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THE SNAPPING OUT BROWNIAN MOTION

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We give a probabilistic representation of a one-dimensional diffusion equation where the solution is discontinuous at 0 with a jump proportional to its flux. This kind of interface condition is usually seen as a semi-permeable barrier. For this, we use a process called here the snapping out Brownian motion, whose properties are studied. As this construction is motivated by applications, for example, in brain imaging or in chemistry, a simulation scheme is also provided.

1. Introduction. Many diffusion phenomena have to deal with interface conditions. Let D be a diffusivity coefficient which is smooth away from a regular surface S, but presents some discontinuity there. In this case, the solution to the diffusion equation

$$\partial_t u(t,x) = \frac{1}{2}\nabla\bigl(D(x)\nabla u(t,x)\bigr) = 0 \qquad \text{with } u(0,x) = f(x) \tag{1}$$

has to be understood as a *weak solution*. However, u is smooth away from S and satisfies

$$\begin{gathered} u(t,x+) = u(t,x-) \quad \text{and} \\ D(x+)n^+(x)\cdot\nabla u(t,x+) = D(x-)n^-(x)\cdot\nabla u(t,x-), \end{gathered} \tag{2}$$

for $x \in S$, when S is assumed to separate locally $\mathbb{R}^d$ into a "+" and a "−" part and where $n^\pm$ is a vector normal to S at x pointing to the "±" side. The second condition is called the *continuity of the flux*.

Now, let us assume that D takes scalar values, and is constant away from a thin layer of width 2ℓ enclosed between two parallel surfaces S^+ and S^-. When the width ℓ of the layer tends to 0, S^+ and S^- merge into a single interface located on a surface S.

When the diffusivity D_0 decreases to 0 with ℓ and $D_0/\ell \to \lambda > 0$, then the solution to (1) converges to a function v satisfying (1) away from S with the interface condition for $x \in S$:

$$\begin{gathered} \nabla v(t,x+) = \nabla v(t,x-) \quad \text{and} \\ \frac{\lambda}{2}\bigl(v(t,x+) - v(t,x-)\bigr) = D(x\pm)\nabla v(t,x\pm). \end{gathered} \tag{3}$$

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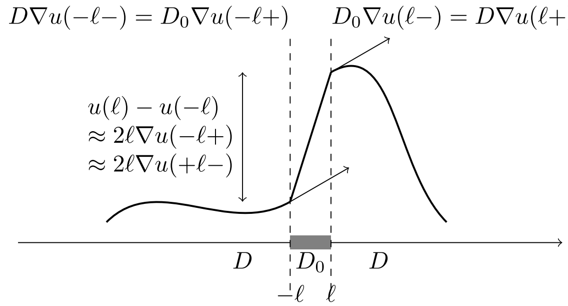

FIG. 1. *The thin layer problem.*

The solution has a continuous flux on S but is discontinuous on S (see, e.g., [33], Chapter 13). A heuristic explanation is given Figure 1.

If D is smooth on $\mathbb{R}^d$, it is well known that

$$u(t,x) = \mathbb{E}_x\big[f(X_t)\big], \tag{4}$$

where X is the diffusion process generated by $\frac{1}{2}\nabla(D\nabla)$ which is solution under $\mathbb{P}_x$ to the stochastic differential equation (SDE)

$$X_t = x + \int_0^t \sigma(X_s)\,\mathrm{d}B_t + \int_0^t \frac{1}{2}\sum_{i=1}^d \frac{D_{i,\cdot}}{\partial x_i}(X_s)\,\mathrm{d}s \qquad \text{with } \sigma\sigma^{\mathrm{T}} = D \tag{5}$$

for a Brownian motion B.

When D presents some discontinuities, (5) has no longer a meaning. However, a Feller processes $(X, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}})$ is associated to $\frac{1}{2}\nabla(D\nabla\cdot)$ for which (4) holds. In particular, the marginal distributions X_t have a density $p(t,x,\cdot)$ under $\mathbb{P}_x$, where $p(t,x,y)$ is the fundamental solution to (1) (see, e.g., [36]).

Let us now assume that the dimension of the space is equal to 1 and that D is discontinuous at some separated points $\{x_i\}$ with left and right limit there, and smooth elsewhere. The process X is solution to a SDE with local time. The Itô–Tanaka formula is the key tool to manipulate it, and several simulation algorithms have been proposed (see the references in [25], e.g.). The process called the *Skew Brownian motion* is the main tool for this construction [22, 24].

Coming back to the thin layer problem, we assume that D is constant and equal to D_1 on $(-\infty,-\ell)$ and (ℓ,∞), and to D_0 on $(-\ell,\ell)$. The associated stochastic process is solution to

$$X_t = x + \int_0^t \sqrt{D(X_s)}\,\mathrm{d}B_s + \frac{D_1 - D_0}{D_1 + D_0} L_t^{\ell}(X) + \frac{D_0 - D_1}{D_1 + D_0} L_t^{-\ell}(X),$$

where $L_t^{\pm\ell}(X)$ is the local time of X at $\pm\ell$ [24].

Letting D_0/ℓ converging to 2κ with $\ell \to 0$, one may expect that X converges in distribution to a stochastic process Y such that the solution to (1) with the interface condition (3) is given by $v(t,x) = \mathbb{E}_x[f(Y_t)]$.

The article then aims at constructing and giving several properties related to the process Y which we call a *snapping out Brownian motion* (SNOB). This process is Feller on $\mathbb{G} = (-\infty, 0-] \cup [0+, +\infty)$ but not on $\mathbb{R}$. The intervals in the definition of $\mathbb{G}$ are disjoint so that 0 corresponds either to $0+$ or $0-$ seen as distinct points.

The behavior of this process is the following: Assume that its starting point is $x \geq 0$. It behaves as a positively reflected Brownian motion until its local time is greater than an independent exponential random variable of parameter 2κ. Then its decides its sign with probability $1/2$ and starts afresh as a new reflected Brownian motion, until its local time is greater than a new exponential random variable, and so on. Using the properties of the exponential random variable, it is equivalent to assert that the particle changes its sign when its local time is greater than an exponential random variable with parameter κ, and behaves like a positively or negatively reflected Brownian motion between these switching times.

Its name is justified by the following fact: As the time at which the particle possibly changes it signs is the same as for the *elastic Brownian motion* [10, 15, 18, 19] (also called the *partially reflected Brownian motion*), it could also be seen as some elastic Brownian motion which is reborn once killed.

