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RATE OF CONVERGENCE AND ASYMPTOTIC ERROR DISTRIBUTION OF EULER APPROXIMATION SCHEMES FOR FRACTIONAL DIFFUSIONS

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For a stochastic differential equation(SDE) driven by a fractional Brownian motion(fBm) with Hurst parameter $H > \frac{1}{2}$, it is known that the existing (naive) Euler scheme has the rate of convergence n^{1-2H}. Since the limit $H \to \frac{1}{2}$ of the SDE corresponds to a Stratonovich SDE driven by standard Brownian motion, and the naive Euler scheme is the extension of the classical Euler scheme for Itô SDEs for $H = \frac{1}{2}$, the convergence rate of the naive Euler scheme deteriorates for $H \to \frac{1}{2}$. In this paper we introduce a new (modified Euler) approximation scheme which is closer to the classical Euler scheme for Stratonovich SDEs for $H = \frac{1}{2}$, and it has the rate of convergence γ_n^{-1}, where $\gamma_n = n^{2H-1/2}$ when $H < \frac{3}{4}$, $\gamma_n = n/\sqrt{\log n}$ when $H = \frac{3}{4}$ and $\gamma_n = n$ if $H > \frac{3}{4}$. Furthermore, we study the asymptotic behavior of the fluctuations of the error. More precisely, if $\{X_t, 0 \le t \le T\}$ is the solution of a SDE driven by a fBm and if $\{X_t^n, 0 \le t \le T\}$ is its approximation obtained by the new modified Euler scheme, then we prove that $\gamma_n(X^n - X)$ converges stably to the solution of a linear SDE driven by a matrix-valued Brownian motion, when $H \in (\frac{1}{2}, \frac{3}{4}]$. In the case $H > \frac{3}{4}$, we show the L^p convergence of $n(X_t^n - X_t)$, and the limiting process is identified as the solution of a linear SDE driven by a matrix-valued Rosenblatt process. The rate of weak convergence is also deduced for this scheme. We also apply our approach to the naive Euler scheme.

1. Introduction. Consider the following stochastic differential equation (SDE) on $\mathbb{R}^d$:

$$X_t = x + \int_0^t b(X_s)\,ds + \sum_{j=1}^m \int_0^t \sigma^j(X_s)\,dB_s^j, \qquad t \in [0, T], \tag{1.1}$$

where $x \in \mathbb{R}^d$, $B = (B^1, \dots, B^m)$ is an m-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $b, \sigma^1, \dots, \sigma^m : \mathbb{R}^d \to \mathbb{R}^d$ are continuous functions. The above stochastic integrals are pathwise Riemann–Stieltjes integrals. If $\sigma^1, \dots, \sigma^m$ are continuously differentiable and their partial derivatives

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are bounded and locally Hölder continuous of order $\delta > \frac{1}{H} - 1$ and b is Lipschitz, then equation (1.1) has a unique solution which is Hölder continuous of order γ for any $0 < \gamma < H$. This result was first proved by Lyons [14], using Young integrals (see [33]) and p-variation estimates, and later by Nualart and Rascanu [25], using fractional calculus; see [34].

We are interested in numerical approximations for the solution to equation (1.1). For simplicity of presentation, we consider uniform partitions of the interval $[0, T]$, $t_i = \frac{iT}{n}$, $i = 0, \dots, n$. For every positive integer n, we define $\eta(t) = t_i$ when $t_i \le t < t_i + \frac{T}{n}$. The following naive Euler numerical approximation scheme has been previously studied:

$$X_t^n = x + \int_0^t b(X_{\eta(s)}^n)\,ds + \sum_{j=1}^m \int_0^t \sigma^j(X_{\eta(s)}^n)\,dB_s^j, \qquad t \in [0, T]. \tag{1.2}$$

This scheme can also be written as

$$X_t^n = X_{t_k}^n + b(X_{t_k}^n)(t - t_k) + \sum_{j=1}^m \sigma^j(X_{t_k}^n)(B_t^j - B_{t_k}^j),$$

for $t_k \le t \le t_{k+1}$, $k = 0, 1, \dots, n-1$ and $X_0^n = x$. It was proved by Mishura [17] that for any real number $\varepsilon > 0$ there exists a random variable C_ε such that almost surely,

$$\sup_{0 \le t \le T} |X_t^n - X_t| \le C_\varepsilon n^{1-2H+\varepsilon}.$$

Moreover, the convergence rate n^{1-2H} is sharp for this scheme, in the sense that $n^{2H-1}[X_t^n - X_t]$ converges almost surely to a finite and nonzero limit. This has been proved in the one-dimensional case by Nourdin and Neuenkirch in [18] using the Doss representation of the solution; see also Theorem 10.1 below. Notice that while H tends to $\frac{1}{2}$, the convergence rate $2H - 1$ of the numerical scheme (1.2) deteriorates, and so it is not a proper extension of the Euler–Maruyama scheme for the case $H = \frac{1}{2}$; see, for example, [7, 12]. This is not surprising because the limit $H \to \frac{1}{2}$ of the SDE (1.1) corresponds to a Stratonovich SDE driven by standard Brownian motion, while the Euler scheme (1.2) is the extension of the classical Euler scheme for the Itô SDEs. It is then natural to ask the following question: Can we find a numerical scheme that generalizes the Euler–Maruyama scheme to the fBm case?

In this paper we introduce the following new approximation scheme that we call a *modified Euler scheme*:

$$\begin{aligned} X_t^n = x &+ \int_0^t b(X_{\eta(s)})\,ds + \sum_{j=1}^m \int_0^t \sigma^j(X_{\eta(s)}^n)\,dB_s^j \\ &+ H \sum_{j=1}^m \int_0^t (\nabla\sigma^j \sigma^j)(X_{\eta(s)}^n)(s - \eta(s))^{2H-1}\,ds, \end{aligned} \tag{1.3}$$

or

$$X_t^n = X_{t_k}^n + b(X_{t_k}^n)(t-t_k) + \sum_{j=1}^m \sigma^j(X_{t_k}^n)(B_t^j - B_{t_k}^j)$$
$$+ \frac{1}{2}\sum_{j=1}^m (\nabla\sigma^j\sigma^j)(X_{t_k}^n)(t-t_k)^{2H},$$

for any $t \in [t_k, t_{k+1}]$ and $X_0^n = x$. Here $\nabla\sigma^j$ denotes the $d \times d$ matrix $(\frac{\partial\sigma^{j,i}}{\partial x_k})_{1\le i,k\le d}$, and $(\nabla\sigma^j\sigma^j)^i = \sum_{k=1}^d \frac{\partial\sigma^{j,i}}{\partial x_k}\sigma^{j,k}$.

Notice that if we formally set $H = \frac{1}{2}$ and replace B by a standard Brownian motion W, this is the classical Euler scheme for the Stratonovich SDE,

$$X_t = x + \int_0^t b(X_s)\,ds + \sum_{j=1}^m \int_0^t \sigma^j(X_s)\,dW_s^j$$
$$= x + \int_0^t b(X_s)\,ds + \sum_{j=1}^m \int_0^t \sigma^j(X_s)\delta W_s^j + \frac{1}{2}\int_0^t \sum_{j=1}^m (\nabla\sigma^j\sigma^j)(X_s)\,ds.$$

In the above and throughout this paper, d denotes the Stratonovich integral, and δ denotes the Itô (or Skorohod) integral.

For our new modified Euler scheme (1.3) we shall prove the following estimate:

$$\sup_{0\le t\le T} (\mathbb{E}|X_t - X_t^n|^p)^{1/p} \le C\gamma_n^{-1}, \tag{1.4}$$

for any $p \ge 1$, where

$$\gamma_n = \begin{cases} n^{2H-1/2}, & \text{if } \frac{1}{2} < H < \frac{3}{4}, \\ \frac{n}{\sqrt{\log n}}, & \text{if } H = \frac{3}{4}, \\ n, & \text{if } \frac{3}{4} < H < 1. \end{cases} \tag{1.5}$$

Note that in (1.4), if we formally set $H = \frac{1}{2}$, then the convergence rate is $n^{-1/2}$, which is exactly the convergence rate of the classical Euler–Maruyama scheme in the Brownian motion case. This suggests that the modified Euler scheme should be viewed as an authentic modified version of the Euler–Maruyama scheme (1.2). The cutoff of the convergence rate for the Euler scheme has already been observed in a simpler context in [19]. The Lévy area corresponds to the simple SDE with $b = 0$, $\sigma^1(x,y) = (1,0)$, $\sigma^2(x,y) = (0,x)$. In particular, one has $\nabla\sigma^j\sigma^j = 0$, $j = 1,2$ here, that is, no diagonal noise.

The proof of this result combines the techniques of Malliavin calculus with classical fractional calculus. On the other hand, we make use of uniform estimates for the moments of all orders of the processes X, X^n and their first and second-order Malliavin derivatives, which can be obtained using techniques of fractional calculus, following the approach used, for instance, by Hu and Nualart [8]. The idea of the proof is to properly decompose the error $X_t - X_t^n$ into a weighted quadratic variation term plus a higher order term, that is,

$$X_t - X_t^n = \sum_{i,j=1}^{m} \sum_{k=0}^{\lfloor nt/T \rfloor} f^{i,j}(t_k) \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \delta B_u^i \delta B_s^j + R_t^n, \tag{1.6}$$

where $\lfloor x \rfloor$ denotes the integer part of a real number x. The weighted quadratic variation term provides the desired rate of convergence in L^p.

To further study this new scheme and compare it to the classical Brownian motion case, it is natural to ask the following questions: Is the above rate of convergence (1.4) exact or not? Namely, does the quantity $\gamma_n(X_t - X_t^n)$ have a nonzero limit? If yes, how does one identify the limit, and is there a similarity to the classical Brownian motion case (see [10, 13])? In the second part of the paper, we give a complete answer to these questions. The weighted variation term in (1.6) is still a key ingredient in our study of the scheme. As in the Breuer–Major theorem, there is a different behavior in the cases $H \in (\frac{1}{2}, \frac{3}{4}]$ and $H \in (\frac{3}{4}, 1)$. If $H \in (\frac{1}{2}, \frac{3}{4}]$, we show that $\gamma_n(X_t - X_t^n)$ converges stably to the solution of a linear stochastic differential equation driven by a matrix-valued Brownian motion W independent of B. The main tools in this case are Malliavin calculus and the fourth moment theorem. We will also make use of a recent limit theorem in law for weighted sums proved in [3]. In the case $H \in (\frac{3}{4}, 1)$, we show the convergence of $\gamma_n(X_t - X_t^n)$ in L^p to the solution of a linear stochastic differential equation driven by a matrix-valued Rosenblatt process. Again we use the technique of Malliavin calculus and the convergence in L^p of weighted sums, which is obtained applying the approach introduced in [3]. We refer to [20] for a discussion on the asymptotic behavior of some weighted Hermite variations of one-dimensional fBm, which are related with the results proved here.

We also consider a weak approximation result for our new numerical scheme. In this case, the rate is n^{-1} for all values of H. More precisely, we are able to show that $n[\mathbb{E}(f(X_t)) - \mathbb{E}(f(X_t^n))]$ converges to a finite nonzero limit which can be explicitly computed. This extends the result of [31] to $H > \frac{1}{2}$. Let us mention that the techniques of Malliavin calculus also allow us to provide an alternative and simpler proof of the fact that the rate of convergence of the numerical scheme (1.2) is of the order n^{1-2H}, and this rate is optimal, extending to the multidimensional case the results by Neuenkirch and Nourdin [18].

If the driven process is a standard Brownian motion, similar problems have been studied in [10, 13] and the references therein. See also [2] for the precise L^2-limit and also for a discussion on the "best" partition. In the case $\frac{1}{4} < H < \frac{1}{2}$ the SDE (1.1) can be solved using the theory of rough paths introduced by Lyons; see [15]. There are also a number of results on the rate of convergence of Euler-type numerical schemes in this case; see, for instance, the paper by Deya, Neuenkirch and Tindel [4] for a Milstein-type scheme without Lévy area in the case $\frac{1}{3} < H <$

$\frac{1}{2}$, the paper by Friz and Riedel [5] for the N-step Euler scheme without involving iterated integrals and the monograph by Friz and Victoir [6].

The paper is organized as follows. The next section contains some basic materials on fractional calculus and Malliavin calculus that will be used throughout the paper, and introduces a matrix-valued Brownian motion and a generalized Rosenblatt process, both of which are key ingredients in our results on the asymptotic behavior of the error; see Section 6 and Section 8. In Section 3, we derive the necessary estimates for the uniform norms and Hölder seminorms of the processes X, X^n and their Malliavin derivatives. In Section 4, we prove our result on the rate of convergence in L^p for the numerical scheme (1.3). In Section 5, we prove a central limit theorem for weighted quadratic sums, and then in Section 6 we apply this result to the study of the asymptotic behavior of the error $\gamma_n(X_t - X_t^n)$ in case $H \in (\frac{1}{2}, \frac{3}{4}]$. In Section 7, we study the L^p-convergence of some weighted random sums. In Section 8, we apply the results of Section 7 to establish the L^p-limit of $n(X_t - X_t^n)$ in case $H \in (\frac{3}{4}, 1)$. The weak approximation result is discussed in Section 9. In Section 10, we deal with the numerical scheme (1.2). In the Appendix, we prove some auxiliary results.

2. Preliminaries and notation. Throughout the paper we consider a fixed time interval $[0, T]$. To simplify the presentation we only deal with the uniform partition of this interval; that is, for each $n \geq 1$ and $i = 0, 1, \dots, n$, we set $t_i = \frac{iT}{n}$. We use C and K to represent constants that are independent of n and whose values may change from line to line.

2.1. *Elements of fractional calculus.* In this subsection we introduce the definitions of the fractional integral and derivative operators, and we review some properties of these operators.

Let $a, b \in [0, T]$ with $a < b$, and let $\beta \in (0, 1)$. We denote by $C^\beta(a, b)$ the space of β-Hölder continuous functions on the interval $[a, b]$. For a function $x : [0, T] \to \mathbb{R}$, $\|x\|_{a,b,\beta}$ denotes the β-Hölder seminorm of x on $[a, b]$, that is,

$$\|x\|_{a,b,\beta} = \sup\left\{\frac{|x_u - x_v|}{(v-u)^\beta}; a \leq u < v \leq b\right\}.$$

We will also make use of the following seminorm:

$$\|x\|_{a,b,\beta,n} = \sup\left\{\frac{|x_u - x_v|}{(v-u)^\beta}; a \leq u < v \leq b, \eta(u) = u\right\}. \tag{2.1}$$

Recall that for each $n \geq 1$ and $i = 0, 1, \dots, n$, $t_i = \frac{iT}{n}$ and $\eta(t) = t_i$ when $t_i \leq t < t_i + \frac{T}{n}$.

We will denote the uniform norm of x on the interval $[a, b]$ as $\|x\|_{a,b,\infty}$. When $a = 0$ and $b = T$, we will simply write $\|x\|_\infty$ for $\|x\|_{0,T,\infty}$ and $\|x\|_\beta$ for $\|x\|_{0,T,\beta}$.

Let $f \in L^1([a,b])$ and $\alpha > 0$. The left-sided and right-sided fractional Riemann–Liouville integrals of f of order α are defined, for almost all $t \in (a,b)$, by

$$I_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds$$

and

$$I_{b-}^{\alpha} f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\,ds,$$

respectively, where $(-1)^{\alpha} = e^{-i\pi\alpha}$ and $\Gamma(\alpha) = \int_0^{\infty} r^{\alpha-1} e^{-r}\,dr$ is the Gamma function. Let $I_{a+}^{\alpha}(L^p)$ [resp., $I_{b-}^{\alpha}(L^p)$] be the image of $L^p([a,b])$ by the operator I_{a+}^{α} (resp., I_{b-}^{α}). If $f \in I_{a+}^{\alpha}(L^p)$ [resp., $f \in I_{b-}^{\alpha}(L^p)$] and $0 < \alpha < 1$, then the fractional Weyl derivatives are defined as

$$D_{a+}^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t)-f(s)}{(t-s)^{\alpha+1}}\,ds \right) \tag{2.2}$$

and

$$D_{b-}^{\alpha} f(t) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(b-t)^{\alpha}} + \alpha \int_t^b \frac{f(t)-f(s)}{(s-t)^{\alpha+1}}\,ds \right), \tag{2.3}$$

where $a < t < b$.

Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Then, according to Young [33], the Riemann–Stieltjes integral $\int_a^b f\,dg$ exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral $\int_a^b f\,dg$ in terms of fractional derivatives. We refer to [34] for additional details.

PROPOSITION 2.1. *Suppose that $f \in C^{\lambda}(a,b)$ and $g \in C^{\mu}(a,b)$ with $\lambda + \mu > 1$. Let $\lambda > \alpha$ and $\mu > 1 - \alpha$. Then the Riemann–Stieltjes integral $\int_a^b f\,dg$ exists, and it can be expressed as*

$$\int_a^b f\,dg = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(t) D_{b-}^{1-\alpha} g_{b-}(t)\,dt, \tag{2.4}$$

where $g_{b-}(t) = \mathbf{1}_{(a,b)}(t)(g(t) - g(b-))$.

The notion of Hölder continuity and the above result on the existence of Riemann–Stieltjes integrals can be generalized to functions taking values in some normed spaces. We fix a probability space $(\Omega, \mathscr{F}, P)$ and denote by $\|\cdot\|_p$ the norm in the space $L^p := L^p(\Omega)$, where $p \geq 1$.

DEFINITION 2.1. Let $f = \{f(t), t \in [0, T]\}$ be a stochastic process such that $f(t) \in L^p$ for all $t \in [0, T]$. We say that f is Hölder continuous of order $\beta > 0$ in L^p if

$$\|f(t) - f(s)\|_p \le C|t - s|^\beta, \tag{2.5}$$

for all $s, t \in [0, T]$.

The following result shows that with proper Hölder continuity assumptions on f and g, the Riemann–Stieltjes integral $\int_0^T f\,dg$ exists, and equation (2.4) holds.

PROPOSITION 2.2. *Let the positive numbers p_0, λ, μ, p, q satisfy $p_0 \ge 1$, $\lambda + \mu > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $p_0 p > \frac{1}{\mu}$, $p_0 q > \frac{1}{\lambda}$. Assume that $f = \{f(t), t \in [0, T\}$ and $g = \{g(t), t \in [0, T]\}$ are Hölder continuous stochastic processes of order μ and λ in $L^{p_0 p}$ and $L^{p_0 q}$, respectively, and $f(0) \in L^{p_0 p}$. Let $\pi : 0 = t_0 < t_1 < \cdots < t_N = T$ be a partition on $[0, T]$, and $\xi_i : t_{i-1} \le \xi_i \le t_i$. Then the sum $\sum_{i=1}^N f(\xi_i)[g(t_i) - g(t_{i-1})]$ converges in L^{p_0} to the Riemann–Stieltjes integral $\int_0^T f\,dg$ as $|\pi|$ tends to zero, where $|\pi| = \max_{1\le i\le N} |t_i - t_{i-1}|$, and equation (2.4) holds.*

Proposition 2.2 can be proved through a slight modification of Zähle's proof in the real-valued case [34] using Hölder's inequality.

2.2. *Elements of Malliavin calculus.* We briefly recall some basic facts about the stochastic calculus of variations with respect to an fBm. We refer the reader to [22] for further details. Let $B = \{(B_t^1, \dots, B_t^m), t \in [0, T]\}$ be an m-dimensional fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$, defined on some complete probability space $(\Omega, \mathscr{F}, P)$. Namely, B is a mean zero Gaussian process with covariance

$$\mathbb{E}(B_t^i B_s^j) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})\delta_{ij}, \qquad i, j = 1, \dots, m,$$

for all $s, t \in [0, T]$, where δ_{ij} is the Kronecker symbol.

Let $\mathcal{H}$ be the Hilbert space defined as the closure of the set of step functions on $[0, T]$ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \rangle_{\mathcal{H}} = \tfrac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

It is easy to see that the covariance of fBm can be written as

$$\alpha_H \int_0^t \int_0^s |u - v|^{2H-2}\,du\,dv,$$

where $\alpha_H = H(2H - 1)$. This implies that

$$\langle \psi, \phi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T \psi_u \phi_v |u - v|^{2H-2}\,du\,dv$$

for any pair of step functions ϕ and ψ on $[0, T]$.

