < 0\}$ is the number of inversions in π_n, and $q > 0$ is a scalar parameter. In this case an exact formula is known for the normalizing constant $Z_{n,q}$, and expectation and variance formulas for $Inv(\pi_n)$ are easy to derive (see for e.g. [9]). In [6] Borodin et al. asked the question of behavior of permutation statistics such as cycle structure and longest increasing sub-sequence for general class of Mallows models which includes the Mallows model with Kendall's Tau. This question was partially answered by Mueller-Starr in [14], where they derived the weak law of the length of the longest increasing sub-sequence. Specifically, for the scaling $n(1-q(n)) \to \beta$ they showed that

$$\frac{1}{\sqrt{n}} LIS(\pi_n) \stackrel{p}{\to} \mathcal{L}(\beta),$$

where $\mathcal{L}(\beta) := 2\beta^{-1/2}\sinh^{-1}(\sqrt{e^\beta - 1})$ for $\beta > 0$, and $2|\beta|^{-1/2}\sin^{-1}(\sqrt{1-e^\beta})$ for $\beta < 0$. For the scaling when $n(1-q(n)) \to \infty$, it was shown by Bhatnagar-Peled ([5]) that

$$\frac{1}{n\sqrt{1-q(n)}} LIS(\pi_n) \stackrel{p}{\to} 1.$$

The more recent work of Basu-Bhatnagar ([2]) consider the case $q(n) = q \neq 1$ is fixed, and prove a weak law for $LIS(\pi_n)$ (they also derive a central limit theorem for $q < 1$). This answers the question of LIS for Mallows model with Kendall's Tau for all parameter scalings, at least at the level of weak limits. On the other hand, the question of the cycle structure still remains largely unanswered. See however the recent work of Gladkich-Peled, who derive the order of expected number of cycles in a Mallows random permutation in [10, Theorem 1.1], when the underlying parameter $q(n) \in (0,1)$ is arbitrary.

In a different direction, in [11] the authors Hoppen et al. proposed a framework where a permutation can be viewed as a measure. This is described below in brief:

For a permutation $\pi_n \in S_n$ define the measure ν_π on $[0,1]^2$ as

$$\nu_{\pi_n} := \frac{1}{n}\sum_{i=1}^n \delta_{(i/n, \pi_n(i)/n)}.$$

A sequence of permutations $\{\pi_n\}_{n\ge 1}$ with $\pi_n \in S_n$ is said to converge to a measure μ, if the sequence of probability measures ν_{π_n} converge weakly to μ. Any such limit is in $\mathcal{M}$, the set of probability distribution on the unit square with uniform marginals. Any $\mu \in \mathcal{M}$ is called a permuton (following [11]), and it is shown in [11, Theorem 1.6] that any $\mu \in \mathcal{M}$ can indeed arise as a limit of a sequence of permutations in this manner. See [3, 11] for a more detailed introduction to permutation limits.

If $\{\pi_n\}_{n\ge 1}$ is a sequence of random permutations (not necessarily in the same probability space), the sequence is said to converge to a deterministic measure $\mu \in \mathcal{M}$ in probability, if the sequence of measures ν_{π_n} converge weakly to the measure μ in probability. Equivalently, for any continuous function f on the unit square, one has

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f\Big(\frac{i}{n}, \frac{\pi_n(i)}{n}\Big) \stackrel{p}{\to} \int_{[0,1]^2} f(x,y)d\mu.$$

Using the topology of permutation limits in [15] the author gave a new proof for a large deviation principle (originally proved in [19]), and used it to analyze a class of exponential families on the space of permutations. The large deviation principle was re-derived in [12], where Kenyon et al. study permutation ensembles constrained to have fixed densities of finite number of patterns.

It was shown by Starr in [17] that if π_n is generated from a Mallows model with Kendall's Tau with parameter $q(n)$ such that $n(1-q(n)) \to \beta$, then the sequence of measures ν_{π_n} converge weakly in probability to a measure $\mu_{\rho_\beta} \in \mathcal{M}$ induced by the density

$$\rho_\beta(x,y) := \frac{(\beta/2)\sinh(\beta/2)}{e^{\beta/4}\cosh(\beta(x-y)/2) - e^{-\beta/4}\cosh(\beta(x+y-1)/2)}, \tag{1.2}$$

which is the Frank's Copula (see [16]). Since π_n converges weakly to the measure μ_{ρ_β}, in an attempt to understand the marginal distribution of $\pi_n(i)$ one might conjecture that $\mathbb{P}_n(\pi_n(i) = j) \approx \frac{1}{n}\rho_\beta(i/n, j/n)$. We will show that this is indeed true, under certain regularity of the law of the random permutations. We start by introducing some notations.

Definition 1.1. *For $l \in [n] := \{1, 2, \cdots, n\}$ let*

$$\mathcal{S}(n,l) := \{\mathbf{p} := (p_1, p_2, \cdots, p_l) \in [n]^l : p_a \neq p_b \text{ for all } a \neq b, a, b \in [l]\}.$$

Then we have $|\mathcal{S}(n,l)| = \binom{n}{l} l!$. For $\mathbf{p}, \mathbf{q} \in \mathcal{S}(n,l)$ let $||\mathbf{p} - \mathbf{q}||_\infty := \max_{a \in [l]} |p_a - q_a|$. Also for $\mathbf{p} \in \mathcal{S}(n,l)$ let $\pi_n(\mathbf{p})$ denote the vector $(\pi_n(p_1), \cdots, \pi_n(p_k))$.

For every $n \geq 1$ let π_n be a random permutation on S_n with law $\mathbb{P}_n$. In [3, Def 6.2] the authors define a notion of equi-continuity of random permutations, which they show is implied by the condition

$$\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{\mathbf{p},\mathbf{q},\mathbf{r} \in \mathcal{S}(n,l) : ||\mathbf{p}-\mathbf{r}||_\infty \leq n\delta} \Big| \frac{\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r}) = \mathbf{q})} - 1 \Big| = 0 \tag{1.3}$$

(see [3, Prop 6.2]). In particular for $l = 1$ condition (1.3) in spirit demands that the function $\mathbb{P}_n(\pi_n(p) = q)$ is equi-continuous in p. In this paper we will need an extra notion of equi-continuity which demands that the function $\mathbb{P}_n(\pi_n(p) = q)$ is jointly equi-continuous in p, q. This is stated below:

Definition 1.2. *A sequence of random permutations π_n is said to be equi-continuous in both co-ordinates if*

$$\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s} \in \mathcal{S}(n,l) : ||\mathbf{p}-\mathbf{r}||_\infty \leq n\delta, ||\mathbf{q}-\mathbf{s}|| \leq n\delta} \Big| \frac{\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r}) = \mathbf{s})} - 1 \Big| = 0. \tag{1.4}$$

Definition 1.3. *Let $\mathcal{C}$ denote the set of all strictly positive continuous functions ρ on $[0,1]^2$ with uniform marginals, i.e.*

$$\int_0^1 \rho(x,y)dx = \int_0^1 \rho(x,y)dy = 1.$$

Denote by $\mu_\rho \in \mathcal{M}$ the measure induced by ρ.

Our first theorem now proves an estimate of $\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})$ for vectors $\mathbf{p}, \mathbf{q}$ if π_n is equi-continuous in both co-ordinates, and converges in the sense of permutation limits to μ_ρ.

Theorem 1.4. *Suppose $\{\pi_n\}_{n \geq 1}$ is a sequence of random permutations with $\pi_n \in S_n$, such that the sequence is equi-continuous in both co-ordinates, i.e. it satisfies (1.4). If $\{\pi_n\}_{n \geq 1}$ converges to μ_ρ for some $\rho \in \mathcal{C}$, we have*

$$\lim_{n \to \infty} \sup_{\mathbf{p},\mathbf{q} \in \mathcal{S}(n,l)} \Big| \frac{n^l \mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})}{\prod_{a=1}^l \rho\left(\frac{p_a}{n}, \frac{q_a}{n}\right)} - 1 \Big| = 0. \tag{1.5}$$

As an immediate corollary of Theorem 1.4 we obtain limiting distribution of the vector $\pi_n(\mathbf{p})$. A more general version of this corollary was already derived in [3, Proposition 6.1].