The elastic Brownian motion, also called a *partially reflected Brownian motion*, is associated to the Robin boundary condition and has then many applications [8, 15, 35]. This process is the "basic brick" for constructing the SNOB.

The behavior of the SNOB justifies also the old heuristic that the interface condition (3) corresponds to a *semi-permeable barrier*, which arises, for example, in diffusion magnetic resonance imaging [11] or in chemistry [1, 8]. The interface condition (3) is different from (2), to which is associated a Skew Brownian motion and where the particle crosses the interface when it reaches it, and which corresponds to a *permeable barrier* (see references in [22, 25]).

Here, we work under the condition of a single interface at 0. In short time, it is sufficient to describe the behavior of the process even in a more complex media, since other interface or boundary conditions far enough have "exponentially small" influence on the distribution of the process. This is sufficient for simulation purposes, where particles positions are represented by the stochastic process and move according to its dynamic.

Using similar computations, one may generalize our work to the case where $D(x) = D^+$ if $x \geq 0$, D^- if $x \leq 0$ and an interface condition

$$\nabla u(t, 0+) = \beta \nabla u(t, 0-) \quad \text{and} \quad \lambda u(t, 0+) - \mu u(t, x-) = \nabla u(t, x+)$$

with $\lambda, \mu > 0$. Diffusions on graphs specified by a condition at each vertex could also be considered, which could be of interest in several applications. This process has been described without proof by Bobrowski in [6], which have studied its limit behavior when the diffusion coefficients increase.

Although the SNOB may be seen as a diffusion on a graph, it is not a diffusion on a metric graph, where the edges are joined by vertices. Such diffusions have been classified by Freidlin and Wentzell in [12, 13]. The conditions that are required at the vertices of the graphs are some extension of the possible boundary conditions for a Markov process studied by Feller [10]. See also [21], for example, for the related problem of pasting diffusions.[2]

Our interface condition does not fall in these categories. Our process is best thought as a kind of *random evolution process* which switches back and forth randomly among a collection of processes (see, e.g., [16, 34]).

Outline. In Sections 2 and 3, we present quickly the main results related to the elastic Brownian motions and the piecing out procedure. The SNOB is constructed in Section 4 through its resolvent. In Section 5, we show the relationship between the SNOB and the thin layer problem. Finally, in Section 6, we show how to simulate this process.

2. Elastic Brownian motion. Let $(R_t)_{t\geq 0}$ a reflected Brownian motion, and denote by $(L_t)_{t\geq 0}$ its symmetric local time at 0. We add a cemetery point $\dagger$ to $\mathbb{R}_+$. For a constant $\kappa > 0$, we consider an exponential random variable ξ with parameter κ independent from B. Set

$$Z_t = \begin{cases} R_t, & \text{if } L_t \leq \xi, \\ \dagger, & \text{if } L_t > \xi. \end{cases}$$

Thanks to the properties of the local time, this process, called the *elastic Brownian motion* (EBM), is still a strong Markov process. Its semi-group is

$$P_t^{\mathrm{e}} f(x) = \mathbb{E}_x\big[\exp(-\kappa L_t) f(X_t)\big]$$

for f in the set $\mathcal{C}_0(\mathbb{R}_+, \mathbb{R})$ of continuous functions that vanishes at infinity. Closed form expressions of the density transition function are given in [14, 35].

Let $\mathfrak{k}$ be the time at which the EBM is killed, which means $\mathfrak{k} = \inf\{t > 0 | L_t \geq \xi\}$. This is a stopping time. Since the local time increases only on the closure of $\mathcal{Z} = \{t > 0 | X_t = 0\}$, it holds that $Z_{\mathfrak{k}} = 0$ almost surely. Using standard computations in the inverse of the local time of the Brownian motion,

$$\psi(x, \alpha) = \mathbb{E}_x\big[\exp(-\alpha\mathfrak{k})\big] = \frac{\kappa}{\sqrt{2\alpha} + \kappa} \exp(-\sqrt{2\alpha} x). \tag{6}$$

Using the Itô formula, it is easily shown that $u(t, x) = P_t^{\mathrm{e}} f(x)$ is solution to the heat equation with Robin (or third kind) boundary condition [3, 15, 31]

$$\begin{cases} \dfrac{\partial u(t, x)}{\partial t} = \dfrac{1}{2}\triangle u(t, x), & \text{on } (0, +\infty)^2, \\ \dfrac{\partial u(t, 0)}{\partial x} = \kappa u(t, 0). \end{cases}$$

[2]The article [32] defines a notion of *semipermeable membrane* which is different from ours, where the solution is continuous with a discontinuous gradient.

For a Markov process X, let us recall that its resolvent $(G_\alpha)_{\alpha>0}$ is a family of operators defined by $G_\alpha f(x) = \mathbb{E}_x[\int_0^{+\infty} e^{-\alpha s} f(X_s)\,\mathrm{d}s]$ for any $f \in \mathcal{C}_0$ and any $\alpha > 0$. It has a density g_α when $G_\alpha f(x) = \int g_\alpha(x, y) f(y)\,\mathrm{d}y$.

Using standard computations on the Green functions, the density $g_\alpha^{\mathrm{e}}(x, y)$ of the resolvent of the EBM is for $x, y \geq 0$,

$$g_\alpha^{\mathrm{e}}(x, y) = \frac{1}{\sqrt{2\alpha}} \begin{cases} \dfrac{\sqrt{2\alpha} - \kappa}{\sqrt{2\alpha} + \kappa} e^{-\sqrt{2\alpha}(y+x)} + e^{-\sqrt{2\alpha}(x-y)}, & \text{for } y \in [0, x], \\ e^{\sqrt{2\alpha}(x-y)} + \dfrac{\sqrt{2\alpha} - \kappa}{\sqrt{2\alpha} + \kappa} e^{-\sqrt{2\alpha}(x+y)}, & \text{for } y \geq x. \end{cases}$$

We extend the EBM to a process on $\mathbb{G}$ by symmetry, so that its resolvent becomes

$$G_\alpha^{\mathrm{e}} f(x) := \mathbb{E}_x\left[\int_0^{\mathfrak{k}} e^{-\alpha s} f(X_s)\,\mathrm{d}s\right] = \int_0^{+\infty} g_\alpha^{\mathrm{e}}\big(|x|, y\big) f\big(\mathrm{sgn}(x) y\big)\,\mathrm{d}y \tag{7}$$

for $x \in \mathbb{G}$. This process evolves either on $\mathbb{R}_-$ or $\mathbb{R}_+$ but never crosses 0 and is naturally identified with a process on $\mathbb{G}$.

