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Characterization of the law for 3D stochastic hyperviscous fluids

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Abstract

We consider the 3D hyperviscous Navier-Stokes equations in vorticity form, where the dissipative term $-\Delta\vec{\xi}$ of the Navier-Stokes equations is substituted by $(-\Delta)^{1+c}\vec{\xi}$. We investigate how big the correction term c has to be in order to prove, by means of Girsanov transform, that the vorticity equations are equivalent (in law) to easier reference equations obtained by neglecting the stretching term. This holds as soon as $c > \frac{1}{2}$, improving previous results obtained with $c > \frac{3}{2}$ in a different setting in [5, 14].

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1 Introduction

The stochastic Navier-Stokes equations, governing the motion of a homogeneous and incompressible viscous fluid, are

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} - \nu\Delta\vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f} + \vec{n} \\ \nabla\cdot\vec{v} = 0 \end{cases} \tag{1.1}$$

where the unknown are the velocity $\vec{v}$ and the pressure p; the data are the viscosity $\nu > 0$, the deterministic forcing term $\vec{f}$ and the random one $\vec{n}$.

Working in a bounded three dimensional spatial domain with suitable boundary conditions, it is known that for initial velocity of finite energy and suitable forcing terms there exists a weak solution to (1.1) defined for any positive time, but uniqueness is an open problem. On the other side, more regular initial velocities provide existence and uniqueness of a solution, which is only local in time. For these results we refer to [20] for the deterministic equations (the case $\vec{n} = \vec{0}$) and to [9] for the stochastic ones (the case $\vec{n} \neq \vec{0}$).

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However, suitable modifications of the first equation in (1.1) provide better results. Let us consider the hyperviscous model

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \nu(-\Delta)^{1+c}\vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f} + \vec{n} \\ \nabla \cdot \vec{v} = 0 \end{cases} \tag{1.2}$$

We consider $c > 0$, whereas it reduces to the Navier-Stokes system for $c = 0$. This model is widely used in computer simulations (see e.g. [11], [18] and references therein). It turns out that large enough values of the parameter c provide better mathematical properties of system (1.2).

As far as the well posedness of (1.2) is concerned, the condition $c \geq \frac{1}{4}$ allows to prove that there exists a unique global solution for the hyperviscous Navier-Stokes equations (1.2). This is based on the fact that the operator $(-\Delta)^{1+c}$ has a more regularizing effect than the Laplacian itself and $c \geq \frac{1}{4}$ provides a sufficient regularity to prove uniqueness of the global weak solution. The result has been proved first for integer values of $c \geq 1$, both in the stochastic (see [19]) and deterministic case (see [15]). Then, these results have been improved allowing c to be non integer (see [6] for the stochastic case and [16] for the deterministic one).

A further question concerns the characterization of the law of the process solving (1.2) with a stochastic force. When $\vec{f} = \vec{0}$ and $\vec{n}$ is a Gaussian random field, white in time and coloured in space, Gallavotti (see [12], Ch 6.1) suggested to use Girsanov transform to relate the law of the stochastic Navier-Stokes equations with that of the stochastic Stokes equations, which are linear equations obtained from the Navier-Stokes ones by neglecting the non linear term $(\vec{v}\cdot\nabla)\vec{v}$. The formula given in [12] when $c = 0$ is formal, but this idea can be used also for the hyperviscous fluids. Actually, a rigorous result has been proved in [14], [5]: for $c > \frac{3}{2}$ the law of the process $\vec{v}$ solving

$$\begin{cases} \dfrac{\partial \vec{v}}{\partial t} + \nu(-\Delta)^{1+c}\vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{n} \\ \nabla \cdot \vec{v} = 0 \end{cases} \tag{1.3}$$

is equivalent to the law of the process $\vec{z}$ solving the stochastic hyperviscous Stokes system

$$\begin{cases} \dfrac{\partial \vec{z}}{\partial t} + \nu(-\Delta)^{1+c}\vec{z} + \nabla p = \vec{n} \\ \nabla \cdot \vec{z} = 0 \end{cases} \tag{1.4}$$

This holds in the 2D and in the 3D setting and implies that all what holds a.s. for the hyperviscous Stokes problem (1.4) holds a.s. for the hyperviscous Navier-Stokes problem (1.3) as well. In other words: the advection term $(\vec{v}\cdot\nabla)\vec{v}$ takes second place to the dissipative term $(-\Delta)^{1+c}\vec{v}$ for c large enough. This means that hyperviscosity with $c > \frac{3}{2}$ changes drastically the nature of the equations of motion of the fluid. This remark already appeared in [11], where the authors discuss *artifacts* arising in numerical simulation of hyperviscous fluids. The mathematical representation of the law of $\vec{v}$ by means of Girsanov transform, which reduces the analysis of the law of $\vec{v}$ to the analysis of the law of the linear problem for $\vec{z}$, gives evidence in support of the fact that hyperviscous fluid models with $c > \frac{3}{2}$ are far away from the real turbulent fluids.

But, what happens for smaller values of the correction term, i.e. for $c \leq \frac{3}{2}$? To answer this question, we change the auxiliary process. First of all we write the Navier-Stokes system in vorticity form

$$\begin{cases} \dfrac{\partial \vec{\xi}}{\partial t} + \nu(-\Delta)^{1+c}\vec{\xi} + (\vec{v}\cdot\nabla)\vec{\xi} - (\vec{\xi}\cdot\nabla)\vec{v} = \nabla\times\vec{n} \\ \nabla\cdot\vec{v} = 0 \\ \vec{\xi} = \nabla\times\vec{v} \end{cases} \tag{1.5}$$

Notice that the first equation can be rewritten as

$$\frac{\partial \vec{\xi}}{\partial t} + \nu(-\Delta)^{1+c}\vec{\xi} + P[(\vec{v}\cdot\nabla)\vec{\xi}] - P[(\vec{\xi}\cdot\nabla)\vec{v}] = \nabla\times\vec{n}$$

where P is the projection operator onto the space of divergence free vector fields (see details in Section 2).

The idea is to simplify the vorticity equation by neglecting only the vorticity stretching term, getting

$$\begin{cases} \dfrac{\partial \vec{\eta}}{\partial t} + \nu(-\Delta)^{1+c}\vec{\eta} + P[(\vec{v}\cdot\nabla)\vec{\eta}] = \nabla\times\vec{n} \\ \nabla\cdot\vec{v} = 0 \\ \vec{\eta} = \nabla\times\vec{v} \end{cases} \tag{1.6}$$

This system has the same structure as the 2D vorticity system, but we consider it in the 3D setting. Indeed, in the 2D setting the vorticity is a vector orthogonal to the plane where the fluid moves and therefore the term $(\vec{\xi}\cdot\nabla)\vec{v}$ vanishes. Therefore, systems (1.5) and (1.6) are different only in the 3D setting. Let us compare them.

From the mathematical point of view we shall prove that system (1.6) is well posed for any $c \geq 0$, whereas the well posedness of the full system (1.5) has been proved by assuming $c \geq \frac{1}{4}$.

On the other hand, the vorticity stretching term $(\vec{\xi}\cdot\nabla)\vec{v}$ is essential in 3D fluids (see e.g. [10] Ch 9); it is responsible of the peculiar features of 3D turbulence, which is very different from and more involved than 2D turbulence. Thus one expects the dynamics of

$$\begin{cases} \dfrac{\partial \vec{\xi}}{\partial t} - \nu\Delta\vec{\xi} + P[(\vec{v}\cdot\nabla)\vec{\xi} - (\vec{\xi}\cdot\nabla)\vec{v}] = \nabla\times\vec{n} \\ \nabla\cdot\vec{v} = 0 \\ \vec{\xi} = \nabla\times\vec{v} \end{cases}$$

to be very different from that of

$$\begin{cases} \dfrac{\partial \vec{\eta}}{\partial t} - \nu\Delta\vec{\eta} + P[(\vec{v}\cdot\nabla)\vec{\eta}] = \nabla\times\vec{n} \\ \nabla\cdot\vec{v} = 0 \\ \vec{\eta} = \nabla\times\vec{v} \end{cases}$$

Now, the question is: what happens if we introduce hyperviscosity $(-\Delta)^{1+c}$? Our main theorem states the equivalence of laws of the solution processes of systems (1.5) and (1.6) under the assumption $c > \frac{1}{2}$. Again our result gives evidence that the hyperviscous models with $c > \frac{1}{2}$ do not well represent the real 3D turbulence, since the effect of the vorticity stretching term are not relevant when $c > \frac{1}{2}$.

Finally, we present this paper. In the next section we define the functional spaces and the noise term. Section 3 presents various technical results. Then we start to analyze the main equations: the linear problem in Section 4, the auxiliary problem (1.6) in Section 5 and the full vorticity problem (1.5) in Section 6. The main result on the equivalence of the laws is proved in Section 7.

2 Mathematical setting

We denote a 3D vector as $\vec{k} = (k^{(1)}, k^{(2)}, k^{(3)})$; we define $\mathbb{Z}_0^3 = \mathbb{Z}^3 \setminus \{\vec{0}\}$ and $\mathbb{Z}_+^3 = \{k^{(1)} > 0\} \cup \{k^{(1)} = 0, k^{(2)} > 0\} \cup \{k^{(1)} = 0, k^{(2)} = 0, k^{(3)} > 0\}$. Then for any $\vec{k} \in \mathbb{Z}_0^3$, there exist two unit vectors $\vec{b}_{\vec{k},1}$ and $\vec{b}_{\vec{k},2}$, orthogonal to each other and belonging to the plane orthogonal to $\vec{k}$; we choose these vectors in such a way that $(\vec{b}_{\vec{k},1}, \vec{b}_{\vec{k},2}, \frac{\vec{k}}{|\vec{k}|})$ is a right-handed orthonormal frame and $\vec{b}_{\vec{k},j} = -\vec{b}_{-\vec{k},j}$.

We work on the 3D torus, that is we deal with functions defined on $\mathbb{R}^3$ and $[-\pi,\pi]^3$-periodic. We set $D = [-\pi,\pi]^3$. As usual, in the periodic case we assume that the mean value of the vectors we are dealing with is zero. This gives a simplification in the mathematical treatment, but it does not prevent to consider non zero mean value vectors. Actually, if we can analyse the problem for zero mean vectors then the problem without this assumption can be dealt with in a similar way (see [21]).

The velocity vector $\vec{v}$ is divergence free by assumption and the vorticity vector $\vec{\xi}$ is divergence free by construction. We can write any zero mean, periodic, divergence free vector $\vec{u}$ in Fourier series as

$$\vec{u}(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}_0^3} [u_{\vec{k},1}\vec{b}_{\vec{k},1} + u_{\vec{k},2}\vec{b}_{\vec{k},2}] e^{i\vec{k}\cdot\vec{x}}, \qquad \vec{x} \in \mathbb{R}^3$$

where $u_{\vec{k},1}, u_{\vec{k},2} \in \mathbb{C}$, with the condition $\overline{u}_{\vec{k},j} = -u_{-\vec{k},j}$ in order to have a real vector $\vec{u}(\vec{x})$.