The elements of the Hilbert space $\mathcal{H}$, or more generally, of the space $\mathcal{H}^{\otimes l}$ may not be functions, but distributions; see [28] and [29]. We can find a linear space of functions contained in $\mathcal{H}^{\otimes l}$ in the following way: Let $|\mathcal{H}|^{\otimes l}$ be the linear space of measurable functions ϕ on $[0,T]^l \subset \mathbb{R}^l$ such that

$$\|\phi\|^2_{|\mathcal{H}|^{\otimes l}} := \alpha_H^l \int_{[0,T]^{2l}} |\phi_{\mathbf{u}}||\phi_{\mathbf{v}}||u_1 - v_1|^{2H-2} \cdots |u_l - v_l|^{2H-2} \, d\mathbf{u} \, d\mathbf{v} < \infty,$$

where $\mathbf{u} = (u_1, \dots, u_l), \mathbf{v} = (v_1, \dots, v_l) \in [0,T]^l$. Suppose $\phi \in L^{1/H}([0,T]^l)$. The following estimate holds:

$$\|\phi\|_{|\mathcal{H}|^{\otimes l}} \le b_{H,l} \|\phi\|_{L^{1/H}([0,T]^l)} \tag{2.6}$$

for some constant $b_{H,l} > 0$; the case $l = 1$ was proved in [16], and the extension to general case is easy; see [9], equation (2.5).

The mapping $\mathbf{1}_{[0,t_1]} \times \cdots \times \mathbf{1}_{[0,t_m]} \mapsto (B^1_{t_1}, \dots, B^m_{t_m})$ can be extended to a linear isometry between $\mathcal{H}^m$ and the Gaussian space spanned by B. We denote this isometry by $h \mapsto B(h)$. In this way, $\{B(h), h \in \mathcal{H}^m\}$ is an isonormal Gaussian process indexed by the Hilbert space $\mathcal{H}^m$.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B_{s_1}, \dots, B_{s_N}),$$

where $N \ge 1$ and $f \in C_b^\infty(\mathbb{R}^{m \times N})$. For each $j = 1, \dots, m$ and $t \in [0,T]$, the derivative operator $D^j F$ on $F \in \mathcal{S}$ is defined as the $\mathcal{H}$-valued random variable

$$D_t^j F = \sum_{i=1}^N \frac{\partial f}{\partial x_i^j}(B_{s_1}, \dots, B_{s_N}) \mathbf{1}_{[0,s_i]}(t), \qquad t \in [0,T].$$

We can iterate this procedure to define higher order derivatives $D^{j_1,\dots,j_l} F$ which take values on $\mathcal{H}^{\otimes l}$. For any $p \ge 1$ and any integer $k \ge 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|^p_{k,p} = \mathbb{E}[|F|^p] + \mathbb{E}\left[\sum_{l=1}^k \left(\sum_{j_1,\dots,j_l=1}^m \|D^{j_1,\dots,j_l} F\|^2_{\mathcal{H}^{\otimes l}}\right)^{p/2}\right].$$

If V is a Hilbert space, $\mathbb{D}^{k,p}(V)$ denotes the corresponding Sobolev space of V-valued random variables.

For any $j = 1, \dots, m$, we denote by δ^j the adjoint of the derivative operator D^j. We say $u \in \operatorname{Dom} \delta^j$ if there is a $\delta^j(u) \in L^2(\Omega)$ such that for any $F \in \mathbb{D}^{1,2}$ the following duality relationship holds:

$$\mathbb{E}(\langle u, D^j F \rangle_{\mathcal{H}}) = \mathbb{E}(\delta^j(u) F). \tag{2.7}$$

The random variable $\delta^j(u)$ is also called the Skorohod integral of u with respect to the fBm B^j, and we use the notation $\delta^j(u) = \int_0^T u_t \delta B_t^j$.

Let $F \in \mathbb{D}^{1,2}$ and u be in the domain of δ^j such that $Fu \in L^2(\Omega; \mathcal{H})$. Then (see [23]) Fu belongs to the domain of δ^j, and the following equality holds:

$$\delta^j(Fu) = F\delta^j(u) - \langle D^j F, u\rangle_{\mathcal{H}}, \tag{2.8}$$

provided the right-hand side of (2.8) is square integrable.

Suppose that $u = \{u_t, t \in [0, T]\}$ is a stochastic process whose trajectories are Hölder continuous of order $\gamma > 1 - H$. Then, for any $j = 1, \dots, m$, the Riemann–Stieltjes integral $\int_0^T u_t \, dB_t^j$ exists. On the other hand, if $u \in \mathbb{D}^{1,2}(\mathcal{H})$ and the derivative $D_s^j u_t$ exists and satisfies almost surely

$$\int_0^T \int_0^T |D_s^j u_t||t - s|^{2H-2} \, ds \, dt < \infty,$$

and $\mathbb{E}(\|D^j u\|^2_{L^{1/H}([0,T]^2)}) < \infty$, then (see Proposition 5.2.3 in [23]) $\int_0^T u_t \delta B_t^j$ exists, and we have the following relationship between these two stochastic integrals:

$$\int_0^T u_t \, dB_t^j = \int_0^T u_t \delta B_t^j + \alpha_H \int_0^T \int_0^T D_s^j u_t |t - s|^{2H-2} \, ds \, dt. \tag{2.9}$$

The following result is Meyer's inequality for the Skorohod integral; see, for example, Proposition 1.5.7 of [23]. Given $p > 1$ and an integer $k \geq 1$, there is a constant $c_{k,p}$ such that

$$\|\delta^k(u)\|_p \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathcal{H}^{\otimes k})} \qquad \text{for all } u \in \mathbb{D}^{k,p}(\mathcal{H}^{\otimes k}). \tag{2.10}$$

Applying (2.6) and then the Minkowski inequality to the right-hand side of (2.10) yields

$$\begin{aligned} \|\delta^k(u)\|_p &\leq C \big\| \|u\|_p \big\|_{L^{1/H}([0,T]^p)} \\ &\quad + C \sum_{l=1}^k \sum_{j_1,\dots,j_l=1}^m \big\| \|D^{j_1,\dots,j_l} u\|_p \big\|_{L^{1/H}([0,T]^{p+l})} \end{aligned} \tag{2.11}$$

for all $u \in \mathbb{D}^{k,p}(\mathcal{H}^{\otimes k})$, provided $pH \geq 1$.

2.3. *Stable convergence.* Let Y_n, $n \in \mathbb{N}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ with values in a Polish space $(E, \mathscr{E})$. We say that Y_n *converges stably* to the limit Y, where Y is defined on an extension of the original probability space $(\Omega', \mathscr{F}', P')$, if and only if for any bounded $\mathscr{F}$-measurable random variable Z, it holds that

$$(Y_n, Z) \Rightarrow (Y, Z)$$

as $n \to \infty$, where $\Rightarrow$ denotes the convergence in law.

Note that stable convergence is stronger than weak convergence but weaker than convergence in probability. We refer to [11] and [1] for more details on this concept.

2.4. *A matrix-valued Brownian motion.* The aim of this subsection is to define a matrix-valued Brownian motion that will play a fundamental role in our central limit theorem. First, we introduce two constants Q and R which depend on H.

Denote by μ the measure on $\mathbb{R}^2$ with density $|s-t|^{2H-2}$. Define, for each $p \in \mathbb{Z}$,

$$Q(p) = T^{4H} \int_0^1 \int_p^{p+1} \int_0^t \int_p^s \mu(dv\, du)\mu(ds\, dt)$$

and

$$R(p) = T^{4H} \int_0^1 \int_p^{p+1} \int_t^1 \int_p^s \mu(dv\, du)\mu(ds\, dt).$$

It is not difficult to check that for $\frac{1}{2} < H < \frac{3}{4}$, the series $\sum_{p\in\mathbb{Z}} Q(p)$ and $\sum_{p\in\mathbb{Z}} R(p)$ are convergent, and for $H = \frac{3}{4}$, they diverge at the rate $\log n$. Then we set (we omit the explicit dependence of Q and R on H to simplify the notation)

$$Q = \sum_{p\in\mathbb{Z}} Q(p), \qquad R = \sum_{p\in\mathbb{Z}} R(p), \tag{2.12}$$

for the case $H \in (\frac{1}{2}, \frac{3}{4})$, and

$$Q = \lim_{n\to\infty} \frac{\sum_{|p|\le n} Q(p)}{\log n} = \frac{T^{4H}}{2}, \qquad R = \lim_{n\to\infty} \frac{\sum_{|p|\le n} R(p)}{\log n} = \frac{T^{4H}}{2},$$

for the case $H = \frac{3}{4}$.

LEMMA 2.1. *The constants Q and R satisfy $R \le Q$.*

PROOF. If $H = \frac{3}{4}$, we see from (2.12) that these two constants are both equal to $\frac{T^{4H}}{2}$. Suppose $H \in (\frac{1}{2}, \frac{3}{4})$. Consider the functions on $\mathbb{R}^2$ defined by $\varphi_p(v,s) = \mathbf{1}_{\{p\le v\le s\le p+1\}}$, $\psi_p(v,s) = \mathbf{1}_{\{p\le s\le v\le p+1\}}$, $p \in \mathbb{Z}$. Then

$$\begin{aligned}
&\frac{1}{n}\left\| \sum_{p=0}^{n-1} (\varphi_p - \psi_p) \right\|^2_{L^2(\mathbb{R}^2,\mu)} \\
&\quad = \frac{2}{n} \sum_{p,q=0}^{n-1} \big(\langle \mathbf{1}_{\{p\le v\le s\le p+1\}}, \mathbf{1}_{\{q\le v\le s\le q+1\}} \rangle_{L^2(\mathbb{R}^2,\mu)} \\
&\qquad\qquad - \langle \mathbf{1}_{\{p\le v\le s\le p+1\}}, \mathbf{1}_{\{q\le s\le v\le q+1\}} \rangle_{L^2(\mathbb{R}^2,\mu)} \big) \\
&\quad = \frac{2}{n} \sum_{p,q=0}^{n-1} \big(Q(p-q) - R(p-q) \big).
\end{aligned}$$

It is easy to see that the above is equal to

$$\frac{2}{n}\sum_{j=0}^{n-1}\sum_{k=-j}^{j}\big(Q(k)-R(k)\big).$$

It then follows from a Cesàro limit argument that the quantity in the right-hand side of the above converges to $2(Q-R)$ as n tends to infinity. Therefore, $Q \geq R$. $\square$

Let $\widetilde{W}^{0,ij} = \{\widetilde{W}_t^{0,ij}, t \in [0,T]\}$, $i \leq j, i, j = 1,\dots,m$ and $\widetilde{W}^{1,ij} = \{\widetilde{W}_t^{1,ij}, t \in [0,T]\}$, $i, j = 1,\dots,m$ be independent standard Brownian motions. When $i > j$, we define $\widetilde{W}_t^{0,ij} = \widetilde{W}_t^{0,ji}$. The matrix-valued Brownian motion $(W^{ij})_{1\leq i,j\leq m}$, $i, j = 1,\dots,m$ is defined as follows:

$$W^{ii} = \frac{\alpha_H}{\sqrt{T}}\big(\sqrt{Q+R}\,\widetilde{W}^{1,ii}\big)$$

and

$$W^{ij} = \frac{\alpha_H}{\sqrt{T}}\big(\sqrt{Q-R}\,\widetilde{W}^{1,ij} + \sqrt{R}\,\widetilde{W}^{0,ij}\big) \qquad \text{when } i \neq j.$$

Notice that this definition makes sense because $R \leq Q$. The random matrix W_t is not symmetric when $H < \frac{3}{4}$; see the plot and table below. For $i, j, i', j' = 1,\dots,m$, the covariance $\mathbb{E}(W_t^{ij} W_s^{i'j'})$ is equal to

$$\frac{\alpha_H^2(t\wedge s)}{T}(R\delta_{ji'}\delta_{ij'} + Q\delta_{jj'}\delta_{ii'}),$$

where δ is the Kronecker function.

In the Figure 1 and Table 1, we consider two quantities for $H \in (\frac{1}{2}, \frac{3}{4})$,

$$q = \frac{\alpha_H^2}{T^{4H}}Q \quad \text{and} \quad r = \frac{\alpha_H^2}{T^{4H}}R.$$

We see that the values of q and r approach 0.5 and 0 as H tends to $\frac{1}{2}$, respectively, and both of them tend to infinity when H gets closer to $\frac{3}{4}$.

2.5. *A matrix-valued generalized Rosenblatt process.* In this subsection we introduce a generalized Rosenblatt process which will appear in the limiting result proved in Section 8 when $H > \frac{3}{4}$. Consider an m-dimensional fBm $B_t = (B_t^1,\dots,B_t^m)$ with Hurst parameter $H \in (\frac{3}{4}, 1)$. Define for $i_1, i_2 \in 1,\dots,m$,

$$Z_n^{i_1,i_2}(t) := n\sum_{j=1}^{\lfloor nt/T\rfloor}\int_{t_j}^{t_{j+1}}\big(B_s^{i_1} - B_{t_j}^{i_1}\big)\delta B_s^{i_2}.$$

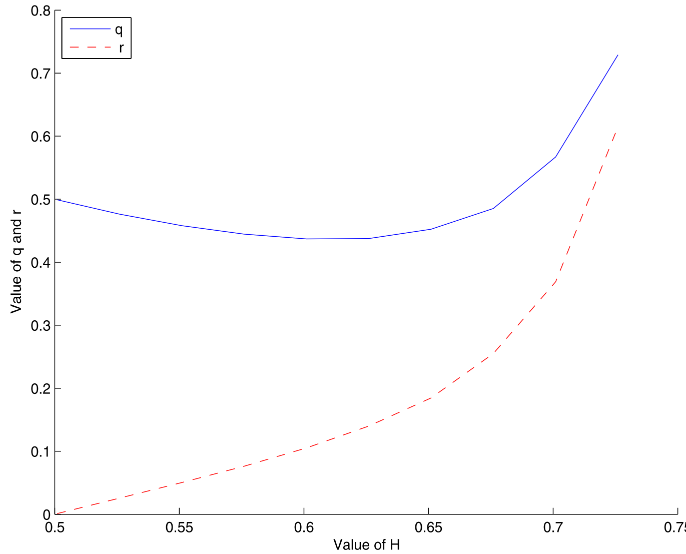

FIG. 1. *Simulation of q and r.*

When $i_1 = i_2 = i$, we can write

$$Z_n^{i,i}(t) = \frac{T^{2H}}{2n^{2H-1}} \sum_{j=1}^{\lfloor nt/T \rfloor} H_2(\xi_j^{n,i}),$$

where $H_2(x) = x^2 - 1$ is the second degree Hermite polynomial and $\xi_j^{n,i} = T^{-H} n^H (B_{t_{j+1}}^i - B_{t_j}^i)$. It is well known (see [20]) that for each $i = 1, \dots, m$, the process $Z_n^{i,i}(t)$ converges in L^2 to the *Rosenblatt process* $R(t)$. We refer the reader to [30] and [32] for further details on the Rosenblatt process.

When $i_1 \neq i_2$, the stochastic integral $\int_{t_j}^{t_{j+1}} (B_s^{i_1} - B_{t_j}^{i_1}) \delta B_s^{i_2}$ cannot be written as the second Hermite polynomial of a Gaussian random variable. Nevertheless, the process $Z_n^{i_1,i_2}(t)$ is still convergent in L^2. Indeed, for any positive integers n

TABLE 1
Simulation of q and r

H	0.5010	0.5260	0.5510	0.6010	0.6260	0.6510	0.7010	0.7260
q	0.4990	0.4763	0.4580	0.4369	0.4375	0.4522	0.5669	0.7290
r	9.9868×10^{-4}	0.0256	0.0503	0.1053	0.1400	0.1845	0.3689	0.6149

and n', we have

$$\begin{aligned}
&\mathbb{E}\big(Z_n^{i_1 i_2}(t) Z_{n'}^{i_1 i_2}(t)\big) \\
&\quad = nn' \sum_{k=0}^{\lfloor nt/T \rfloor} \sum_{k'=0}^{\lfloor n't/T \rfloor} \mathbb{E}\bigg[\int_{(k/n)T}^{((k+1)/n)T} \big(B_s^{i_1} - B_{(k/n)T}^{i_1}\big)\delta B_s^{i_2} \\
&\qquad\qquad \times \int_{(k/n')T}^{((k+1)/n')T} \big(B_s^{i_1} - B_{(k/n')T}^{i_1}\big)\delta B_s^{i_2}\bigg] \\
&\quad = nn'\alpha_H^2 \sum_{k=0}^{\lfloor nt/T \rfloor} \sum_{k'=0}^{\lfloor n't/T \rfloor} \int_{(k/n)T}^{((k+1)/n)T} \int_{(k'/n')T}^{((k'+1)/n')T} \int_{(k/n)T}^{t} \int_{(k'/n')T}^{s} \mu(dv\,du) \\
&\qquad\qquad \times \mu(ds\,dt) \\
&\quad \to \frac{T^2\alpha_H^2}{4} \int_0^t \int_0^t |u-v|^{4H-4}\,du\,dv \\
&\quad = c_H t^{4H-2},
\end{aligned}$$

as $n', n \to +\infty$, where $c_H = \frac{T^2 H^2(2H-1)}{4(4H-3)}$. This allows us to conclude that $Z_n^{i_1 i_2}(t)$ is a Cauchy sequence in L^2. We denote by $Z_t^{i_1 i_2}$ the L^2-limit of $Z_n^{i_1 i_2}(t)$. Then $Z_t^{i_1 i_2}$ can be considered a *generalized Rosenblatt process*.

It is easy to show that

$$\mathbb{E}\big[\big|Z_t^{i_1 i_2} - Z_s^{i_1 i_2}\big|^2\big] \le C|t-s|^{4H-2},$$

and by the hypercontractivity property, we deduce

$$\mathbb{E}\big[\big|Z_t^{i_1 i_2} - Z_s^{i_1 i_2}\big|^p\big] \le C_p |t-s|^{p(2H-1)} \tag{2.13}$$

for any $p \ge 2$ and $s, t \in [0, T]$. By the Kolmogorov continuity criterion this implies that $Z^{i_1 i_2}$ has a Hölder continuous version of exponent λ for any $\lambda < 2H - 1$.

3. Estimates for solutions of some SDEs. The purpose of this section is to provide upper bounds for the Hölder seminorms of solutions of two types of SDEs. The first type [see (3.1)] covers equation (1.1) and its Malliavin derivatives, as well as all the other SDEs involving only continuous integrands which we will encounter in this paper. The second type [see (3.13)] deals with the case where the integrands are step processes. These SDEs arise from approximation schemes such as (1.2) and (1.3).

For any integers $k, N, M \ge 1$, we denote by $C_b^k(\mathbb{R}^M; \mathbb{R}^N)$ the space of k times continuously differentiable functions $f : \mathbb{R}^M \to \mathbb{R}^N$ which are bounded together with their first k partial derivatives. If $N = 1$, we simply write $C_b^k(\mathbb{R}^M)$.

In order to simplify the notation we only consider the case when the fBm is one-dimensional, that is, $m = 1$. All results of this section can be generalized to the case

$m > 1$. Throughout the remainder of the paper we let β be any number satisfying $\frac{1}{2} < \beta < H$. The first two lemmas are path-wise results, and they will still hold when B is replaced by general Hölder continuous functions of index $\gamma > \beta$. The constants appearing in the lemmas depend on β, H, T and the uniform and Hölder seminorms of the coefficients. We fix a time interval $[\tau, T]$, and to simplify we omit the dependence on τ and T of the uniform norm and β-Hölder seminorm on the interval $[\tau, T]$.

LEMMA 3.1. *Fix $\tau \in [0, T)$. Let $V = \{V_t, t \in [\tau, T]\}$ be an $\mathbb{R}^M$-valued processes satisfying*

$$V_t = S_t + \int_\tau^t \left[g_1(V_u) + U_u^1 V_u\right] du + \int_\tau^t \left[g_2(V_u) + U_u^2 V_u\right] dB_u, \tag{3.1}$$

where $g_1 \in C_b(\mathbb{R}^M; \mathbb{R}^M)$, $g_2 \in C_b^1(\mathbb{R}^M; \mathbb{R}^M)$ and $U^i = \{U_t^i, t \in [\tau, T]\}$, $i = 1, 2$, and $S = \{S_t, \in [\tau, T]\}$ are $\mathbb{R}^{M \times M}$-valued and $\mathbb{R}^M$-valued processes, respectively. We assume that S has β-Hölder continuous trajectories, and the processes U^i, $i = 1, 2$, are uniformly bounded by a constant C.