3. Piecing out Markov processes. The procedure of piecing out is a way to construct a Markov process from a killed one. We present in this section a result due to Ikeda, Nagasawa and Watanabe [17] (similar considerations are given in [29]).

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a state space $\mathbb{S}$, let $((X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\overline{\mathbb{S}}})$ be a right continuous strong Markov process living in the extended state space $\mathbb{S}^\dagger = \mathbb{S} \cup \{\dagger\}$ with a death point $\dagger$. The lifetime of X is denoted by $\mathfrak{k}$.

The shift operator associated to X is denoted by $(\theta_t)_{t\geq 0}$.

We also consider a family μ defined on $\Omega \times \mathbb{S}^\dagger$ such that $\mu(\omega, \cdot)$ is a probability measure on $\mathbb{S}^\dagger$ and for any fixed Borel subset A, $\mu(\cdot, A)$ is $\sigma(X_t, t \geq 0)$-measurable. We assume additionally that $\mu(\omega, \mathrm{d}y) = \delta_\dagger(\mathrm{d}y)$ when $\mathfrak{k}(\omega) = 0$ and

$$\mathbb{P}_x\big[\mu(\omega, \mathrm{d}y) = \mu(\theta_{\mathfrak{t}(\omega)}\omega, \mathrm{d}y), \mathfrak{t}(\omega) < \mathfrak{k}(\omega)\big] = \mathbb{P}_x[\mathfrak{t} < \mathfrak{k}]$$

for any stopping time $\mathfrak{t}$. The family μ, called an *instantaneous distribution*, describes the way the process is reborn once killed.

Let $\widehat{\Omega}$ be the product of an infinite, countable, number of copies of $\Omega \times \mathbb{S}^\dagger$. We define X on $\widehat{\Omega}$ by

$$X_t(\widehat{\omega}) = \begin{cases} x_t(\omega_1), & \text{if } t \in \big[0, \mathfrak{k}(\omega_1)\big), \\ y_1, & \text{if } t = \mathfrak{k}(\omega_1), \\ x_{t-\mathfrak{k}(\omega_1)}(\widetilde{\omega}_2), & \text{if } t \in \big(\mathfrak{k}(\omega_1), \mathfrak{k}(\omega_1) + \mathfrak{k}(\omega_2)\big), \\ y_2, & \text{if } t = \mathfrak{k}(\omega_2), \\ \cdots & \\ \dagger, & \text{if } t \geq \mathfrak{k}(\omega_1) + \cdots + \mathfrak{k}_N(\omega_N) \end{cases}$$

with $\widehat{\omega} = (\omega_1, y_1, \omega_2, y_2, \dots) \in \widehat{\Omega}$ and $N = \inf\{k \geq 0; \mathfrak{k}(\omega_k) = 0\}$.

We consider the probability measure

$$\widehat{\mathbb{P}}_x[\mathrm{d}\omega_1, \mathrm{d}x^1, \dots, \mathrm{d}\omega_n, \mathrm{d}x^n] = \mathbb{P}_x[\mathrm{d}\omega^1]\mu(\omega^1, \mathrm{d}x^1)\mathbb{P}_{x^1}[\mathrm{d}\omega^2]\mu(\omega^1, \mathrm{d}x^2) \cdots \mathbb{P}_{x^n}[\mathrm{d}\omega^2]\mu(\omega^n, \mathrm{d}x^n).$$

Under this measure $\widehat{\mathbb{P}}_x$, when the path $X(\omega)$ is killed, we let it reborn by placing it at the point x_1 with probability $\mu(\omega, \mathrm{d}x_1)$ and then start again.

We left the technical details about the construction of the probability space and the filtration and presents the main result on piecing out Markov process.

THEOREM 1 ([17]). *Using the above defined notation, there exists a probability space* $(\widehat{\Omega}, \widehat{\mathcal{B}}, \widehat{\mathbb{P}})$ *and a filtration* $(\widehat{\mathcal{B}}_t)_{t\geq 0}$ *on which* $(X, (\widehat{\mathcal{B}}_t)_{t\geq 0}, (\widehat{\mathbb{P}}_x)_{x\in\mathbb{S}^\dagger})$ *is a strong Markov process on* $\mathbb{S}^\dagger$ *with* $\mathbb{P}_\dagger[X_t = \dagger, \forall t \geq 0] = 1$.

4. The snapping out Brownian motion.

DEFINITION 1. A *snapping out Brownian motion* (SNOB) X is a strong Markov stochastic process living on $\mathbb{G}$ constructed by making EBM reborn on $0+$ or $0-$ with probability $1/2$ using the piecing-out procedure.

The sign of X changes with probability $1/2$ when its local time L_t at 0 is greater than $\mathfrak{u}_k$ with $\mathfrak{u}_0 = 0$, $\mathfrak{u}_k - \mathfrak{u}_{k-1} \sim \exp(\kappa)$ is independent from $(\mathfrak{u}_i)_{i\leq k-1}$. From the properties of the exponential and binomial distributions, the sign of X changes when its local time is greater than $\mathfrak{s}_k$ with $\mathfrak{s}_0 = 0$, $\mathfrak{s}_k - \mathfrak{s}_{k-1} \sim \exp(\kappa/2)$ is independent from $(\mathfrak{s}_i)_{i\leq k-1}$.

It is also immediate that $|X|$ is a reflected Brownian motion, where $|\cdot|$ is the canonical projection of $\mathbb{G}$ onto $[0, +\infty)$.

PROPOSITION 1. *The resolvent family* $(G_\alpha)_{\alpha>0}$ *of the SNOB is solution to*

$$\left(\alpha - \frac{1}{2}\triangle\right)G_\alpha f(x) = f(x) \qquad \text{for } x \in \mathbb{G}$$

$$\text{with } \nabla G_\alpha f(0+) = \nabla G_\alpha f(0-) \text{ and } \frac{\kappa}{2}\big(G_\alpha f(0+) - G_\alpha f(0-)\big) = \nabla G_\alpha f(0)$$

for any bounded, continuous function f *on* $\mathbb{G}$ *that vanishes at infinity.*

This proposition identifies the infinitesimal generator of the process X. The points $0+$ and $0-$ are then interpreted as the sides of a semi-permeable barrier.