Corollary 1.5. *Suppose* $\mathbf{p}_n \in \mathcal{S}(n,l)$ *is such that*

$$\lim_{n\to\infty} \frac{1}{n}\mathbf{p}_n = \mathbf{x} \in [0,1]^l.$$

If $\{\pi_n\}_{n\geq 1}$ *is a sequence of random permutations with* $\pi_n \in S_n$ *which satisfies* (1.5) *for some* $\rho \in \mathcal{C}$*, then*

$$\frac{1}{n}\pi_n(\mathbf{p_n}) \xrightarrow{d} \{Y(x_1), \cdots, Y(x_l)\},$$

where $\{Y(x_a)\}_{a=1}^l$ *are mutually independent with* $Y(x_a)$ *having the density* $\rho(x_a, .)$*.*

Having proved Theorem 1.4 we now turn our focus on the number of fixed points, or more generally the statistic

$$N_n(\pi_n, \sigma_n) := \sum_{i=1}^n 1\{\pi_n(i) = \sigma_n(i)\}$$

for any $\sigma_n \in S_n$, which denotes the number of overlaps between π_n and σ_n. In this notation the number of fixed points of π_n equals $N(\pi_n, e_n)$, where e_n is the identity permutation in S_n. By (1.5) $N_n(\pi_n, \sigma_n)$ is approximately the sum of n independent variables, and so should be approximately distributed as Poisson. Our next theorem confirms this conjecture, showing convergence to Poisson distribution of $N_n(\pi_n, \sigma_n)$ in distribution and in moments.

Theorem 1.6. *Suppose* $\{\pi_n\}_{n\geq 1}$ *is a sequence of random permutations with* $\pi_n \in S_n$ *which satisfies* (1.5) *for some* $\rho \in \mathcal{C}$*. If* σ_n *converges to* μ*, then* $\lim_{n\to\infty} \mathbb{E}N_n(\pi_n, \sigma_n)^k = \mathbb{E}Poi(\mu[\rho])^k$ *for any* $k \in \mathbb{N}$*, where* $\mu[\rho] := \int_{[0,1]^2} \rho(x,y)d\mu$*, and* $Poi(\lambda)$ *is the Poisson distribution with parameter* λ*. In particular this implies* $N_n(\pi_n, \sigma_n) \xrightarrow{d} Poi(\mu[\rho])$*.*

Remark 1.7. Setting $\sigma_n = e_n$ it follows by Theorem 1.6 that the number of fixed points in π_n has a limiting Poisson distribution with mean $\int_0^1 \rho(x,x)dx$, provided $\{\pi_n\}_{n\geq 1}$ satisfies (1.5) for some $\rho \in \mathcal{C}$.

The random variable $N_n(\pi_n, e_n) = \sum_{i=1}^n 1\{\pi_n(i) = i\}$ is essentially the number of cycles of length 1, and a similar intuition for Poisson approximation holds for cycles of length l for any $l \geq 1$. In order to make this precise, we introduce a few more notations.

Definition 1.8. *For any* $l \in [n]$ *setting* $\mathcal{U}(n,l) := \{\mathbf{p} \in \mathcal{S}(n,l) : p_1 = \min(p_a, a \in [l])\}$ *note that* $\mathcal{U}(n,l) \subset \mathcal{S}(n,l)$*, and* $|\mathcal{S}(n,l)| = l \times |\mathcal{U}(n,l)|$*. For* $\mathbf{p} \in \mathcal{S}(n,l)$ *let* $T(\mathbf{p}) \in \mathcal{S}(n,l)$ *denote the vector* $(p_2, p_3, \cdots, p_l, p_1)$*.*

As an example if $l = 3$ *and* $n = 6$ *then the vector* $\mathbf{p} = (2,5,4) \in \mathcal{U}(n,l)$*, as* $2 = \min(2,5,4)$*. In this case* $T(\mathbf{p}) = (5,4,2) \in \mathcal{S}(n,l)$ *but does not belong to* $\mathcal{U}(n,l)$*, as* $5 \neq \min(2,5,4)$*. Thus* T *is the shift operator which shifts every co-ordinate by* 1*.*

For any $l \geq 1$ *let*

$$C_n(l) := \sum_{\mathbf{p}\in\mathcal{U}(n,l)} 1\{\pi_n(\mathbf{p}) = T(\mathbf{p})\} = \frac{1}{l}\sum_{\mathbf{p}\in\mathcal{S}(n,l)} 1\{\pi_n(\mathbf{p}) = T(\mathbf{p})\}.$$

Then $C_n(l)$ *is the number of cycles of length* l*, where the factor* l *in the second definition accounts for the fact that every cycle is counted* l *times in the second sum. In particular we have* $C_n(1) = N_n(\pi_n, e_n)$ *to be the number of fixed points. Also let*

$$c_\rho(l) := \frac{1}{l}\int_{[0,1]^l} \rho(x_1, x_2)\cdots, \rho(x_l, x_1)dx_1\cdots dx_l.$$

The following theorem derives the limiting distribution for $C_n(l)$ under condition (1.5).

Theorem 1.9. *Suppose $\{\pi_n\}_{n\geq 1}$ is a sequence of random permutations with $\pi_n \in S/_n$ which satisfies* (1.5) *for some $\rho \in \mathcal{C}$. Then for any $\{k_1, \cdots, k_l\} \in \mathbb{N}^l$ we have*

$$\lim_{n\to\infty} \mathbb{E}\prod_{a=1}^{l} C_n(a)^{k_a} = \prod_{a=1}^{l} \mathbb{E}Poi(c_\rho(a))^{k_a}.$$

In particular this implies

$$\Big\{C_n(1), \cdots, C_n(l)\Big\} \xrightarrow{d} \Big\{Poi(c_\rho(1)), \cdots, Poi(c_\rho(l))\Big\},$$

where $\{Poi(c_\rho(i))\}_{i=1}^{l}$ are mutually independent.

Remark 1.10. Thus the number of cycles of length l has a limiting Poisson distribution with parameter $c_\rho(l)$, whenever the sequence of permutations π_n satisfies (1.5) for some $\rho \in \mathcal{C}$. In particular if π_n is uniformly random then (1.5) holds for the function $\rho \equiv 1$, in which case $c_\rho(l) = \frac{1}{l}$ for all $l \geq 1$. In this case we get back the classical result that the number of cycles of length l is asymptotically $Poi(1/l)$, and the random variables $\{C_n(1), \cdots, C_n(l)\}$ are mutually asymptotically independent for any $l \in \mathbb{N}$.

1.1 Applications

As applications of Theorem 1.6 and Theorem 1.9, we will now derive the limit distributions of the number of fixed points and cycle structures for three classes of non uniform distributions on S_n.

(i) The first result in this direction is the next corollary, which deals with the Mallows model with Kendall's Tau.

Corollary 1.11. *Suppose π_n is a random permutation on S_n generated from the Mallows model with Kendall's Tau defined in* (1.1)*, such that $n(1-q(n)) \to \beta \in (-\infty, \infty)$. In this case the following conclusions hold with ρ_β as defined in* (1.2).

(a) If $\{\sigma_n\}_{n\geq 1}$ is a sequence of non random permutations with $\sigma_n \in S_n$ converging to μ, then $N_n(\pi_n, \sigma_n)$ converges to $Poi(\mu[\rho_\beta])$ in distribution and in moments.

(b) $\Big\{C_n(1), \cdots, C_n(l)\Big\}$ converges to $\Big\{Poi(c_{\rho_\beta}(1)), \cdots, Poi(c_{\rho_\beta}(l))\Big\}$ in distribution and in moments, where $\{Poi(c_{\rho_\beta}(i))\}_{i=1}^{l}$ are mutually independent.

As an illustration of the Poisson approximation, in figure 1 we compare the histogram of the number of fixed points in a permutation of size $n = 100$ with the limiting Poisson prediction. We used 10000 independent observations from the Mallows model with Kendall's Tau with parameter $q(n) = e^{-20/n}$. From the picture it seems that the Poisson prediction is fairly accurate for $n = 100$. Since $q(n) < 1$ it is expected that this model will have more fixed points than a uniformly random permutation, which is reflected in the fact that the mean of the Poisson distribution is much larger than 1.