When needed, we use the notation $\vec{v}$ and $\vec{\xi}$ to make precise that we deal with the velocity or vorticity vector. For instance, we have $\vec{\xi} = \nabla \times \vec{v}$, but we can also express the velocity in terms of the vorticity, solving

$$\begin{cases} -\Delta \vec{v} = \nabla \times \vec{\xi} \\ \nabla \cdot \vec{v} = 0 \\ \vec{v} \text{ periodic} \end{cases} \tag{2.1}$$

More explicitly

$$\vec{\xi}(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}_0^3} (\xi_{\vec{k},1}\vec{b}_{\vec{k},1} + \xi_{\vec{k},2}\vec{b}_{\vec{k},2}) e^{i\vec{k}\cdot\vec{x}}$$

$$\Longrightarrow \quad \vec{v}(\vec{x}) = i \sum_{\vec{k} \in \mathbb{Z}_0^3} \frac{1}{|\vec{k}|} (\xi_{\vec{k},1}\vec{b}_{\vec{k},2} - \xi_{\vec{k},2}\vec{b}_{\vec{k},1}) e^{i\vec{k}\cdot\vec{x}} \tag{2.2}$$

We now define the functional spaces. Let L_2 denote the subspace of $[L^2(D)]^3$ consisting of zero mean, periodic, divergence free vectors (this condition has to be understood in the distributional sense):

$$L_2 = \Big\{ \vec{u}(\vec{x}) = \sum_{\vec{k} \in \mathbb{Z}_0^3} [u_{\vec{k},1}\vec{b}_{\vec{k},1} + u_{\vec{k},2}\vec{b}_{\vec{k},2}] e^{i\vec{k}\cdot\vec{x}} : \sum_{\vec{k} \in \mathbb{Z}_0^3} (|u_{\vec{k},1}|^2 + |u_{\vec{k},2}|^2) < \infty \Big\}$$

This is a Hilbert space with scalar product

$$\langle \vec{u}, \vec{v} \rangle = (2\pi)^3 \sum_{\vec{k} \in \mathbb{Z}_0^3} (u_{\vec{k},1}\overline{v}_{\vec{k},1} + u_{\vec{k},2}\overline{v}_{\vec{k},2})$$

The space L_2 is a closed subspace of $[L^2(D)]^3$; we decide to put the subindex in L_2 in order to distinguish them.

Moreover, for any integer n we define the projection operator Π_n as a linear bounded operator in L_2 such that

$$\Pi_n\left(\sum_{\vec{k}\in\mathbb{Z}^3_0}[u_{\vec{k},1}\vec{b}_{\vec{k},1}+u_{\vec{k},2}\vec{b}_{\vec{k},2}]e^{i\vec{k}\cdot\vec{x}}\right)=\sum_{0<|\vec{k}|\le n}[u_{\vec{k},1}\vec{b}_{\vec{k},1}+u_{\vec{k},2}\vec{b}_{\vec{k},2}]e^{i\vec{k}\cdot\vec{x}}$$

and we set $H_n=\Pi_n L_2$.

For $p>2$ we define the Banach spaces

$$L_p=L_2\cap[L^p(D)]^3$$

These are Banach spaces with norms inherited from $[L^p(D)]^3$.
We denote by P the projection operator from $[L^p(D)]^3$ onto L_p. We have that $P[(\vec{v}\cdot\nabla)\vec{\xi}-(\vec{\xi}\cdot\nabla)\vec{v}]=0$. Indeed, the vorticity transport term $(\vec{v}\cdot\nabla)\vec{\xi}$ and the vorticity stretching term $(\vec{\xi}\cdot\nabla)\vec{v}$ are not divergence free vector fields; so $P[(\vec{v}\cdot\nabla)\vec{\xi}]\neq(\vec{v}\cdot\nabla)\vec{\xi}$ and $P[(\vec{\xi}\cdot\nabla)\vec{v}]\neq(\vec{\xi}\cdot\nabla)\vec{v}$. However, their difference is divergence free, being given by the curl form $\nabla\times[(\vec{v}\cdot\nabla)\vec{v}]$. Moreover, if $\vec{\phi}$ is a divergence free vector field (i.e. $P\vec{\phi}=\vec{\phi}$), then

$$\langle P[(\vec{\xi}\cdot\nabla)\vec{v}],\vec{\phi}\rangle=\langle(\vec{\xi}\cdot\nabla)\vec{v},\vec{\phi}\rangle$$

For any $a\in\mathbb{R}$ we define the fractional powers of the Laplace operator; formally, if

$$\vec{u}(\vec{x})=\sum_{\vec{k}\in\mathbb{Z}^3_0}[u_{\vec{k},1}\vec{b}_{\vec{k},1}+u_{\vec{k},2}\vec{b}_{\vec{k},2}]e^{i\vec{k}\cdot\vec{x}}$$

then

$$(-\Delta)^a\vec{u}(\vec{x})=\sum_{\vec{k}\in\mathbb{Z}^3_0}|\vec{k}|^{2a}[u_{\vec{k},1}\vec{b}_{\vec{k},1}+u_{\vec{k},2}\vec{b}_{\vec{k},2}]e^{i\vec{k}\cdot\vec{x}}$$

Thus, for $b\in\mathbb{R}$ we define the Hilbert spaces

$$H^b=\{\vec{u}(\vec{x})=\sum_{\vec{k}\in\mathbb{Z}^3_0}[u_{\vec{k},1}\vec{b}_{\vec{k},1}+u_{\vec{k},2}\vec{b}_{\vec{k},2}]e^{i\vec{k}\cdot\vec{x}}:\sum_{\vec{k}\in\mathbb{Z}^3_0}|\vec{k}|^{2b}(|u_{\vec{k},1}|^2+|u_{\vec{k},2}|^2)<\infty\}$$

with scalar product

$$\langle\vec{u},\vec{v}\rangle_b=(2\pi)^3\sum_{\vec{k}\in\mathbb{Z}^3_0}|\vec{k}|^{2b}(u_{\vec{k},1}\overline{v}_{\vec{k},1}+u_{\vec{k},2}\overline{v}_{\vec{k},2})\equiv\langle(-\Delta)^{\frac{b}{2}}\vec{u},(-\Delta)^{\frac{b}{2}}\vec{v}\rangle$$

The duality between H^b and H^{-b} (or between $[H^b(D)]^3$ and $[H^{-b}(D)]^3$) is again denoted by $\langle\cdot,\cdot\rangle$.

For $b>0$ and $p>2$, we define the generalized Sobolev spaces H^b_p

$$H^b_p=\{\vec{u}\in L_p:(-\Delta)^{\frac{b}{2}}\vec{u}\in L_p\}$$

which are Banach spaces with norms

$$\|\vec{u}\|_{H^b_p}=\|(-\Delta)^{\frac{b}{2}}\vec{u}\|_{L_p}$$

When $b\in\mathbb{N}$, H^b_p are the Sobolev spaces. We recall the Sobolev embedding theorem (see [17] Ch 1 §8)

- if $1<p<q<\infty$ with $\frac{1}{q}=\frac{1}{p}-\frac{a-b}{3}$, then the following inclusion holds

  $$H^a_p\subset H^b_q$$

  and there exists a constant C (depending on $a-b,p,q$) such that

  $$\|\vec{v}\|_{H^b_q}\le C\|\vec{v}\|_{H^a_p}$$

- if $1 < p < \infty$ with $3 < ap$, then the following inclusion holds
$$H^a_p \subset L_\infty$$
and there exists a constant C (depending on a, p) such that
$$\|\vec{v}\|_{L_\infty} \le C\|\vec{v}\|_{H^a_p}$$

The Poincaré inequality holds, because of the zero mean value assumption, and therefore $\|\vec{u}\|_{H^b_p}$ is equivalent to $(\|\vec{u}\|^p_{L_p} + \|\vec{u}\|^p_{H^b_p})^{1/p}$, which appears usually in the definition of the generalized Sobolev spaces.

Moreover for $\vec{\xi} = \nabla \times \vec{v}$, the norms $\|\vec{v}\|_{H^b_p}$ and $\|\vec{\xi}\|_{H^{b-1}_p}$ are equivalent (see (2.2)).

For any $t > 0$ and $b > 0$, the linear operator $e^{-t(-\Delta)^b}$, formally defined as

$$e^{-t(-\Delta)^b}\left(\sum_{\vec{k}\in\mathbb{Z}^3_0}[u_{\vec{k},1}\vec{b}_{\vec{k},1} + u_{\vec{k},2}\vec{b}_{\vec{k},2}]e^{i\vec{k}\cdot\vec{x}}\right) = \sum_{\vec{k}\in\mathbb{Z}^3_0} e^{-t|\vec{k}|^{2b}}[u_{\vec{k},1}\vec{b}_{\vec{k},1} + u_{\vec{k},2}\vec{b}_{\vec{k},2}]e^{i\vec{k}\cdot\vec{x}}$$

is a contraction operator in L_p for any $p \ge 2$.

Next, we define the random forcing term. We consider a noise $d\vec{n}$ of the form $d(-\Delta)^{-b}\vec{w}$, where $\vec{w}$ is a cylindrical Wiener process in L_2 (see, e.g., [4]). We can represent it as follows. Suppose we are given a Brownian stochastic basis, i.e. a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\ge0}$; we denote by $\mathbb{E}$ the mathematical expectation with respect to $\mathbb{P}$. Let $\{\beta_{\vec{k},1}, \beta_{\vec{k},2}\}_{\vec{k}\in\mathbb{Z}^3_+}$ be a double sequence of complex valued independent Brownian motions on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P}\right)$; namely, the sequence $\{\Re\beta_{\vec{k},j}, \Im\beta_{\vec{k},j}\}_{\vec{k}\in\mathbb{Z}^3_+; j=1,2}$ consists of real valued processes that are independent, adapted to $(\mathcal{F}_t)_{t\ge0}$, continuous for $t \ge 0$ and null at $t = 0$, with increments on any time interval $[s, t]$ that are $N(0, t-s)$-distributed and independent of $\mathcal{F}_s$.

Moreover, for $-\vec{k} \in \mathbb{Z}^3_+$ let $\beta_{\vec{k},j} = -\overline{\beta}_{-\vec{k},j}$. Then

$$\vec{w}(t, \vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^3_0}[\vec{b}_{\vec{k},1}\beta_{\vec{k},1}(t) + \vec{b}_{\vec{k},2}\beta_{\vec{k},2}(t)]e^{i\vec{k}\cdot\vec{x}} \tag{2.3}$$

is a cylindrical Wiener process in L_2. Its paths do not live in the space $C(\mathbb{R}_+; L_2)$; they are less regular in space. Indeed

$$\mathbb{E}\|(-\Delta)^a\vec{w}(t)\|^2_{L_2} = 2t\sum_{\vec{k}\in\mathbb{Z}^3_0}|\vec{k}|^{2a}$$

which is finite if and only if $a < -\frac{3}{2}$.