PROOF OF PROPOSITION 1. From this very construction and the strong Markov property, for any continuous function f on $\mathbb{G}$ which vanishes at infinity,

$$G_\alpha f(x) = G^{\mathrm{e}}_\alpha f(x) + \frac{\psi(|x|, \alpha)}{2}\big(G_\alpha f(0+) + G_\alpha f(0-)\big), \tag{8}$$

where G^{e}_α is defined by (7).

Using $x=0+$ and $x=0-$ in (8) and summing the two resulting equations leads to

$$G_\alpha f(x) = G^{\mathrm{e}}_\alpha f(x) + \frac{\kappa e^{-\sqrt{2\alpha}|x|}}{2\sqrt{2\alpha}}\beta(f) \tag{9}$$
$$\text{with } \beta(f) = G^{\mathrm{e}}_\alpha f(0+) + G^{\mathrm{e}}_\alpha f(0-).$$

Then

$$G_\alpha f(x) + G_\alpha f(-x) = G^{\mathrm{e}}_\alpha f(x) + G^{\mathrm{e}}_\alpha f(-x) + \frac{\kappa}{\sqrt{2\alpha}} e^{-\sqrt{2\alpha}|x|}\beta(f), \tag{10}$$

$$G_\alpha f(x) - G_\alpha f(-x) = G^{\mathrm{e}}_\alpha f(x) - G^{\mathrm{e}}_\alpha f(-x). \tag{11}$$

Derivating (10) and setting $x=0+$, since $\nabla G^{\mathrm{e}}_\alpha f(0\pm) = \pm\kappa G^{\mathrm{e}}_\alpha f(0\pm)$,

$$\nabla G_\alpha f(0+) - \nabla G_\alpha f(0-) = 0.$$

Derivating (11),

$$\begin{aligned}2\nabla G_\alpha f(0\pm) &= \nabla G_\alpha f(0+) + \nabla G_\alpha f(0-) = \nabla G^{\mathrm{e}}_\alpha f(0+) + \nabla G^{\mathrm{e}}_\alpha f(0-)\\ &= \kappa\big(G^{\mathrm{e}}_\alpha f(0+) - G^{\mathrm{e}}_\alpha f(0-)\big) = \kappa\big(G_\alpha f(0+) - G_\alpha f(0-)\big).\end{aligned}$$

In addition, it is easily seen that $(\alpha - \frac{1}{2}\triangle)G_\alpha f = f$ since $\psi(x,\alpha)$ is solution to $(\alpha - \frac{1}{2}\triangle)\psi(x,\alpha) = 0$. The resolvent is then identified. $\square$

PROPOSITION 2. *The semi-group $(P_t)_{t\geq 0}$ of the SNOB has the following representation*:

$$\begin{aligned}P_t f(x) = {}&\mathbb{E}_x\left[\left(\frac{1+e^{-\kappa L_t}}{2}\right) f\big(\mathrm{sgn}(x)|B_t|\big)\right]\\ &+ \mathbb{E}_x\left[\left(\frac{1-e^{-\kappa L_t}}{2}\right) f\big(-\mathrm{sgn}(x)|B_t|\big)\right]\end{aligned} \tag{12}$$

for a Brownian motion B.

PROOF. Let us decompose a function f as its even and odd parts:

$$\hat f(x) = \tfrac{1}{2}\big(f(x) + f(-x)\big) \quad \text{and} \quad \check f(x) = \tfrac{1}{2}\big(f(x) - f(-x)\big).$$

Then $G^{\mathrm{e}}_\alpha \hat f(-x) = G^{\mathrm{e}}_\alpha \hat f(x)$ and $G^{\mathrm{e}}_\alpha \check f(-x) = -G^{\mathrm{e}}_\alpha \check f(x)$, so that $\beta(\check f) = 0$ for β defined by (9). Thus $G_\alpha \check f(x) = G^{\mathrm{e}}_\alpha \check f(x)$. In addition, since $\hat f(|x|) = \hat f(x)$ and the SNOB has the same distribution as the reflected Brownian motion $|B|$,

$$G_\alpha \hat f(x) = G^{\mathrm{r}}_\alpha \hat f(x) := \mathbb{E}_x\left[\int_0^{+\infty} e^{-\alpha s}\hat f\big(|B_s|\big)\,\mathrm{d}s\right].$$

This gives an alternative representation for the resolvent of the SNOB: $G_\alpha f(x) = G^{\mathrm{r}}_\alpha \hat{f}(x) + G^{\mathrm{e}}_\alpha \check{f}(x)$. Inverting the resolvent to recover the semi-group $(P_t)_{t\geq 0}$,

$$P_t f(x) = P^{\mathrm{r}}_t \hat{f}(x) + P^{\mathrm{e}}_t \check{f}(x) = \mathbb{E}_x\big[\hat{f}\big(|B_t|\big)\big] + \mathbb{E}_x\big[\exp(-\kappa L_t)\check{f}\big(\operatorname{sgn}(x)|B_t|\big)\big].$$

This expression could be arranged as (12). $\square$

5. The thin layer problem. We now fix $\varepsilon > 0$ and we consider the process X^ε generated by (see, e.g., [36] for general considerations on this process)

$$\mathcal{L}^\varepsilon := \frac{1}{2}\frac{\partial}{\partial x}\left(a^\varepsilon(x)\frac{\partial}{\partial x}\right) \qquad \text{with } a^\varepsilon(x) := \begin{cases} 1, & \text{when } x \notin [-\varepsilon, \varepsilon], \\ \kappa\varepsilon, & \text{when } x \in [-\varepsilon, \varepsilon] \end{cases}$$

whose domain $\mathrm{Dom}(\mathcal{L}^\varepsilon) = \{f \in \mathrm{L}^2(\mathbb{R}) | \mathcal{L}^\varepsilon f \in \mathrm{L}^2(\mathbb{R})\}$ is a subset of the Sobolev space $\mathrm{H}^1(\mathbb{R})$ [hence, any function in $\mathrm{Dom}(\mathcal{L}^\varepsilon)$ is identified with a continuous function], where $\mathrm{L}^2(\mathbb{R})$ is the set of square integrable functions on $\mathbb{R}$ with scalar product $\langle f, g\rangle = \int_{\mathbb{R}} f(x)g(x)\,\mathrm{d}x$. Let us set $[h](x) := h(x-) - h(x+)$ and

$$D^\varepsilon := \left\{ f \in \mathcal{C}^2\big((-\infty, -\varepsilon) \cup (-\varepsilon, \varepsilon) \cup (\varepsilon, \infty)\big) \;\middle|\; \begin{array}{l} f, f'' \in \mathrm{L}^2(\mathbb{R}), \\ [f](\pm\varepsilon) = 0, \\ \big[a^\varepsilon \nabla f\big](\pm\varepsilon) = 0 \end{array} \right\}. \tag{13}$$