(ii) Another class of non uniform measures on permutations introduced by the author in [15] is the following:

For any continuous function f on the unit square, let $\mathbb{Q}_{n,\theta}$ be a one parameter exponential family with sufficient statistic

$$\sum_{i=1}^{n} f\Big(\frac{i}{n}, \frac{\pi_n(i)}{n}\Big) = n\nu_{\pi_n}[f].$$

More precisely, the p.m.f. is given by

$$\mathbb{Q}_{n,\theta,f}(\pi) = e^{n\theta\nu_{\pi_n}[f] - Z_n(f,\theta)}, \tag{1.6}$$

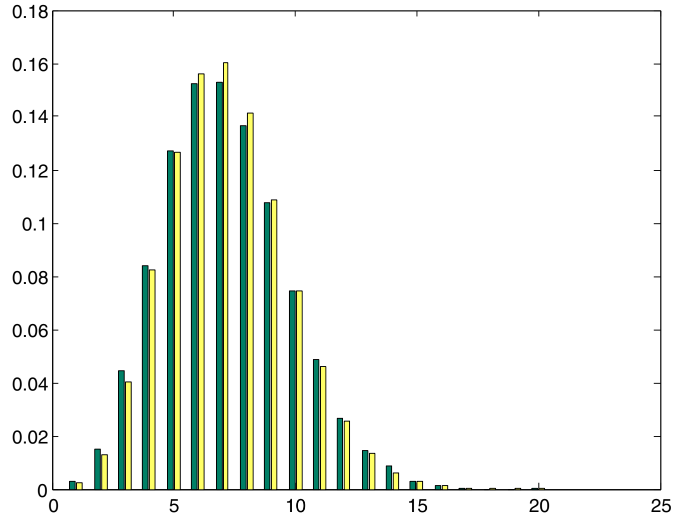

Figure 1: Bar plot of empirical distribution (from 10000 observations) of number of fixed points in a permutation of size $n = 100$ from the Mallows model with Kendall's Tau with parameter $q(n) = e^{-20/n}$ in green, compared to the Poisson prediction in yellow.

where $Z_n(f, \theta)$ is the log normalizing constant of the model. In particular the permutation model obtained by the following two specific choices have been studied in the Statistics literature:

(a) $f(x, y) = |x - y|$, which gives the statistic $\sum_{i=1}^n |\pi(i) - i|$ known as the Spearman's Footrule.
(b) $f(x, y) = (x-y)^2$, which gives the statistic $\sum_{i=1}^n (\pi(i)-i)^2$ known as Spearman's rank correlation Statistic.

See [8, Chapter 5,6] for more on these and other non uniform permutation models considered in the Statistics literature. The convergence of a sequence of random permutations π_n generated from $\mathbb{Q}_{n,f,\theta}$ of (1.6) was shown in [15, Theorem 1.4]. Building on this result, the next corollary derives the limiting distributions of the number of fixed points and cycle structure for a permutation π_n generated from this model.

Corollary 1.12. *Suppose π_n is a random permutation on S_n generated from the model $\mathbb{Q}_{n,f,\theta}$ defined in* (1.6) *for some function f which is continuous on the unit square. In this case the following conclusions hold:*

(a) The sequence $\{\pi_n\}_{n\geq 1}$ converges weakly to a non random measure $\mu_{f,\theta} \in \mathcal{M}$ with a continuous density $g_{f,\theta}(.,.)$.
(b) If $\{\sigma_n\}_{n\geq 1}$ is a sequence of non random permutations with $\sigma_n \in S_n$ converging to μ, then $N_n(\pi_n, \sigma_n)$ converges to $Poi(\mu[g_{f,\theta}])$ in distribution and in moments.
(c) $\Big\{C_n(1), \cdots, C_n(l)\Big\}$ converges to $\Big\{Poi(c_{g_{f,\theta}}(1)), \cdots, Poi(c_{g_{f,\theta}}(l))\Big\}$ in distribution and in moments, where $\{Poi(c_{g_{f,\theta}}(i))\}_{i=1}^l$ are mutually independent.

(iii) The final class of permutation models that we consider is a non parametric model with a measure as the parameter, as opposed to the previous two models which are one parameter models. This class of models will be referred to as μ random permutations, and was first introduced in [11].

Given any $\mu \in \mathcal{M}$ let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be i.i.d. random vectors with law μ. Define a permutation $\pi_n^\mu \in S_n$ as follows:

If there exists $l \in [n]$ such that $X_l = X_{(i)}, Y_l = Y_{(j)}$, then set $\pi_n^\mu(i) = j$. To visualize this definition differently, let σ_x and σ_y be the permutations of order n such that $x_{\sigma_x(1)} < x_{\sigma_x(2)} < \cdots x_{\sigma_x(n)}$ and $y_{\sigma_y(1)} < y_{\sigma_y(2)} < \cdots y_{\sigma_y(n)}$, respectively (since the marginals of μ are uniform, ties do not occur with probability 1). Then the above definiton is equivalent to setting $\pi_n^\mu = \sigma_y^{-1} \circ \sigma_x$. It is easy to see that if μ has density ρ, then for any permutation $\pi_n \in S_n$ one has

$$\mathbb{P}_n(\pi_n^\mu = \pi_n) = n! \int_{0<u_1<\cdots<u_n<1, 0<v_1<\cdots<v_n<1} \prod_{i=1}^n \rho\Big(u_i, v_{\pi_n(i)}\Big) du_i dv_i. \tag{1.7}$$

By [11, Lemma 4.2] it follows that π_n^μ converges weakly to μ in probability.

Our next corollary derives limiting distributions for μ random permutations, when the measure μ has a continuous density function with respect to Lebesgue measure.

Corollary 1.13. *Suppose π_n is a μ_ρ random permutation in S_n for some $\rho \in \mathcal{C}$. In this case the following conclusions hold:*

(a) If $\{\sigma_n\}_{n\geq 1}$ is a sequence of non random permutations with $\sigma_n \in S_n$ which converges to μ, then $N_n(\pi_n, \sigma_n)$ converges to $Poi(\mu[\rho])$ in distribution and in moments.

(b) $\Big\{C_n(1), \cdots, C_n(l)\Big\}$ converges to $\Big\{Poi(c_\rho(1)), \cdots, Poi(c_\rho(l))\Big\}$ in distribution and in moments, where $\{Poi(c_\rho(i))\}_{i=1}^l$ are mutually independent.

Even though the weak convergence of the random permutation sequence is the main ingredient in all the above results, the equi-continuity in both co-ordinates is not just a technical requirement. The following example shows that the conclusions of Theorems 1.4 and 1.6 might not hold if the equi-continuity condition fails.

Proposition 1.14. *Let $\mathbb{R}_{n,\theta}$ be a probability distributon on S_n with the p.m.f.*

$$\mathbb{R}_{n,\theta}(\pi_n) = e^{\theta N_n(\pi_n, e_n) - Z_n(\theta)}$$

where e_n is the identity permutation, and $N_n(\pi_n, e_n)$ is the number of fixed points in π_n. Then for every $\theta \neq 0$ the following conclusions hold:

(a) The random variable $N_n(\pi_n, e_n)$ converges to a Poisson random variable with mean e^θ in distribution and in moments.

(b) π_n converges weakly to u, the uniform distribution on $[0,1]^2$ which is free of θ.

(c)

$$\frac{\mathbb{R}_{n,\theta}(\pi_n(1) = 1, \pi_n(2) = 2)}{\mathbb{R}_{n,\theta}(\pi_n(1) = 2, \pi_n(2) = 1)} = e^{2\theta} \neq 1.$$

Remark 1.15. Thus even though the sequence of random permutations under $\mathbb{R}_{n,\theta}$ converge to Lebesgue measure (which is free of θ and has a continuous density), the number of fixed points has a limiting Poisson distribution which depends on θ. This is the case as equi-continuity in both coordinates does not hold here, as demonstrated by part (c) of the proposition.

1.2 Scope of future research

For the Mallows model with Kendall's Tau, the results of this paper only apply for the case $n(1-q(n)) = O(1)$. If $n(1-q(n)) \to \infty$, one should expect the number of fixed points to go to $+\infty$, and computing the weak limits/limiting distribution after centering/scaling in this case remain open. In another vein, one might expect that convergence in the sense of permutations along with "mild" regularity conditions imply the weak convergence of LIS, as worked out for the Mallows model with Kendall's Tau in [14]. Finally, computing the limiting density for the model defined in (1.6) might help give a more explicit description for the parameters of the limiting distributions of Corollary 1.12, as well as give non trivial copulas (bivariate distributions with uniform marginals) which constitute a subject area of its own in Finance.

1.3 Outline of the paper

Section 2 gives the proof of Theorem 1.4, Corollary 1.5, and Theorems 1.6 and 1.9. Section 3 concludes the paper by proving Corollaries 1.11-1.13, and Proposition 1.14.