Within this setting, we write system (1.5) for the vorticity as

$$\begin{cases} d\vec{\xi} + \Big((-\Delta)^{1+c}\vec{\xi} + P[(\vec{v}\cdot\nabla)\vec{\xi}] - P[(\vec{\xi}\cdot\nabla)\vec{v}]\Big)\ dt = (-\Delta)^{-b}d\vec{w} \\ \nabla\cdot\vec{v} = 0 \\ \vec{\xi} = \nabla\times\vec{v} \end{cases} \tag{2.4}$$

We have put $\nu = 1$ for simplicity and consider $b, c \ge 0$.

We give the following definition of solution: this is a weak (or distributional) solution from the point of view of PDE's and a strong solution from the point of view of stochastic equations.

Definition 2.1. *Given* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *and an* L_2*-cylindrical Wiener process* $\vec{w}$*, we say that a process* $\vec{\xi}$ *is a basic solution to system* (2.4) *on the finite time interval* $[0,T]$ *with initial condition* $\vec{\xi}(0) = \vec{\xi}_0 \in L_2$ *if*

$$\vec{\xi} \in C([0,T]; L_2) \cap L^1(0,T; L_3) \qquad \mathbb{P} - a.s. \tag{2.5}$$

and it satisfies the first equation of (2.4) *in the following sense:*
for any $t \in [0,T]$*, for any* $\vec{\phi} \in H^{2+2c} \cap H^{4-2b}$

$$\langle \vec{\xi}(t), \vec{\phi}\rangle + \int_0^t \langle \vec{\xi}(s), (-\Delta)^{1+c}\vec{\phi}\rangle ds - \int_0^t \langle (\vec{v}(s)\cdot\nabla)\vec{\phi}, \vec{\xi}(s)\rangle ds \\ + \int_0^t \langle (\vec{\xi}(s)\cdot\nabla)\vec{\phi}, \vec{v}(s)\rangle ds = \langle \vec{\xi}_0, \vec{\phi}\rangle + \langle (-\Delta)^{-2}\vec{w}(t), (-\Delta)^{2-b}\vec{\phi}\rangle \tag{2.6}$$

$\mathbb{P}$*-a.s.*

The latter relationship is obtained by multiplying the first equation of (2.4) by $\vec{\phi}$, integrating in space and time and finally by integration by part in the trilinear terms. Indeed, $-\langle (\vec{v}(s)\cdot\nabla)\vec{\phi}, \vec{\xi}(s)\rangle = \langle (\vec{v}(s)\cdot\nabla)\vec{\xi}(s), \vec{\phi}\rangle = \langle P[(\vec{v}(s)\cdot\nabla)\vec{\xi}(s)], \vec{\phi}\rangle$ and $\langle (\vec{\xi}(s)\cdot\nabla)\vec{\phi}, \vec{v}(s)\rangle = -\langle (\vec{\xi}(s)\cdot\nabla)\vec{v}(s), \vec{\phi}\rangle = -\langle P[(\vec{\xi}(s)\cdot\nabla)\vec{v}(s)], \vec{\phi}\rangle$, since $\vec{\phi}$ is a divergence free vector.

Remark 2.2. We remark that all the terms in (2.5) are meaningful. We show the basic estimates for the trilinear terms, by means of Hölder and Sobolev inequalities:

$$\begin{aligned} \left| \int_0^t \langle (\vec{v}(s)\cdot\nabla)\vec{\phi}, \vec{\xi}(s)\rangle ds \right| &\leq \|\vec{\phi}\|_{H^1} \int_0^t \|\vec{v}(s)\|_{L_6} \|\vec{\xi}(s)\|_{L_3} ds \\ &\leq C\|\vec{\phi}\|_{H^1} \int_0^t \|\vec{v}(s)\|_{H^1} \|\vec{\xi}(s)\|_{L_3} ds \\ &\leq C\|\vec{\phi}\|_{H^1} \int_0^t \|\vec{\xi}(s)\|_{L_2} \|\vec{\xi}(s)\|_{L_3} ds \\ &\leq C\|\vec{\phi}\|_{H^1} \|\vec{\xi}\|_{L^\infty(0,T;L_2)} \|\vec{\xi}\|_{L^1(0,T;L_3)} \end{aligned}$$

and similarly

$$\begin{aligned} \left| \int_0^t \langle (\vec{\xi}(s)\cdot\nabla)\vec{\phi}, \vec{v}(s)\rangle ds \right| &\leq C\|\vec{\phi}\|_{H^1} \int_0^t \|\vec{\xi}(s)\|_{L_3} \|\vec{v}(s)\|_{L_6} ds \\ &\leq C\|\vec{\phi}\|_{H^1} \|\vec{\xi}\|_{L^1(0,T;L_3)} \|\vec{\xi}\|_{L^\infty(0,T;L_2)} \end{aligned}$$

Here and in the following, we denote by C a generic constant, which may vary from line to line. However a subscript denotes that the constant depends on the specified parameters.

Remark 2.3. To prove the well posedness of system (2.4), we shall exploit the pathwise technique used the first time in [2] and later on in a more useful way in [8]. We shall transform the stochastic equation of Itô type (2.4) into a random equation which behaves like a deterministic equation when studied for $\mathbb{P}$-a.e. $\omega \in \Omega$, that is we find estimates for the paths of the solution process.

The solution process will enjoy more properties as a stochastic process; as in the 2D setting, we shall prove pathwise uniqueness and continuous dependence on the initial data in L_2. Thus our solution will be a strong solution from the point of view of stochastic differential equations (see e.g. [13]), and a Feller and Markov process in L_2. For these details, see [9] and references therein.

3 Estimates of the nonlinearities

This is a technical section, where we present the estimates to be used in proving the well posedness of system (2.4) and (1.6).

First, we present a classical result.

Lemma 3.1. *Let $\vec{u}, \vec{v}, \vec{w} : \mathbb{R}^3 \to \mathbb{R}^3$ be smooth D-periodic and divergence free vector fields. Then*

$$\langle P[(\vec{u} \cdot \nabla)\vec{v}], \vec{w}\rangle = -\langle P[(\vec{u} \cdot \nabla)\vec{w}], \vec{v}\rangle \tag{3.1}$$

In particular

$$\langle P[(\vec{u} \cdot \nabla)\vec{v}], \vec{v}\rangle = 0 \tag{3.2}$$

Proof. First

$$\langle P[(\vec{u} \cdot \nabla)\vec{v}], \vec{w}\rangle = \langle (\vec{u} \cdot \nabla)\vec{v}, \vec{w}\rangle = \sum_{i,j=1}^{3} \int_D u^{(i)}(\vec{x})\partial_i v^{(j)}(\vec{x}) w^{(j)}(\vec{x}) \, d\vec{x}$$

Then by integration by parts we get (3.1). The relationship (3.2) is obtained from (3.1) by taking $\vec{w} = \vec{v}$. □

By density, the above results hold for all vectors giving meaning to the above expressions. One can find estimates on the trilinear term in [21]. Here we present particular estimates, not included in [21], and useful in the sequel. Their proofs are based on Sobolev embeddings theorems and Hölder inequalities.

Lemma 3.2. *Let $c \geq 0$. Then there exists a positive constant C (depending on c) such that for any $\epsilon > 0$ we have*

$$|\langle (\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_3\rangle| \leq \epsilon \|\vec{u}_2\|^2_{H^{1+c}} + \frac{C}{\epsilon} \|\vec{u}_1\|^2_{H^1} \|\vec{u}_3\|^2_{L_3} \tag{3.3}$$

$$|\langle (\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_3\rangle| \leq \epsilon \|\vec{u}_3\|^2_{H^{1+c}} + \frac{C}{\epsilon} \|\vec{u}_1\|^2_{L_3} \|\vec{u}_2\|^2_{H^{1+c}} \tag{3.4}$$

$$|\langle (\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_3\rangle| \leq \epsilon \big\|\vec{u}_3\big\|^2_{H^{1+c}} + \frac{C}{\epsilon} \|\vec{u}_1\|^2_{H^1} \|\vec{u}_2\|^2_{L_3} \tag{3.5}$$

for all vectors making finite each r.h.s.

Proof. We begin with the first inequality:

$$\begin{aligned}
|\langle (\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_3\rangle| &\leq \|\vec{u}_1\|_{L_6} \|\nabla \vec{u}_2\|_{L_2} \|\vec{u}_3\|_{L_3} \text{ by Hölder inequality} \\
&\leq C \|\vec{u}_1\|_{H^1} \|\vec{u}_2\|_{H^1} \|\vec{u}_3\|_{L_3} \text{ by Sobolev embedding } H^1 \subset L_6 \\
&\leq C_c \|\vec{u}_1\|_{H^1} \|\vec{u}_2\|_{H^{1+c}} \|\vec{u}_3\|_{L_3} \\
&\leq \epsilon \|\vec{u}_2\|^2_{H^{1+c}} + \frac{C_c^2}{4\epsilon} \|\vec{u}_1\|^2_{H^1} \|\vec{u}_3\|^2_{L_3} \text{ by Cauchy inequality}
\end{aligned}$$

For the second inequality, we proceed in a similar way:

$$\begin{aligned}
|\langle (\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_3\rangle| &\leq \|\vec{u}_1\|_{L_3} \|\nabla \vec{u}_2\|_{L_2} \|\vec{u}_3\|_{L_6} \\
&\leq C \|\vec{u}_1\|_{L_3} \|\vec{u}_2\|_{H^1} \|\vec{u}_3\|_{H^1} \\
&\leq C_c \|\vec{u}_1\|_{L_3} \|\vec{u}_2\|_{H^{1+c}} \|\vec{u}_3\|_{H^{1+c}}
\end{aligned}$$

Then we apply Cauchy inequality to get the desired result.

For the third inequality, we have

$$\langle (\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_3\rangle = -\langle (\vec{u}_1 \cdot \nabla)\vec{u}_3, \vec{u}_2\rangle$$

from (3.1). Then we get (3.5) from (3.3). □

Lemma 3.3. *Let $c \geq \frac{1}{4}$. Then there exists a positive constant C (depending on c) such that for any $\epsilon > 0$ we have*

$$|\langle(\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_1\rangle| \leq \epsilon \|\vec{u}_1\|^2_{H^{1+c}} + \frac{C}{\epsilon}\|\vec{u}_2\|^2_{H^{1+c}}\|\vec{u}_1\|^2_{L_2}$$

for all vectors making finite the r.h.s..