For $k \geq 0$, we write $\mathcal{C}^k_{\mathrm{c}}(\mathbb{R})$ the set of functions with compact support and continuous derivatives up to order k. With an integration by parts, for $f \in D^\varepsilon$ and $g \in \mathcal{C}^2_{\mathrm{c}}(\mathbb{R})$,

$$\begin{aligned} \big\langle (\alpha - L)f, g\big\rangle = \alpha\langle f, g\rangle &+ \int_{\mathbb{R}} a^\varepsilon(x)\nabla f(x)\nabla g(x)\,\mathrm{d}x \\ &+ \big[a^\varepsilon \nabla f\big](-\varepsilon)g(-\varepsilon) - \big[a^\varepsilon \nabla f\big](\varepsilon)g(\varepsilon). \end{aligned}$$

Using this formula and the regularity of the solution to $(\alpha - L)f = g$ when $g \in \mathcal{C}^\infty(I, \mathbb{R})$ with $-\varepsilon, \varepsilon \notin I$, we easily get that D^ε contains $(\alpha - \mathcal{L}^\varepsilon)^{-1}(\mathcal{C}^\infty_{\mathrm{c}}(\mathbb{R}))$ and is then dense in $\mathrm{Dom}(\mathcal{L}^\varepsilon)$ for the operator norm $(\langle f, f\rangle + \langle Lf, Lf\rangle)^{1/2}$.

A fundamental solution may be associated to $\mathcal{L}^\varepsilon$, as well as a resolvent density g^ε_α, which we will compute explicitly.

This operator is self-adjoint with respect to $\langle\cdot,\cdot\rangle$, so that its resolvent density satisfies $g^\varepsilon_\alpha(x, y) = g^\varepsilon_\alpha(y, x)$. This process is a Feller process, and is a strong solution to the SDE with local time

$$X^\varepsilon_t = x + \int_0^t \sqrt{a^\varepsilon\big(X^\varepsilon_s\big)}\,\mathrm{d}B_s + \eta_\varepsilon L^\varepsilon_t\big(X^\varepsilon\big) - \eta_\varepsilon L^{-\varepsilon}_t\big(X^\varepsilon\big) \qquad \text{with } \eta_\varepsilon = \frac{1 - \kappa\varepsilon}{1 + \kappa\varepsilon},$$

where B is a Brownian motion and $L^x_t(X^\varepsilon)$ is the symmetric local time at x of X^ε (see, e.g., [24], and [4, 28] among others for general results on SDEs with local time).

In [10], Section 11, the elastic Brownian motion is constructed as the limit of a process which either jumps at ε or is killed with probability $\kappa\varepsilon$ when it arrives at 0.

Using the piecing out procedure, we construct a strong Markov process Z^ε by considering the process X^ε which is instantaneously replaced at $-\varepsilon$ or ε with probability $1/2$ when it reaches 0, and then behaving again as X^ε until it reaches 0, and so on. This process Z^ε could be identified as a process living in $\mathbb{G}$ by defining $\mathbb{P}_{0+}$ as $\mathbb{P}_\varepsilon$ and $\mathbb{P}_{0-}$ as $\mathbb{P}_{-\varepsilon}$, since the process is instantaneously killed when at 0.

THEOREM 2. *The process Z^ε with $Z^\varepsilon_0 = x$ converges in distribution to the SNOB starting from x in the Skorohod topology.*

The proof relies on the next two results.

PROPOSITION 3. *Let g^ε_α be the resolvent density of X^ε. Then $g^\varepsilon_\alpha(x,y)$ converges to $g(x,y)$ for any $x,y \neq 0$ and any $\alpha > 0$ as $\varepsilon \to 0$.*

REMARK 1. This result follows from classical results in deterministic homogenization theory (see, e.g., [33]) where the convergence holds in Sobolev spaces. Here, we consider a direct computational proof for the convergence of the Green kernel, which we use later.

PROOF OF PROPOSITION 3. We assume that $x > 0$ and we set $\mu := \sqrt{2\alpha}$ for some $\alpha > 0$. The resolvent density g^ε_α of X^ε has the form, for $x > \varepsilon$,

$$g^\varepsilon_\alpha(x,y) = \begin{cases} C_\varepsilon(x)e^{-\mu y}, & \text{for } y > x, \\ A_\varepsilon(x)e^{-\mu y} + B_\varepsilon(x)e^{\mu y}, & \text{for } y \in [\varepsilon, x], \\ H_\varepsilon(x)e^{\mu y/\sqrt{\kappa\varepsilon}} + E_\varepsilon(x)e^{-\mu y/\sqrt{\kappa\varepsilon}}, & \text{for } y \in [-\varepsilon, \varepsilon], \\ F_\varepsilon(x)e^{\mu y}, & \text{for } y < -\varepsilon. \end{cases}$$

By this, we mean that for any bounded, measurable function f,

$$\mathbb{E}_x\left[\int_0^{+\infty} e^{-\alpha t} f(X^\varepsilon_s)\,\mathrm{d}s\right] = \int_{\mathbb{R}} g^\varepsilon_\alpha(x,y) f(y)\,\mathrm{d}y.$$

The kernel g^α_ε satisfies the conditions

$$g^\varepsilon_\alpha(x,\varepsilon+) = g^\varepsilon_\alpha(x,\varepsilon-), \qquad g^\varepsilon_\alpha(x,\varepsilon-) = g^\varepsilon_\alpha(x,\varepsilon+),$$
$$\nabla_y g^\varepsilon_\alpha(x,-\varepsilon-) = \kappa\varepsilon\nabla_y g^\varepsilon_\alpha(x,-\varepsilon+),$$
$$\kappa\varepsilon\nabla_y g^\varepsilon_\alpha(x,\varepsilon-) = \nabla_y g^\varepsilon_\alpha(x,\varepsilon+),$$
$$\nabla_y g^\varepsilon_\alpha(x,x+) - \nabla_y g^\varepsilon_\alpha(x,x-) = 2.$$