2 Proofs of main results

2.1 Proof of Theorem 1.4 and Corollary 1.5

Proof of Theorem 1.4. For $k \in \mathbb{N}$ setting

$$\epsilon_n(k) := \sup_{\mathbf{p},\mathbf{q},\mathbf{r},\mathbf{s} \in \mathcal{S}(n,l):||\mathbf{p}-\mathbf{r}||_\infty \le n/k} \Big| \frac{\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r}) = \mathbf{s})} - 1 \Big|,$$

condition (1.4) can be stated as

$$\lim_{k\to\infty} \lim_{n\to\infty} \epsilon_n(k) = 0. \tag{2.1}$$

Fix $k \in \mathbb{N}$ and partition $(0,1]$ as $\cup_{a=1}^k I_i$ with $I_a := \left(\frac{i-1}{k}, \frac{i}{k}\right]$. Setting

$$A_n^k := \prod_{a=1}^l I_{\lceil kp_a/n \rceil}, \quad B_n^k := \prod_{a=1}^l I_{\lceil kq_a/n \rceil}$$

note that $\frac{1}{n}\mathbf{p} \in A_n^k$, $\frac{1}{n}\mathbf{q} \in B_n^k$. Now for any $\mathbf{r}, \mathbf{s} \in \mathcal{S}(n,l)$ such that $\frac{1}{n}\mathbf{r} \in A_n^k$, $\frac{1}{n}\mathbf{s} \in B_n^k$ we have

$$\Big| \frac{\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r}) = \mathbf{s})} - 1 \Big| \le \epsilon_n(k),$$

which on summing over $\mathbf{r} \in A_n^k, \mathbf{s} \in B_n^k$ and noting that the number of terms summed is at least $(n-1)^{2l}k^{-2l}$ gives

$$\begin{aligned}
\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q}) \le & (1+\epsilon_n(k)) \frac{k^{2l}}{(n-1)^{2l}} \sum_{\mathbf{r}\in A_n^k, \mathbf{s}\in B_n^k} \mathbb{P}_n(\pi_n(\mathbf{r}) = \mathbf{s}) \\
= & (1+\epsilon_n(k)) \frac{k^{2l} n^l}{(n-1)^{2l}} \mathbb{E}\nu_{\pi_n}^{(l)}[A_n^k \times B_n^k] \\
= & (1+\epsilon_n(k)) \frac{k^{2l} n^l}{(n-1)^{2l}} \mathbb{E} \prod_{a=1}^l \nu_{\pi_n}[I_{\lfloor kp_a/n \rfloor} \times I_{\lfloor kq_a/n \rfloor}],
\end{aligned}$$

where $\nu_{\pi_n}^l$ denotes the l fold product measure of ν_{π_n}. Using the fact that ρ is the density for μ_ρ this readily gives

$$\sup_{\mathbf{p},\mathbf{q}\in\mathcal{S}(n,l)} \frac{n^l\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\prod_{a=1}^l \rho(\frac{p_a}{n},\frac{q_a}{n})} \leq (1+\epsilon_n(k)) \times \frac{n^{2l}}{(n-1)^{2l}} \tag{2.2}$$

$$\times \mathbb{E} \sup_{\mathbf{i},\mathbf{j}\in[k]^l} \frac{\prod_{a=1}^l \nu_{\pi_n}[I_{i_a}\times I_{j_a}]}{\prod_{i=1}^l \mu_\rho[I_{i_a}\times I_{j_a}]} \tag{2.3}$$

$$\times \sup_{\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{w}\in[0,1]^l:||\mathbf{x}-\mathbf{z}||_\infty\leq 1/k,||\mathbf{y}-\mathbf{w}||_\infty\leq 1/k} \frac{\prod_{a=1}^l \rho(x_a,y_a)}{\prod_{a=1}^l \rho(z_a,w_a)}. \tag{2.4}$$

The term in the r.h.s. of (2.2) converges to 1 on letting $n\to\infty$ followed by $k\to\infty$, using (2.1).

Since ν_{π_n} converges to μ_ρ, by [11, Theorem 5.2] we have

$$\max_{a\in[l]}\max_{i_a\in[k]}\Big|\nu_{\pi_n}[I_{i_a}\times I_{j_a}]-\mu_\rho[I_{i_a}\times I_{j_a}]\Big| \stackrel{p}{\to} 0,$$

which along with the observation that $\mu_\rho[I_{i_a}\times I_{j_a}]$ is uniformly bounded away from 0 gives

$$\max_{a\in[l]}\max_{i_a\in[k]} \frac{\prod_{a=1}^l \nu_{\pi_n}[I_{i_a}\times I_{j_a}]}{\prod_{a=1}^l \mu_\rho[I_{i_a}\times I_{j_a}]} \stackrel{p}{\to} 1$$

as $n\to\infty$, for k fixed. An application of Dominated Convergence theorem implies that the term in (2.3) converges to 1 as well. Finally (2.4) is free of n, and converges to 1 as $k\to\infty$ by continuity of ρ. Combining this gives

$$\limsup_{k\to\infty}\limsup_{n\to\infty}\sup_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{S}(n,l):||\mathbf{p}-\mathbf{r}||_\infty\leq n/k} \frac{n^l\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\prod_{a=1}^l \rho(\frac{p_a}{n},\frac{q_a}{n})} \leq 1,$$

thus giving the upper bound in (1.5). A similar proof gives the lower bound, thus completing the proof of the theorem. □

We now introduce some auxiliary variables, to be used in the proofs of Corollary 1.5, and Theorems 1.6 and 1.9.

Definition 2.1. *For every $n\geq 1$ let $\{Z_n(1),\cdots,Z_n(n)\}$ be mutually independent random variables supported on $[n]$ such that the marginal laws are given by*

$$\mathbb{Q}_n(Z_n(p)=q)=\frac{\rho(p/n,q/n)}{\sum_{s=1}^n \rho(p/n,q/n)}$$

for some $\rho\in\mathcal{C}$. Also set

$$M_n(\sigma_n):=\sum_{p=1}^n 1\{Z_n(p)=\sigma_n(p)\},$$

and for $l\geq 1$ set

$$D_n(l):=\sum_{\mathbf{p}\in\mathcal{U}(n,l)} 1\{Z_n(\mathbf{p})=T(\mathbf{p})\}.$$

Proof of Corollary 1.5. With Z_n as constructed in definition 2.1 we have

$$\begin{aligned}
&\sum_{\mathbf{q}\in\mathcal{S}(n,l)} \Big|\mathbb{P}_n(\pi_n(\mathbf{p}_n)=\mathbf{q}) - \mathbb{Q}_n(Z_n(\mathbf{p}_n)=\mathbf{q})\Big| \\
\leq &\max_{\mathbf{p},\mathbf{q}\in\mathcal{S}(n,l)} \Big|\frac{\mathbb{P}_n(\pi_n(\mathbf{p}_n)=\mathbf{q})}{\mathbb{Q}_n(Z(\mathbf{p}_n)=\mathbf{q})} - 1\Big| \sum_{\mathbf{q}\in\mathcal{S}(n,l)} \mathbb{Q}_n(Z_n(\mathbf{p}_n)=\mathbf{q}) \\
= &\max_{\mathbf{p},\mathbf{q}\in\mathcal{S}(n,l)} \Big|\frac{\mathbb{P}_n(\pi_n(\mathbf{p}_n)=\mathbf{q})}{\mathbb{Q}_n(Z_n(\mathbf{p}_n)=\mathbf{q})} - 1\Big|,
\end{aligned}$$

which goes to 0 by (1.5). This implies that the laws of $\pi_n(\mathbf{p}_n)$ and $Z_n(\mathbf{p}_n)$ are close in total variation. Since the desired conclusion can be verified easily for $Z_n(\mathbf{p}_n)$, the proof is complete. ☐

2.2 Proofs of Theorem 1.6 and 1.9

We will use Stein's method based on dependency graphs to prove Poisson limit theorems, as explained below:

Let $\{X_\alpha\}_{\alpha\in I}$ be a finite set of Bernoulli random variables. A dependency graph for $\{X_\alpha\}_{\alpha\in I}$ is a graph with node set I and edge set E, such that if I_1, I_2 are disjoint subsets of I with no edges connecting them, then $\{X_\alpha\}_{\alpha\in I_1}$ and $\{X_\beta\}_{\beta\in I_2}$ are independent. Let N_α be the neighborhood of vertex α, i.e. $N(\alpha) := \{\beta\in I : (\alpha,\beta)\in E\}\cup\{\alpha\}$. Then one has the following Poisson approximation result, first proved in [1].