Proof. First we consider the range of values $\frac{1}{4} \leq c < \frac{1}{2}$. We have $\frac{1-2c}{6} + \frac{3-2c}{6} + \frac{1}{2} \leq 1$ and $H^{1+c} \subset L_{\frac{6}{1-2c}}$, $H^c \subset L_{\frac{6}{3-2c}}$. Thus, Hölder and Sobolev inequalities give

$$\begin{aligned} |\langle(\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_1\rangle| &\leq \|\vec{u}_1\|_{L_{\frac{6}{1-2c}}}\|\nabla\vec{u}_2\|_{L_{\frac{6}{3-2c}}}\|\vec{u}_1\|_{L_2} \\ &\leq C_c\|\vec{u}_1\|_{H^{1+c}}\|\nabla\vec{u}_2\|_{H^c}\|\vec{u}_1\|_{L_2} \\ &\leq C_c\|\vec{u}_1\|_{H^{1+c}}\|\vec{u}_2\|_{H^{1+c}}\|\vec{u}_1\|_{L_2} \end{aligned}$$

Otherwise, for $c \geq \frac{1}{2}$, we use the Sobolev embeddings $H^{\frac{1}{2}} \subset L_3$ and $H^1 \subset L_6$. Therefore, again we estimate

$$\begin{aligned} |\langle(\vec{u}_1 \cdot \nabla)\vec{u}_2, \vec{u}_1\rangle| &\leq \|\vec{u}_1\|_{L_6}\|\nabla\vec{u}_2\|_{L_3}\|\vec{u}_1\|_{L_2} \\ &\leq C\|\vec{u}_1\|_{H^1}\|\nabla\vec{u}_2\|_{H^{\frac{1}{2}}}\|\vec{u}_1\|_{L_2} \\ &\leq C_c\|\vec{u}_1\|_{H^{1+c}}\|\vec{u}_2\|_{H^{1+c}}\|\vec{u}_1\|_{L_2} \end{aligned}$$

Applying Cauchy inequality we conclude the proof. □

4 The linear equation

When we neglect the non linearites in system (2.4) for the vorticity, we get

$$\begin{cases} d\vec{\zeta} + (-\Delta)^{1+c}\vec{\zeta}\, dt = (-\Delta)^{-b} d\vec{w} \\ \nabla \cdot \vec{\zeta} = 0 \end{cases} \tag{4.1}$$

Here the second equation keeps track of the fact that the vorticity vector is divergence free. So $\vec{\zeta}$ is the usual Ornstein-Uhlenbeck process, well studied in the literature. Here we assume $\vec{\zeta}(0) = \vec{0}$. Therefore the mild solution of (4.1) is

$$\vec{\zeta}(t) = \int_0^t e^{-(-\Delta)^{1+c}(t-s)}(-\Delta)^{-b} d\vec{w}(s) \tag{4.2}$$

(see e.g. [4]). We have

Proposition 4.1. *Let*

$$2b + c > a + \frac{1}{2} \tag{4.3}$$

Then, for any $m \in \mathbb{N}$

$$\vec{\zeta} \in C(\mathbb{R}_+; H^a_{2m}) \qquad \mathbb{P} - a.s$$

Proof. The proof is basically the same as that in [3] proving that $\vec{\zeta}$ has $\mathbb{P}$-a.e. path in $C(\mathbb{R}_+; H^a)$. Working on the torus, we can improve that result getting $\vec{\zeta} \in C(\mathbb{R}_+; H^a_{2m})$.

The factorization method uses that

$$\vec{\zeta}(t) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t \frac{1}{(t-s)^{1-\alpha}} e^{-(-\Delta)^{1+c}(t-s)}\vec{Y}_\alpha(s) ds \tag{4.4}$$

for $0 < \alpha < 1$, with

$$\vec{Y}_\alpha(s) = \int_0^s \frac{1}{(s-r)^\alpha} e^{-(-\Delta)^{1+c}(s-r)}(-\Delta)^{-b} d\vec{w}(r)$$

Now we prove that under assumption (4.3) there exists $\alpha \in (0, \frac{1}{2})$ such that

$$\mathbb{E}\|\vec{Y}_\alpha\|^{2m}_{L^{2m}(0,T;H^a_{2m})} < \infty \tag{4.5}$$

for any $m \in \mathbb{N}$.

For fixed $\vec{x}$ and t, $[(-\Delta)^{a/2}\vec{Y}_\alpha](t,\vec{x})$ is a Gaussian random variable given by the sum of independent Gaussian random variables

$$(-\Delta)^{a/2}\vec{Y}_\alpha(t,\vec{x}) = \sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^a \sum_{j=1}^{2} \int_0^t \frac{1}{(t-s)^\alpha} e^{-|\vec{k}|^{2(1+c)}(t-s)} |\vec{k}|^{-2b} \vec{b}_{\vec{k},j} d\beta_{\vec{k},j}(s) e^{i\vec{k}\cdot\vec{x}}$$

Therefore the variance of $(-\Delta)^{a/2}\vec{Y}_\alpha(t,\vec{x})$ is the sum of the variance of each addend:

$$\begin{aligned}
\mathbb{E}|(-\Delta)^{a/2}\vec{Y}_\alpha(t,\vec{x})|^2 &= \sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^{2a-4b} \int_0^t \frac{1}{(t-s)^{2\alpha}} e^{-2|\vec{k}|^{2(1+c)}(t-s)} ds \\
&= \sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^{2a-4b} \int_0^t \frac{1}{r^{2\alpha}} e^{-2|\vec{k}|^{2(1+c)}r} dr \\
&= \sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^{2a-4b} |\vec{k}|^{2(1+c)(2\alpha-1)} \int_0^{t|\vec{k}|^{2(1+c)}} \frac{1}{u^{2\alpha}} e^{-2u} du \\
&\le \sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^{2a-4b} |\vec{k}|^{2(1+c)(2\alpha-1)} \int_0^{\infty} \frac{1}{u^{2\alpha}} e^{-2u} du \\
&= C_\alpha \sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^{2a-4b+2(1+c)(2\alpha-1)}
\end{aligned}$$

where the constant C_α is finite for any $\alpha < \frac{1}{2}$.

Since $(-\Delta)^{a/2}\vec{Y}_\alpha(t,\vec{x})$ is a centered Gaussian random variable, for any integer m we have

$$\mathbb{E}|(-\Delta)^{a/2}\vec{Y}_\alpha(t,\vec{x})|^{2m} = C_m \left(\mathbb{E}|(-\Delta)^{a/2}\vec{Y}_\alpha(t,\vec{x})|^2\right)^m \le C_{m,\alpha} \left(\sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^{2a-4b+2(1+c)(2\alpha-1)}\right)^m$$

Integrating with respect to the variables $t \in [0,T]$ and $\vec{x} \in D$ we get

$$\mathbb{E}\|\vec{Y}_\alpha\|^{2m}_{L^{2m}(0,T;H^a_{2m})} \le C_{m,\alpha} T (2\pi)^3 \left(\sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^{2a-4b+2(1+c)(2\alpha-1)}\right)^m$$

The series in the r.h.s. converges if and only if

$$2a - 4b + 2(1+c)(2\alpha-1) < -3$$

i.e.

$$2b + c > a + \frac{1}{2} + 2\alpha(1+c) \tag{4.6}$$

If (4.3) holds then there exists $\alpha > 0$ small enough to get (4.6) and thus for such an α we have proved (4.5).

Now, given (4.5), with a trivial modification of the proof of Lemma 2.7 in [3], from (4.4) we get

$$\mathbb{E} \sup_{0\le t\le T} \|\vec{\zeta}(t)\|^{2m}_{H^a_{2m}} \le C_{m,T} \|\vec{Y}_\alpha\|^{2m}_{L^{2m}(0,T;L_{2m})}$$

and the continuity result. □

5 The vorticity transport equation

As explained before, we consider the system obtained from (2.4) by neglecting the term $P[(\vec{\xi} \cdot \nabla)\vec{v}]$ in the first equation. This is

$$\begin{cases} d\vec{\eta} + (-\Delta)^{1+c}\vec{\eta}\, dt + P[(\vec{v} \cdot \nabla)\vec{\eta}]\, dt = (-\Delta)^{-b} d\vec{w} \\ \nabla \cdot \vec{v} = 0 \\ \vec{\eta} = \nabla \times \vec{v} \end{cases} \tag{5.1}$$

We call it the vorticity transport system, since its first equation is a reduced form of the vorticity equation in (2.4): in (5.1) vorticity is only transported, not stretched.

Let us point out a feature of the equation of $\vec{\eta}$. The nonlinearity $(\vec{v} \cdot \nabla)\vec{\eta}$ has a peculiar form similar to that appearing in the regularized form of Leray-α models for fluids (see e.g. [1]), that is the first entry of the bilinear term $P[(\vec{v} \cdot \nabla)\vec{\eta}]$ is not the unknown $\vec{\eta}$ itself but indeed $\vec{v}$, which has one order more of regularity with respect to $\vec{\eta}$ (recall that if $\vec{\eta} \in H^b_p$ then $\vec{v} \in H^{b+1}_p$). Therefore, even if $\vec{\eta}$ satisfies a nonlinear equation, the quadratic term $(\vec{v} \cdot \nabla)\vec{\eta}$ in (5.1) (with $\vec{\eta} = \nabla \times \vec{v}$) behaves better than $(\vec{v} \cdot \nabla)\vec{v}$ in (1.1) and this makes the difference in the analysis of systems (5.1) and (1.1).

As far as the technique is concerned, we point out that in order to get existence and uniqueness results, we could look for mean estimates. However, for our purpose it is enough to get pathwise estimates (see Theorem 7.1). Moreover, the advantage of the pathwise approach is twofold: the existence result is obtained asking weaker assumption on the covariance of the noise and the regularity results are easily obtained. To see the first advantage, thanks to (3.1), with the usual techniques (see e.g. [2], [9]) we can get

$$\mathbb{E}\left[\|\vec{\eta}(t)\|^2_{L_2} + 2\int_0^t \|\vec{\eta}(s)\|^2_{H^{1+c}} ds\right] \le \|\vec{\eta}(0)\|^2_{L_2} + Tr\left((-\Delta)^{-2b}\right)\ t$$

This requires $Tr\left((-\Delta)^{-2b}\right) < \infty$, i.e.

$$\sum_{\vec{k}\in\mathbb{Z}^3_0} |\vec{k}|^{-4b} < \infty$$

which holds when $b > \frac{3}{4}$. But Theorem 5.2 allows to get existence of a basic solution $\vec{\eta}$ for $b > \frac{1}{4} - \frac{c}{2}$. Since our task in Theorem 7.1 will be to estimate

$$\|(-\Delta)^b P[(\vec{\eta} \cdot \nabla)\vec{v}]\|_{L_2}$$

it is clear than the smaller is b the easier is our task.