With $\mu = \sqrt{2\alpha}$, the coefficients A_ε, B_ε, C_ε, H_ε and F_ε are then expressed with the help of

$$G_\varepsilon := \big(2e^{4\mu\varepsilon/\sqrt{\kappa\varepsilon}}\sqrt{\kappa\varepsilon} + e^{4\mu\varepsilon/\sqrt{\kappa\varepsilon}}\kappa\varepsilon + 2\sqrt{\kappa\varepsilon} - \kappa\varepsilon + e^{4\mu\varepsilon/\sqrt{\kappa\varepsilon}} - 1\big)\mu.$$

Since $\varepsilon \to 0$, $G_\varepsilon = 4\sqrt{\kappa\varepsilon}(1 + \frac{\mu}{\kappa} + \mathrm{O}(\kappa\varepsilon))\mu$. After tedious computations,

$$\begin{aligned}
A_\varepsilon(x) &= -\big(e^{4\mu\varepsilon/\sqrt{\kappa\varepsilon}}\kappa\varepsilon - \kappa\varepsilon - e^{4\mu\varepsilon/\sqrt{\kappa\varepsilon}} + 1\big)e^{\mu(2\varepsilon - x)}/G_\varepsilon \xrightarrow[\varepsilon\to 0]{} A_0(x) \\
&:= -\frac{e^{-\mu x}}{\kappa+\mu}, \\
B_\varepsilon(x) &= B_0(x) := -\frac{e^{-\mu x}}{\mu}, \\
C_\varepsilon(x) &= -2\sinh(2\varepsilon/\sqrt{\kappa\varepsilon})\big(de^{-\mu x + 2\mu\varepsilon} - e^{-\mu x + 2\mu\varepsilon} + de^{\mu x} + e^{-\mu x}\big) \\
&\qquad \times e^{2\mu\varepsilon/\sqrt{\kappa\varepsilon}}/G_\varepsilon + 4\sqrt{\kappa\varepsilon}e^{\mu x}\cosh(2\varepsilon/\sqrt{\kappa\varepsilon})e^{2\mu\varepsilon/\sqrt{\kappa\varepsilon}}/G_\varepsilon \\
&\xrightarrow[\varepsilon\to 0]{} C_0(x) := \frac{\kappa e^{\mu x}}{\mu(\kappa+\mu)}, \\
H_\varepsilon(x) &= -2e^{\mu(3\varepsilon + \varepsilon\sqrt{\kappa\varepsilon} - x\sqrt{\kappa\varepsilon})/\sqrt{\kappa\varepsilon}}(1+\sqrt{\kappa\varepsilon})\sqrt{\kappa\varepsilon}/G_\varepsilon \xrightarrow[\varepsilon\to 0]{} H_0(x) \\
&:= -\frac{\kappa e^{-\mu x}}{2\mu(\kappa+\mu)}, \\
E_\varepsilon(x) &= -2e^{\mu(\varepsilon + \varepsilon\sqrt{\kappa\varepsilon} - x\sqrt{\kappa\varepsilon})/\sqrt{\kappa\varepsilon}}(1-\sqrt{\kappa\varepsilon})\sqrt{\kappa\varepsilon}/G_\varepsilon \xrightarrow[\varepsilon\to 0]{} H_0(x), \\
F_\varepsilon(x) &= -4\sqrt{\kappa\varepsilon}e^{\mu(2\varepsilon + 2\varepsilon\sqrt{\kappa\varepsilon} - x\sqrt{\kappa\varepsilon})/\sqrt{\kappa\varepsilon}}/G_\varepsilon \xrightarrow[\varepsilon\to 0]{} F_0(x) := \frac{-\kappa e^{-\mu x}}{\mu(\kappa+\mu)} \\
&= -C_0(-x).
\end{aligned}$$

Let g_α be the function

$$g_\alpha(x,y) := \begin{cases} C_0(x)e^{-\mu y}, & \text{if } y > x, \\ A_0(x)e^{-\mu y} + B_0(x)e^{\mu y}, & \text{if } y \in [0,x], \\ F_0(x)e^{\mu y}, & \text{if } y < 0. \end{cases}$$

A similar work may be performed for $x < 0$. Thus, we easily obtain that $g_\alpha^\varepsilon(x,y) \longrightarrow_{\varepsilon\to 0} g_\alpha(x,y)$ converges to g_α and that g_α is the density resolvent of the SNOB by checking it satisfies the appropriate conditions at the interface. □

PROPOSITION 4. *Let $\mathfrak{h}_0^\varepsilon$ be the first hitting time of* 0 *for X^ε. Under $\mathbb{P}_x$, $\mathfrak{h}_0^\varepsilon$ converges in distribution to a random variable $\mathfrak{k}$ distributed as the lifetime of the EBM of parameter κ.*

PROOF. As in [9, 24], we introduce $\Phi^\varepsilon(x)$ as the piecewise linear function defined by

$$\frac{\mathrm{d}\Phi^\varepsilon}{\mathrm{d}x}(x) = \begin{cases} 1/\sqrt{\kappa\varepsilon}, & \text{if } x \in [-\varepsilon,\varepsilon], \\ 1, & \text{otherwise.} \end{cases}$$

Set $Y^\varepsilon = \Phi^\varepsilon(X^\varepsilon)$ so that Y^ε is solution to the SDE [9, 24]

$$Y^\varepsilon_t = \Phi^\varepsilon(x) + B_t + \theta^\varepsilon L_t^{y_\varepsilon}(Y^\varepsilon) - \theta^\varepsilon L_t^{-y_\varepsilon}(Y^\varepsilon)$$
$$\text{with } \theta^\varepsilon = \frac{1-\sqrt{\kappa\varepsilon}}{1+\sqrt{\kappa\varepsilon}} \text{ and } y^\varepsilon := \Phi^\varepsilon(\varepsilon) = \sqrt{\frac{\varepsilon}{\kappa}}.$$

The infinitesimal generator of Y^ε is $\mathcal{L}^\varepsilon := \frac{1}{2}\triangle$ whose domain contains as a dense subset [it is similar to the discussion on D^ε in (13)]

$$\left\{ f \in \mathcal{C}^2\big((-\infty, -y_\varepsilon) \cup (-y_\varepsilon, y_\varepsilon) \cup (y_\varepsilon, \infty)\big) \,\middle|\, \begin{array}{l} f, f'' \in \mathrm{L}^2(\mathbb{R}), \\ [f](\pm y^\varepsilon) = 0, \\ (1-\theta^\varepsilon) f'(y^\varepsilon-) \\ \qquad = (1+\theta^\varepsilon) f'(y^\varepsilon+), \\ (1+\theta^\varepsilon) f'(-y^\varepsilon-) \\ \qquad = (1-\theta^\varepsilon) f'(-y^\varepsilon+), \end{array} \right\}.$$

From now, we assume for the sake of simplicity that $x > 0$.