(i) *If $U^1 = U^2 = 0$, then we can find constants K and K' such that $(t - s)^\beta \|B\|_\beta \le K$, $\tau \le s < t \le T$ implies*

$$\|V\|_{s,t,\beta} \le K'\big(\|B\|_\beta + 1\big) + 2\|S\|_\beta.$$

(ii) *Suppose that there exist constants K_0 and K_0' such that $(t-s)^\beta \|B\|_\beta \le K_0$, $\tau \le s < t \le T$ implies*

$$\big\|U^2\big\|_{s,t,\beta} \le K_0'\big(\|B\|_\beta + 1\big). \tag{3.2}$$

Then there exists a positive constant K such that

$$\max\big\{\|V\|_\infty, \|V\|_\beta\big\} \le K e^{K\|B\|_\beta^{1/\beta}}\big(|S_\tau| + \|S\|_\beta + 1\big). \tag{3.3}$$

PROOF. The proof follows the approach used, for instance, by Hu and Nualart [8]. Let $\tau \le s < t \le T$. By the definition of V,

$$V_t - V_s = S_t - S_s + \int_s^t \left[g_1(V_u) + U_u^1 V_u\right] du + \int_s^t \left[g_2(V_u) + U_u^2 V_u\right] dB_u. \tag{3.4}$$

Applying Lemma A.1(ii) to the vector valued function $f : (u, v) \to g_2(v) + uv$ and the integrator $z = B$, and taking $\beta' = \beta$ yields

$$\begin{aligned} |V_t - V_s| \le {} & \|S\|_\beta (t-s)^\beta + \big(\|g_1\|_\infty + C\|V\|_{s,t,\infty}\big)(t-s) \\ & + K_1\big(\|g_2\|_\infty + C\|V\|_{s,t,\infty}\big)\|B\|_\beta (t-s)^\beta \\ & + K_2\big(\|\nabla g_2\|_\infty + C\big)\|V\|_{s,t,\beta}\|B\|_\beta (t-s)^{2\beta} \\ & + K_2\|V\|_{s,t,\infty}\big\|U^2\big\|_{s,t,\beta}\|B\|_\beta (t-s)^{2\beta}. \end{aligned} \tag{3.5}$$

Step 1. In the case $U^1 = U^2 = 0$ (which means that we can take $C = 0$ and $\|U^2\|_{s,t,\beta} = 0$), dividing both sides of (3.5) by $(t-s)^\beta$ and taking the Hölder seminorm on the left-hand side, we obtain

$$\begin{aligned}\|V\|_{s,t,\beta} &\le \|S\|_\beta + c_1(t-s)^{1-\beta} + K_1 c_1 \|B\|_\beta \\ &\quad + K_2 c_1 \|V\|_{s,t,\beta}\|B\|_\beta (t-s)^\beta,\end{aligned} \tag{3.6}$$

where (and throughout this section) we denote

$$c_1 = \max\{C, \|g_1\|_\infty, \|g_2\|_\infty, \|\nabla g_2\|_\infty\}. \tag{3.7}$$

Take $K = \frac{1}{2}(K_2 c_1)^{-1}$. Then for any $\tau \le s < t \le T$ such that $(t-s)^\beta \|B\|_\beta \le K$, we have

$$\|V\|_{s,t,\beta} \le 2\|S\|_\beta + 2c_1(t-s)^{1-\beta} + 2K_1 c_1 \|B\|_\beta,$$

which implies (i).

Step 2. As in step 1, we divide (3.5) by $(t-s)^\beta$ and then take the Hölder seminorm on the left-hand side to obtain

$$\begin{aligned}\|V\|_{s,t,\beta} &\le \|S\|_\beta + c_1\big(1 + \|V\|_{s,t,\infty}\big)(t-s)^{1-\beta} \\ &\quad + K_1 c_1\big(1 + \|V\|_{s,t,\infty}\big)\|B\|_\beta \\ &\quad + 2K_2 c_1 \|V\|_{s,t,\beta}\|B\|_\beta (t-s)^\beta \\ &\quad + K_2 \|V\|_{s,t,\infty}\big\|U^2\big\|_{s,t,\beta}\|B\|_\beta (t-s)^\beta.\end{aligned} \tag{3.8}$$

If $(t-s)^\beta \|B\|_\beta \le \frac{1}{4}(K_2 c_1)^{-1}$, then the coefficient of $\|V\|_{s,t,\beta}$ on the right-hand side of (3.8) is less or equal than $\frac{1}{2}$. Thus we obtain

$$\begin{aligned}\|V\|_{s,t,\beta} &\le 2\|S\|_\beta + 2c_1\big(1 + \|V\|_{s,t,\infty}\big)(t-s)^{1-\beta} \\ &\quad + 2K_1 c_1\big(1 + \|V\|_{s,t,\infty}\big)\|B\|_\beta \\ &\quad + 2K_2 \|V\|_{s,t,\infty}\big\|U^2\big\|_{s,t,\beta}\|B\|_\beta (t-s)^\beta.\end{aligned}$$

On the other hand, assuming $(t-s)^\beta \|B\|_\beta \le K_0$ and applying (3.2), we obtain

$$\|V\|_{s,t,\beta} \le 2\|S\|_\beta + C_1\big(1 + \|B\|_\beta\big)\big(1 + \|V\|_{s,t,\infty}\big), \tag{3.9}$$

for some constant C_1. This implies

$$\|V\|_{s,t,\infty} \le |V_s| + 2(t-s)^\beta \|S\|_\beta + C_1(t-s)^\beta\big(1 + \|B\|_\beta\big)\big(1 + \|V\|_{s,t,\infty}\big).$$

Assuming $(t-s)^\beta \|B\|_\beta \le \frac{1}{4C_1}$ and $(t-s)^\beta \le \frac{1}{4C_1} \wedge \frac{1}{2}$, we obtain

$$\|V\|_{s,t,\infty} \le 2|V_s| + 2\|S\|_\beta + 1. \tag{3.10}$$

Take $\Delta = [\|B\|_\beta^{-1}\min(\frac{1}{4K_2c_1}, K_0, \frac{1}{4C_1})]^{1/\beta} \wedge (\frac{1}{4C_1} \wedge \frac{1}{2})^{1/\beta}$. We divide the interval $[\tau, T]$ into $N = \lfloor \frac{T-\tau}{\Delta} \rfloor + 1$ subintervals and denote by $s_1, s_2, \dots, s_N$ the left endpoints of these intervals and $s_{N+1} = T$. Applying inequality (3.10) to each interval $[s_i, s_{i+1}]$ for $i = 1, \dots, N$ yields

$$\|V\|_\infty \le 2^{N+1}\big(|S_\tau| + 2\|S\|_\beta + 1\big). \tag{3.11}$$

From the definition of Δ we get

$$N \le 1 + \frac{T}{\Delta} \le 1 + T\max\big(C_2, C_3\|B\|_\beta^{1/\beta}\big), \tag{3.12}$$

for some constants C_2 and C_3. From inequalities (3.11) and (3.12) we obtain the desired estimate for $\|V\|_\infty$.

If $t, s \in [\tau, T]$ satisfy $0 \le t - s \le \Delta$, then from (3.9) and from the upper bound of $\|V\|_\infty$ we can estimate $\frac{V_t - V_s}{(t-s)^\beta}$ by the right-hand side of (3.3) for some constant K. On the other hand, if $t - s > \Delta$, then

$$\frac{|V_t - V_s|}{(t-s)^\beta} \le 2\|V\|_\infty \Delta^{-1}.$$

We can obtain a similar estimate from the upper bound of $\|V\|_\infty$ and from the definition of Δ. This gives then the desired estimate for $\|V\|_\beta$, and hence we complete the proof of (ii). $\square$

For the second lemma we fix n and consider the partition of $[0, T]$ given by $t_i = i\frac{T}{n}$, $i = 0, 1, \dots, n$. Define $\eta(t) = t_i$ if $t_i \le t < t_i + \frac{T}{n}$ and $\varepsilon(t) = t_i + \frac{T}{n}$ if $t_i < t \le t_i + \frac{T}{n}$.

LEMMA 3.2. *Suppose that* S, g_i, U^i, $i = 1, 2$ *are the same as in Lemma* 3.1. *Let* $g \in C([0, T])$. *Let* $V = \{V_t, t \in [\tau, T]\}$ *be an* $\mathbb{R}^M$*-valued processes satisfying the equation*

$$\begin{aligned} V_t = S_t &+ \int_{\varepsilon(\tau)}^{t\vee\varepsilon(\tau)} \big[g_1(V_{\eta(u)}) + U^1_{\eta(u)} V_{\eta(u)}\big] g\big(u - \eta(u)\big)\, du \\ &+ \int_{\varepsilon(\tau)}^{t\vee\varepsilon(\tau)} \big[g_2(V_{\eta(u)}) + U^2_{\eta(u)} V_{\eta(u)}\big]\, dB_u. \end{aligned} \tag{3.13}$$

(i) *If* $U^1 = U^2 = 0$, *then we can find constants* K *and* K' *such that* $(t - s)^\beta \|B\|_\beta \le K$, $\tau \le s < t \le T$ *implies*

$$\|V\|_{s,t,\beta,n} \le K'\big(\|B\|_\beta + 1\big) + 2\|S\|_\beta.$$

(ii) *Suppose that there exist constants* K_0 *and* K_0' *such that* $(t-s)^\beta\|B\|_\beta \le K_0$, $\tau \le s < t \le T$ *implies*

$$\|U^2\|_{s,t,\beta,n} \le K_0'\big(\|B\|_\beta + 1\big). \tag{3.14}$$

Then there exists a constant K such that

$$\max\{\|V\|_\infty, \|V\|_\beta\} \le K e^{K\|B\|_\beta^{1/\beta}} (|S_\tau| + \|S\|_\beta + 1).$$

REMARK 3.1. The proof of this result is similar to that of Lemma 3.1. Nevertheless, since the integral is discrete, we need to replace the Hölder seminorm $\|\cdot\|_{s,t,\beta}$ by the seminorm $\|\cdot\|_{s,t,\beta,n}$ introduced in (2.1).

PROOF OF LEMMA 3.2. Let $s, t \in [\tau, T]$ be such that $s < t$ and $s = \eta(s)$. This implies $s \ge \varepsilon(\tau)$. As in the proof of (3.5), applying Lemma A.1(i) [instead of Lemma A.1(ii)] yields

$$\begin{aligned} &|V_t - V_s| \\ &\quad \le \|S\|_\beta (t-s)^\beta + (\|g_1\|_\infty + C\|V\|_{s,t,\infty})\|g\|_\infty (t-s) \\ &\qquad + K_1(\|g_2\|_\infty + C\|V\|_{s,t,\infty})\|B\|_\beta (t-s)^\beta \\ &\qquad + K_3[(\|\nabla g_2\|_\infty + C)\|V\|_{s,t,\beta,n} + \|V\|_{s,t,\infty}\|U^2\|_{s,t,\beta,n}]\|B\|_\beta (t-s)^{2\beta}. \end{aligned}$$

Dividing both sides of the above inequality by $(t-s)^\beta$ and taking the Hölder seminorm on the left-hand side, we obtain

$$\begin{aligned} \|V\|_{s,t,\beta,n} \le{}& \|S\|_\beta + (\|g_1\|_\infty + C\|V\|_{s,t,\infty})\|g\|_\infty (t-s)^{1-\beta} \\ &+ K_1(\|g_2\|_\infty + C\|V\|_{s,t,\infty})\|B\|_\beta \\ &+ K_3(\|\nabla g_2\|_\infty + C)\|V\|_{s,t,\beta,n}\|B\|_\beta (t-s)^\beta \\ &+ K_3\|V\|_{s,t,\infty}\|U^2\|_{s,t,\beta,n}\|B\|_\beta (t-s)^\beta. \end{aligned} \tag{3.15}$$

Step 1. In the case $U^1 = U^2 = 0$, (3.15) becomes

$$\begin{aligned} &\|V\|_{s,t,\beta,n} \\ &\quad \le \|S\|_\beta + c_1\|g\|_\infty (t-s)^{1-\beta} + K_1 c_1 \|B\|_\beta + K_3 c_1 \|V\|_{s,t,\beta,n}\|B\|_\beta (t-s)^\beta, \end{aligned}$$

where c_1 is defined in (3.7). Taking $K = \frac{1}{2}(K_3 c_1)^{-1}$, for any $\tau \le s < t \le T$ such that $(t-s)^\beta \|B\|_\beta \le K$, we have

$$\|V\|_{s,t,\beta,n} \le 2\|S\|_\beta + 2c_1\|g\|_\infty (t-s)^{1-\beta} + 2K_1 c_1 \|B\|_\beta.$$

This completes the proof of (i).

Step 2. In the general case, we follow the proof of Lemma 3.1, except that we assume $s = \eta(s)$ and use the seminorm $\|\cdot\|_{s,t,\beta,n}$ instead of $\|\cdot\|_{s,t,\beta}$. We also apply (3.14) instead of (3.2). In this way we obtain inequality (3.9) with $\|V\|_{s,t,\beta}$ replaced by $\|V\|_{s,t,\beta,n}$, that is,

$$\|V\|_{s,t,\beta,n} \le 2\|S\|_\beta + C_1(1 + \|B\|_\beta)(1 + \|V\|_{s,t,\infty}) \tag{3.16}$$

for some constant C_1. Inequality (3.10) remains the same,

$$\|V\|_{s,t,\infty} \le 2|V_s| + 2\|S\|_\beta + 1, \tag{3.17}$$

provided $s = \eta(s)$, and both $t - s$ and $(t-s)^\beta \|B\|_\beta$ are bounded by some constant C_4.

Take $\Delta = (C_4^{1/\beta}\|B\|_\beta^{-1/\beta}) \wedge C_4$. We are going to consider two cases depending on the relation between Δ and $\frac{2T}{n}$.

If $\Delta > \frac{2T}{n}$, we take $N = \lfloor \frac{2(T-\varepsilon(\tau))}{\Delta} \rfloor$ and divide the interval $[\varepsilon(\tau), \varepsilon(\tau) + N\frac{\Delta}{2}]$ into N subintervals of length $\frac{\Delta}{2}$. Since the length of each of these subintervals is larger than $\frac{T}{n}$, we are able to choose N points $s_1, s_2, \dots, s_N$ from each of these intervals such that $s_1 = \varepsilon(\tau)$ and $\eta(s_i) = s_i$, $i = 1, 2, \dots, N$. On the other hand, we have $s_{i+1} - s_i \le \Delta$ for all $i = 1, \dots, N-1$. Applying inequality (3.17) to each of the intervals $[s_1, s_2], [s_2, s_3], \dots, [s_{N-1}, s_N], [s_N, T]$ yields

$$\|V\|_{\varepsilon(\tau),T,\infty} \le 2^{N+1}\big(|S_{\varepsilon(\tau)}| + 2\|S\|_\beta + 1\big). \tag{3.18}$$

From the definition of Δ we have

$$N \le \frac{2T}{\Delta} \le K + K\|B\|_\beta^{1/\beta}, \tag{3.19}$$

for some constant K depending on T and C_4. From (3.18) and (3.19) and taking into account that

$$\|V\|_{\tau,\varepsilon(\tau),\infty} = \|S\|_{\tau,\varepsilon(\tau),\infty} \le |S_\tau| + T^\beta\|S\|_\beta, \tag{3.20}$$

we obtain the desired estimate for $\|V\|_\infty$.

If $\Delta \le \frac{2T}{n}$, that is, when $n \le \frac{2T}{\Delta} \le K + K\|B\|_\beta^{1/\beta}$, then by equation (3.13) we have

$$\begin{aligned}|V_t| &\le |V_{\eta(t)}| + |S_t - S_{\eta(t)}| + \big(c_1 + C|V_{\eta(t)}|\big)\|g\|_\infty(T/n) \\ &\quad + \big(c_1 + C|V_{\eta(t)}|\big)\|B\|_\beta(T/n)^\beta \\ &\le A_n + B_n\big|V_{\eta(t)}\big|,\end{aligned}$$

for any $t \in [\tau, T]$, where

$$A_n = \|S\|_\beta(T/n)^\beta + c_1\|g\|_\infty(T/n) + c_1\|B\|_\beta(T/n)^\beta$$

and

$$B_n = 1 + C\|g\|_\infty(T/n) + C\|B\|_\beta(T/n)^\beta.$$

Iterating this estimate, we obtain

$$\begin{aligned}\|V\|_{\varepsilon(\tau),T,\infty} &\le |S_{\varepsilon(\tau)}|B_n^n + nA_nB_n^{n-1} \\ &\le K\big(|S_{\varepsilon(\tau)}| + \|S\|_\beta + 1\big)e^{K\|B\|_\beta^{1/\beta}},\end{aligned} \tag{3.21}$$

for some constant K independent of n, where we have used the inequality

$$B_n^n \le e^{K(1+\|B\|_\beta)n^{1-\beta}},$$

and the fact that $n \le K + K\|B\|_\beta^{1/\beta}$ for some constant K. Taking (3.20) into account, we obtain the desired upper bound for $\|V\|_\infty$.

In order to show the upper bound for $\|V\|_{\tau,T,\beta}$, we notice that if $0 \le t - s \le \Delta$, then from (3.16) and from the upper bound of $\|V\|_{\tau,T,\infty}$, we have

$$\|V\|_{\varepsilon(s),t,\beta,n} \le K\big(|S_\tau| + \|S\|_\beta + 1\big)e^{K\|B\|_\beta^{1/\beta}},$$

for some constant K. Thus

$$\begin{aligned}\frac{|V_t - V_s|}{(t-s)^\beta} &\le \|V\|_{\varepsilon(s),t,\beta,n} + \frac{|V_{\varepsilon(s)} - V_s|}{(\varepsilon(s) - s)^\beta} \\ &\le K\big(|S_\tau| + \|S\|_\beta + 1\big)e^{K\|B\|_\beta^{1/\beta}}.\end{aligned}$$

If $t - s \ge \Delta$, we can obtain the upper bound of $\|V\|_\beta$ by an argument similar to that in the proof of Lemma 3.1. The proof of (ii) is now complete. □

The following result gives upper bounds for the norm of Malliavin derivatives of the solutions of the two types of SDEs, (3.1) and (3.13). Given a process $P = \{P_t, t \in [\tau, T]\}$ such that $P_t \in \mathbb{D}^{N,2}$, for each t and some $N \ge 1$, we denote by $\mathscr{D}_N^* P$ the maximum of the supnorms of the functions $P_{r_0}, D_{r_1} P_{r_0}, \dots, D^N_{r_1,\dots,r_N} P_{r_0}$ over $r_0, \dots, r_N \in [\tau, T]$, and denote by $\mathscr{D}_N P$ the maximum of the random variable $\mathscr{D}_N^* P$ and the supnorms of $\|P\|_\beta$, $\|D_{r_1} P\|_{r_1,T,\beta}, \dots, \|D^N_{r_1,\dots,r_N} P\|_{r_1 \vee\cdots\vee r_N,T,\beta}$ over $r_0, \dots, r_N \in [\tau, T]$. If $N = 0$, we simply write $\mathscr{D}_0^* P = \|P\|_\infty$ and $\mathscr{D}_0 P = \max(\|P\|_\infty, \|P\|_\beta)$.

LEMMA 3.3. (i) *Let V be the solution of equation* (3.1). *Assume that $g_1 = g_2 = 0$. Suppose that U^1 are U^2 are uniformly bounded by a constant C, and assume that there exist constants K_0 and K_0' such that $(t-s)^\beta\|B\|_\beta \le K_0$, $\tau \le s < t \le T$ implies*

$$\|U^2\|_{s,t,\beta} \le K_0'\big(\|B\|_\beta + 1\big). \tag{3.22}$$

Suppose that $S, U^1, U^2 \in \mathbb{D}^{N,2}$, where $N \ge 0$ is an integer, and $D_r S_t = D_r U_t^i = 0$, $i = 1, 2$, if $0 \le t < r \le T$, and suppose that there exists a constant $K > 0$ such that the random variables $\mathscr{D}_N S$, $\mathscr{D}_N^* U^1$, $\mathscr{D}_N U^2$ are less than or equal to $Ke^{K\|B\|_\beta^{1/\beta}}$. Then there exists a constant $K' > 0$ such that $\mathscr{D}_N V$ is less than $K'e^{K'\|B\|_\beta^{1/\beta}}$.