For this aim, we set $\vec{\beta} = \vec{\eta} - \vec{\zeta}$ and exploit that the noise is independent of the unknowns; then

$$\begin{cases} \dfrac{\partial\vec{\beta}}{\partial t} + (-\Delta)^{1+c}\vec{\beta} + P[(\vec{v} \cdot \nabla)(\vec{\beta} + \vec{\zeta})] = \vec{0} \\ \nabla \cdot \vec{v} = 0 \\ \nabla \times \vec{v} = \vec{\beta} + \vec{\zeta} \end{cases} \tag{5.2}$$

System (5.2) is studied pathwise. We have the following result

Proposition 5.1. *i) Assume that*

$$\begin{cases} c \ge 0 \\ 2b + c > \frac{1}{2} \end{cases}$$

Then, for any $\vec\beta(0) \in L_2$ *there exists a solution to* (5.2) *such that*

$$\vec\beta \in C([0,T]; L_2) \cap L^2(0,T; H^{1+c}) \qquad \mathbb{P}-a.s.$$

ii) Assume that

$$\begin{cases} c \geq 0 \\ 2b + c > \frac{3}{2} \end{cases}$$

Then, for any $\vec\beta(0) \in H^1$ *the solution given in i) enjoys also*

$$\vec\beta \in C([0,T]; H^1) \cap L^2(0,T; H^{2+c}) \qquad \mathbb{P}-a.s.$$

iii) Assume that

$$\begin{cases} c \geq 0 \\ 2b + c > \frac{5}{2} \end{cases}$$

Then, for any $\vec\beta(0) \in H^2$ *the solution given in i) enjoys also*

$$\vec\beta \in C([0,T]; H^2) \cap L^2(0,T; H^{3+c}) \qquad \mathbb{P}-a.s.$$

Proof. We proceed pathwise. The technique to prove existence is to consider first the finite dimensional problem, obtained by applying the projection operator Π_n to (5.2). The goal is to find suitable a priori estimates, uniformly in n. Thus, when any finite dimensional (Galerkin) problem has a solution we pass to the limit as $n \to \infty$ to get an existence result for (5.2). This technique, based on finite dimensional approximation, is well known (see e.g. [20, 21]). Therefore we look for a priori estimates for the full system (5.2); they hold for any Galerkin approximation as well, but we skip the details for the limit as $n \to \infty$.

i) We multiply the l.h.s. of the first equation of (5.2) by $\vec\beta(t)$ and integrate over D. Using (3.1)-(3.2) and then Hölder and Sobolev inequalities, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\vec\beta(t)\|_{L_2}^2 + \|\vec\beta(t)\|_{H^{1+c}}^2 &= -\langle P[(\vec v(t)\cdot\nabla)\vec\zeta(t)], \vec\beta(t)\rangle \\
&= \langle (\vec v(t)\cdot\nabla)\vec\beta(t), \vec\zeta(t)\rangle \\
&\leq C\|\vec v(t)\|_{L_6}\|\vec\beta(t)\|_{H^1}\|\vec\zeta(t)\|_{L_3} \\
&\leq C_c\|\vec v(t)\|_{H^1}\|\vec\beta(t)\|_{H^{1+c}}\|\vec\zeta(t)\|_{L_3} \\
&\leq C\|\vec\beta(t)+\vec\zeta(t)\|_{L_2}\|\vec\beta(t)\|_{H^{1+c}}\|\vec\zeta(t)\|_{L_3}
\end{aligned}$$

Cauchy inequality gives

$$\frac{1}{2}\frac{d}{dt}\|\vec\beta(t)\|_{L_2}^2 + \|\vec\beta(t)\|_{H^{1+c}}^2 \leq \frac{1}{2}\|\vec\beta(t)\|_{H^{1+c}}^2 + C\|\vec\zeta(t)\|_{L_3}^2\|\vec\beta(t)\|_{L_2}^2 + C\|\vec\zeta(t)\|_{L_3}^4 \tag{5.3}$$

Therefore, Gronwall inequality applied to

$$\frac{d}{dt}\|\vec\beta(t)\|_{L_2}^2 \leq C\|\vec\zeta(t)\|_{L_3}^2\|\vec\beta(t)\|_{L_2}^2 + C\|\vec\zeta(t)\|_{L_3}^4$$

gives

$$\sup_{0\leq t\leq T}\|\vec\beta(t)\|_{L_2}^2 \leq C(b,c,T,\|\vec\beta(0)\|_{L_2}, \|\vec\zeta\|_{L^\infty(0,T;L_3)})$$

Integrating in time (5.3) we get

$$\int_0^T \|\vec\beta(t)\|_{H^{1+c}}^2 dt \leq \tilde C(b,c,T,\|\vec\beta(0)\|_{L_2}, \|\vec\zeta\|_{L^\infty(0,T;L_3)})$$

We remind that $\vec{\zeta} \in C([0,T];L_3)$ if $2b+c > \frac{1}{2}$, according to Proposition 4.1. Then these a priori estimates give $\vec{\beta} \in L^\infty(0,T;L_2) \cap L^2(0,T;H^{1+c})$.

Moreover,

$$\frac{\partial \vec{\beta}}{\partial t} = -(-\Delta)^{1+c}\vec{\beta} - P[(\vec{v}\cdot\nabla)\vec{\beta}] - P[(\vec{v}\cdot\nabla)\vec{\zeta}]$$

Given the regularity of $\vec{\beta}$ we have that the r.h.s. belongs to $L^2(0,T;H^{-1-c})$; indeed $(-\Delta)^{1+c}\vec{\beta} \in L^2(0,T;H^{-1-c})$ and the two latter terms belong to $L^2(0,T;H^{-1})$. Let us see this; we proceed as before

$$|\langle(\vec{v}\cdot\nabla)\vec{\beta},\vec{u}\rangle| = |\langle(\vec{v}\cdot\nabla)\vec{u},\vec{\beta}\rangle| \le \|\vec{v}\|_{L_6}\|\nabla\vec{u}\|_{L_2}\|\vec{\beta}\|_{L_3}$$

This gives

$$\begin{aligned}
\|(\vec{v}\cdot\nabla)\vec{\beta}\|_{H^{-1}} = \sup_{\|\vec{u}\|_{H^1}>0} \frac{|\langle(\vec{v}\cdot\nabla)\vec{\beta},\vec{u}\rangle|}{\|\vec{u}\|_{H^1}} &\le \|\vec{v}\|_{L_6}\|\vec{\beta}\|_{L_3} \\
&\le C\|\vec{v}\|_{H^1}\|\vec{\beta}\|_{H^1} \\
&\le C(\|\vec{\beta}\|_{L_2} + \|\vec{\zeta}\|_{L_2})\|\vec{\beta}\|_{H^1}
\end{aligned}$$

Similarly we deal with $(\vec{v}\cdot\nabla)\vec{\zeta}$:

$$\|(\vec{v}\cdot\nabla)\vec{\zeta}\|_{H^{-1}} \le \|\vec{v}\|_{L_6}\|\vec{\zeta}\|_{L_3} \le C\|\vec{\zeta}\|^2_{L_3} + \|\vec{\zeta}\|_{L_3}\|\vec{\beta}\|_{L_2}$$

We recall that the space $\{\vec{\beta} \in L^2(0,T;H^{1+c}) : \frac{\partial\vec{\beta}}{\partial t} \in L^2(0,T;H^{-1-c})\}$ is compactly embedded in $L^2(0,T;L_2)$.

These are the basic results to implement the Galerkin approximation.

As far as the continuity is concerned, the fact that $\vec{\beta} \in L^2(0,T;H^{1+c})$ and $\frac{\partial\vec{\beta}}{\partial t} \in L^2(0,T;H^{-1-c})$ implies $\vec{\beta} \in C([0,T];L_2)$ (see Ch III Lemma 1.2 of [20]).

ii) We need a priori estimates and we proceed as in the previous step. We multiply the l.h.s. of the first equation of (5.2) by $-\Delta\vec{\beta}(t)$ and integrate on D. We get

$$\frac{1}{2}\frac{d}{dt}\|\vec{\beta}(t)\|^2_{H^1} + \|\vec{\beta}(t)\|^2_{H^{2+c}} = \langle(\vec{v}(t)\cdot\nabla)(\vec{\beta}(t)+\vec{\zeta}(t)),\Delta\vec{\beta}(t)\rangle$$

We estimate the r.h.s. as follows

$$\begin{aligned}
\langle(\vec{v}\cdot\nabla)(\vec{\beta}+\vec{\zeta}),\Delta\vec{\beta}\rangle &\le \|(\vec{v}\cdot\nabla)(\vec{\beta}+\vec{\zeta})\|_{L_2}\|\Delta\vec{\beta}\|_{L_2} \\
&\le \|\vec{v}\|_{L_\infty}\|\vec{\beta}+\vec{\zeta}\|_{H^1}\|\vec{\beta}\|_{H^2} \\
&\le C\|\vec{v}\|_{H^2}\|\vec{\beta}+\vec{\zeta}\|_{H^1}\|\vec{\beta}\|_{H^2} \quad \text{since } H^2 \subset L_\infty \\
&\le C_c\|\vec{\beta}+\vec{\zeta}\|^2_{H^1}\|\vec{\beta}\|_{H^{2+c}} \\
&\le \frac{1}{2}\|\vec{\beta}\|^2_{H^{2+c}} + C\|\vec{\beta}\|^4_{H^1} + \|\vec{\zeta}\|^4_{H^1}
\end{aligned}$$

This gives

$$\frac{d}{dt}\|\vec{\beta}(t)\|^2_{H^1} + \|\vec{\beta}(t)\|^2_{H^{2+c}} \le C\|\vec{\beta}\|^4_{H^1} + \|\vec{\zeta}\|^4_{H^1}$$

and we conclude as before using Gronwall Lemma and the fact that $\vec{\beta} \in L^2(0,T;H^1)$ from i) and $\vec{\zeta} \in C([0,T];H^1)$ from Proposition 4.1, getting

$$\sup_{0\le t\le T}\|\vec{\beta}(t)\|^2_{H^1} \le C(b,c,T,\|\vec{\beta}(0)\|_{H^1},\|\vec{\zeta}\|_{L^\infty(0,T;H^1)})$$

$$\int_0^T \|\vec{\beta}(t)\|^2_{H^{2+c}}dt \le \tilde{C}(b,c,T,\|\vec{\beta}(0)\|_{H^1},\|\vec{\zeta}\|_{L^\infty(0,T;H^1)})$$

Continuity in time is obtained as before.