The hitting time $\mathfrak{h}_0^\varepsilon$ is also the first hitting time of zero by Y^ε. Since by symmetry $\psi(-x,\alpha) = \psi(x,\alpha)$ for any $x \geq 0$, we consider only that $x \geq 0$.

Since the Feynman–Kac formula is valid for the process Y^ε, $\psi^\varepsilon(x,\alpha) := \mathbb{E}_x[e^{-\alpha \mathfrak{h}_0^\varepsilon}]$ is solution to

$$\begin{cases} \frac{1}{2}\triangle \psi^\varepsilon(x,\alpha) = \alpha \psi^\varepsilon(x,\alpha), & \text{for } x \neq y^\varepsilon, \\ \psi^\varepsilon(0,\alpha) = 1, \\ \psi^\varepsilon(y^\varepsilon-,\alpha) = \psi^\varepsilon(y^\varepsilon+,\alpha), \\ (1-\theta^\varepsilon)\nabla_x \psi^\varepsilon(y^\varepsilon-,\alpha) = (1+\theta^\varepsilon)\nabla_x \psi^\varepsilon(y^\varepsilon+,\alpha). \end{cases}$$

Hence, $\psi^\varepsilon(x,\alpha)$ is sought as

$$\psi^\varepsilon(x,\alpha) = \begin{cases} \gamma^\varepsilon \exp(-\sqrt{2\alpha}x), & \text{if } x > y^\varepsilon, \\ \cos(\sqrt{2\alpha}x) + \beta^\varepsilon \sin(\sqrt{2\alpha}x), & \text{if } x \in [0, y^\varepsilon]. \end{cases}$$

After some computations,

$$\beta^\varepsilon = \frac{-\cos(\sqrt{2\alpha}y^\varepsilon) + \sqrt{\kappa\varepsilon}\sin(\sqrt{2\alpha}y^\varepsilon)}{\sin(\sqrt{2\alpha}y^\varepsilon) + \sqrt{\kappa\varepsilon}\cos(\sqrt{2\alpha}y^\varepsilon)} \quad \text{and} \quad \sqrt{\varepsilon}\beta^\varepsilon \underset{\varepsilon\to 0}{\sim} \frac{-\sqrt{\kappa}}{\kappa+\sqrt{2\alpha}}.$$

Besides,

$$\gamma^\varepsilon = e^{\sqrt{2\alpha}y^\varepsilon}\sqrt{\kappa\varepsilon}\big(\beta^\varepsilon \cos(\sqrt{2\alpha}y^\varepsilon) - \beta^\varepsilon \sin(\sqrt{2\alpha}y^\varepsilon)\big) \underset{\varepsilon\to 0}{\sim} \frac{\kappa}{\kappa+\sqrt{2\alpha}}.$$

Hence, for any $x > 0$,

$$\psi^\varepsilon(x,\alpha) \xrightarrow[\varepsilon\to 0]{} \psi(x,\alpha) := \frac{\kappa}{\kappa+\sqrt{2\alpha}} e^{-\sqrt{2\alpha}x} \tag{14}$$

with ψ defined by (6).

This proves that under $\mathbb{P}_x$, $\mathfrak{h}_0^\varepsilon$ converges to a random variable $\mathfrak{k}$ whose Laplace transform is $\psi(x,\alpha)$ under $\mathbb{P}_x$. This random variable $\mathfrak{k}$ is then the lifetime of an EBM. $\square$

PROOF OF THEOREM 2. Using the properties of the resolvent, for $\alpha > 0$ and a bounded, measurable function f,

$$\begin{aligned} G_\alpha^\varepsilon f(x) &:= \mathbb{E}_x\left[\int_0^{+\infty} e^{-\alpha t} f(X_s^\varepsilon)\,\mathrm{d}s\right] \\ &= R_\alpha^\varepsilon f(x) + \mathbb{E}_x\big[e^{-\alpha \mathfrak{h}_0^\varepsilon}\big]\frac{1}{2}\big(G_\alpha^\varepsilon f(\varepsilon) + G_\alpha^\varepsilon f(-\varepsilon)\big) \end{aligned}$$

with

$$R_\alpha^\varepsilon f(x) := \mathbb{E}_x\left[\int_0^{\mathfrak{h}_0^\varepsilon} e^{-\alpha t} f(X_s^\varepsilon)\,\mathrm{d}s\right].$$

Since $\psi^\varepsilon(x,\alpha) = \psi^\varepsilon(-x,\alpha)$,

$$G_\alpha^\varepsilon f(x) = R_\varepsilon^\alpha f(x) + \frac{\psi^\varepsilon(x,\alpha)}{1-\psi^\varepsilon(\varepsilon,\alpha)}\frac{R_\varepsilon^\alpha f(\varepsilon) + R_\varepsilon^\alpha f(-\varepsilon)}{2}.$$

For the sake of simplicity, we assume that $x > 0$. Using the symmetry properties of $\mathcal{L}^\varepsilon$,

$$R_\alpha^\varepsilon f(x) = \int_0^{+\infty} \big(g_\alpha^\varepsilon(x,y) - g_\alpha^\varepsilon(x,-y)\big) f(y)\,\mathrm{d}y.$$

But

$$g_\alpha^\varepsilon(x,y) - g_\alpha^\varepsilon(x,-y) \underset{\varepsilon\to 0}{\longrightarrow} g_\alpha(x,y) - g_\alpha(x,-y) = g_\alpha^{\mathrm{e}}(x,y),$$

where $g_\alpha^{\mathrm{e}}(x,y)$ is the resolvent density of the EBM. Thus, $R_\varepsilon^\alpha f(x) \longrightarrow_{\varepsilon\to 0} G_\alpha^{\mathrm{e}} f(x)$ for any $x > 0$. It is also easily obtained that

$$R_\varepsilon^\alpha f(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} G_\alpha^{\mathrm{e}} f(0+) \quad \text{and} \quad R_\varepsilon^\alpha f(-\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} G_\alpha^{\mathrm{e}} f(0-).$$

Using (9) and (14), $G_\alpha^\varepsilon f(x) \longrightarrow_{\varepsilon\to 0} G_\alpha f(x)$. The Trotter–Kato theorem (see, e.g., [20], Theorem IX.2.16, page 504) and the Markov property imply the convergence in finite-dimensional distributions of Z^ε to X under $\mathbb{P}_x$ for $x \geq 0$. By symmetry, this could be extended to $x \leq 0$.