(ii) *Let V be the solution of equation* (3.13). *Then the conclusion in* (i) *still holds true under the same assumptions, except that in* (3.22) *we replace $\|U^2\|_{s,t,\beta}$ by $\|U^2\|_{s,t,\beta,n}$.*

PROOF. We first show point (i). The upper bounds of $\|V\|_\infty$ and $\|V\|_\beta$ follow from Lemma 3.1(ii). The Malliavin derivative $D_r V_t$ satisfies the equation (see Proposition 7 in [26])

$$D_r V_t = S_t^{(1)} + \int_r^t U_u^1 D_r V_u \, du + \int_r^t U_u^2 D_r V_u \, dB_u$$

while $t \in [r \vee \tau, T]$ and $D_r V_t = 0$ otherwise, where

$$S_t^{(1)} := D_r S_t + U_r^2 V_r + \int_r^t [D_r U_u^1] V_u \, du + \int_r^t [D_r U_u^2] V_u \, dB_u \tag{3.23}$$

for $t \in [r \vee \tau, T]$. Lemma 3.1(ii) applied to the time interval $[r, T]$, where $r \geq \tau$, implies that

$$\max\{\|D_r V\|_{r,T,\infty}, \|D_r V\|_{r,T,\beta}\} \leq K e^{K\|B\|_\beta^{1/\beta}} (|S_r^{(1)}| + \|S^{(1)}\|_{r,T,\beta} + 1).$$

Therefore, to obtain the desired upper bound it suffices to show that there exists a constant K independent of r such that both $\|S^{(1)}\|_{r,T,\infty}$ and $\|S^{(1)}\|_{r,T,\beta}$ are less than or equal to $K e^{K\|B\|_\beta^{1/\beta}}$. Applying Lemma A.1(ii) to the second integral in (3.23) and noticing that $\|D_r U^2\|_\infty$, $\|D_r U^2\|_{r,T,\beta}$, $\|V\|_\infty$, $\|V\|_{r,T,\beta}$ are bounded by $K e^{K\|B\|_\beta^{1/\beta}}$, we see that the upper bound of $\|S^{(1)}\|_\infty$ is bounded by $K e^{K\|B\|_\beta^{1/\beta}}$. On the other hand, in order to show the upper bound for $\|S^{(1)}\|_{r,T,\beta}$, we calculate $\frac{S_t^{(1)} - S_s^{(1)}}{(t-s)^\beta}$ using (3.23) to obtain

$$\begin{aligned}\frac{S_t^{(1)} - S_s^{(1)}}{(t-s)^\beta} &\leq \|D_r S\|_{r,T,\beta} + (t-s)^{-\beta} \int_s^t [D_r U_u^1] V_u \, du \\ &\quad + (t-s)^{-\beta} \int_s^t [D_r U_u^2] V_u \, dB_u.\end{aligned}$$

Now we can estimate each term of the above right-hand side as before. Taking the supremum over $s, t \in [r, T]$ yields the upper bound of $\|S^{(1)}\|_{r,T,\beta}$.

We turn to the second derivative. As before, we are able to find the equation of $D^2_{r_1,r_2} V_t$; see Proposition 7 in [26]. The estimates of $D^2_{r_1,r_2} V_t$ can then be obtained in the same way as above by applying Lemma 3.1(ii) and the estimates that we just obtained for V_t and $D_s V_t$, as well as the assumptions on S and U^i. The estimates of the higher order derivatives of V can be obtained analogously.

The proof of (ii) follows along the same lines, except that we use Lemma 3.2(ii) and Lemma A.1(i) instead of Lemma 3.1(ii) and Lemma A.1(ii). □

REMARK 3.2. Since $\beta > \frac{1}{2}$, from Fernique's theorem we know that $K e^{K\|B\|_\beta^{1/\beta}}$ has finite moments of any order. So Lemma 3.3 implies that the uniform norms and Hölder seminorms of the solutions of (3.1) and (3.13) and their Malliavin derivatives have finite moments of any order. We will need this fact in many of our arguments.

The next proposition is an immediate consequence of Lemma 3.3. Recall that the random variables $\mathscr{D}_N^* P$ and $\mathscr{D}_N P$ are defined in Section 3.

PROPOSITION 3.1. *Let X be the solution of equation (1.1), and let X^n be the solution of the Euler scheme (1.2). Fix $N \geq 0$, and suppose that $b \in C_b^N(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \in C_b^{N+1}(\mathbb{R}^d, \mathbb{R}^d)$ (recall that we assume $m = 1$). Then there exists a positive constant K such that the random variables $\mathscr{D}_N X$ and $\mathscr{D}_N X^n$ are bounded by $Ke^{K\|B\|_\beta^{1/\beta}}$ for all $n \in \mathbb{N}$. If we further assume $\sigma \in C_b^{N+2}(\mathbb{R}^d, \mathbb{R}^d)$, then the same upper bound holds for the modified Euler scheme (1.3).*

PROOF. We first consider the process X, the solution to equation (1.1). The upper bounds for $\|X\|_\infty$ and $\|X\|_\beta$ follow from Lemma 3.1(ii). The Malliavin derivative $D_r X_t$ satisfies the following linear stochastic differential equation:

$$D_r X_t = \sigma(X_r) + \int_r^t \nabla b(X_u) D_r X_u \, du + \int_r^t \nabla\sigma(X_u) D_r X_u \, dB_u, \tag{3.24}$$

while $0 < r \leq t \leq T$, and $D_r X_t = 0$ otherwise. Then it suffices to show that

$$\sup_{r\in[0,T]} \mathscr{D}_M(D_r X) \leq Ke^{K\|B\|_\beta^{1/\beta}}, \tag{3.25}$$

for $M = N - 1$. We can prove estimate (3.25) by induction on $N \geq 1$. Set $S_t = \sigma(X_r)$, $U_t^1 = \nabla b(X_t)$ and $U_t^2 = \nabla\sigma(X_t)$. Applying Lemma 3.1(i) to X we obtain that U^2 satisfies (3.22). Therefore, Lemma 3.3 implies that (3.25) holds for $M = 0$. Now we assume that

$$\sup_{r\in[0,T]} \mathscr{D}_M(D_r X) \leq Ke^{K\|B\|_\beta^{1/\beta}}$$

for some $0 \leq M \leq N - 2$. It is then easy to see that

$$\mathscr{D}_{M+1}^*(U^1) \vee \mathscr{D}_{M+1}(U^2) \vee \mathscr{D}_{M+1}(S) \leq Ke^{K\|B\|_\beta^{1/\beta}},$$

taking into account that $b \in C_b^N(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C_b^{N+1}(\mathbb{R}^d; \mathbb{R}^d)$, which enables us to apply Lemma 3.3 to (3.24) to obtain the upper bound of the quantity $\sup_{r\in[0,T]} \mathscr{D}_{M+1}(D_r X)$.

The estimates of the Euler scheme and the modified Euler scheme and their derivatives can be obtained in the same way. We omit the proof, and we only point out that one more derivative of σ is needed for the modified Euler scheme because the function $\nabla\sigma$ is involved in its equation. □

4. Rate of convergence for the modified Euler scheme and related processes. The main result of this section is the convergence rate of the scheme defined by (1.3) to the solution of the SDE (1.1). Recall that γ_n is the function of n defined in (1.5).

THEOREM 4.1. *Let X and X^n be solutions to equations* (1.1) *and* (1.3), *respectively. We assume $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$. Then for any $p \geq 1$ there exists a constant C independent of n* (*but dependent on p*) *such that*

$$\sup_{0\leq t\leq T} \mathbb{E}\big[|X_t^n - X_t|^p\big]^{1/p} \leq C\gamma_n^{-1}.$$

PROOF. Denote $Y := X - X^n$. Notice that Y depends on n, but for notational simplicity we shall omit the explicit dependence on n for Y and some other processes when there is no ambiguity. The idea of the proof is to decompose Y into seven terms [see (4.7) below] and then study their convergence rate individually.

Step 1. By the definitions of the processes X and X^n, we have

$$\begin{aligned} Y_t = {} & \int_0^t \big[b(X_s) - b(X_s^n) + b(X_s^n) - b(X_{\eta(s)}^n)\big]\,ds \\ & + \sum_{j=1}^m \int_0^t \big[\sigma^j(X_s) - \sigma^j(X_s^n) + \sigma^j(X_s^n) - \sigma^j(X_{\eta(s)}^n)\big]\,dB_s^j \\ & - H\sum_{j=1}^m \int_0^t (\nabla\sigma^j\sigma^j)(X_{\eta(s)}^n)(s-\eta(s))^{2H-1}\,ds. \end{aligned}$$

By denoting

$$\sigma_0^j(s) = (\nabla\sigma^j\sigma^j)(X_{\eta(s)}^n), \qquad b_1(s) = \int_0^1 \nabla b(\theta X_s + (1-\theta)X_s^n)\,d\theta,$$

$$\sigma_1^j(s) = \int_0^1 \nabla\sigma^j(\theta X_s + (1-\theta)X_s^n)\,d\theta,$$

we can write

$$\begin{aligned} Y_t = {} & \int_0^t b_1(s)Y_s\,ds + \sum_{j=1}^m \int_0^t \sigma_1^j(s)Y_s\,dB_s^j + \int_0^t \big[b(X_s^n) - b(X_{\eta(s)}^n)\big]\,ds \\ & + \sum_{j=1}^m \int_0^t \big[\sigma^j(X_s^n) - \sigma^j(X_{\eta(s)}^n)\big]\,dB_s^j - H\sum_{j=1}^m \int_0^t \sigma_0^j(s)(s-\eta(s))^{2H-1}\,ds. \end{aligned}$$

Let $\Lambda^n = \{\Lambda_t^n, t\in[0,T]\}$ be the $d\times d$ matrix-valued solution of the following linear SDE:

$$\Lambda_t^n = I + \int_0^t b_1(s)\Lambda_s^n\,ds + \sum_{j=1}^m \int_0^t \sigma_1^j(s)\Lambda_s^n\,dB_s^j, \tag{4.1}$$

where I is the $d\times d$ identity matrix. Applying the chain rule for the Young integral to $\Gamma_t^n\Lambda_t^n$, where Γ_t^n, $t\in[0,T]$ is the unique solution of the equation

$$\Gamma_t^n = I - \int_0^t \Gamma_s^n b_1(s)\,ds - \sum_{j=1}^m \int_0^t \Gamma_s^n\sigma_1^j(s)\,dB_s^j, \tag{4.2}$$

for $t \in [0, T]$, we see that $\Gamma_t^n \Lambda_t^n = \Lambda_t^n \Gamma_t^n = I$ for all $t \in [0, T]$. Therefore, $(\Lambda_t^n)^{-1}$ exists and coincides with Γ_t^n.

We can express the process Y_t in terms of Λ_t^n as follows:

$$
\begin{aligned}
Y_t = & \int_0^t \Lambda_t^n \Gamma_s^n \big[b(X_s^n) - b(X_{\eta(s)}^n)\big]\, ds \\
& + \sum_{j=1}^m \int_0^t \Lambda_t^n \Gamma_s^n \big[\sigma^j(X_s^n) - \sigma^j(X_{\eta(s)}^n)\big]\, dB_s^j \\
& - H \sum_{j=1}^m \int_0^t \Lambda_t^n \Gamma_s^n \sigma_0^j(s)\big(s - \eta(s)\big)^{2H-1}\, ds.
\end{aligned} \tag{4.3}
$$

The first two terms in the right-hand side of equation (4.3) can be further decomposed as follows:

$$
\begin{aligned}
& \int_0^t \Lambda_t^n \Gamma_s^n \big[\sigma^j(X_s^n) - \sigma^j(X_{\eta(s)}^n)\big]\, dB_s^j \\
& \quad = \int_0^t \Lambda_t^n \Gamma_s^n b_2^j(s)\big(s - \eta(s)\big)\, dB_s^j \\
& \qquad + \sum_{i=1}^m \int_0^t \Lambda_t^n \Gamma_s^n \sigma_2^{j,i}(s)\big(B_s^i - B_{\eta(s)}^i\big)\, dB_s^j \\
& \qquad + \int_0^t \Lambda_t^n \Gamma_s^n \sigma_3^j(s)\big(s - \eta(s)\big)^{2H}\, dB_s^j \\
& \quad := I_{2,j}(t) + \sum_{i=1}^m I_{3,j,i}(t) + I_{4,j}(t),
\end{aligned} \tag{4.4}
$$

where

$$
\begin{aligned}
b_2^j(s) &= \int_0^1 \nabla\sigma^j\big(\theta X_s^n + (1-\theta) X_{\eta(s)}^n\big) b\big(X_{\eta(s)}^n\big)\, d\theta, \\
\sigma_2^{j,i}(s) &= \int_0^1 \nabla\sigma^j\big(\theta X_s^n + (1-\theta) X_{\eta(s)}^n\big) \sigma^i\big(X_{\eta(s)}^n\big)\, d\theta, \\
\sigma_3^j(s) &= \frac{1}{2}\int_0^1 \nabla\sigma^j\big(\theta X_s^n + (1-\theta) X_{\eta(s)}^n\big) \sum_{l=1}^m \sigma_0^l(s)\, d\theta
\end{aligned}
$$

and

$$
\begin{aligned}
& \Lambda_t^n \int_0^t \Gamma_s^n \big[b(X_s^n) - b(X_{\eta(s)}^n)\big]\, ds \\
& \quad = \Lambda_t^n \int_0^t \Gamma_s^n b_3(s)\Bigg[b\big(X_{\eta(s)}^n\big)\big(s - \eta(s)\big) + \sum_{j=1}^m \sigma^j\big(X_{\eta(s)}^n\big)\big(B_s^j - B_{\eta(s)}^j\big)
\end{aligned} \tag{4.5}
$$

$$+\frac{1}{2}\sum_{j=1}^{m}\sigma_0^j(s)\bigl(s-\eta(s)\bigr)^{2H}\Biggr]\,ds$$

$$:= I_{11}(t)+\sum_{j=1}^{m}I_{12,j}(t)+I_{13}(t),$$

where $b_3(s)=\int_0^1 \nabla b(\theta X_s^n+(1-\theta)X_{\eta(s)}^n)\,d\theta$. We also denote

$$I_{5,j}(t)=-H\Lambda_t^n\int_0^t \Gamma_s^n\sigma_0^j(s)\bigl(s-\eta(s)\bigr)^{2H-1}\,ds. \tag{4.6}$$

Substituting equations (4.4), (4.5) and (4.6) into (4.3) yields

$$Y=I_{11}+\sum_{j=1}^{m}I_{12,j}+I_{13}+\sum_{j=1}^{m}I_{2,j}+\sum_{j,i=1}^{m}I_{3,j,i}+\sum_{j=1}^{m}I_{4,j}+\sum_{j=1}^{m}I_{5,j}. \tag{4.7}$$

Step 2. Denote by $(\Lambda^n)_i$, $i=1,\dots,d$, the ith columns of Λ^n. We claim that $(\Lambda^n)_i$ satisfy the conditions in Lemma 3.3 with $M=d$, $\tau=0$, $U_t^1=b_1(t)$, $U_t^2=\sigma_1^j(t)$ and $N=2$. We first show that U^2 satisfies (3.22). Taking into account that $b\in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma\in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$, it suffices to show that both X and X^n satisfy (3.22). This is clear for X because of Lemma 3.1(i). It follows from Lemma 3.2(i) that there exist constants K and K' such that $(t-s)^\beta\|B\|_\beta\le K$, $0\le s<t\le T$ implies

$$\|X^n\|_{s,t,\beta,n}\le K'\bigl(\|B\|_\beta+1\bigr).$$

Notice that

$$\begin{aligned}\frac{|X_t^n-X_s^n|}{(t-s)^\beta}&\le\frac{|X_t^n-X_{\varepsilon(s)}^n|}{(t-\varepsilon(s))^\beta}+\frac{|X_{\varepsilon(s)}^n-X_s^n|}{(\varepsilon(s)-s)^\beta}\\&\le\|X^n\|_{s,t,\beta,n}+\frac{|X_{\varepsilon(s)}^n-X_s^n|}{\varepsilon(s)-s}\end{aligned}$$

for $t,s:t\ge\varepsilon(s)$, where we recall that $\varepsilon(s)=t_{k+1}$ when $s\in(t_k,t_{k+1}]$. Therefore, to verify (3.22) for X^n it suffices to show that

$$\|X^n\|_{s,t,\beta}\le K'\bigl(\|B\|_\beta+1\bigr)$$

for $s,t\in[t_k,t_{k+1}]$ for some k. But this follows immediately from (1.3). On the other hand, the fact that $\mathscr{D}_2^*U^1$ and $\mathscr{D}_2U^2$ are less than $Ke^{K\|B\|_\beta^{1/\beta}}$ for some K follows from Proposition 3.1, and the assumption that $b\in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma\in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$, where $\mathscr{D}_2^*$ and $\mathscr{D}_2$ are defined in Section 3.

In the same way we can show that the columns of Γ^n satisfy the assumptions of Lemma 3.3. As a consequence, it follows from Lemma 3.3 that

$$\mathscr{D}_2\Lambda^n\vee\mathscr{D}_2\Gamma^n\le Ke^{K\|B\|_\beta^{1/\beta}}. \tag{4.8}$$

Step 3. From (4.8) and from the fact that $b \in C_b^3(\mathbb{R}^d; \mathbb{R}^d)$ and $\sigma \in C_b^4(\mathbb{R}^d; \mathbb{R}^{d\times m})$, it follows that

$$\mathbb{E}\big(|I_{11}(t)|^p\big)^{1/p} \le Cn^{-1} \quad \text{and} \quad \mathbb{E}\big(|I_{13}(t)|^p\big)^{1/p} \le Cn^{-2H}. \tag{4.9}$$

Notice that n^{-1} and n^{-2H} are bounded by γ_n^{-1}. Applying estimates (A.4) and (A.5), inequality (4.8) and Proposition 3.1, we have for any j

$$\begin{aligned} &\mathbb{E}\big(|I_{12,j}(t)|^p\big)^{1/p} \le Cn^{-1}, \qquad \mathbb{E}\big(|I_{2,j}(t)|^p\big)^{1/p} \le Cn^{-1}, \\ &\mathbb{E}\big(|I_{4,j}(t)|^p\big)^{1/p} \le Cn^{-2H}. \end{aligned} \tag{4.10}$$

Now to complete the proof of the theorem it suffices to show that for any j, $\mathbb{E}(|\sum_{i=1}^m I_{3,j,i}(t) + I_{5,j}(t)|^p)^{1/p} \le C\gamma_n^{-1}$. For any fixed j we make the decomposition

$$\sum_{i=1}^m I_{3,j,i} + I_{5,j} = E_{1,j} + E_{2,j} + E_{3,j}, \tag{4.11}$$

where

$$E_{1,j}(t) = \Lambda_t^n \sum_{i=1}^m \int_0^t \big[\Gamma_s^n \sigma_2^{j,i}(s) - \Gamma_{\eta(s)}^n (\nabla\sigma^j \sigma^i)\big(X_{\eta(s)}^n\big)\big]\big(B_s^i - B_{\eta(s)}^i\big)\, dB_s^j,$$

$$\begin{aligned} E_{2,j}(t) &= \Lambda_t^n \sum_{i=1}^m \int_0^t \Gamma_{\eta(s)}^n (\nabla\sigma^j \sigma^i)\big(X_{\eta(s)}^n\big)\big(B_s^i - B_{\eta(s)}^i\big)\, dB_s^j \\ &\quad - H\Lambda_t^n \int_0^t \Gamma_{\eta(s)}^n \sigma_0^j(s)\big(s-\eta(s)\big)^{2H-1}\, ds, \end{aligned}$$

$$E_{3,j}(t) = H\Lambda_t^n \int_0^t \big(\Gamma_{\eta(s)}^n - \Gamma_s^n\big)\sigma_0^j(s)\big(s-\eta(s)\big)^{2H-1}\, ds.$$

Applying (4.8) for the quantities $\|\Lambda^n\|_\infty$ and $\|\Gamma^n\|_\beta$, it is easy to see that $\mathbb{E}(|E_{3,j}(t)|^p)^{1/p} \le Cn^{1-2H-\beta}$ for any $\frac{1}{2} < \beta < H$. On the other hand, applying estimate (A.15) from Lemma A.5 to $E_{1,j}$, we obtain $\mathbb{E}(|E_{1,j}(t)|^p)^{1/p} \le Cn^{1-3\beta}$ for any $\frac{1}{2} < \beta < H$. Notice that the exponents $n^{1-2H-\beta}$ and $n^{1-3\beta}$ are bounded by γ_n^{-1} if β is sufficiently close to H.