iii) We multiply the l.h.s. of the first equation of (5.2) by $(-\Delta)^2\vec\beta(t)$ and integrate on D. We get

$$\frac12\frac{d}{dt}\|\vec\beta(t)\|^2_{H^2}+\|\vec\beta(t)\|^2_{H^{3+c}}=-\langle(\vec v(t)\cdot\nabla)(\vec\beta(t)+\vec\zeta(t)),(-\Delta)^2\vec\beta(t)\rangle$$

We estimate the r.h.s. as follows. First, we use the estimate for the product; by means of the Sobolev embedding $H^2\subset L_\infty$ we get

$$\begin{aligned}\|fg\|_{H^1}\le\|g\nabla f\|_{L_2}+\|f\nabla g\|_{L_2}&\le\|\nabla f\|_{L_\infty}\|g\|_{L_2}+\|f\|_{L_\infty}\|\nabla g\|_{L_2}\\&\le C\|f\|_{H^3}\|g\|_{L_2}+C\|f\|_{H^2}\|g\|_{H^1}\end{aligned}$$

Hence, for the trilinear term we get

$$\begin{aligned}\langle(\vec v\cdot\nabla)(\vec\beta+\vec\zeta),(-\Delta)^2\vec\beta\rangle&=\langle(-\Delta)^{\frac12}[(\vec v\cdot\nabla)(\vec\beta+\vec\zeta)],(-\Delta)^{\frac32}\vec\beta\rangle\\&\le\|(\vec v\cdot\nabla)(\vec\beta+\vec\zeta)\|_{H^1}\|\vec\beta\|_{H^3}\\&\le C\Big(\|\vec v\|_{H^3}\|\vec\beta+\vec\zeta\|_{H^1}+\|\vec v\|_{H^2}\|\vec\beta+\vec\zeta\|_{H^2}\Big)\|\vec\beta\|_{H^3}\\&\le C\|\vec\beta+\vec\zeta\|_{H^1}\|\vec\beta+\vec\zeta\|_{H^2}\|\vec\beta\|_{H^3}\\&\le C_c\|\vec\beta+\vec\zeta\|_{H^1}\|\vec\beta+\vec\zeta\|_{H^2}\|\vec\beta\|_{H^{3+c}}\\&\le\frac12\|\vec\beta\|^2_{H^{3+c}}+C\|\vec\beta+\vec\zeta\|^2_{H^1}\|\vec\beta\|^2_{H^2}+C\|\vec\beta+\vec\zeta\|^2_{H^1}\|\vec\zeta\|^2_{H^2}\end{aligned}$$

This gives

$$\frac{d}{dt}\|\vec\beta(t)\|^2_{H^2}+\|\vec\beta(t)\|^2_{H^{3+c}}\le C\|\vec\beta(t)+\vec\zeta(t)\|^2_{H^1}\|\vec\beta(t)\|^2_{H^2}+C\|\vec\zeta(t)\|^2_{H^2}\|\vec\beta(t)\|^2_{H^1}+C\|\vec\zeta(t)\|^4_{H^2}$$

Since $\vec\beta\in C([0,T];H^1)$ from step ii) and $\vec\zeta\in C([0,T];H^2)$ from Proposition 4.1, we get first

$$\sup_{0\le t\le T}\|\vec\beta(t)\|^2_{H^2}\le C(b,c,T,\|\vec\beta(0)\|_{H^2},\|\vec\zeta\|_{L^\infty(0,T;H^2)})$$

and then

$$\int_0^T\|\vec\beta(t)\|^2_{H^{3+c}}dt\le\tilde C(b,c,T,\|\vec\beta(0)\|_{H^2},\|\vec\zeta\|_{L^\infty(0,T;H^2)})$$

Continuity in time is obtained as before. This concludes the proof. □

Now we come back to the unknown $\vec\eta=\vec\beta+\vec\zeta$. The definition of basic solution is the same as that for $\vec\xi$ given at the end of Section 2, with the obvious modification of the equation by neglecting $P[(\vec\xi\cdot\nabla)\vec v]$.

Theorem 5.2. *i) Assume that*

$$\begin{cases}c\ge0\\2b+c>\frac12\end{cases}$$

Then, for any $\vec\eta(0)\in L_2$ there exists a unique process $\vec\eta$ which is a basic solution to (5.1) *such that*

$$\vec\eta\in C([0,T];L_2)\cap L^2(0,T;L_6)$$

$\mathbb P$*-a.s.*

Moreover there is continuous dependence on the initial data: given two initial data $\vec\eta(0),\vec\eta_\star(0)\in L_2$ we have

$$\|\vec\eta(0)-\vec\eta_\star(0)\|_{L_2}\to0\Longrightarrow\|\vec\eta-\vec\eta_\star\|_{C([0,T];L_2)}\to0$$

ii) Assume that

$$\begin{cases} c \geq 0 \\ 2b + c > \frac{3}{2} \end{cases}$$

Then, for any $\vec{\eta}(0) \in H^1$ the solution given in i) enjoys also

$$\vec{\eta} \in C([0,T]; H^1) \qquad \mathbb{P} - a.s.$$

iii) Assume that

$$\begin{cases} c \geq 0 \\ 2b + c > \frac{5}{2} \end{cases}$$

Then, for any $\vec{\eta}(0) \in H^2$ the solution given in i) enjoys also

$$\vec{\eta} \in C([0,T]; H^2) \qquad \mathbb{P} - a.s.$$

Proof. The existence comes from the existence results on $\vec{\beta}$, $\vec{\zeta}$. Moreover

$$\vec{\zeta} \in C([0,T]; L_q) \qquad \forall q < \infty$$

and by Sobolev embedding

$$\vec{\beta} \in L^2(0,T; H^{1+c}) \subset L^2(0,T; H^1) \subset L^2(0,T; L_6)$$

Merging toghether the regularity of these processes we get our three results for $\vec{\eta}$.

As far as continuous dependence on the initial data is concerned, let us take two basic solutions $\vec{\eta}_1$ and $\vec{\eta}_2$ with $\vec{\eta}_1(0) = \vec{\eta}_2(0) \in L_2$; at least we have

$$\vec{\eta}_1, \vec{\eta}_2 \in C([0,T]; L_2) \cap L^2(0,T; L_3)$$

We define $\vec{y} = \vec{\eta}_1 - \vec{\eta}_2$; then the system fulfilled by $\vec{y}$ can be written as

$$\begin{cases} \dfrac{\partial \vec{y}}{\partial t} + (-\Delta)^{1+c}\vec{y} + P[(\vec{v}_1 \cdot \nabla)\vec{y}] + P[((\vec{v}_1 - \vec{v}_2) \cdot \nabla)\, \vec{\eta}_2] = \vec{0} \\ \nabla \cdot \vec{v}_1 = \nabla \cdot \vec{v}_2 = 0 \\ \vec{y} = \nabla \times (\vec{v}_1 - \vec{v}_2) \end{cases}$$

We estimate the following term, as usual:

$$\begin{aligned} |\langle [(\vec{v}_1 - \vec{v}_2) \cdot \nabla]\vec{\eta}_2, \vec{y}\rangle| &= |\langle [(\vec{v}_1 - \vec{v}_2) \cdot \nabla]\vec{y}, \vec{\eta}_2\rangle| \\ &\leq \frac{1}{2}\|\vec{y}\|^2_{H^{1+c}} + C\|\vec{\eta}_2\|^2_{L_3}\|\vec{v}_1 - \vec{v}_2\|^2_{H^1} \text{ from (3.3)} \\ &\leq \frac{1}{2}\|\vec{y}\|^2_{H^{1+c}} + C\|\vec{\eta}_2\|^2_{L_3}\|\vec{y}\|^2_{L_2} \end{aligned}$$

Then taking the scalar product of the the first equation for $\vec{y}$ with $\vec{y}$, integrating on the spatial domain and using (3.1), we get

$$\frac{d}{dt}\|\vec{y}(t)\|^2_{L_2} + \|\vec{y}(t)\|^2_{H^{1+c}} \leq C\|\vec{\eta}_2(t)\|^2_{L_3}\|\vec{y}(t)\|^2_{L_2}$$

Recall that $\vec{\eta}_2 \in L^2(0,T; L_3)$. Applying Gronwall lemma to

$$\frac{d}{dt}\|\vec{y}(t)\|^2_{L_2} \leq C\|\vec{\eta}_2(t)\|^2_{L_3}\|\vec{y}(t)\|^2_{L_2}$$

we get

$$\sup_{0 \leq t \leq T} \|\vec{y}(t)\|_{L_2} \leq \|\vec{y}(0)\|_{L_2} e^{C\int_0^T \|\vec{\eta}_2(t)\|^2_{L_3} dt}$$

This gives the continuous dependence on the initial data; uniqueness is obtained when $\vec{y}(0) = \vec{0}$. $\square$

6 The vorticity equation

Now we consider the full nonlinear system (2.4). If the initial velocity is more regular, say $\vec{v}(0) \in H^1$ (i.e. $\vec{\xi}(0) \in L_2$), one can prove a local existence and uniqueness result for $c = 0$; global existence holds only for $c \geq \frac{1}{4}$ (see [6]). In this paper we improve the results for $c \geq \frac{1}{4}$ considering initial data $\vec{\xi}(0) \in H^1$ and H^2.

We need a preliminary result for the velocity, fulfilling (1.3) with the noise obtained from a Wiener process $\vec{w}_{vel}$ such that $\nabla \times \vec{w}_{vel} = (-\Delta)^{-b}\vec{w}$, that is

$$\vec{w}_{vel}(t, \vec{x}) = \sum_{\vec{k} \in \mathbb{Z}_0^3} |\vec{k}|^{-2b-1}[-\vec{b}_{\vec{k},1}\beta_{\vec{k},1}(t) + \vec{b}_{\vec{k},2}\beta_{\vec{k},2}(t)]e^{i\vec{k}\cdot\vec{x}}$$

Therefore (1.3) becomes

$$\begin{cases} d\vec{v} + (-\Delta)^{1+c}\vec{v}\, dt + (\vec{v} \cdot \nabla)\vec{v}\, dt + \nabla p\, dt = d\vec{w}_{vel} \\ \nabla \cdot \vec{v} = 0 \end{cases} \tag{6.1}$$

Proposition 6.1. *Assume that*

$$\begin{cases} c \geq 0 \\ b > \frac{1}{4} \end{cases}$$

Then for any $\vec{v}(0) \in L_2$ there exists a process $\vec{v}$ with $\mathbb{P}$-a.e. path in $L^\infty(0, T; L_2) \cap L^2(0, T; H^{1+c})$, solving (6.1)*.*

Proof. We know the result for $c = 0$ (see [9]); the case $c > 0$ does not provide any difficulty. But we show the shortest way to get it, by means of mean value estimates. Only here we use mean value estimates instead of the pathwise ones.