Theorem 2.2. *[7, Theorem 15] Let $\{X_\alpha\}_{\alpha\in I}$ be a finite set of Bernoulli random variables with dependency graph (I,E). Then setting $\lambda := \sum_{\alpha\in I} p_\alpha$, $W := \sum_{\alpha\in I} X_\alpha$ we have*

$$||\mathcal{L}(W) - \mathcal{L}(Poi(\lambda))||_{TV} \leq \sum_{\alpha\in I}\sum_{\beta\in N(\alpha)/\{\alpha\}} \mathbb{E}X_\alpha X_\beta + \sum_{\alpha\in I}\sum_{\beta\in N(\alpha)} \mathbb{E}X_\alpha \mathbb{E}X_\beta.$$

The following lemma uses Theorem 2.2 to prove two Poisson limits which will be used in the proofs of Theorems 1.6 and 1.9.

Lemma 2.3. *Let $M_n(\sigma_n)$ and $D_n(l)$ be as in definition 2.1.*

(a) If σ_n converges to $\mu\in\mathcal{M}$ in the sense of permutation limits, then we have $M_n(\sigma_n) \stackrel{d}{\to} P_{\mu[\rho]}$, and

$$\lim_{n\to\infty} \mathbb{E}M_n(\sigma_n)^k = \mathbb{E}Poi(\mu[\rho])^k, \text{ for all } k\in\mathbb{N}.$$

(b) For any $l\in\mathbb{N}$ we have $D_n(l) \stackrel{d}{\to} P_{c_\rho(l)}$, and

$$\lim_{n\to\infty} \mathbb{E}D_n(l)^k = \mathbb{E}Poi(c_\rho(l))^k, \text{ for all } k\in\mathbb{N}.$$

Proof. Setting $m := \inf_{0\leq x,y\leq 1}\rho(x,y)$, $M := \sup_{0\leq x,y\leq 1}\rho(x,y)$ we have $0 < m \leq M < \infty$.

(a) Since the random variables $X_p = 1\{Z_n(p) = \sigma_n(p)\}$ for $p = 1,2,\cdots,n$ are mutually independent, the dependency graph of $\{X_1, X_2, \cdots, X_n\}$ is empty. It then follows by Theorem 2.2 that

$$||\mathcal{L}(M_n(\sigma_n)) - \mathcal{L}(Poi(\lambda_n))||_{TV} \leq \sum_{p=1}^n \Big[\frac{\rho(p/n,\sigma_n(p)/n)}{\sum_{q=1}^n \rho(p/n,q/n)}\Big]^2 \leq \frac{1}{n}\times\frac{M^2}{m^2}$$

where

$$\lambda_n = \sum_{p=1}^n \frac{\rho(p/n,\sigma_n(p)/n)}{\sum_{q=1}^n \rho(p/n,q/n)} \stackrel{n\to\infty}{\to} \int_{[0,1]^2} \rho(x,y)d\mu = \mu[\rho],$$

and so $M_n(\sigma_n)$ converges to $Poi(\mu[\rho])$ in distribution. To conclude convergence in moments it suffices to show that $\limsup_{n\to\infty}\mathbb{E}M_n(\sigma_n)^k<\infty$ for every $k\in\mathbb{N}$. To see this, set

$$\widetilde{\mathcal{S}}(n,l):=\{\mathbf{p}\in\mathcal{S}(n,l):p_1<p_2<\cdots<p_l\}$$

denote the set of all n tuples in increasing order, and note that

$$\mathbb{E}M_n(\sigma_n)^k=\sum_{\mathbf{p}\in[n]^k}\mathbb{Q}_n(Z_n(\mathbf{p})=\sigma_n(\mathbf{p}))\le\sum_{l=1}^k k^l\sum_{\mathbf{p}\in\widetilde{\mathcal{S}}(n,l)}\mathbb{Q}_n(Z_n(\mathbf{p})=\sigma_n(\mathbf{p})).$$

Here the factor k^l in the r.h.s. above accounts for the fact that a specific term $\{Z_n(\mathbf{p})=\sigma_n(\mathbf{p})\}$ with $\mathbf{p}\in\widetilde{\mathcal{S}}(n,l)$ can arise from at most k^l terms in $[n]^k$. Since $|\widetilde{\mathcal{S}}(n,l)|=\binom{n}{l}$, we can bound the r.h.s. above by

$$\sum_{l=1}^k k^l\sum_{\mathbf{p}\in\widetilde{\mathcal{S}}(n,l)}\prod_{a=1}^l\frac{\rho(p_a/n,\sigma_n(p_a)/n)}{\sum_{q_a=1}^n\rho(p_a/n,q_a/n)}\le\sum_{l=1}^k\frac{k^l}{l!}\Big(\frac{M}{m}\Big)^l<\infty.$$

(b) The proof of part (b) is similar to the proof of part (a). For $\mathbf{p}\in\mathcal{U}(n,l)$ setting $X_{\mathbf{p}}=1\{Z_n(\mathbf{p})=T(\mathbf{p})\}$ note that $X_{\mathbf{p}}$ is independent of $X_{\mathbf{q}}$ whenever the indices $\mathbf{p}$ and $\mathbf{q}$ have no overlap. Thus the dependency graph of the random variables $\{X_{\mathbf{p}},\mathbf{p}\in\mathcal{U}_{n,l}\}$ has maximum degree at most $\binom{n-1}{l-1}l!$. Also for any $\mathbf{p},\mathbf{q}$ which overlap we have $\mathbb{E}X_{\mathbf{p}}X_{\mathbf{q}}=0$ unless $\mathbf{p}=\mathbf{q}$. Thus an application of Theorem 2.2 gives

$$||\mathcal{L}(D_n(l))-\mathcal{L}(P_{\lambda_n})||\le\binom{n}{l}(l-1)!\times\binom{n-1}{l-1}l!\times\frac{M^{2l}}{n^{2l}m^{2l}}\le\frac{1}{n}\times\frac{M^{2l}}{m^{2l}},$$

with

$$\lambda_n=\frac{1}{l}\sum_{\mathbf{p}\in\mathcal{S}_{n,l}}\frac{\rho(p_1/n,p_2/n)}{\sum_{q_1=1}^n\rho(p_1/n,q_1/n)}\times\cdots\times\frac{\rho(p_l/n,p_1/n)}{\sum_{q_l=1}^n\rho(p_l/n,q_l/n)}\overset{n\to\infty}{\to}c(l),$$

and so $D_n(l)$ converges to $Poi(c(l))$ in distribution. Convergence in moments follows by a similar calculation as before. □

Proof of Theorem 1.6. Let $\{Z_n(1),\cdots,Z_n(n)\}$ and $M_n(\sigma_n)$ be as defined in 2.1. Then using part (a) of Lemma 2.3 and the fact that the Poisson distribution is characterized by its moments, it suffices to show that for every $k\in\mathbb{N}$ we have

$$\lim_{n\to\infty}|\mathbb{E}N_n(\pi_n,\sigma_n)^k-\mathbb{E}M_n(\sigma_n)^k|=0.$$

To this effect setting $Z_n(\mathbf{p})=(Z_n(p_1),\cdots,Z_n(p_k))$ for $\mathbf{p}\in[n]^k$ we have

$$|\mathbb{E}N_n(\pi_n,\sigma_n)^k-\mathbb{E}M_n(\sigma_n)^k|\le\sum_{\mathbf{p}\in[n]^k}\Big|\Big\{\mathbb{P}_n\Big(\pi_n(\mathbf{p})=\sigma_n(\mathbf{p})\Big)-\mathbb{Q}_n\Big(Z_n(\mathbf{p})=\sigma_n(\mathbf{p})\Big)\Big\}\Big|$$