Taking into account the relationship between the Skorohod and path-wise integral, we can express the term $E_{2,j}$ as follows:

$$E_{2,j}(t) = \Lambda_t^n \sum_{i=1}^m \sum_{k=0}^{\lfloor nt/T \rfloor} F_{t_k}^{n,i,j} \int_{t_k}^{t_{k+1}\wedge t} \int_{t_k}^s \delta B_u^i \delta B_s^j, \tag{4.12}$$

for $t \in [0,T]$, where $F_t^{n,i,j} = \Gamma_t^n (\nabla\sigma^j\sigma^i)(X_t^n)$, and we define $t_{n+1} = (n+1)\frac{T}{n}$. From (4.8) and Proposition 3.1, we have

$$\max\big\{|F_t^{n,i,j}|, |D_{r_1} F_t^{n,i,j}|, |D_{r_2} D_{r_1} F^{n,i,j}|\big\} \le K e^{K\|B\|_\beta^{1/\beta}}. \tag{4.13}$$

Hence, applying estimate (A.8) from Lemma A.4 to $E_{2,j}(t)$, we obtain $\mathbb{E}(|E_{2,j}(t)|^p)^{1/p} \le C\gamma_n^{-1}$. The proof is now complete. □

The following result provides a rate of convergence for the Malliavin derivatives of the modified scheme and some related processes. Recall that β satisfies $\frac{1}{2} < \beta < H$.

LEMMA 4.1. *Let X and X^n be the processes defined by* (1.1) *and* (1.3), *respectively. Suppose that $\sigma \in C_b^5(\mathbb{R}^d;\mathbb{R}^{d\times m})$, $b \in C_b^4(\mathbb{R}^d;\mathbb{R}^d)$. Let $p \ge 1$. Then*:

(i) *There exists a constant C such that the quantities $\|D_s X_t - D_s X_t^n\|_p$, $\|D_r D_s X_t - D_r D_s X_t^n\|_p$, $\|D_u D_r D_s X_t - D_u D_r D_s X_t^n\|_p$ are less than $Cn^{1-2\beta}$ for all $u, r, s, t \in [0, T]$ and $n \in \mathbb{N}$.*

(ii) *Let V and V^n be d-dimensional processes satisfying the equations*

$$V_t = V_0 + \int_0^t f_1(X_u, X_u)V_u\,du + \sum_{j=1}^m \int_0^t f_2^j(X_u, X_u)V_u\,dB_u^j,$$

$$V_t^n = V_0 + \int_0^t f_1(X_u, X_u^n)V_u^n\,du + \sum_{j=1}^m \int_0^t f_2^j(X_u, X_u^n)V_u^n\,dB_u^j,$$

where $f_1 \in C_b^3(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R}^{d\times d})$ and $f_2^j \in C_b^4(\mathbb{R}^d \times \mathbb{R}^d;\mathbb{R}^{d\times d})$. Then there exists a constant C such that the quantities $\|V_t - V_t^n\|_p$, $\|D_s V_t - D_s V_t^n\|_p$, $\|D_r D_s V_t - D_r D_s V_t^n\|_p$ are less than $Cn^{1-2\beta}$ for all $r, s, t \in [0, T]$ and $n \in \mathbb{N}$.

REMARK 4.1. The above results still hold when the approximation process X^n is replaced by the one defined by the recursive scheme (1.2). The proof follows exactly along the same lines.

PROOF OF LEMMA 4.1. (i) Taking the Malliavin derivative in both sides of (4.3), we obtain

$$\begin{aligned} D_r(X_t - X_t^n) &= \int_0^t D_r\big[\Lambda_t^n\Gamma_s^n\big(b(X_s^n) - b(X_{\eta(s)}^n)\big)\big]\,ds \\ &\quad + \sum_{j=1}^m \int_0^t D_r\big[\Lambda_t^n\Gamma_s^n\big(\sigma^j(X_s^n) - \sigma^j(X_{\eta(s)}^n)\big)\big]\,dB_s^j \\ &\quad + \sum_{j=1}^m \Lambda_t^n\Gamma_r^n\big(\sigma^j(X_r^n) - \sigma^j(X_{\eta(r)}^n)\big) \\ &\quad - H\sum_{j=1}^m \int_0^t D_r\big[\Lambda_t^n\Gamma_s^n\sigma_0^j(s)\big]\big(s - \eta(s)\big)^{2H-1}\,ds. \end{aligned}$$

Proposition 3.1 and equation (4.8) imply that the first, third and last terms of the above right-hand side have L^p-norms bounded by Cn^{1-2H}. Applying estimate (A.16) from Lemma A.5 to the second term and noticing that $\|X\|_\beta$ and $\sup_{r\in[0,T]}\|D_rX\|_\beta$ have finite moments of any order, we see that its L^p-norm is also bounded by $Cn^{1-2\beta}$.

Similarly, we can take the second derivative in (4.3) and then estimate each term individually as before to obtain that the upper bound of $\|D_rD_sX_t - D_rD_sX_t^n\|_p$ is bounded by $Cn^{1-2\beta}$.

(ii) Using the chain rule for Young's integral we derive the following explicit expression for $V_t - V_t^n$:

$$
\begin{aligned}
V_t - V_t^n &= \int_0^t \Upsilon_t\Upsilon_s^{-1}\big(f_1(X_s,X_s) - f_1\big(X_s,X_s^n\big)\big)V_s^n\,ds \\
&\quad + \sum_{j=1}^m \int_0^t \Upsilon_t\Upsilon_s^{-1}\big(f_2^j(X_s,X_s) - f_2^j\big(X_s,X_s^n\big)\big)V_s^n\,dB_s^j,
\end{aligned}
\tag{4.14}
$$

where $\Upsilon = \{\Upsilon_t, t\in[0,T]\}$ is the $\mathbb{R}^{d\times d}$-valued process that satisfies

$$
\Upsilon_t = I + \int_0^t f_1(X_s,X_s)\Upsilon_t\,ds + \sum_{j=1}^m \int_0^t f_2^j(X_s,X_s)\Upsilon_t\,dB_s^j.
$$

Lemma 3.3 implies that there exists a constant K such that for all $n\in\mathbb{N}$, $u,r,s,t\in[0,T]$, we have

$$
\max\{\Upsilon_t, D_s\Upsilon_t, D_rD_s\Upsilon_t, D_uD_rD_s\Upsilon_t\} \le Ke^{K\|B\|_\beta^{1/\beta}}.
\tag{4.15}
$$

Therefore, applying estimate (A.4) to the second integral in (4.14) with $\nu = 0$ and taking into account the estimate of Lemma 4.1(i), we obtain

$$
\|V - V^n\|_p \le Cn^{1-2\beta}.
$$

Taking the Malliavin derivative on both sides of (4.14), and then applying estimates (A.4) from Lemmas A.3 and 4.1(i) as before, we can obtain the desired estimate for $\|D_sV_t - D_sV_t^n\|_p$. The estimate for $\|D_rD_sV_t - D_rD_sV_t^n\|_p$ can be obtained in a similar way. □

We define $\{\Lambda_t, t\in[0,T]\}$ as the solution of the limiting equation of (4.1), that is,

$$
\Lambda_t = I + \int_0^t \nabla b(X_s)\Lambda_s\,ds + \sum_{j=1}^m \int_0^t \nabla\sigma^j(X_s)\Lambda_s\,dB_s^j.
\tag{4.16}
$$

The inverse of the matrix Λ_t, denoted by Γ_t, exists and satisfies

$$
\Gamma_t = I - \int_0^t \Gamma_t\nabla b(X_s)\,ds - \sum_{j=1}^m \int_0^t \Gamma_t\nabla\sigma^j(X_s)\,dB_s^j.
$$

It follows from Lemma 4.1 that if we assume that $\sigma \in C_b^5(\mathbb{R}^d; \mathbb{R}^{d\times m})$ and $b \in C_b^4(\mathbb{R}^d; \mathbb{R}^d)$, then the estimate in Lemma 4.1(ii) holds with the pair (V, V^n) being replaced by (Γ_i, Γ_i^n) or (Λ_i, Λ_i^n), $i = 1, \dots, d$, where the subindex i denotes the ith column of each matrix.

5. Central limit theorem for weighted sums. Our goal in this section is to prove a central limit result for weighted sums (see Proposition 5.5 below) that will play a fundamental role in the proof of Theorem 6.1 in the next section. This result has an independent interest and we devote this entire section to it.

We recall that $B = \{B_t, t \in [0, T]\}$ is an m-dimensional fBm, and we assume that the Hurst parameter satisfies $H \in (\frac{1}{2}, \frac{3}{4}]$. For any $n \geq 1$ we set $t_j = \frac{jT}{n}$, $j = 0, \dots, n$. Recall that $\eta(s) = t_k$ if $t_k \leq s < t_{k+1}$. Consider the $d \times d$ matrix-valued process

$$\Xi_t^{n,i,j} = \gamma_n \sum_{k=0}^{\{t\}} \int_{t_k}^{t_{k+1}} (B_s^i - B_{\eta(s)}^i) \delta B_s^j, \qquad 1 \leq i, j \leq m,$$

where we denote $\{t\} = \lfloor \frac{nt}{T} \rfloor$ for $t \in [0, T)$ and $\{T\} = t_{n-1}$.

PROPOSITION 5.1. *The following stable convergence holds as n tends to infinity*

$$(\Xi^n, B) \to (W, B),$$

where $W = \{W_t, t \in [0, T]\}$ is the matrix-valued Brownian motion, introduced in Section 2.4, and W and B are independent.

PROOF. From inequality (A.8) in Lemma A.4 it follows that

$$\mathbb{E}(|\Xi_{t_k}^n - \Xi_{t_j}^n|^4) \leq C\left(\frac{k-j}{n}\right)^2, \tag{5.1}$$

for any $j \leq k$. This implies the tightness of (Ξ^n, B).

Then it remains to show the convergence of the finite dimensional distributions of (Ξ^n, B) to that of (W, B). To do this, we fix a finite set of points $r_1, \dots, r_{L+1} \in [0, T]$ such that $0 = r_1 < r_2 < \cdots < r_{L+1} \leq T$ and define the random vectors $B_L = (B_{r_2} - B_{r_1}, \dots, B_{r_{L+1}} - B_{r_L})$, $\Xi_L^n = (\Xi_{r_2}^n - \Xi_{r_1}^n, \dots, \Xi_{r_{L+1}}^n - \Xi_{r_L}^n)$ and $W_L = (W_{r_2} - W_{r_1}, \dots, W_{r_{L+1}} - W_{r_L})$. We claim that as n tends to infinity, the following convergence in law holds:

$$(\Xi_L^n, B_L) \Rightarrow (W_L, B_L). \tag{5.2}$$

For notational simplicity, we add one term to each component of Ξ_L^n, and we define

$$\Theta_l^n(i, j) := \Xi_{r_{l+1}}^{n,i,j} - \Xi_{r_l}^{n,i,j} + \zeta_{\{r_l\},n}^{i,j} = \gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \zeta_{k,n}^{i,j}, \tag{5.3}$$

for $1 \le l \le L$, $1 \le i, j \le d$, where

$$\zeta_{k,n}^{i,j} = \int_{t_k}^{t_{k+1}} (B_s^i - B_{t_k}^i)\delta B_s^j.$$

Then Slutsky's lemma implies that the convergence in law in (5.2) is equivalent to

$$\big(\Theta_l^n(i,j), 1 \le i, j \le d, 1 \le l \le L, B_L\big) \Rightarrow (W_L, B_L).$$

According to Peccati and Tudor [27] (see also Theorem 6.2.3 in [21]), to show the convergence in law of (Θ_L^n, B_L), it suffices to show the convergence of each component of (Θ_L^n, B_L) to the correspondent component of (W_L, B_L) and the convergence of the covariance matrix.

The convergence of the covariance matrix of Θ_L^n follows from Propositions 5.2 and 5.3 below. The convergence in law of each component to a Gaussian distribution follows from Proposition 5.4 below and the fourth moment theorem; see [24] and also Theorem 5.2.7 in [21]. This completes the proof. $\square$

In order to show the convergence of the covariance matrix and the fourth moment of Θ_n we first introduce the following notation:

$$\begin{aligned} \mathcal{D}_k &= \big\{(s,t,v,u) : t_k \le v \le s \le t_{k+1}, u, t \in [0,T]\big\}, \\ \mathcal{D}_{k_1,k_2} &= \big\{(s,t,v,u) : t_{k_2} \le v \le s \le t_{k_2+1}, t_{k_1} \le u \le t \le t_{k_1+1}\big\}. \end{aligned} \tag{5.4}$$

The next two propositions provide the convergence of the covariance $\mathbb{E}[\Theta_{l'}^n(i',j')\Theta_l^n(i,j)]$ in the cases $l = l'$ and $l \ne l'$, respectively. We denote $\beta_{k/n}(s) = \mathbf{1}_{[t_k,t_{k+1}]}(s)$.

PROPOSITION 5.2. *Let $\Theta_l^n(i,j)$ be defined in (5.3). Then*

$$\mathbb{E}\big[\Theta_l^n(i',j')\Theta_l^n(i,j)\big] \to \alpha_H^2 \frac{r_{l+1} - r_l}{T}(R\delta_{ji'}\delta_{ij'} + Q\delta_{jj'}\delta_{ii'}), \tag{5.5}$$

as $n \to +\infty$. Here $\delta_{ii'}$ is the Kronecker function, $\alpha_H = H(2H-1)$ and Q and R are the constants defined in (2.12).

PROOF. The proof will involve several steps.

Step 1. Applying twice the integration by parts formula (2.7), we have

$$\begin{aligned} &\mathbb{E}\big[\Theta_l^n(i',j')\Theta_l^n(i,j)\big] \\ &\quad = \alpha_H^2 \gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathcal{D}_k} D_u^i D_t^j \Theta_l^n(i',j')\mu(dv\,du)\mu(ds\,dt), \end{aligned} \tag{5.6}$$

where we recall that $\{t\} = \lfloor \frac{nt}{T} \rfloor$ for $t \in [0, T)$ and $\{T\} = t_{n-1}$, and $\mathcal{D}_k$ is defined in (5.4). Since

$$
\begin{aligned}
&D_u^i D_t^j \Theta_l^n(i', j') \\
&\quad = \gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \big(\mathbf{1}_{[t_k,t]}(u)\beta_{k/n}(t)\delta_{jj'}\delta_{ii'} + \mathbf{1}_{[t_k,u]}(t)\beta_{k/n}(u)\delta_{ji'}\delta_{ij'}\big),
\end{aligned}
\tag{5.7}
$$

the left-hand side of (5.5) equals

$$
\begin{aligned}
\alpha_H^2 \gamma_n^2 & \sum_{k,k'=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathcal{D}_k} \{\mathbf{1}_{[t_{k'},t]}(u)\beta_{k'/n}(t)\delta_{jj'}\delta_{ii'} \\
&\qquad\qquad + \mathbf{1}_{[t_{k'},u]}(t)\beta_{k'/n}(u)\delta_{ji'}\delta_{ij'}\}\mu(dv\,du)\mu(ds\,dt) \\
&:= \alpha_H^2\gamma_n^2(G_1\delta_{jj'}\delta_{ii'} + G_2\delta_{ji'}\delta_{ij'}).
\end{aligned}
$$

In the next two steps, we compute the limits of $\gamma_n^2 G_1$ and $\gamma_n^2 G_2$ as n tends to infinity in the case $H \in (\frac{1}{2}, \frac{3}{4})$ and in the case $H = \frac{3}{4}$ separately.

Step 2. In this step, we consider the case $H \in (\frac{1}{2}, \frac{3}{4})$. Recall that

$$
\begin{aligned}
Q(p) &= T^{4H} \int_0^1 \int_p^{p+1} \int_0^t \int_p^s \mu(dv\,du)\mu(ds\,dt) \\
&= n^{4H} \int_{\mathcal{D}_{k',k'+p}} \mu(dv\,du)\mu(ds\,dt),
\end{aligned}
$$

which is independent of n, where the set $\mathcal{D}_{k_1,k_2}$ is defined in (5.4). We can express $\gamma_n^2 G_1$ in terms of $Q(p)$ as follows:

$$
\begin{aligned}
\gamma_n^2 G_1 &= n^{4H-1} \sum_{k,k'=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathcal{D}_{k',k}} \mu(dv\,du)\mu(ds\,dt) \\
&= \frac{1}{n} \sum_{p=\{r_l\}-\{r_{l+1}\}}^{\{r_{l+1}\}-\{r_l\}} \sum_{k'=(\{r_l\}-p)\vee\{r_l\}}^{(\{r_{l+1}\}-p)\wedge\{r_{l+1}\}} Q(p) \\
&= \sum_{p=-\infty}^{\infty} \Psi_l^n(p) Q(p),
\end{aligned}
$$

where

$$
\Psi_l^n(p) = \frac{(\{r_{l+1}\} - p) \wedge \{r_{l+1}\} - (\{r_l\} - p) \vee \{r_l\}}{n} \mathbf{1}_{[\{r_l\}-\{r_{l+1}\},\{r_{l+1}\}-\{r_l\}]}(p).
$$

The term $\Psi_l^n(p)$ is uniformly bounded and converges to $\frac{r_{l+1}-r_l}{T}$ as n tends to infinity for any fixed p. Therefore, taking into account that $\sum_{p=-\infty}^{\infty} Q(p) = Q < \infty$, the dominated convergence theorem implies

$$\lim_{n\to\infty} \gamma_n^2 G_1 = \frac{r_{l+1} - r_l}{T} Q.$$

Similarly, we can show that

$$\lim_{n\to\infty} \gamma_n^2 G_2 = \frac{r_{l+1} - r_l}{T} R.$$

Step 3. In the case $H = \frac{3}{4}$, we can write

$$\begin{aligned}
\gamma_n^2 G_1 &= \frac{n^2}{\log n} \sum_{k,k'=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathcal{D}_{k',k}} \mu(dv\,du)\mu(ds\,dt) \\
&= \frac{1}{n\log n} \sum_{p=\{r_l\}-\{r_{l+1}\}}^{\{r_{l+1}\}-\{r_l\}} \sum_{k'=\{r_l\}}^{\{r_{l+1}\}} Q(p) \\
&\quad - \frac{1}{n\log n} \Bigg\{ \sum_{p=\{r_l\}-\{r_{l+1}\}}^{0} \sum_{k'=\{r_l\}}^{\{r_l\}-p-1} + \sum_{p=1}^{\{r_{l+1}\}-\{r_l\}} \sum_{k'=\{r_{l+1}\}-p+1}^{\{r_{l+1}\}} \Bigg\} Q(p) \\
&:= G_{11} + G_{12}.
\end{aligned}$$

Taking into account that $Q(p)$ behaves like $1/|p|$ as $|p|$ tends to infinity, it is then easy to see that G_{12} converges to zero. On the other hand, recall that $Q = \lim_{n\to+\infty} \frac{\sum_{|p|\le n} Q(p)}{\log n}$. This implies that G_{11} converges to $\frac{Q}{T}(r_{l+1} - r_l)$. This gives the limit of $\gamma_n^2 G_1$. The limit of $\gamma_n^2 G_2$ can be obtained similarly. $\square$

PROPOSITION 5.3. *Let $l, l' \in \{1, \dots, L\}$ be such that $l \neq l'$. Let Θ^n be defined as in* (5.3). *Then*

$$\lim_{n\to\infty} \mathbb{E}[\Theta_{l'}^n(i', j')\Theta_l^n(i, j)] = 0. \tag{5.8}$$

PROOF. Without any loss of generality, we assume $l' < l$. As in (5.6) we have

$$\mathbb{E}[\Theta_{l'}^n(i', j')\Theta_l^n(i, j)] = \alpha_H^2 \gamma_n \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \int_{\mathcal{D}_k} D_u^i D_t^j \Theta_{l'}^n(i', j')\mu(dv\,du)\mu(ds\,dt).$$

Taking into account (5.7), we can write

$$\begin{aligned}
&\mathbb{E}[\Theta_{l'}^n(i', j')\Theta_l^n(i, j)] \\
&\quad = \alpha_H^2 \gamma_n^2 \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \sum_{k'=\{r_{l'}\}}^{\{r_{l'+1}\}} \int_{\mathcal{D}_k} \{\mathbf{1}_{[t_{k'},t]}(u)\beta_{k'/n}(t)\delta_{jj'}\delta_{ii'} \\
&\qquad\qquad + \mathbf{1}_{[t_{k'},u]}(t)\beta_{k'/n}(u)\delta_{ji'}\delta_{ij'}\}\mu(dv\,du)\mu(ds\,dt) \\
&\quad := \alpha_H^2 \gamma_n^2 (\widetilde{G}_1\delta_{jj'}\delta_{ii'} + \widetilde{G}_2\delta_{ji'}\delta_{ij'}).
\end{aligned}$$

In the case $H \in (\frac{1}{2}, \frac{3}{4})$ we have

$$\gamma_n^2 \widetilde{G}_1 = n^{4H-1} \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \sum_{k'=\{r_{l'}\}}^{\{r_{l'+1}\}} \int_{\mathcal{D}_k} \mathbf{1}_{[t_{k'},t]}(u) \beta_{k'/n}(t) \mu(dv\,du)\mu(ds\,dt)$$

$$= \frac{1}{n} \sum_{p=\{r_l\}-\{r_{l'+1}\}}^{\{r_{l+1}\}-\{r_{l'}\}} \sum_{k'=(\{r_l\}-p)\vee\{r_{l'}\}}^{\{r_{l'+1}\}\wedge(\{r_{l+1}\}-p)} Q(p)$$

$$= \sum_{p=-\infty}^{\infty} \Phi_l^n(p) Q(p),$$

where $\Phi_l^n(p)$ is equal to

$$\frac{\max\{(\{r_{l'+1}\}-p) \wedge \{r_{l+1}\} - (\{r_l\}-p) \vee \{r_l'\}, 0\}}{n} \mathbf{1}_{[\{r_l\}-\{r_{l'+1}\}, \{r_{l+1}\}-\{r_{l'}\}]}(p).$$

The term $\Phi_l^n(p)$ is uniformly bounded and converges to 0 as n tends to infinity for any fixed p because $l < l'$. Therefore, taking into account that $\sum_{p=-\infty}^{\infty} Q(p) = Q < \infty$, the dominated convergence theorem implies that $\gamma_n^2 \widetilde{G}_1$ converges to zero as n tends to infinity. Similarly, we can show that $\gamma_n^2 \widetilde{G}_2$ converges to zero as n tends to infinity.