We write the basic energy estimate obtained from Itô formula for $d\|\vec{v}(t)\|^2_{L_2}$; the details can be found in [9]. We have

$$\mathbb{E}\|\vec{v}(t)\|^2_{L_2} + 2\int_0^t \mathbb{E}\|\vec{v}(s)\|^2_{H^{1+c}} ds \leq \|\vec{v}(0)\|^2_{L_2} + t \sum_{\vec{k} \in \mathbb{Z}_0^3} |\vec{k}|^{-2(2b+1)}$$

The series in the r.h.s. converges if and only if $2(2b + 1) > 3$, i.e. $b > \frac{1}{4}$. These estimates improves the regularity: $\vec{v} \in L^2(0, T; H^{1+c})$, $\mathbb{P}$-a.s. □

Now we consider the unknown $\vec{\xi}$. Let $\vec{\delta} := \vec{\xi} - \vec{\zeta}$; bearing in mind the equations for $\vec{\xi}$ and $\vec{\zeta}$ we have that this new unknown satisfies

$$\frac{\partial \vec{\delta}}{\partial t} + (-\Delta)^{1+c}\vec{\delta} + P[(\vec{v} \cdot \nabla)\vec{\delta} - (\vec{\delta} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{\zeta} - (\vec{\zeta} \cdot \nabla)\vec{v}] = \vec{0} \tag{6.2}$$

Now the quantities $\vec{v}$ and $\vec{\delta}$ are linked through $\vec{\delta} = -\vec{\zeta} + \nabla \times \vec{v}$.

Our aim is to find existence and regularity results for $\vec{\delta}$ in order to obtain the same results for $\vec{\xi}$. This requires $c \geq \frac{1}{4}$.

As in the previous section we look for pathwise results.

Proposition 6.2. *i) Assume that*

$$\begin{cases} c \geq \frac{1}{4} \\ b > \frac{1}{4} \end{cases}$$

Then, for any $\vec{\delta}(0) \in L_2$ there exists a solution to (6.2) *such that*

$$\vec{\delta} \in C([0, T]; L_2) \cap L^2(0, T; H^{1+c}) \qquad \mathbb{P} - a.s.$$

ii) Assume that

$$\begin{cases} c \geq \frac{1}{4} \\ b > \frac{1}{4} \\ 2b + c > \frac{3}{2} \end{cases}$$

Then, for any $\vec{\delta}(0) \in H^1$ the solution given in i) enjoys also

$$\vec{\delta} \in C([0,T]; H^1) \cap L^2(0,T; H^{2+c}) \qquad \mathbb{P} - a.s.$$

iii) Assume that

$$\begin{cases} c \geq \frac{1}{4} \\ b > \frac{1}{4} \\ 2b + c > \frac{5}{2} \end{cases}$$

Then, for any $\vec{\delta}(0) \in H^2$ the solution given in i) enjoys also

$$\vec{\delta} \in C([0,T]; H^2) \cap L^2(0,T; H^{3+c}) \qquad \mathbb{P} - a.s.$$

Proof. i) First, notice that if $c \geq \frac{1}{4}$ and $b > \frac{1}{4}$ then $2b + c > \frac{3}{4} > \frac{1}{2}$. Therefore Proposition 4.1 provides that for any finite p we have $\vec{\zeta} \in C([0,T]; L_p)$ a.s..

We deal with (6.2) as we did with (5.2). So

$$\frac{1}{2}\frac{d}{dt}\|\vec{\delta}(t)\|^2_{L_2} + \|\vec{\delta}(t)\|^2_{H^{1+c}} = \langle (\vec{\delta}\cdot\nabla)\vec{v} - (\vec{v}\cdot\nabla)\vec{\zeta} + (\vec{\zeta}\cdot\nabla)\vec{v}, \vec{\delta}\rangle$$

From Lemma 3.3

$$\langle (\vec{\delta}\cdot\nabla)\vec{v}, \vec{\delta}\rangle \leq \tfrac{1}{6}\|\vec{\delta}\|^2_{H^{1+c}} + C\|\vec{v}\|^2_{H^{1+c}}\|\vec{\delta}\|^2_{L_2}$$

From (3.5) of Lemma 3.2

$$|\langle (\vec{v}\cdot\nabla)\vec{\zeta}, \vec{\delta}\rangle| \leq \tfrac{1}{6}\|\vec{\delta}\|^2_{H^{1+c}} + C\|\vec{v}\|^2_{H^1}\|\vec{\zeta}\|^2_{L_3}$$

From (3.4) of Lemma 3.2

$$|\langle (\vec{\zeta}\cdot\nabla)\vec{v}, \vec{\delta}\rangle| \leq \tfrac{1}{6}\|\vec{\delta}\|^2_{H^{1+c}} + C\|\vec{v}\|^2_{H^1}\|\vec{\zeta}\|^2_{L_3}$$

Summing up, we get

$$\frac{d}{dt}\|\vec{\delta}(t)\|^2_{L_2} + \|\vec{\delta}(t)\|^2_{H^{1+c}} \leq C\|\vec{v}(t)\|^2_{H^{1+c}}\|\vec{\delta}(t)\|^2_{L_2} + C\|\vec{\zeta}(t)\|^2_{L_3}\|\vec{v}(t)\|^2_{H^{1+c}}$$

From Proposition 6.1, we know that $\vec{v} \in L^2(0,T; H^{1+c})$; moreover our assumption and Proposition 4.1 give $\vec{\zeta} \in C([0,T]; L_3)$. Then by Gronwall lemma we get

$$\sup_{0\leq t\leq T}\|\vec{\delta}(t)\|^2_{L_2} < \infty$$

and integrating in time

$$\int_0^T \|\vec{\delta}(t)\|^2_{H^{1+c}}\, dt < \infty$$

The continuity in time is obtained as in Proposition 5.1.

ii) We need a priori estimates and we proceed as in the previous step. We multiply the l.h.s. of the first equation of (6.2) by $-\Delta\vec{\delta}(t)$ and integrate on D. We get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}&\|\vec{\delta}(t)\|^2_{H^1} + \|\vec{\delta}(t)\|^2_{H^{2+c}} \\ &= \langle (\vec{v}(t)\cdot\nabla)(\vec{\delta}(t) + \vec{\zeta}(t)), \Delta\vec{\delta}(t)\rangle - \langle ((\vec{\delta}(t) + \vec{\zeta}(t))\cdot\nabla)\vec{v}(t), \Delta\vec{\delta}(t)\rangle\end{aligned}$$

We estimate the latter term in the r.h.s. as usual:

$$\begin{aligned}
|\langle((\vec\delta+\vec\zeta)\cdot\nabla)\vec v,\Delta\vec\delta\rangle| &\le \|\vec\delta+\vec\zeta\|_{L_4}\|\nabla\vec v\|_{L_4}\|\Delta\vec\delta\|_{L_2}\\
&\le C\|\vec\delta+\vec\zeta\|_{H^1}\|\nabla\vec v\|_{H^1}\|\vec\delta\|_{H^2}\\
&\le C_c\|\vec\delta+\vec\zeta\|^2_{H^1}\|\vec\delta\|_{H^{2+c}}\\
&\le \frac14\|\vec\delta\|^2_{H^{2+c}}+C\|\vec\delta\|^4_{H^1}+C\|\vec\zeta\|^4_{H^1}
\end{aligned}$$

With this estimate and dealing with the other trilinear term as in the proof of Proposition 5.1 ii), we obtain

$$\frac{d}{dt}\|\vec\delta(t)\|^2_{H^1}+\|\vec\delta(t)\|^2_{H^{2+c}}\le C\|\vec\delta(t)\|^4_{H^1}+\|\vec\zeta(t)\|^4_{H^1}$$

Since $\vec\delta\in L^2(0,T;H^1)$ from the previous step and $\vec\zeta\in C([0,T];H^1)$ from Proposition 4.1, we conclude as in the proof of Proposition 5.1 ii).

iii) We multiply the l.h.s. of the first equation of (6.2) by $(-\Delta)^2\vec\delta(t)$ and integrate on D. We get

$$\begin{aligned}
\frac12\frac{d}{dt}\|\vec\delta(t)\|^2_{H^2}+\|\vec\delta(t)\|^2_{H^{3+c}} = &-\langle(\vec v(t)\cdot\nabla)(\vec\delta(t)+\vec\zeta(t)),(-\Delta)^2\vec\delta(t)\rangle\\
&+\langle((\vec\delta(t)+\vec\zeta(t))\cdot\nabla)\vec v(t),(-\Delta)^2\vec\delta(t)\rangle
\end{aligned}$$

We are left to estimate the latter trilinear term. First, we use the estimate for the product; by means of the Sobolev embeddings $H^2\subset L_\infty$ and $H^1\subset L_4$ we get

$$\begin{aligned}
\|fg\|_{H^1}\le\|g\nabla f\|_{L_2}+\|f\nabla g\|_{L_2}&\le\|\nabla f\|_{L_2}\|g\|_{L_\infty}+\|f\|_{L_4}\|\nabla g\|_{L_4}\\
&\le C\|f\|_{H^1}\|g\|_{H^2}+C\|f\|_{H^1}\|g\|_{H^2}
\end{aligned}$$

Hence, for the trilinear term we get

$$\begin{aligned}
\langle((\vec\delta+\vec\zeta)\cdot\nabla)\vec v,(-\Delta)^2\vec\delta\rangle &\le \|((\vec\delta+\vec\zeta)\cdot\nabla)\vec v\|_{H^1}\|\vec\delta\|_{H^3}\\
&\le C\|\vec\delta+\vec\zeta\|_{H^1}\|\nabla\vec v\|_{H^2}\|\vec\delta\|_{H^3}\\
&\le C_c\|\vec\delta+\vec\zeta\|_{H^1}\|\vec\delta+\vec\zeta\|_{H^2}\|\vec\delta\|_{H^{3+c}}\\
&\le \frac14\|\vec\delta\|^2_{H^{3+c}}+C\|\vec\delta+\vec\zeta\|^2_{H^1}\|\vec\delta\|^2_{H^2}+C\|\vec\delta+\vec\zeta\|^2_{H^1}\|\vec\zeta\|^2_{H^2}
\end{aligned}$$

Therefore, keeping in mind the proof of Proposition 5.1 iii) to estimate the other trilinear term, we obtain

$$\frac{d}{dt}\|\vec\delta(t)\|^2_{H^2}+\|\vec\delta(t)\|^2_{H^{3+c}}\le C\|\vec\delta(t)+\vec\zeta(t)\|^2_{H^1}\|\vec\delta(t)\|^2_{H^2}+C\|\vec\delta(t)+\vec\zeta(t)\|^2_{H^1}\|\vec\zeta(t)\|^2_{H^2}$$

Since $\vec\delta\in L^2(0,T;H^2)$ from the previous step and $\vec\zeta\in C([0,T];H^2)$ from Proposition 4.1, we conclude as in the proof of Proposition 5.1 iii). □

Now we have the result for $\vec\xi=\vec\delta+\vec\zeta$.