The only remaining point of the tightness. When away from $[-\varepsilon,\varepsilon]$, X^ε behaves like a Brownian motion. Hence, for $0 \leq s \leq t \leq T$, let us set $\mathfrak{f}(s,t) := \inf\{u > s; |X_u^\varepsilon| = \varepsilon\}$ with possibly $\mathfrak{f}(s,t) = +\infty$ and $\mathfrak{l}(s,t) := \sup\{u < t; |X_u^\varepsilon| = \varepsilon\}$ with possibly $\mathfrak{l}(s,t) = -\infty$.

If $\mathfrak{f}(s,t) \geq t$ and $\mathfrak{l}(s,t) \leq s$, then for $\delta < 1/2$, there exists an integrable random variable $C(\omega)$ such that $|X_t^\varepsilon(\omega) - X_s^\varepsilon(\omega)| \leq C(\omega)(t-s)^\delta$ for any $0 \leq s \leq t \leq T$.

If $\mathfrak{f}(s,t) \le t$ and $\mathfrak{l}(s,t) \le s$, then

$$|X_t^\varepsilon - X_s^\varepsilon| \le |X_{\mathfrak{f}(s,t)}^\varepsilon - X_s^\varepsilon| + |X_t^\varepsilon - X_{\mathfrak{f}(s,t)}^\varepsilon| \le C(t-s)^\beta + 2\varepsilon$$

since X_t^ε belongs to $[-\varepsilon, \varepsilon]$. A similar analysis could be carried for the other cases, which means that for some integrable random variable C,

$$\sup_{|t-s|<\delta} |X_t^\varepsilon - X_s^\varepsilon| \le C\delta^\beta + 2\varepsilon.$$

This proves that $(Z^\varepsilon)_{\varepsilon>0}$ is tight is the space $\mathcal{D}([0,T];\mathbb{R})$ of discontinuous functions with the Skorohod topology (see, e.g., [5]) and then on $\mathcal{D}([0,T];\mathbb{G})$. Hence, we easily deduce the convergence of Z^ε to the SNOB in $\mathcal{D}([0,T];\mathbb{G})$. $\square$

6. Simulation of the SNOB. It is easy to simulate a discretized process X in the same way it is easy to simulate the Brownian motion. Following Proposition 2, we draw a random variate with density $p(\delta t, x, \cdot)$ when x is close enough to 0.

For this, we use a Brownian bridge technique to check if the process reaches $0\pm$ before δt (see, e.g., [2] and [25], Section B.2, for an example of application and further references). This involve the inverse Gaussian distribution $\mathcal{IG}(\lambda, \mu)$ whose density is $r_{\mu,\lambda}(x) = \sqrt{\frac{\lambda}{2\pi x^3}}\exp(\frac{-\lambda(x-\mu)^2}{2\mu^2 x})$. Random variates with $\mathcal{IG}$ distribution could be simulated by the methods proposed in [7], page 148 and [30].

We simulate the local time using the following representation under $\mathbb{P}_0$ [26, 27]:

$$(L_t^0(B), |B_t|) \overset{\text{dist}}{=} (\mathfrak{l}, \mathfrak{l} - H) \qquad \text{where } \mathfrak{l} := \tfrac{1}{2}\big(H + \sqrt{V + H^2}\big)$$

with $H \sim \mathcal{N}(0,t)$ and $V \sim \exp(1/2t)$ independent from H.

The generic algorithm to simulate the process at time δt when at point x at time 0 is the following:

1. Set $y := x + \sqrt{\delta t}G$ with G a random variate whose distribution is $\mathcal{N}(0,1)$.
2. If $|x| \ge 4\sqrt{\delta t}$, then return y (here, we neglect the exponentially small probability that the process crosses 0 between the times 0 and δt).
3. If $xy > 0$, then decide with probability $\exp(-2|xy|/\delta t)$ if the path X has crossed 0.
   - If no crossing occurs, then return y.
   - If a crossing occurs, draw $\mathfrak{g} \sim \mathcal{IG}(|x|/|y|, x^2/2\delta t)$, so that $\mathfrak{z} := \delta t\mathfrak{g}/(1+\mathfrak{g})$ is a realization of the first hitting time of 0 for a Brownian bridge with $B_0 = x$ and $B_{\delta t} = y$. Then go the step 5.
4. If $xy < 0$, then draw $\mathfrak{g} \sim \mathcal{IG}(-|x|/|y|, x^2/2\delta t)$ and set $\mathfrak{z} := \delta t\mathfrak{g}/(1+\mathfrak{g})$, the first time the Brownian bridge reaches 0. Go to step 5.
5. Set $\mathfrak{r} := \delta t - \mathfrak{z}$. For two independent random variates $H \sim \mathcal{N}(0,\mathfrak{r})$ and $V \sim \exp(1/2\mathfrak{r})$, set $\mathfrak{l} := (H + \sqrt{V + H^2})/2$.

6. For $U \sim \mathcal{U}(0,1)$ independent from V and H, set $\mathfrak{s} := \operatorname{sgn}(x)$ if $\exp(-\kappa \mathfrak{l}) \geq 2U - 1$. Otherwise, set $\mathfrak{s} := -\operatorname{sgn}(x)$.
7. Return $\mathfrak{s}(\mathfrak{l} - H)$.

An application to the estimation of a macroscopic estimation parameter in the context of a simplified problem related to brain imaging may be found in [23]. The results are satisfactory, unless κ is too small due to a problem of rare event simulation.

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