In the case $H = \frac{3}{4}$, since

$$\gamma_n^2 \widetilde{G}_1 = \frac{n^2}{\ln n} \sum_{k=\{r_l\}}^{\{r_{l+1}\}} \sum_{k'=\{r_{l'}\}}^{\{r_{l'+1}\}} \int_{\mathcal{D}_{k',k}} \mu(dv\,du)\mu(ds\,dt)$$

$$= \frac{1}{n \ln n} \sum_{p=\{r_l\}-\{r_{l'+1}\}}^{\{r_{l+1}\}-\{r_{l'}\}} \sum_{k'=(\{r_l\}-p)\vee\{r_{l'}\}}^{\{r_{l'+1}\}\wedge(\{r_{l+1}\}-p)} Q(p),$$

we have

$$\gamma_n^2 \widetilde{G}_1 \le \frac{1}{n \ln n} \sum_{p=\{r_l\}-\{r_{l'+1}\}}^{\{r_{l+1}\}-\{r_{l'}\}} \sum_{k'=\{r_{l'}\}}^{\{r_{l+1}\}-p} Q(p)$$

$$\le \frac{1}{n \ln n} \sum_{p=-n}^{0} (p+1) Q(p).$$

Noticing that $Q(p) = O(\frac{1}{|p|})$, we conclude that $\gamma_n^2 \widetilde{G}_1 \le \frac{C}{\ln n}$. This shows that $\gamma_n^2 \widetilde{G}_1$ converges to zero as n tends to infinity. In the same way we can show that $\gamma_n^2 \widetilde{G}_2$ converges to zero. $\square$

The following estimate is needed in the calculation of the fourth moment of $\Theta_l^n(i,j)$ in Proposition 5.4.

LEMMA 5.1. *Let $H \in (\frac{1}{2}, \frac{3}{4}]$. We have the following estimate*:

$$\sum_{k_1,k_2,k_3,k_4=0}^{n-1} \langle \beta_{k_1/n}, \beta_{k_2/n}\rangle_{\mathcal{H}} \langle \beta_{k_2/n}, \beta_{k_3/n}\rangle_{\mathcal{H}} \langle \beta_{k_3/n}, \beta_{k_4/n}\rangle_{\mathcal{H}} \langle \beta_{k_1/n}, \beta_{k_4/n}\rangle_{\mathcal{H}} \\ \leq C n^{-2} \gamma_n^{-2}.$$

PROOF. Since the indices k_1, k_2, k_3, k_4 are symmetric, it suffices to consider the case $k_1 \leq k_2 \leq k_3 \leq k_4$. By definition of the inner product we have

$$\begin{aligned}
&\sum_{0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq n-1} \langle \beta_{k_1/n}, \beta_{k_2/n}\rangle_{\mathcal{H}} \langle \beta_{k_2/n}, \beta_{k_3/n}\rangle_{\mathcal{H}} \langle \beta_{k_3/n}, \beta_{k_4/n}\rangle_{\mathcal{H}} \langle \beta_{k_1/n}, \beta_{k_4/n}\rangle_{\mathcal{H}} \\
&\quad = \frac{T^{8H}}{2^4 n^{8H}} \sum_{k_1=0}^{n-1} \sum_{k_2=k_1}^{n-1} \sum_{k_3=k_2}^{n-1} \sum_{k_4=k_3}^{n-1} \big(|k_2 - k_1 + 1|^{2H} + |k_2 - k_1 - 1|^{2H} \\
&\qquad\qquad - 2|k_2 - k_1|^{2H}\big) \\
&\qquad\qquad \times \big(|k_3 - k_2 + 1|^{2H} + |k_3 - k_2 - 1|^{2H} \\
&\qquad\qquad - 2|k_3 - k_2|^{2H}\big) \\
&\qquad\qquad \times \big(|k_4 - k_3 + 1|^{2H} + |k_4 - k_3 - 1|^{2H} \\
&\qquad\qquad - 2|k_4 - k_3|^{2H}\big) \\
&\qquad\qquad \times \big(|k_4 - k_1 + 1|^{2H} + |k_4 - k_1 - 1|^{2H} \\
&\qquad\qquad - 2|k_4 - k_1|^{2H}\big).
\end{aligned}$$

Denote $p_i = k_{i+1} - k_i$, $i = 1, 2, 3$. Then the above sum is bounded by

$$C n^{1-8H} \sum_{p_1,p_2,p_3=1}^{n-1} p_1^{2H-2} p_2^{2H-2} p_3^{2H-2} (p_1 + p_2 + p_3)^{2H-2},$$

which is again bounded by

$$C n^{1-8H} \sum_{p_1,p_2,p_3=1}^{n-1} p_1^{2H-2} p_2^{2H-2} p_3^{4H-4}.$$

In the case $H \in (\frac{1}{2}, \frac{3}{4})$, the series $\sum_{p_3=1}^{n-1} p_3^{4H-4}$ is convergent. When $H = \frac{3}{4}$, it is bounded by $C \log n$. So the above sum is bounded by $C n^{-4H-1}$ if $\frac{1}{2} < H < \frac{3}{4}$ and bounded by $C n^{-4} \log n$ if $H = \frac{3}{4}$. The proof is complete. $\square$

The following proposition contains a result on the convergence of the fourth moment of $\Theta_n^l(i, j)$.

PROPOSITION 5.4. *The fourth moment of $\Theta_l^n(i,j)$ and $3\mathbb{E}(|\Theta_l^n(i,j)|^2)^2$ converge to the same limit as $n\to\infty$.*

PROOF. Applying the integration by parts formula (2.7) yields

$$
\begin{aligned}
\mathbb{E}&\big[\Theta_l^n(i,j)^4\big]\\
&=\alpha_H^2\gamma_n\sum_{k=\{r_l\}}^{\{r_{l+1}\}}\int_{\mathcal{D}_k}\mathbb{E}\big[D_u^iD_t^j\big[\Theta_l^n(i,j)^3\big]\big]\mu(dv\,du)\mu(ds\,dt)\\
&=\alpha_H^2\gamma_n\sum_{k=\{r_l\}}^{\{r_{l+1}\}}\int_{\mathcal{D}_k}\mathbb{E}\big[\big\{3\Theta_l^n(i,j)^2D_u^iD_t^j\big[\Theta_l^n(i,j)\big]\\
&\qquad\qquad+6\Theta_l^n(i,j)D_t^j\big[\Theta_l^n(i,j)\big]\\
&\qquad\qquad\qquad\times D_u^i\big[\Theta_l^n(i,j)\big]\big\}\big]\mu(dv\,du)\mu(ds\,dt)\\
&:=\overline{G}_1+\overline{G}_2.
\end{aligned}
$$

Since $D_u^iD_t^j[\Theta_l^n(i,j)]$ is deterministic, it is easy to see that $\overline{G}_1=3\mathbb{E}(|\Theta_l^n(i,j)|^2)^2$. We have shown the convergence of $\mathbb{E}(|\Theta_n^l(i,j)|^2)$ in Proposition 5.2. It remains to show that $\overline{G}_2\to 0$ as $n\to\infty$.

Applying again the integration by parts formula (2.7) yields

$$
\begin{aligned}
\overline{G}_2=6\alpha_H^4\gamma_n^2&\sum_{k,k'=\{r_l\}}^{\{r_{l+1}\}}\int_{\mathcal{D}_k\times\mathcal{D}_{k'}}D_{u'}^iD_{t'}^j\big\{D_t^j\big[\Theta_l^n(i,j)\big]D_u^i\big[\Theta_l^n(i,j)\big]\big\}\\
&\qquad\times\mu\big(dv'\,du'\big)\mu\big(ds'\,dt'\big)\mu(dv\,du)\mu(ds\,dt).
\end{aligned}
$$

Using equation (5.7) we can derive the inequalities

$$
\begin{aligned}
\overline{G}_2&\le 6\alpha_H^4\gamma_n^4\sum_{k,k',h,h'=\{r_l\}}^{\{r_{l+1}\}}\int_{\mathcal{D}_k\times\mathcal{D}_{k'}}\big\{\beta_{h/n}(t)\beta_{h/n}\big(t'\big)\beta_{h'/n}(u)\beta_{h'/n}\big(u'\big)\\
&\qquad\qquad+\beta_{h/n}(t)\beta_{h/n}\big(u'\big)\beta_{h'/n}(u)\beta_{h'/n}\big(t'\big)\big\}\\
&\qquad\qquad\times\mu\big(dv'du'\big)\mu\big(ds'\,dt'\big)\mu(dv\,du)\mu(ds\,dt)\\
&\le 12\alpha_H^4\gamma_n^4\sum_{k,k',h,h'=\{r_l\}}^{\{r_{l+1}\}}\langle\beta_{h/n},\beta_{k/n}\rangle_{\mathcal{H}}\langle\beta_{h'/n},\beta_{k/n}\rangle_{\mathcal{H}}\langle\beta_{h/n},\beta_{k'/n}\rangle_{\mathcal{H}}\\
&\qquad\qquad\times\langle\beta_{h'/n},\beta_{k'/n}\rangle_{\mathcal{H}}.
\end{aligned}
$$

The convergence of $\overline{G}_2$ to zero now follows from Lemma 5.1. $\square$

We can now establish a central limit theorem for weighted sums based on the previous proposition. Recall that $\zeta_{k,n}^{i,j}=\int_{t_k}^{t_{k+1}}(B_s^i-B_{t_k}^i)\delta B_s^j$, $k=0,\dots,n-1$ and $\zeta_{n,n}^{i,j}=0$.

PROPOSITION 5.5. *Let $f = \{f_t, t \in [0,T]\}$ be a stochastic process with values on the space of $d \times d$ matrices and with Hölder continuous trajectories of index greater than $\frac{1}{2}$. Set, for $i, j = 1, \dots, m$,*

$$\Psi_n^{i,j}(t) = \sum_{k=0}^{\{t\}} f_{t_k}^{i,j} \zeta_{k,n}^{i,j}.$$

Then, the following stable convergence in the space $D([0,T])$ holds as n tends to infinity:

$$\{\gamma_n \Psi_n(t), t \in [0,T]\} \to \left\{ \left(\int_0^t f_s^{i,j} \, dW_s^{ij} \right)_{1 \le i,j \le m}, t \in [0,T] \right\},$$

where W is a matrix-valued Brownian motion independent of B with the covariance introduced in Section 2.4.

PROOF. This proposition is an immediate consequence of the central limit result for weighted random sums proved in [3]. In fact, the process $\Psi_n^{i,j}(t)$ satisfies the required conditions due to Proposition 5.1 and the estimate (5.1). □

6. CLT for the modified Euler scheme in the case $H \in (\frac{1}{2}, \frac{3}{4}]$. The following central limit type result shows that in the case $H \in (\frac{1}{2}, \frac{3}{4}]$, the process $\gamma_n(X - X^n)$ converges stably to the solution of a linear stochastic differential equation driven by a matrix-valued Brownian motion independent of B as n tends to infinity.

THEOREM 6.1. *Let $H \in (\frac{1}{2}, \frac{3}{4}]$, and let X, X^n be the solutions of the SDE (1.1) and recursive scheme (1.3), respectively. Let $W = \{W_t, t \in [0,T]\}$ be the matrix-valued Brownian motion introduced in Section 2.4. Assume $\sigma \in C_b^5(\mathbb{R}^d; \mathbb{R}^{d \times m})$ and $b \in C_b^4(\mathbb{R}^d; \mathbb{R}^d)$. Then the following stable convergence in the space $C([0,T])$ holds as n tends to infinity*:

$$\{\gamma_n(X_t - X_t^n), t \in [0,T]\} \to \{U_t, t \in [0,T]\}, \tag{6.1}$$

where $\{U_t, t \in [0,T]\}$ is the solution of the linear d-dimensional SDE

$$\begin{aligned} U_t = {} & \int_0^t \nabla b(X_s) U_s \, ds + \sum_{j=1}^m \int_0^t \nabla \sigma^j(X_s) U_s \, dB_s^j \\ & + \sum_{i,j=1}^m \int_0^t (\nabla \sigma^j \sigma^i)(X_s) \, dW_s^{ij}. \end{aligned} \tag{6.2}$$

REMARK 6.1. It follows from [13] that when B is replaced by a standard Brownian motion, the process $\sqrt{n}(X - X^n)$ converges in law to the unique solution of the d-dimensional SDE

$$
\begin{aligned}
dU_t &= \nabla b(X_t)U_s\,dt + \sum_{j=1}^{m} \nabla\sigma^j(X_t)U_t\,dB_t^j \\
&\quad + \sqrt{\frac{T}{2}} \sum_{j,i=1}^{m} (\nabla\sigma^j\sigma^i)(X_t)\,dW_t^{ij}
\end{aligned}
\tag{6.3}
$$

with $U_0 = 0$. Here W^{ij}, $i, j = 1, \dots, m$ are independent one-dimensional Brownian motions, independent of B. To compare our Theorem 6.1 with this result, we let the Hurst parameter H converge to $\frac{1}{2}$. Then the constant R will converge to 0, and $\frac{\alpha_H}{\sqrt{T}}\sqrt{Q-R}$ converges to $\sqrt{\frac{T}{2}}$. This formally recovers equation (6.3).

REMARK 6.2. The process U defined in (6.2) is given by

$$
U_t = \sum_{i,j=1}^{m} \int_0^t \Lambda_t \Gamma_s (\nabla\sigma^j\sigma^i)(X_s)\,dW_s^{ij}, \qquad t \in [0, T],
\tag{6.4}
$$

where we recall that Λ is defined in (4.16) and Γ is its inverse.

PROOF OF THEOREM 6.1. Recall that $Y_t = X_t - X_t^n$. We would like to show that the process $\{\gamma_n Y_t, B_t, t \in [0, T]\}$ converges weakly in $C([0, T]; \mathbb{R}^{d+m})$ to $\{U_t, B_t, t \in [0, T]\}$. To do this, it suffices to prove the following:

(i) convergence of the finite dimensional distributions of $\{\gamma_n Y_t, B_t, t \in [0, T]\}$;
(ii) tightness of the process $\{\gamma_n Y_t, B_t, t \in [0, T]\}$.

We first show (i). Recall the decomposition of Y_t given in (4.7) and (4.11), and recall the estimates obtained for each term in the decomposition of Y_t. Since the other terms converge to zero in L^p for $p \ge 1$, from the Slutsky theorem it suffices to consider the convergence of the finite dimensional distributions of $\{\gamma_n \sum_{j=1}^m E_{2,j}(t), B_t, t \in [0, T]\}$, where $E_{2,j}$ is defined in Theorem 4.1 step 3. Set

$$
F_s^{i,j} := \Lambda_t^n \Gamma_s^n (\nabla\sigma^j\sigma^i)(X_s^n) - \Lambda_t \Gamma_s (\nabla\sigma^j\sigma^i)(X_s).
\tag{6.5}
$$

It follows from Lemma 4.1 and Remark 4.1 that

$$
\sup_{r,s,t\in[0,T]} \left(\| F_t^{i,j} \|_p \vee \| D_s F_t^{i,j} \|_p \vee \| D_r D_s F_t^{i,j} \|_p \right) \le C n^{1-2\beta}.
$$

Denote

$$
\widetilde{E}_{2,j}(t) = \Lambda_t \sum_{i=1}^{m} \sum_{k=0}^{\lfloor nt/T \rfloor} \Gamma_{t_k} (\nabla\sigma^j\sigma^i)(X_{t_k}) \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \delta B_u^i \delta B_s^j,
\tag{6.6}
$$

for $t \in [0, T)$, and $\widetilde{E}_{2,j}(T) = \widetilde{E}_{2,j}(T-)$. Then applying Lemma A.4 (A.9) with $F^{i,j}$ defined by (6.5), we obtain that

$$\gamma_n \| E_{2,j}(t) - \widetilde{E}_{2,j}(t) \|_p \le C \gamma_n n^{-H} n^{1-2\beta},$$

which converges to zero as $n \to \infty$ since β can be taken as close as possible to H. By Slutsky's theorem again, it suffices to consider the convergence of the finite dimensional distributions of

$$\left\{ \gamma_n \sum_{j=1}^{m} \widetilde{E}_{2,j}(t), B_t, t \in [0, T] \right\}. \tag{6.7}$$

Applying Proposition 5.5 to the family of processes $f_t^{i,j} = \Gamma_t (\nabla \sigma^j \sigma^i)(X_t)$, we obtain the convergence of the finite dimensional distributions of

$$\left\{ \gamma_n \sum_{j=1}^{m} \Gamma_t \widetilde{E}_{2,j}(t), B_t, t \in [0, T] \right\}$$

to those of $\{\Gamma_t U_t, B_t, t \in [0, T]\}$. This implies the convergence of the finite dimensional distributions of

$$\left\{ \gamma_n \sum_{j=1}^{m} \widetilde{E}_{2,j}(t), B_t, t \in [0, T] \right\}$$

to those of $\{U_t, B_t, t \in [0, T]\}$.