First note that the events $\{\pi_n(\mathbf{p})=\sigma_n(\mathbf{p})\}$ and $\{Z_n(\mathbf{p})=\sigma_n(\mathbf{p})\}$ have positive probability for all $\mathbf{p}\in[n]^k$, and so for any $\mathbf{p}\in[n]^k$ setting $L=L(\mathbf{p})$ denote the number of distinct indices gives the bound

$$\begin{aligned}&\Big|\Big\{\mathbb{P}_n\Big(\pi_n(\mathbf{p})=\sigma_n(\mathbf{p})\Big)-\mathbb{Q}_n\Big(Z_n(\mathbf{p})=\sigma_n(\mathbf{p})\Big)\Big\}\Big|\\ \le&\max_{\mathbf{p},\mathbf{q}\in\mathcal{S}(n,L)}\Big|\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{Q}_n(Z_n(\mathbf{p})=\mathbf{q})}-1\Big|\mathbb{Q}_n\Big(Z_n(\mathbf{p})=\sigma_n(\mathbf{p})\Big).\end{aligned}$$

Since $L(\mathbf{p}) \le k$, taking a maximum over L and summing over $\mathbf{p} \in [n]^k$ gives the bound

$$|\mathbb{E}N_n(\pi_n, \sigma_n)^k - \mathbb{E}M_n(\sigma_n)^k| \le \left\{ \max_{l \in [k]} \max_{\mathbf{p},\mathbf{q} \in \mathcal{S}(n,l)} \Big| \frac{\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})}{\mathbb{Q}_n(Z_n(\mathbf{p}) = \mathbf{q})} - 1 \Big| \right\} \mathbb{E}M_n(\sigma_n)^k. \quad (2.5)$$

By (1.5) we have

$$\max_{l \in [k]} \max_{\mathbf{p},\mathbf{q} \in \mathcal{S}(n,l)} \Big| \frac{\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})}{\mathbb{Q}_n(Z_n(\mathbf{p}) = \mathbf{q})} - 1 \Big| \to 0.$$

Since Lemma 2.3 implies

$$\limsup_{n \to \infty} \mathbb{E}M_n(\sigma_n)^k = \mathbb{E}Poi(\mu[\rho])^k < \infty,$$

the r.h.s. of (2.5) converges to 0 as $n \to \infty$, thus completing the proof of the theorem. □

Proof of Theorem 1.9. Let $\{Z_n(1), \cdots, Z_n(n)\}$ and $\{D_n(a), 1 \le a \le l\}$ be as defined in 2.1. Then by part (b) of Lemma 2.3, for any finite collection of non negative integers $k_1, k_2, \cdots, k_l$ we have

$$\lim_{n \to \infty} \prod_{a=1}^{l} \mathbb{E}D_n(a)^{k_a} = \prod_{a=1}^{l} \mathbb{E}Poi(c_\rho(a))^{k_a}.$$

Thus to complete the proof it suffices to show the following:

$$\lim_{n \to \infty} \left| \mathbb{E} \prod_{a=1}^{l} D_n(a)^{k_a} - \prod_{a=1}^{l} \mathbb{E}D_n(a)^{k_a} \right| = 0, \quad (2.6)$$

$$\lim_{n \to \infty} \left| \mathbb{E} \prod_{a=1}^{l} C_n(a)^{k_a} - \mathbb{E} \prod_{a=1}^{l} D_n(a)^{k_a} \right| = 0. \quad (2.7)$$

For showing (2.6) we have

$$\begin{aligned} |\mathbb{E} \prod_{a=1}^{l} D_n(a)^{k_a} - \prod_{a=1}^{l} \mathbb{E}D_n(a)^{k_a}| \le \sum_{\Gamma} \Big| &\mathbb{Q}_n\Big(\cap_{l=1}^{a} \cap_{b_a=1}^{k_a} \Big\{ Z_n(\mathbf{p}(a,b_a)) = T(\mathbf{p}(a,b_a)) \Big\} \Big) \\ &- \prod_{a=1}^{l} \mathbb{Q}_n\Big(\cap_{b_a=1}^{k_a} \Big\{ Z_n(\mathbf{p}(a,b_a)) = T(\mathbf{p}(a,b_a)) \Big\} \Big) \Big|, \end{aligned} \quad (2.8)$$

where

$$\Gamma := \Big\{ \mathbf{p}(a,b_a) \in \mathcal{U}(n,a), b_a = 1,2,\cdots,k_a, a = 1,2,\cdots,l \Big\}.$$

Proceeding to analyze a generic term in the r.h.s. of (2.8), fix

$$\mathbf{p}(a,b_a) \in \mathcal{U}(n,a), \quad 1 \le b_a \le k_a, 1 \le a \le l.$$

Let $L_a = L_a\{\mathbf{p}(a,b_a), 1 \le b_a \le k_a\}$ denote the set of distinct indices in the set $\{\mathbf{p}(a,b_a), 1 \le b_a \le k_a\}$. First note that if the sets L_a do not overlap across a, both terms in the r.h.s. of (2.8) are the same, and so gets canceled. As an example, this happens for the choice

$$l = 3, k_1 = 0, k_2 = 1, k_3 = 2, \quad \mathbf{p}(2,1) = (1,2), \quad \mathbf{p}(3,1) = (3,4,5), \quad \mathbf{p}(3,2) = (3,4,5).$$

In this case $L_1 = \phi, L_2 = \{1,2\}$ and $L_3 = \{3,4,5\}$ do not overlap, and so the corresponding terms in the r.h.s. of (2.8) get cancelled.

If the sets L_a do overlap across a, then the first term in the r.h.s. of (2.8) is 0. In this case setting $L := \sum_{a=1}^{l} |L_a|$ the total contribution of the second term in the r.h.s. of

(2.8) is bounded by $\left(\frac{M}{mn}\right)^L$. Since there is a repetition among the indices, the number of distinct indices $L(D)$ in the set $\{\mathbf{p}(a,b_a), 1\le b_a\le k_a, 1\le a\le l\}$ is strictly less than L. As an example, this happens for the choice

$$l=3, k_1=0, k_2=1, k_3=2, \quad \mathbf{p}(2,1)=(1,2), \quad \mathbf{p}(3,1)=(3,5,4), \quad \mathbf{p}(3,2)=(1,6,7).$$

In this case $L_1=\phi, L_2=\{1,2\}, L_3=\{1,3,4,5,6,7\}$, and so the number of distinct indices $L(D)=7$ which is less than $L=|L_2|+|L_3|=8$. Setting $K:=\sum_{a=1}^l k_a$, the total number of terms with exactly $L(D)$ distinct indices is at most $\binom{n}{L(D)}K!$. Summing over the possible ranges $L(D)\in[1,L-1], L\in[1,K]$ the total contribution of such terms is at most

$$\sum_{L=1}^{K}\sum_{L(D)=1}^{L-1}\binom{n}{L(D)}K!\Big(\frac{M}{mn}\Big)^L=O\Big(\frac{1}{n}\Big),$$

thus proving (2.6).

Proceeding to prove (2.7) we again have

$$|\mathbb{E}\prod_{a=1}^{l}C_n(a)^{k_a}-\mathbb{E}\prod_{a=1}^{l}D_n(a)^{k_a}|\le\sum_{\Gamma}\Big|\mathbb{P}_n\Big(\cap_{a=1}^{l}\cap_{b_a=1}^{k_a}\pi_n(\mathbf{p}(a,b_a))=T(\mathbf{p}(a,b_a))\Big)$$
$$-\mathbb{Q}_n\Big(\cap_{a=1}^{l}\cap_{b_a=1}^{k_a}Z_n(\mathbf{p}(a,b_a))=T(\mathbf{p}(a,b_a))\Big)\Big| \qquad (2.9)$$

Proceeding to bound the r.h.s. of (2.9), note that in this case if all the indices in the set $\cup_{a=1}^{l}L_a$ are not distinct (i.e. $L(D)\ne L$), then both terms in the r.h.s. of (2.9) are 0. Even if $L(D)=L$, it is possible that both terms are 0, which happens for example for the choice

$$l=3, k_1=0, k_2=1, k_3=2, \quad \mathbf{p}(2,1)=(1,2), \quad \mathbf{p}(3,1)=(3,5,4), \quad \mathbf{p}(3,2)=(3,4,5).$$

In this case $L_1=\{1,2\}, L_2=\{3,4,5\}$ and so $L(D)=L=5$. However both the terms on the r.h.s. of (2.9) have 0 probability. If either of the terms have non zero probability, then a generic term on the r.h.s. of (2.7) is of the form $|\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})-\mathbb{Q}_n(Z_n(\mathbf{p})=\mathbf{q})|$ for some $\mathbf{p},\mathbf{q}\in\mathcal{S}(n,l)$ with $l\in[L]$. Noting that $L\le K$, this can be bounded by

$$\max_{l\in[K]}\max_{\mathbf{p},\mathbf{q}\in\mathcal{S}(n,l)}\Big|\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{Q}_n(Z_n(\mathbf{p})=\mathbf{q})}-1\Big|\mathbb{Q}_n\Big(\cap_{a=1}^{l}\cap_{b_a=1}^{k_a}Z_n(\mathbf{p}(a,b_a))=T(\mathbf{p}(a,b_a))\Big)\Big|.$$