Theorem 6.3. *i) Assume that*

$$\begin{cases} c\ge\frac14\\ b>\frac14\end{cases}$$

Then, for any $\vec\xi(0)\in L_2$ there exists a unique process $\vec\xi$ which is a basic solution to (2.4) *such that*

$$\vec\xi\in C([0,T];L_2)\cap L^2(0,T;L_6)$$

$\mathbb{P}$*-a.s.*

Moreover there is continuous dependence on the initial data: given two initial data $\vec{\xi}(0), \vec{\xi}_\star(0) \in L_2$ *we have*

$$\|\vec{\xi}(0) - \vec{\xi}_\star(0)\|_{L_2} \to 0 \Longrightarrow \|\vec{\xi} - \vec{\xi}_\star\|_{C([0,T];L_2)} \to 0$$

ii) Assume that

$$\begin{cases} c \geq \frac{1}{4} \\ b > \frac{1}{4} \\ 2b + c > \frac{3}{2} \end{cases}$$

Then, for any $\vec{\xi}(0) \in H^1$ *the solution given in i) enjoys also*

$$\vec{\xi} \in C([0,T]; H^1) \qquad \mathbb{P} - a.s.$$

iii) Assume that

$$\begin{cases} c \geq \frac{1}{4} \\ b > \frac{1}{4} \\ 2b + c > \frac{5}{2} \end{cases}$$

Then, for any $\vec{\xi}(0) \in H^2$ *the solution given in i) enjoys also*

$$\vec{\xi} \in C([0,T]; H^2) \qquad \mathbb{P} - a.s.$$

Proof. i) If $c \geq \frac{1}{4}$ and $b > \frac{1}{4}$ then $2b + c > \frac{1}{2}$. Therefore Proposition 4.1 provides that for any finite p we have $\vec{\zeta} \in C([0,T]; L_p)$ a.s.. We merge the results of Proposition 6.2 for $\vec{\delta}$ with those of Proposition 4.1 for $\vec{\zeta}$ to get existence of $\vec{\xi}$ and its regularity. This is the same as in Theorem 5.2.

As far as continuous dependence on the initial data is concerned, we proceed as in the proof of Theorem 5.2. The additional term does not give any problem; we estimate it as follows. Set $\vec{y} = \vec{\xi}_1 - \vec{\xi}_2$; then the system fulfilled by $\vec{y}$ can be written as

$$\begin{cases} \dfrac{\partial \vec{y}}{\partial t} + (-\Delta)^{1+c}\vec{y} + P[(\vec{v}_1 \cdot \nabla)\vec{y} + ((\vec{v}_1 - \vec{v}_2) \cdot \nabla)\,\vec{\xi}_2 - (\vec{\xi}_1 \cdot \nabla)(\vec{v}_1 - \vec{v}_2) - (\vec{y} \cdot \nabla)\vec{v}_2] = \vec{0} \\ \nabla \cdot \vec{v}_1 = \nabla \cdot \vec{v}_2 = 0 \\ \vec{y} = \nabla \times (\vec{v}_1 - \vec{v}_2) \end{cases}$$

Therefore, in the equation fulfilled by $\|\vec{y}(t)\|^2_{L_2}$, in addition to the terms appearing in the proof of Theorem 5.2 we also have

$$\langle (\vec{\xi}_1 \cdot \nabla)(\vec{v}_1 - \vec{v}_2), \vec{y} \rangle + \langle (\vec{y} \cdot \nabla)\vec{v}_2, \vec{y} \rangle$$

We have

$$\begin{aligned} |\langle (\vec{\xi}_1 \cdot \nabla)(\vec{v}_1 - \vec{v}_2), \vec{y} \rangle| &\leq \|\vec{\xi}_1\|_{L_3} \|\nabla(\vec{v}_1 - \vec{v}_2)\|_{L_2} \|\vec{y}\|_{L_6} \\ &\leq C\|\vec{\xi}_1\|_{L_3} \|\vec{y}\|_{L_2} \|\vec{y}\|_{H^1} \\ &\leq C_c\|\vec{\xi}_1\|_{L_3} \|\vec{y}\|_{L_2} \|\vec{y}\|_{H^{1+c}} \\ &\leq \frac{1}{6}\|\vec{y}\|^2_{H^{1+c}} + C\|\vec{\xi}_1\|^2_{L_3} \|\vec{y}\|^2_{L_2} \end{aligned}$$

and

$$\begin{aligned} |\langle (\vec{y} \cdot \nabla)\vec{v}_2, \vec{y} \rangle| &\leq \|\vec{y}\|_{L_2} \|\nabla \vec{v}_2\|_{L_3} \|\vec{y}\|_{L_6} \\ &\leq C\|\vec{y}\|_{L_2} \|\vec{\xi}_2\|_{L_3} \|\vec{y}\|_{H^1} \\ &\leq C_c\|\vec{y}\|_{L_2} \|\vec{\xi}_2\|_{L_3} \|\vec{y}\|_{H^{1+c}} \\ &\leq \frac{1}{6}\|\vec{y}\|^2_{H^{1+c}} + C\|\vec{\xi}_2\|^2_{L_3} \|\vec{y}\|^2_{L_2} \end{aligned}$$

Therefore

$$\frac{d}{dt}\|\vec{y}(t)\|^2_{L_2} \leq C\big(\|\vec{\xi}_1(t)\|^2_{L_3} + \|\vec{\xi}_2(t)\|^2_{L_3}\big)\|\vec{y}(t)\|^2_{L_2}$$

By Gronwall lemma, we get continuous dependence on the initial data. Uniqueness is obtained when $\vec{y}(0) = \vec{0}$ □

7 Equivalence of measures

Let $\mathcal{T} : \vec{\xi} \mapsto \vec{v}$ be the mapping giving the solution to (2.1).

We write system (2.4) as

$$\begin{cases} d\vec{\xi} + (-\Delta)^{1+c}\vec{\xi}\,dt + P[(\mathcal{T}\vec{\xi}\cdot\nabla)\vec{\xi}]\,dt - P[(\vec{\xi}\cdot\nabla)\mathcal{T}\vec{\xi}]\,dt = (-\Delta)^{-b}d\vec{w} \\ \nabla\cdot\vec{\xi} = 0 \end{cases} \tag{7.1}$$

and system (5.1) as

$$\begin{cases} d\vec{\eta} + (-\Delta)^{1+c}\vec{\eta}\,dt + P[(\mathcal{T}\vec{\eta}\cdot\nabla)\vec{\eta}]\,dt = (-\Delta)^{-b}d\vec{w} \\ \nabla\cdot\vec{\eta} = 0 \end{cases} \tag{7.2}$$

Denote by $\mathcal{L}_{\vec{\xi}}$ and $\mathcal{L}_{\vec{\eta}}$ the laws of the processes $\vec{\xi}$ and $\vec{\eta}$ respectively, when defined on a finite time interval $[0,T]$. Let $\sigma_T(\vec{\eta})$ denote the σ-algebra generated by $\{\vec{\eta}(t)\}_{0\leq t\leq T}$.

We recall the main result of [5], [7], in a form adapted to our context; indeed in those papers it was sufficient to assume weak existence (without uniqueness) for system (7.1).

Theorem 7.1. *Assume* (7.2) *and* (7.1) *have a unique basic solution with the same initial data in* H^2*. If*

$$\mathbb{P}\{\textstyle\int_0^T \|(-\Delta)^b P[(\vec{\eta}(t)\cdot\nabla)\mathcal{T}\vec{\eta}(t)]\|^2_{L_2}dt < \infty\} = 1, \tag{7.3}$$

$$\mathbb{P}\{\textstyle\int_0^T \|(-\Delta)^b P[(\vec{\xi}(t)\cdot\nabla)\mathcal{T}\vec{\xi}(t)]\|^2_{L_2}dt < \infty\} = 1, \tag{7.4}$$

then the laws $\mathcal{L}_{\vec{\xi}}$ *and* $\mathcal{L}_{\vec{\eta}}$*, defined as measures on the Borel subsets of* $C([0,T];H^2)$*, are equivalent.*

In particular for the Radon-Nykodim derivative we have

$$\frac{d\mathcal{L}_{\vec{\xi}}}{d\mathcal{L}_{\vec{\eta}}}(\vec{\eta}) = \mathbb{E}\left[e^{\int_0^T\langle(-\Delta)^bP[(\vec{\eta}(t)\cdot\nabla)\mathcal{T}\vec{\eta}(t)],d\vec{w}(s)\rangle - \frac{1}{2}\int_0^T\|(-\Delta)^bP[(\vec{\eta}(t)\cdot\nabla)\mathcal{T}\vec{\eta}(t)]\|^2_{L_2}ds}\Big|\sigma_T(\vec{\eta})\right] \tag{7.5}$$

$\mathbb{P}$*-a.s.*

From this we get our main result.

Theorem 7.2. *Let*

$$\begin{cases} c > \frac{1}{2} \\ b = 1 \end{cases}$$

If $\vec{\eta}(0) = \vec{\xi}(0) \in H^2$*, then the laws* $\mathcal{L}_{\vec{\xi}}$ *and* $\mathcal{L}_{\vec{\eta}}$ *are equivalent and* (7.5) *holds.*

Proof. We use Theorems 6.3, iii); notice that the conditions on b and c are fulfilled if $b = 1$ and $c > \frac{1}{2}$. We have only to check estimates (7.3) - (7.4) with $b = 1$. This follows easily, since H^2 is a multiplicative algebra and $\|\mathcal{T}\vec{\xi}\|_{H^3} \leq C\|\vec{\xi}\|_{H^2}$; therefore

$$\|P[(\vec{\xi}\cdot\nabla)\mathcal{T}\vec{\xi}]\|_{H^2} \leq C\|\vec{\xi}\|_{H^2}\|\nabla\mathcal{T}\vec{\xi}\|_{H^2} \leq C\|\vec{\xi}\|_{H^2}\|\mathcal{T}\vec{\xi}\|_{H^3} \leq C\|\vec{\xi}\|^2_{H^2}$$

and finally we use that the paths are in $C([0,T];H^2)$. □

We point out that the restriction $c > \frac{1}{2}$ cannot be weakened with this technique using

$$\|(-\Delta)^b P[(\vec{\zeta}\cdot\nabla)\mathcal{T}\vec{\zeta}]\|^2_{L_2} \le \|(\vec{\zeta}\cdot\nabla)\mathcal{T}\vec{\zeta}\|^2_{H^{2b}} \le C\|\vec{\zeta}\|^2_{H^{2b}}$$

for b large enough. Indeed, Proposition 4.1 provides $\zeta \in C([0,T];H^{2b})$ a.s. if $c > \frac{1}{2}$. And the paths of $\vec{\xi},\vec{\eta}$ cannot have better behavior than those of $\vec{\zeta}$.

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