To show (ii), we prove the following tightness condition:

$$\sup_{n \ge 1} \mathbb{E}\big(\big| \gamma_n \big(X_t - X_t^n\big) - \gamma_n \big(X_s - X_s^n\big) \big|^4 \big) \le C (t - s)^2. \tag{6.8}$$

Taking into account (4.7) and (4.11), we only need to show the above inequality for $\gamma_n I_{11}$, $\gamma_n I_{12,j}$, $\gamma_n I_{13}$, $\gamma_n I_{2,j}$, $\gamma_n I_{4,j}$, $\gamma_n E_{1,j}$, $\gamma_n E_{2,j}$ and $\gamma_n E_{3,j}$. The tightness for the terms $\gamma_n I_{11}$, $\gamma_n I_{13}$ and $\gamma_n E_{3,j}$ is clear. Now we consider the tightness of the term $I_{2,j}$. We write

$$\begin{aligned} I_{2,j}(t) - I_{2,j}(s) &= \big(\Lambda_t^n - \Lambda_s^n\big) \int_0^t \Gamma_s^n b_2^j(s) \big(s - \eta(s)\big) \, dB_s^j \\ &\quad + \int_s^t \Lambda_s^n \Gamma_u^n b_2^j(u) \big(u - \eta(u)\big) \, dB_u^j. \end{aligned}$$

Then it follows from Lemma A.3 (A.4) that

$$\begin{aligned} \mathbb{E}\Big(\Big| \gamma_n \big(\Lambda_t^n - \Lambda_s^n\big) \int_0^t \Gamma_s^n b_2^j(s) \big(s - \eta(s)\big) \, dB_s^j \Big|^4 \Big) &\le C (t - s)^{4\beta} \big(\mathbb{E} \| \Lambda^n \|_\beta^8 \big)^{1/2} \\ &\le C (t - s)^{4\beta}. \end{aligned}$$

Lemma A.3 (A.4) also implies that the fourth moment of the second term is bounded by $C(t - s)^{4H}$. The tightness for $\gamma_n I_{12,j}$, $\gamma_n I_{4,j}$, $\gamma_n E_{1,j}$, $\gamma_n E_{2,j}$ can be obtained in a similar way by applying the estimates (A.5) and (A.4) from Lemma A.3, (A.15) from Lemma A.5, and (A.8) from Lemma A.4, respectively. $\square$

7. A limit theorem in L^p for weighted sums. Following the methodology used in [3], we can show the following limit result for random weighted sums. The proof uses the techniques of fractional calculus and the classical decompositions in large and small blocks.

Consider a double sequence of random variables $\zeta = \{\zeta_{k,n}, n \in \mathbb{N}, k = 0, 1, \dots, n\}$, and for each $t \in [0, T]$, we denote

$$g_n(t) := \sum_{k=0}^{\lfloor nt/T \rfloor} \zeta_{k,n}. \tag{7.1}$$

PROPOSITION 7.1. *Fix $\lambda > 1 - \beta$, where $0 < \beta < 1$. Let $p \geq 1$ and $p', q' > 1$ such that $\frac{1}{p'} + \frac{1}{q'} = 1$ and $pp' > \frac{1}{\beta}$, $pq' > \frac{1}{\lambda}$. Let g_n be the sequence of processes defined in (7.1). Suppose that the following conditions hold true*:

(i) *for each $t \in [0, T]$, $g_n(t) \to z(t)$ in $L^{pq'}$*;
(ii) *for any $j, k = 0, 1, \dots, n$ we have*

$$\mathbb{E}\big(\big|g_n(kT/n) - g_n(jT/n)\big|^{pq'}\big) \leq C\big(|k - j|/n\big)^{\lambda pq'}.$$

Let $f = \{f(t), t \in [0, T]\}$ be a process such that $\mathbb{E}(\|f\|_\beta^{pp'}) \leq C$ and $\mathbb{E}(|f(0)|^{pp'}) \leq C$. Then for each $t \in [0, T]$,

$$F^n(t) := \sum_{k=0}^{\lfloor nt/T \rfloor} f(t_k)\zeta_{k,n} \to \int_0^t f(s)\,dz(s) \qquad \text{in } L^p \text{ as } n \to \infty. \tag{7.2}$$

REMARK 7.1. The integral $\int_0^t f(s)\,dz(s)$ is interpreted as a Young integral in the sense of Proposition 2.2, which is well defined because f and z, as functions on $[0, T]$ with values in $L^{pp'}$ and $L^{pq'}$, are Hölder continuous [conditions (i) and (ii) together imply the Hölder continuity of z] of order β and λ, respectively. Recall that the Hölder continuity of a function with values in L^p is defined in (2.5).

REMARK 7.2. Convergence (7.2) still holds true if the condition $\mathbb{E}(\|f\|_\beta^{pp'}) \leq C$ is weakened by assuming that f is Hölder continuous of order β in $L^{pp'}$. The proof will be similar to that of Proposition 7.1.

PROOF OF PROPOSITION 7.1. Given two natural numbers $m < n$ we consider the associated partitions of the interval $[0, T]$ given by $t_k = \frac{kT}{n}$, $k = 0, 1, \dots, n$ and $u_l = \frac{lT}{m}$, $l = 0, 1, \dots, m$. Then we have the decomposition

$$F^n(t) = \sum_{l=0}^{\lfloor mt/T \rfloor} f(u_l) \sum_{k \in I_m(l)} \zeta_{k,n} + \sum_{l=0}^{\lfloor mt/T \rfloor} \sum_{k \in I_m(l)} \big[f(t_k) - f(u_l)\big]\zeta_{k,n}, \tag{7.3}$$

where $I_m(l) := \{k : 0 \leq k \leq \lfloor \frac{nt}{T} \rfloor, t_k \in [u_l, u_{l+1})\}$.

Because of condition (i) and the assumption that $\mathbb{E}(|f(t)|^{pp'}) \le C$ for all $t \in [0,T]$, the first term on the right-hand side of the above expression converges in L^p, as n tends to infinity, to

$$\sum_{l=0}^{\lfloor mt/T \rfloor} f(u_l)\big[z(u_{l+1}) - z(u_l)\big].$$

Applying Proposition 2.2 to f and z we obtain that the above Riemann–Stieltjes sum converges to the Young integral $\int_0^t f(s)\,dz(s)$ in L^p as m tends to infinity. To show convergence (7.2) it suffices to show that

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} \mathbb{E}\left(\left|\sum_{l=0}^{\lfloor mt/T \rfloor} \sum_{k\in I_m(l)} \big[f(t_k) - f(u_l)\big]\zeta_{k,n}\right|^p\right) = 0. \tag{7.4}$$

Notice that k belongs to $I_m(l)$ if and only if $u_l \le t_k < \varepsilon(u_{l+1})$ and $t_k \le \eta(t)$. Recall that $\varepsilon(u) = t_{k+1}$ if $t_k < u \le t_{k+1}$ and $\eta(u) = t_k$ if $t_k \le u < t_{k+1}$. As a consequence, we can write

$$\begin{aligned}
&\sum_{l=0}^{\lfloor mt/T \rfloor} \sum_{k\in I_m(l)} \big[f(t_k) - f(u_l)\big]\zeta_{k,n} \\
&\qquad = \sum_{l=0}^{\lfloor mt/T \rfloor} \int_{(a_l,b_l)} \big[f(s) - f(a_l)\big]\,dg_n(s),
\end{aligned}$$

where $a_l = u_l$ and $b_l = \varepsilon(u_{l+1}) \wedge (\eta(t) + \frac{T}{n})$. By the fractional integration by parts formula,

$$\begin{aligned}
&\int_{(a_l,b_l)} \big[f(s) - f(a_l)\big]\,dg_n(s) \\
&\qquad = (-1)^\alpha \int_{a_l}^{b_l} D_{a_l+}^{\alpha}\big[f(s) - f(a_l)\big] D_{b_l-}^{1-\alpha}\big[g_n(s) - g_n(b_l-)\big]\,ds,
\end{aligned} \tag{7.5}$$

where we take $\alpha \in (1-\lambda, \beta)$. By (2.2), it is easy to show that

$$\begin{aligned}
\big|D_{a_l+}^{\alpha}\big[f(s) - f(a_l)\big]\big| &\le \frac{1}{\Gamma(1-\alpha)} \frac{\beta}{\beta-\alpha} \|f\|_\beta (s-a_l)^{\beta-\alpha} \\
&\le C\|f\|_\beta m^{\alpha-\beta}.
\end{aligned} \tag{7.6}$$

On the other hand, by (2.3) we have

$$\begin{aligned}
&\big|D_{b_l-}^{1-\alpha}\big[g_n(s) - g_n(b_l-)\big]\big| \\
&\qquad = \frac{1}{\Gamma(\alpha)} \left|\frac{g_n(s) - g_n(b_l-)}{(b_l-s)^{1-\alpha}} + (1-\alpha)\int_s^{b_l} \frac{g_n(s) - g_n(u)}{(u-s)^{2-\alpha}}\,du\right|.
\end{aligned} \tag{7.7}$$

We can calculate the integral in the above equation explicitly.

$$\begin{aligned}
&\int_s^{b_l} \frac{g_n(s)-g_n(u)}{(u-s)^{2-\alpha}}\,du \\
&\quad = \int_{\varepsilon(s)}^{b_l} \frac{g_n(s)-g_n(u)}{(u-s)^{2-\alpha}}\,du \\
&\quad = \sum_{k\,:\,t_k\in[\varepsilon(s),b_l)} \big[g_n(s)-g_n(t_k)\big]\int_{t_k}^{t_{k+1}} (u-s)^{\alpha-2}\,du \\
&\quad = \sum_{k\,:\,t_k\in[\varepsilon(s),b_l)} \big[g_n(s)-g_n(t_k)\big]\frac{1}{1-\alpha}\big[(t_k-s)^{\alpha-1}-(t_{k+1}-s)^{\alpha-1}\big].
\end{aligned} \tag{7.8}$$

Substituting (7.6), (7.7) and (7.8) into (7.5), we obtain

$$\begin{aligned}
&\left|\int_{(a_l,b_l)} \big[f(s)-f(a_l)\big]\,dg_n(s)\right| \\
&\quad \le C\|f\|_\beta m^{\alpha-\beta}\int_{a_l}^{b_l} \big|D_{b_l-}^{1-\alpha}\big[g_n(s)-g_n(b_l-)\big]\big|\,ds \\
&\quad \le C\|f\|_\beta m^{\alpha-\beta}\sum_{k\,:\,t_k\in[\eta(a_l),b_l)} \int_{t_k}^{t_{k+1}} \big|D_{b_l-}^{1-\alpha}\big[g_n(s)-g_n(b_l-)\big]\big|\,ds \\
&\quad \le C\|f\|_\beta m^{\alpha-\beta}\sum_{k\,:\,t_k\in[\eta(a_l),b_l)} \big|g_n(t_k)-g_n(b_l-)\big|\int_{t_k}^{t_{k+1}} (b_l-s)^{\alpha-1}\,ds \\
&\qquad + C\|f\|_\beta m^{\alpha-\beta}\sum_{k,j\,:\,\eta(a_l)\le t_k<t_j<b_l} \big|g_n(t_k)-g_n(t_j)\big| \\
&\qquad \times \int_{t_k}^{t_{k+1}} \big[(t_j-s)^{\alpha-1}-(t_{j+1}-s)^{\alpha-1}\big]\,ds.
\end{aligned}$$

We denote the first term in the right-hand side of the above expression by $A_{1,l}$ and the second one by $A_{2,l}$.

Applying the Minkowski inequality, we see that the quantity

$$\mathbb{E}\left(\left|\sum_{l=0}^{\lfloor mt/T\rfloor} A_{1,l}\right|^p\right)^{1/p} \tag{7.9}$$

is less than

$$Cm^{\alpha-\beta}\left|\sum_{l=0}^{\lfloor mt/T\rfloor}\sum_{k\,:\,t_k\in[\eta(a_l),b_l)} \mathbb{E}\big(\|f\|_\beta^p\big|g_n(t_k)-g_n(b_l-)\big|^p\big)^{1/p}\int_{t_k}^{t_{k+1}} (b_l-s)^{\alpha-1}\,ds\right|,$$

so by applying the Hölder inequality, condition (ii) and the assumption $\mathbb{E}(\|f\|_\beta^{pp'}) \le C$ to the above, we can show that quantity (7.9) is less than

$$C m^{\alpha-\beta} \left| \sum_{l=0}^{\lfloor mt/T \rfloor} \sum_{k:\, t_k \in [\eta(a_l), b_l)} (b_l - t_k)^\lambda \int_{t_k}^{t_{k+1}} (b_l - s)^{\alpha-1}\, ds \right|. \tag{7.10}$$

Since

$$\begin{aligned}
&\sum_{k:\, t_k \in [\eta(a_l), b_l)} (b_l - t_k)^\lambda \int_{t_k}^{t_{k+1}} (b_l - s)^{\alpha-1}\, ds \\
&\quad = \frac{1}{\alpha}\left(\frac{T}{n}\right)^{\lambda+\alpha} + \sum_{k:\, t_k \in [\eta(a_l), b_l - T/n)} (b_l - t_k)^\lambda \int_{t_k}^{t_{k+1}} (b_l - s)^{\alpha-1}\, ds \\
&\quad \le \frac{1}{\alpha}\left(\frac{T}{n}\right)^{\lambda+\alpha} + \frac{T}{n} \sum_{k:\, t_k \in [\eta(a_l), b_l - T/n)} (b_l - t_k)^\lambda (b_l - t_{k+1})^{\alpha-1} \\
&\quad \le \frac{1}{\alpha}\left(\frac{T}{n}\right)^{\lambda+\alpha} + C \frac{T}{n} \frac{n}{m} m^{-\alpha+1-\lambda} \\
&\quad \le C m^{-\alpha-\lambda},
\end{aligned}$$

where in the second inequality we used the assumption that $\alpha > 1 - \lambda$ and the fact that the number of partition points $\{t_k, k = 0, 1, \ldots, n\}$ in $[\eta(a_l), b_l - \frac{T}{n})$ is bounded by $\frac{n}{m}$, the estimate (7.10) of (7.9) implies that

$$\mathbb{E}\left(\left| \sum_{l=0}^{\lfloor mt/T \rfloor} A_{1,l} \right|^p \right)^{1/p} \le C m^{\alpha-\beta} \sum_{l=0}^{\lfloor mt/T \rfloor} m^{-\alpha-\lambda} \le C m^{1-\beta-\lambda} \to 0 \tag{7.11}$$

as m tends to ∞.

Using an argument similar to the estimate of quantity (7.9), it can be shown that the quantity

$$\mathbb{E}\left(\left| \sum_{l=0}^{\lfloor mt/T \rfloor} A_{2,l} \right|^p \right)^{1/p}$$

is less than

$$\begin{aligned}
C \Bigg| m^{\alpha-\beta} \sum_{l=0}^{\lfloor mt/T \rfloor} &\sum_{k,j:\, \eta(a_l) \le t_k < t_j < b_l} |t_k - t_j|^\lambda \\
&\times \int_{t_k}^{t_{k+1}} \left[(t_j - s)^{\alpha-1} - (t_{j+1} - s)^{\alpha-1} \right] ds \Bigg|.
\end{aligned}$$

The summand in the above can be estimated as follows:

$$\begin{aligned}
&\sum_{k,j\,:\,\eta(a_l)\le t_k<t_j<b_l} |t_k-t_j|^\lambda \int_{t_k}^{t_{k+1}} \big[(t_j-s)^{\alpha-1}-(t_{j+1}-s)^{\alpha-1}\big]\,ds \\
&\quad\le C\frac{n}{m}\left(\frac{T}{n}\right)^{\alpha+\lambda} \\
&\qquad + \sum_{k,j\,:\,\eta(a_l)\le t_{k+1}<t_j<b_l} |t_k-t_j|^\lambda \int_{t_k}^{t_{k+1}} \big[(t_j-s)^{\alpha-1}-(t_{j+1}-s)^{\alpha-1}\big]\,ds \\
&\quad\le C\frac{n}{m}\left(\frac{T}{n}\right)^{\alpha+\lambda} + C\left(\frac{T}{n}\right)^2 \sum_{k,j\,:\,\eta(a_l)\le t_{k+1}<t_j<b_l} |t_{k+1}-t_j|^\lambda (t_j-t_{k+1})^{\alpha-2} \\
&\quad\le C\frac{n}{m}\left(\frac{T}{n}\right)^{\alpha+\lambda} + Cn^{-2}n^{2-\lambda-\alpha} \sum_{k,j\,:\,\eta(a_l)\le t_{k+1}<t_j<b_l} (j-k-1)^{\alpha-2+\lambda} \\
&\quad\le C\frac{n}{m}\left(\frac{T}{n}\right)^{\alpha+\lambda} + Cn^{-2}n^{2-\lambda-\alpha}\frac{n}{m}\sum_{p=2}^{n/m}(p-1)^{\alpha-2+\lambda} \\
&\quad\le Cm^{-\alpha-\lambda}.
\end{aligned}$$

Therefore, we have

$$\mathbb{E}\left(\left|\sum_{l=0}^{\lfloor mt/T\rfloor} A_{2,l}\right|^p\right)^{1/p} = Cm^{\alpha-\beta}\left|\sum_{l=0}^{\lfloor mt/T\rfloor} m^{-\alpha-\lambda}\right| \to 0 \qquad \text{as } m\to\infty.$$

The above convergence and equality (7.11) together imply convergence (7.4). The proof is now complete. $\square$

This result has the following two consequences.

COROLLARY 7.1. *Let $B=\{B_t, t\in[0,T]\}$ be an m-dimensional fBm with Hurst parameter $H>3/4$. Define*

$$\zeta_{k,n}^{ij} = n\int_{t_k}^{t_{k+1}} (B_s^i - B_{\eta(s)}^i)\,\delta B_s^j,$$

for $i,j=1,\dots,m$ and $k=0,\dots,n-1$, where we recall that $t_k=\frac{kT}{n}$. Set also $\zeta_{n,n}=0$. Let $\lambda=\frac{1}{2}$, and β, p, p', q', f satisfy the assumptions in Proposition 7.1. Then

$$\sum_{k=0}^{\lfloor nt/T\rfloor} f(t_k)\zeta_{k,n}^{ij} \to \int_0^t f(s)\,dZ_s^{ij} \qquad \text{in } L^p,$$

where Z^{ij} is the generalized Rosenblatt process defined in Section 2.5.

PROOF. To prove the corollary, it suffices to show that the conditions in Proposition 7.1 are all satisfied here. We have shown in Section 2.5 the L^2 convergence of $g_n(t) = \sum_{k=0}^{\lfloor nt/T \rfloor} \zeta_{k,n}^{ij}$ to Z_t^{ij}. This convergence also holds in L^p due to the equivalence of all the L^p-norms in a finite Wiener chaos. Applying (A.8) in Lemma A.4 with $F \equiv 1$ and taking into account that $\gamma_n = n$ when $H > \frac{3}{4}$, we obtain condition (ii) in Proposition 7.1 with $\lambda = \frac{1}{2}$. □

The following result will also be useful later.

COROLLARY 7.2. *Let $B = \{B_t, t \in [0,T]\}$ be one-dimensional fBm with Hurst parameter $H \in (\frac{1}{2}, 1)$. Define*

$$\zeta_{k,n} = \int_{t_k}^{t_{k+1}} \big(s - \eta(s)\big)\, dB_s, \tag{7.12}$$

for $k = 0, \dots, n-1$. Set also $\zeta_{n,n} = 0$. Let $\lambda = H$, and β, p, p', q', f satisfy the assumptions in Proposition 7.1. Then for each $t \in [0,T]$,

$$n \sum_{k=0}^{\lfloor nt/T \rfloor} f(t_k)\zeta_{k,n} \to \frac{T}{2} \int_0^t f(s)\, dB_s,$$

in L^p, as n tends to infinity. This convergence still holds true when we replace the above $\zeta_{k,n}$ by

$$\tilde{\zeta}_{k,n} = \int_{t_k}^{t_{k+1}} (B_s - B_{\eta(s)})\, ds.$$

PROOF. As before, to prove the corollary it suffices to show that the conditions in Proposition 7.1 are all satisfied here. Let us first consider the convergence for $\zeta_{k,n}$. Set

$$g_n(t) := n \sum_{k=0}^{\lfloor nt/T \rfloor} \zeta_{k,n},$$

where $\zeta_{k,n}$ is defined in (7.12). Condition (ii) follows from estimate (A.4) in Lemma A.3 by taking $F \equiv 1$ and $\nu = 1$. The covariance of the process g_n is given by

$$\begin{aligned}
\mathbb{E}\big(g_n(t) g_{n'}(t)\big) &= \alpha_H n n' \int_0^t \int_0^t \big(u - \eta_n(u)\big)\big(v - \eta_{n'}(v)\big) \mu(du\, dv) \\
&\to \frac{T^2}{4} \alpha_H \int_0^t \int_0^t |u - v|^{2H-2}\, du\, dv \\
&= \frac{T^2}{4} t^{2H}
\end{aligned}$$

as $n, n' \to \infty$, which implies that $g_n(t)$ is a Cauchy sequence in L^2. Here $\eta_n(t) = \frac{T}{n}i$ when $\frac{T}{n}i \le t < \frac{T}{n}(i+1)$ and $\eta_{n'}(t) = \frac{T}{n'}i$ when $\frac{T}{n'}i \le t < \frac{T}{n'}(i+1)$. In fact, we can also calculate the kernel of the limit of $z_n(t)$. Suppose that $\phi_n \in \mathcal{H}$ satisfies $g_n(t) = \delta(\phi_n(t))$. Then for any $\psi \in \mathcal{H}$,

$$\begin{aligned}\langle n\phi_n, \psi\rangle_{\mathcal{H}} &= n\alpha_H \int_0^T \int_0^{\eta(t)} \big(u - \eta(u)\big)\psi(v)|u-v|^{2H-2}\,du\,dv \\ &\to \frac{T}{2}\langle \psi, \mathbf{1}_{[0,t]}\rangle_{\mathcal{H}},\end{aligned}$$

as $n \to +\infty$. This implies that the kernel of the limit of $g_n(t)$ is $\frac{T}{2}\mathbf{1}_{[0,t]}$; in other words, the random variable $g_n(t)$ converges in L^2 to $\frac{T}{2}B_t$.