On summing over Γ using (2.9) gives

$$|\mathbb{E}\prod_{a=1}^{l}C_n(a)^{k_a}-\mathbb{E}\prod_{a=1}^{l}D_n(a)^{k_a}|\le\max_{l\in[K]}\max_{\mathbf{p},\mathbf{q}\in\mathcal{S}(n,l)}\Big|\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{Q}_n(Z_n(\mathbf{p})=\mathbf{q})}-1\Big|\mathbb{E}\prod_{a=1}^{l}D_n(a)^{k_a},$$

from which (2.7) follows on using (1.5) along with (2.6). □

3 Proof of Corollaries 1.11-1.13 and Proposition 1.14

Proof of Corollary 1.11. By [17, Theorem 1] it follows that π_n converges weakly in probability to the measure μ_{ρ_β} induced by the density ρ_β defined in (1.2). Given Theorems 1.4, 1.6 and 1.9, for proving both parts (a) and (b) it suffices to verify the equi-continuity condition (1.4), which is equivalent to the following two conditions:

$$\lim_{\delta\to0}\lim_{n\to\infty}\sup_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{S}(n,l):||\mathbf{p}-\mathbf{r}||_\infty\le n\delta}\Big|\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})}-1\Big|=0, \qquad (3.1)$$

$$\lim_{\delta\to0}\lim_{n\to\infty}\sup_{\mathbf{q},\mathbf{r},\mathbf{s}\in\mathcal{S}(n,l):||\mathbf{q}-\mathbf{s}||_\infty\le n\delta}\Big|\frac{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{s})}-1\Big|=0. \qquad (3.2)$$

Recall that (3.1) was already verified in [3, Corollary 6.3+Lemma 7.1]. By repeating the argument presented there, we prove both (3.1) and (3.2) here for completeness. To show (3.1), fix $\mathbf{p},\mathbf{q},\mathbf{r}$ such that $\|\mathbf{p}-\mathbf{r}\|_\infty \le n\delta$. Let $\Omega(\mathbf{p},\mathbf{q})$ denote the set of all permutations in S_n such that $\pi_n(\mathbf{p})=\mathbf{q}$, and $\Omega(\mathbf{r},\mathbf{q})$ be defined likewise. We will now define a bijection $\Phi=\Phi[(\mathbf{p},\mathbf{q});(\mathbf{r},\mathbf{q})]$ from $\Omega(\mathbf{p},\mathbf{q})$ to $\Omega(\mathbf{r},\mathbf{q})$. For any $\pi_n\in\Omega(\mathbf{p},\mathbf{q})$ set

$$\Phi(\pi_n)(\mathbf{r})=\mathbf{q},\quad \Phi(\pi_n)(\mathbf{p}):=\pi_n(\mathbf{r}),\quad \Phi(\pi_n)(i)=\pi_n(i)\text{ otherwise.}$$

It is easy to see that Φ is indeed a bijection, and

$$\frac{M_{n,q(n)}(\pi_n)}{M_{n,q(n)}(\Phi(\pi_n))}=q(n)^{Inv(\pi_n)-Inv(\Phi(\pi_n))}\le \max\Big(q(n),q(n)^{-1}\Big)^{nl\delta},$$

where we use the fact that the inversion status of a pair (i,j) in π_n is the same as its inversion status in $\Phi(\pi_n)$ unless $i\in\cup_{a=1}^l[p_a,r_a]$ and $j\in\mathbf{q}$. Summing over $\pi_n\in\Omega(\mathbf{p},\mathbf{q})$ gives

$$\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})}\le \max\Big(q(n),q(n)^{-1}\Big)^{nl\delta},$$

and since the bound in the r.h.s. above is free of $\mathbf{p},\mathbf{q},\mathbf{r}$, taking a sup gives

$$\sup_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{S}(n,l):||\mathbf{p}-\mathbf{r}||_\infty\le n\delta}\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})}\le \max\Big(q(n),q(n)^{-1}\Big)^{nl\delta}.$$

On letting $n\to\infty$ followed by $\delta\to 0$ and noting that $n(1-q(n))\to\beta\in(-\infty,\infty)$, we get

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\sup_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{S}(n,l):||\mathbf{p}-\mathbf{r}||_\infty\le n\delta}\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})}\le 1,$$

thus giving the upper bound in (3.1). By symmetry we have

$$\liminf_{\delta\to 0}\liminf_{n\to\infty}\sup_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{S}(n,l):||\mathbf{p}-\mathbf{r}||_\infty\le n\delta}\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})}\ge 1,$$

thus giving the lower bound, and hence proving (3.1). For proving (3.2) a similar argument works, except now we set up the bijection $\widetilde{\Phi}_n=\widetilde{\Phi}_n[(\mathbf{r},\mathbf{q});(\mathbf{r},\mathbf{s})]$ between $\Omega_{\mathbf{r},\mathbf{q}}$ to $\Omega_{\mathbf{r},\mathbf{s}}$ by setting

$$\widetilde{\Phi}(\pi_n)(\mathbf{r})=\mathbf{s},\quad \widetilde{\Phi}(\pi_n)(\pi_n^{-1}\mathbf{s}):=\mathbf{q},\quad \Phi(\pi_n)(i)=\pi_n(i)\text{ otherwise.}$$

The rest of the argument repeats itself, and we omit the details. □

Proof of Corollary 1.12. (a) It follows from [15, Theorem 1.4] that π_n converges to a unique measure $\mu_{f,\theta}$ weakly in probability, which is the solution of the optimization problem

$$\mu\mapsto\{\theta\mu[f]-D(\mu||u)\},$$

where u is the uniform measure on the unit square, and $D(.||.)$ is the Kullback Leibler divergence. It was further shown there that $\mu_{f,\theta}$ has a density of the form $g_{f,\theta}(x,y)=e^{\theta f(x,y)+a_{f,\theta}(x)+b_{f,\theta}(y)}$, where $a_{f,\theta}(.)$ and $b_{f,\theta}(.)$ are unique almost surely. To complete the proof of part (a), it suffices to show that the function $g_{f,\theta}$ is continuous on the unit square, or equivalently that $e^{-a_{f,\theta}(.)}$ is continuous. To this effect, using the fact that $\mu_{f,\theta}$ has uniform marginals, we have

$$e^{-a_{f,\theta}(x)}=\int_0^1 e^{\theta f(x,y)+b_{f,\theta}(y)}dy,$$

which readily gives

$$\int_0^1 e^{b_{f,\theta}(y)}dy \le e^{-a_{f,\theta}(x)-\inf_{x,y\in[0,1]}\{\theta f(x,y)\}}$$

for almost all $x \in [0,1]$, and consequently $e^{b_{f,\theta}(.)}$ is integrable. But then we have

$$\left|e^{-a_{f,\theta}(x_1)} - e^{-a_{f,\theta}(x_2)}\right| \le \sup_{y\in[0,1]} \left|e^{\theta f(x_1,y)} - e^{\theta f(x_2,y)}\right| \int_0^1 e^{b_{f,\theta}(y)}dy,$$

from which continuity of $e^{-a_{f,\theta}(.)}$ follows from continuity of $f(.,.)$.

(b),(c) As in the proof of Corollary 1.11 it suffices to verify the conditions (3.1) and (3.2). Using the same notations as in the proof of Corollary 1.11, we have

$$\frac{\mathbb{Q}_{n,f,\theta}(\pi_n)}{\mathbb{Q}_{n,f,\theta}(\Phi(\pi_n))} = e^{\theta\sum_{a=1}^{l} f(p_a/n,q_a/n)-f(r_a/n,q_a/n)},$$

and the exponent in the r.h.s. above is bounded by

$$|\theta| \sup_{x_1,x_2,y\in[0,1]:|x_1-x_2|\le\delta} |f(x_1,y) - f(x_2,y)|.$$

Since this goes to 0 as $\delta \to 0$, a similar proof as before verifies (3.1). The proof of (3.2) is similar, and again we omit the details. □

Proof of Corollary 1.13. Since a sequence of μ_ρ random permutations converge to μ_ρ weakly in probability, it suffices to verify (3.1) and (3.2).

To this effect, with $(X_1,Y_1),\cdots,(X_n,Y_n) \overset{i.i.d.}{\sim} \mu_\rho$ first note that marginally both $(X_1,\cdots,X_n)$ and $(Y_1,\cdots,Y_n)$ are i.i.d. $U(0,1)$. Thus if $(U_1,\cdots,U_n)$ and $(V_1,\cdots,V_n)$ are the order statistics of $(X_1,\cdots,X_n)$ and $(Y_1,\cdots,Y_n)$ respectively, for any $\delta > 0$ we have

$$\mathbb{P}_n\Big(\Big|U_i - \frac{i}{n}\Big| > \delta\Big) = \mathbb{P}_n\left(Bin\Big(n, \frac{i}{n} - \delta\Big) \ge i\right) + \mathbb{P}_n\left(Bin\Big(n, \frac{i}{n} + \delta\Big) \le i\right) \le 2e^{-\delta^2 n} \tag{3.3}$$

by Hoeffding's inequality. Also using (1.7), for any $\mathbf{p},\mathbf{q} \in \mathcal{S}(n,l)$ we have

$$\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q}) = n! \sum_{\pi_n\in\Omega(\mathbf{p},\mathbf{q})} \int_{u_1<u_2<\cdots,u_n, v_1<v_2<\cdots<v_n} \prod_{i=1}^{n} f\Big(u_i, v_{\pi_n(i)}\Big) du_i dv_i,$$

which, for $\mathbf{r} \in \mathcal{S}(n,l)$ gives

$$\frac{\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{r})} \le \sup_{\mathbf{x},\mathbf{y},\mathbf{z}\in[0,1]^l} \prod_{a=1}^{l} \frac{\rho(x_a,y_a)}{\rho(x_a,z_a)} \le \Big(\frac{M}{m}\Big)^l.$$

Noting that $|\mathcal{S}(n,l)| \le n^l$, summing over $\mathbf{r}$ this implies

$$\mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{q}) \ge \Big(\frac{m}{Mn}\Big)^l \sum_{\mathbf{r}\in\mathcal{S}(n,l)} \mathbb{P}_n(\pi_n(\mathbf{p}) = \mathbf{r}) = \Big(\frac{m}{nM}\Big)^l \tag{3.4}$$

Finally, for any $\mathbf{p},\mathbf{q},\mathbf{r} \in \mathcal{S}(n,l)$ such that $||\mathbf{p}-\mathbf{r}||_\infty \le n\delta$, setting $A_n := \{\max_{i\in[n]} \Big|U_i - \frac{i}{n}\Big| \le$

$\delta\}$ we have

$$\begin{aligned}
\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})} \leq & \frac{\mathbb{P}_n(A_n^c)+\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q},A_n)}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q},A_n)} \\
= & \frac{\mathbb{P}_n(A_n^c)+n!\sum_{\pi_n\in\Omega(\mathbf{p},\mathbf{q})}\int_{u_1<u_2<\cdots<u_n,v_1<v_2<\cdots,<v_n,A_n}\prod_{i=1}^n\rho\Big(u_i,v_{\pi_n(i)}\Big)}{n!\sum_{\pi_n\in\Omega(\mathbf{r},\mathbf{q})}\int_{u_1<u_2<\cdots<u_n,v_1<v_2<\cdots,<v_n,A_n}\prod_{i=1}^n\rho\Big(u_i,v_{\pi_n(i)}\Big)} \\
\leq & \max_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{S}(n,l):||\mathbf{p}-\mathbf{r}||_\infty\leq n\delta}\ \sup_{\mathbf{u},\mathbf{v}\in[0,1]^n:|u_i-\frac{i}{n}|\leq\delta}\frac{\prod_{a=1}^l\rho\Big(u_{p_a},v_{q_a}\Big)}{\prod_{a=1}^l\rho\Big(u_{r_a},v_{q_a}\Big)} \qquad (3.5)\\
& +\frac{\mathbb{P}_n(A_n^c)}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})-\mathbb{P}_n(A_n^c)}. \qquad (3.6)
\end{aligned}$$

Since $|u_{p_a}-u_{r_a}|\leq 2\delta+\frac{|p_a-r_a|}{n}\leq 3\delta$, the expression in (3.5) can be bounded by

$$\sup_{\mathbf{x},\mathbf{y},\mathbf{z}\in[0,1]^l:||\mathbf{x}-\mathbf{z}||_\infty\leq 3\delta}\frac{\prod_{a=1}^l\rho(x_a,y_a)}{\prod_{a=1}^l\rho(z_a,y_a)}$$

which is free of n, and goes to 0 as $\delta\to 0$ by continuity of ρ. Also, using (3.3) and (3.4) it follows that the expression in (3.6) is bounded above by

$$\frac{2e^{-n\delta^2}}{\left(\frac{M}{nm}\right)^l-2e^{-n\delta^2}},$$

which converges to 0 as $n\to\infty$, for every δ fixed. Thus, taking a maximum over $\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{S}(n,l)$ such that $||\mathbf{p}-\mathbf{r}||_\infty\leq n\delta$ we have

$$\limsup_{\delta\to 0}\limsup_{n\to\infty}\max_{\mathbf{p},\mathbf{q},\mathbf{r}\in\mathcal{S}(n,l),||\mathbf{p}-\mathbf{r}||_\infty\leq n\delta}\frac{\mathbb{P}_n(\pi_n(\mathbf{p})=\mathbf{q})}{\mathbb{P}_n(\pi_n(\mathbf{r})=\mathbf{q})}\leq 1,$$

thus giving the upper bound in (3.1). Similar arguments give the lower bound in (3.1), as well as (3.2), thus completing the proof of the corollary. □

Proof of Proposition 1.14. With $\mathbb{P}_n=\mathbb{R}_{n,0}$ denoting the uniform measure on S_n and D_n denoting the number of derangements of n, we have

$$\frac{1}{n!}e^{Z_n(\theta)}=\mathbb{E}_{\mathbb{P}_n}e^{\theta N_n(\pi_n,e_n)}=\sum_{k=0}^\infty e^{\theta k}\frac{\binom{n}{k}D_{n-k}}{n!}\to\exp\{e^\theta-1\}, \qquad (3.7)$$

where we use the fact that $D_n/n!$ converges to e^{-1}.

(a) For any $\lambda>0$ we have

$$\mathbb{E}_{\mathbb{R}_{n,\theta}}e^{\lambda N_n(\pi_n,e_n)}=e^{Z_n(\theta+\lambda)-Z_n(\theta)}\to\ \exp\{e^\theta(e^\lambda-1)\},$$

and so $N_n(\pi_n,e_n)$ converges to $Poi(e^\theta)$ in distribution and in moments.

(b) With $D(.||.)$ denoting the Kullback-Leibler divergence we have

$$D(\mathbb{R}_{n,0}||\mathbb{R}_{n,\theta})=\log\Big(\frac{e^{Z_n(\theta)}}{n!}\Big)-\theta\mathbb{E}_{\mathbb{P}_n}N(\pi_n,e_n)\to e^\theta-1-\theta,$$

and so by [4, Prop 5.1] we have that the two probability distributions $\mathbb{R}_{n,\theta}$ and $\mathbb{R}_{n,0}=\mathbb{P}_n$ are mutually contiguous. Since π_n converges weakly to u under $\mathbb{P}_n=\mathbb{R}_{n,0}$, by contiguity the same happens for $\mathbb{R}_{n,\theta}$.

(c) Let $A_n := \{\pi_n \in S_n : \pi_n(1) = 1, \pi_n(2) = 2\}$, and $B_n := \{\pi_n \in S_n : \pi_n(1) = 2, \pi_n(2) = 1\}$. Define a bijection ω from A_n to B_n by setting $\omega(\pi_n)(i) = i$ for $3 \le i \le n$, and note that

$$\frac{\mathbb{R}_{n,\theta}(\pi_n)}{\mathbb{R}_{n,\theta}(\omega(\pi_n))} = e^{2\theta},$$

and so summing over $\pi_n \in A_n$ gives

$$\frac{\mathbb{R}_{n,\theta}(\pi_n(1) = 1, \pi_n(2) = 2)}{\mathbb{R}_{n,\theta}(\pi_n(1) = 2, \pi_n(2) = 1)} = e^{2\theta} \neq 1,$$

thus proving part (c). □

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