The convergence result for $\tilde{\zeta}_{k,n}$ can be shown by noticing that

$$\tilde{\zeta}_{k,n} = \int_{t_k}^{t_{k+1}} (B_s - B_{\eta(s)})\,ds = \frac{T}{n}(B_{t_{k+1}} - B_{t_k}) - \int_{t_k}^{t_{k+1}} \big(s - \eta(s)\big)\,dB_s.$$

This completes the proof of the corollary. $\square$

8. Asymptotic error of the modified Euler scheme in case $H \in (\frac{3}{4}, 1)$.

The limit theorems for weighted sums proved in the previous section allow us to derive the L^p-limit of the quantity $n(X_t - X_t^n)$ in the case $H \in (\frac{3}{4}, 1)$.

THEOREM 8.1. *Let $H \in (\frac{3}{4}, 1)$. Suppose that X and X^n are defined by* (1.1) *and* (1.3), *respectively. Let Z^{ij}, $i, j = 1, \dots, m$ be the matrix-valued generalized Rosenblatt process defined in Section* 2.5. *Assume $\sigma \in C_b^5(\mathbb{R}^d; \mathbb{R}^{d\times m})$ and $b \in C_b^4(\mathbb{R}^d; \mathbb{R}^d)$. Then*

$$n\big(X_t - X_t^n\big) \to \overline{U}_t$$

in $L^p(\Omega)$ as n tends to infinity, where $\{\bar{U}_t, t \in [0,T]\}$ is the solution of the following linear stochastic differential equation:

$$\begin{aligned}\overline{U}_t = {}& \int_0^t \nabla b(X_s)\overline{U}_s\,ds + \sum_{j=1}^m \int_0^t \nabla\sigma^j(X_s)\overline{U}_s\,dB_s^j \\ & + \sum_{i,j=1}^m \int_0^t \big(\nabla\sigma^j\sigma^i\big)(X_s)\,dZ_s^{ij} \\ & + \frac{T}{2}\int_0^t (\nabla bb)(X_s)\,ds + \frac{T}{2}\int_0^t (\nabla b\sigma)(X_s)\,dB_s \\ & + \frac{T}{2}\sum_{j=1}^m \int_0^t \big(\nabla\sigma^j b\big)(X_s)\,dB_s^j.\end{aligned} \tag{8.1}$$

PROOF. Recall the decomposition $Y_t = X_t - X_t^n$ given in (4.7) and (4.11). We have shown that $nI_{13}(t)$, $nI_{4,j}(t)$, $nE_{1,j}(t)$ and $nE_{3,j}(t)$ converge in L^p to zero for each $t \in [0,T]$. It remains to show the L^p convergence of $nI_{11}(t)$, $nI_{12,j}(t)$, $nI_{2,j}(t)$ and $nE_{2,j}(t)$ and identify their limits.

Step 1. Recall $\widetilde{E}_{2,j}(t)$ is defined in (6.6). It has been shown in the proof of Theorem 6.1 that $n(E_{2,j}(t) - \widetilde{E}_{2,j}(t))$ converges to zero in L^p. On the other hand, applying Corollary 7.1 to $n\widetilde{E}_{2,j}(t)$ yields

$$n\widetilde{E}_{2,j}(t) \to \sum_{i=1}^m \int_0^t \Lambda_t \Gamma_s (\nabla\sigma^j \sigma^i)(X_s)\, dZ_s^{ij} \qquad \text{in } L^p.$$

Therefore, $nE_{2,j}(t)$ converges in L^p, and the limit is the same as $n\widetilde{E}_{2,j}(t)$.

Step 2. Denote

$$\tilde{I}_{2,j}(t) = \Lambda_t \sum_{k=0}^{\lfloor nt/T \rfloor} \Gamma_{t_k} (\nabla\sigma^j b)(X_{t_k}) \int_{t_k}^{t_{k+1}\wedge t} (s - \eta(s))\, dB_s^j,$$

for $t \in [0,T]$ [as before, we define $t_{n+1} = \frac{T}{n}(n+1)$]. Applying Corollary 7.2 to $n\tilde{I}_{2,j}(t)$ yields

$$n\tilde{I}_{2,j}(t) \to \frac{T}{2} \int_0^t \Lambda_t \Gamma_s (\nabla\sigma^j b)(X_s)\, dB_s^j \qquad \text{in } L^p.$$

We want to show that $nI_{2,j}(t)$ and $n\tilde{I}_{2,j}(t)$ have the same limit in L^p. Write

$$\begin{aligned} & n\big(I_{2,j}(t) - \tilde{I}_{2,j}(t)\big) \\ & \quad = n\int_0^t \big(\Lambda_t^n \Gamma_s^n b_2^j(s) - \Lambda_t \Gamma_s \tilde{b}_2^j(s)\big)(s - \eta(s))\, dB_s^j \\ & \qquad + n\int_0^t \Lambda_t \big(\Gamma_s \tilde{b}_2^j(s) - \Gamma_{\eta(s)}(\nabla\sigma^j b)(X_{\eta(s)})\big)(s - \eta(s))\, dB_s^j, \end{aligned} \tag{8.2}$$

where $\tilde{b}_2^j(s) = \int_0^1 \nabla\sigma^j(\theta X_s + (1-\theta)X_{\eta(s)}) b(X_{\eta(s)})\, d\theta$. It suffices to show that the two terms on the right-hand side of (8.2) both converge to zero in L^p. The convergence of the second term follows from estimate (A.16) of Lemma A.5. Lemma 4.1 implies that the L^p-norms of $[\Lambda_t^n \Gamma_s^n b_2^j(s) - \Lambda_t \Gamma_s \tilde{b}_2^j(s)]$ and its Malliavin derivative converge to zero as $n \to \infty$. So applying Lemma A.3 (A.4) with $\nu = 1$ and $F_s = \Lambda_t^n \Gamma_s^n b_2^j(s) - \Lambda_t \Gamma_s \tilde{b}_2^j(s)$, we obtain the convergence of the first term.

Step 3. Following the lines in step 2 we can show that $nI_{12,j}(t)$ converges in L^p to

$$\frac{T}{2} \int_0^t \Lambda_t \Gamma_s (\nabla b \sigma^j)(X_s)\, dB_s^j.$$

Instead of (A.4) and (A.16) in step 2, we need to use estimates (A.5) and (A.15) here.

Similarly, it can be shown that nI_{11} converges in L^p to

$$\frac{T}{2}\int_0^t \Lambda_t \Gamma_s (\nabla bb)(X_s)\,ds.$$

Step 4. We have shown that $n(X_t - X_t^n)$ converges in L^p to $\overline{U}_t$, where we define, for each $t \in [0, T]$,

$$\overline{U}_t = \sum_{i,j=1}^m \int_0^t \Lambda_t \Gamma_s \big(\nabla \sigma^j \sigma^i\big)(X_s)\,dZ_s^{ij} + \frac{T}{2}\int_0^t \Lambda_t \Gamma_s (\nabla bb)(X_s)\,ds$$
$$+ \frac{T}{2}\int_0^t \Lambda_t \Gamma_s (\nabla b\sigma)(X_s)\,dB_s + \frac{T}{2}\sum_{j=1}^m \int_0^t \Lambda_t \Gamma_s \big(\nabla \sigma^j b\big)(X_s)\,dB_s^j.$$

The theorem follows from the fact that the process $\overline{U}$ satisfies equation (8.1). □

9. Weak approximation of the modified Euler scheme. The next result provides the weak rate of convergence for the modified Euler scheme (1.3).

THEOREM 9.1. *Let X and X^n be the solution to equations* (1.1) *and* (1.3), *respectively. Suppose that $b \in C_b^3(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C_b^4(\mathbb{R}^d; \mathbb{R}^{d\times m})$. Then for any function $f \in C_b^3(\mathbb{R}^d)$ there exists a constant C independent of n such that*

$$\sup_{0\le t\le T} \big|\mathbb{E}[f(X_t)] - \mathbb{E}\big[f\big(X_t^n\big)\big]\big| \le Cn^{-1}. \tag{9.1}$$

If we further assume that $b \in C^4$, $\sigma \in C^5$ and $f \in C^4$, then for each $t \in [0, T]$, the sequence

$$n\big\{\mathbb{E}[f(X_t)] - \mathbb{E}\big[f\big(X_t^n\big)\big]\big\}, \qquad n \in \mathbb{N},$$

converges as n tends to infinity, and the limit is equal to the sum of the following two quantities:

$$\frac{\alpha_H^2 T}{2}\sum_{j,i=1}^m \int_0^t\int_0^t\int_0^t \mathbb{E}\big\{D_u^i D_r^j\big[\nabla f(X_t)\Lambda_t \Gamma_s\big(\nabla\sigma^j\sigma^i\big)(X_s)\big]\big\}$$
$$\times |u-s|^{2H-2}|s-r|^{2H-2}\,du\,ds\,dr \tag{9.2}$$

and

$$\frac{T}{2}\mathbb{E}\Bigg\{\nabla f(X_t)\Lambda_t\Bigg[\int_0^t \Gamma_s(\nabla bb)(X_s)\,ds + \int_0^t \Gamma_s(\nabla b\sigma)(X_s)\,dB_s$$
$$+ \sum_{j=1}^m \int_0^t \Gamma_s\big(\nabla\sigma^j b\big)(X_s)\,dB_s^j\Bigg]\Bigg\}. \tag{9.3}$$

PROOF. We use again decompositions (4.7) and (4.11) of $Y_t = X_t - X_t^n$, $t \in [0,T]$, and we continue to use the notation there. Given a function $f \in C_b^3(\mathbb{R}^d)$, we can write

$$n\{\mathbb{E}[f(X_t)] - \mathbb{E}[f(X_t^n)]\} = n\int_0^1 \mathbb{E}[\nabla f(Z_t^\theta)Y_t]\,d\theta,$$

where we denote $Z_t^\theta = \theta X_t + (1-\theta)X_t^n$, $0 \le t \le T$.

Step 1. In this step, we show that $\sup_{0\le t\le T}|\mathbb{E}[\nabla f(Z_t^\theta)Y_t]| \le Cn^{-1}$, which implies (9.1). From estimates (4.9) and (4.10) it follows that this inequality is true when Y is replaced by I_{11}, I_{13}, $I_{12,j}$, $I_{2,j}$ or $I_{4,j}$. Therefore, it suffices to show that $|\mathbb{E}[\nabla f(Z_t^\theta)E_{i,j}(t)]| \le Cn^{-1}$ for $i = 1,2,3$ and $j = 1,\dots,m$, where $E_{ij}(t)$ are defined in Theorem 4.1 step 3. Consider first the term $i = 2$. The use of expression (4.12) and an application of the integration by parts formula yield

$$\begin{aligned}
&\mathbb{E}[\nabla f(Z_t^\theta)E_{2,j}(t)]\\
&\quad= \mathbb{E}\left[\nabla f(Z_t^\theta)\Lambda_t^n\sum_{i=1}^m\sum_{k=0}^{\lfloor nt/T\rfloor}F_{t_k}^{n,i,j}\int_{t_k}^{t_{k+1}\wedge t}\int_{t_k}^s \delta B_u^i\delta B_s^j\right]\\
&\quad= \alpha_H^2\sum_{i=1}^m\sum_{k=0}^{\lfloor nt/T\rfloor}\mathbb{E}\biggl[\int_0^t\int_{t_k}^{t_{k+1}\wedge t}\int_0^t\int_{t_k}^s D_v^iD_r^j[\nabla f(Z_t^\theta)\Lambda_t^nF_{t_k}^{n,i,j}]\\
&\qquad\qquad\times\mu(du\,dv)\mu(ds\,dr)\biggr],
\end{aligned}\tag{9.4}$$

where we recall that $F_t^{n,i,j} = \Gamma_t^n(\nabla\sigma^j\sigma^i)(X_t^n)$. [As before, in the above equation we set $t_{n+1} = \frac{T}{n}(n+1)$.] Therefore,

$$\begin{aligned}
|\mathbb{E}[\nabla f(Z_t^\theta)E_{2,j}(t)]| &\le C\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\int_0^t\int_{t_k}^{t_{k+1}}\int_0^t \mu(du\,dv)\mu(dr\,ds)\\
&\le Cn^{-1}.
\end{aligned}$$

For the term containing $E_{1,j}$ we can write

$$\mathbb{E}[\nabla f(Z_t^\theta)E_{1,j}(t)] = \sum_{i=1}^m\mathbb{E}\left[\int_0^t H_s^{n,i,j}(B_s^i - B_{\eta(s)}^i)\,dB_s^j\right],$$

where $H_s^{n,i,j} = \nabla f(Z_t^\theta)\Lambda_t^n[\Gamma_s^n\sigma_2^{j,i}(s) - \Gamma_{\eta(s)}^n(\nabla\sigma^j\sigma^i)(X_{\eta(s)}^n)]$. An application of the relation between the Skorohod and path-wise integrals (2.9) yields

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t H_s^{n,i,j}(B_s^i - B_{\eta(s)}^i)\,dB_s^j\right]\\
&\quad= \alpha_H\int_0^T\int_0^t \mathbb{E}[D_u^j(H_s^{n,i,j}(B_s^i - B_{\eta(s)}^i))]|s-u|^{2H-2}\,ds\,du
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_H \int_0^T \int_0^t \mathbb{E}\big[D_u^j H_s^{n,i,j}\big(B_s^i - B_{\eta(s)}^i\big)\big]|s-u|^{2H-2}\,ds\,du \\
&\quad + \alpha_H \int_0^T \int_0^t \mathbb{E}\big[H_s^{n,i,j}\big]\mathbf{1}_{[\eta(s),s]}(u)\delta_{ij}|s-u|^{2H-2}\,ds\,du \\
&:= A_1 + A_2.
\end{aligned}
$$

By the integration by parts we see that A_1 is equal to

$$
\alpha_H^2 \int_0^T \int_0^t \int_0^T \int_0^s \mathbb{E}\big[D_r^i D_u^j H_s^{n,i,j}\big]\mathbf{1}_{[\eta(s),s]}(v)|v-r|^{2H-2}|s-u|^{2H-2}\,dv\,dr\,ds\,du.
$$

Using $\sup_{r,u,s} |\mathbb{E}[D_r^i D_u^j H_s^{n,i,j}]| \le Cn^{-\beta}$ for any $\frac{1}{2} < \beta < H$ we obtain

$$
|A_1| \le Cn^{-1-\beta}. \tag{9.5}
$$

On the other hand, it is easy to show by the definitions of Γ^n, X^n and X that the quantity $[\Gamma_s^n \sigma_2^{j,i}(s) - \Gamma_{\eta(s)}^n (\nabla\sigma^j\sigma^i)(X_{\eta(s)}^n)]$ can be expressed as the sum of integrals over the interval $[\eta(s), s]$. So by applying (2.9) and integration by parts we can show that $|\mathbb{E}[H_s^{n,i,j}]| \le Cn^{-1}$, which implies

$$
|A_2| \le Cn^{-2H}. \tag{9.6}
$$

From (9.5) and (9.6) we conclude that $|\mathbb{E}[\nabla f(Z_t^\theta)E_{1,j}(t)]| \le Cn^{-1}$. Finally, for the term containing $E_{3,j}$ we have

$$
\mathbb{E}\big[\nabla f\big(Z_t^\theta\big)E_{3,j}(t)\big] = \int_0^t \mathbb{E}\big[J_s^{n,i,j}\big](s-\eta(s)^{2H-1}\,ds,
$$

where $J_s^{n,i,j} = H\nabla f(Z_t^\theta)\Lambda_t^n(\Gamma_{\eta(s)}^n - \Gamma_s^n)\sigma_0^j(s)$. By expressing the term $(\Gamma_{\eta(s)}^n - \Gamma_s^n)$ as the sum of integrals over the interval $[\eta(s), s]$ and then applying (2.9) and integration by parts, we can show that $\sup_{s\in[0,T]} \mathbb{E}[J_s^{n,i,j}] \le Cn^{-1}$. This implies

$$
\big|\mathbb{E}\big[\nabla f\big(Z_t^\theta\big)E_{3,j}(t)\big]\big| \le Cn^{-2H}, \tag{9.7}
$$

which completes the proof of (9.1).

Step 2. Now we show the second part of the theorem. From estimates (4.9), (4.10), (9.5), (9.6) and (9.7) we see that the expression $n\int_0^1 \mathbb{E}[\nabla f(Z_t^\theta)Y_t]\,d\theta$ converges to zero as n tends to infinity when Y_t is replaced by $I_{13}(t)$, $I_{4,j}(t)$, $E_{1,j}(t)$ or $E_{3,j}(t)$. Therefore, it suffices to consider $n\int_0^1 \mathbb{E}[\nabla f(Z_t^\theta)Y_t]\,d\theta$ when Y_t is replaced by the remaining terms in the decomposition of Y_t.

Consider first the term $E_{2,j}(t)$, and denote

$$
G_{s,r,v}^{i,j} = D_v^i D_r^j\big[\nabla f(X_t)\Lambda_t\Gamma_s\big(\nabla\sigma^j\sigma^i\big)(X_s)\big].
$$

It is clear that

$$
\begin{aligned}
&n\alpha_H^2 \sum_{i,j=1}^{m} \sum_{k=0}^{\lfloor nt/T \rfloor} \int_0^t \int_{t_k}^{t_{k+1}\wedge t} \int_0^t \int_{t_k}^{s} G_{t_k,r,v}^{i,j} \mu(du\,dv)\mu(ds\,dr) \\
&\quad \to \frac{\alpha_H^2 T}{2} \sum_{i,j=1}^{m} \int_0^t \int_0^t \int_0^t G_{s,r,v}^{i,j} |s-v|^{2H-2}|r-s|^{2H-2}\,ds\,dv\,dr,
\end{aligned}
\tag{9.8}
$$

almost surely. Therefore, by the dominated convergence theorem, the expectation of the left-hand side of the above expression converges to the expectation of the right-hand side, which is term (9.2). From Lemma 4.1, we have

$$
\begin{aligned}
&\| D_v^i D_r^j [\nabla f(Z_t^\theta) \Lambda_t^n F_{t_k}^{n,i,j}] - D_v^i D_r^j [\nabla f(X_t) \Lambda_t \Gamma_{t_k} (\nabla \sigma^j \sigma^i)(X_{t_k})] \|_p \\
&\quad \le C n^{1-2\beta},
\end{aligned}
\tag{9.9}
$$

which, together with equation (9.4), implies that $n \sum_{j=1}^m \mathbb{E}[\nabla f(Z_t^\theta) E_{2,j}(t)]$ converges to the same limit as the expectation of the left-hand side of (9.8).

The results in steps 2 and 3 of the proof of Theorem 8.1 imply that the terms $n\mathbb{E}[\nabla f(Z_t^\theta) I_{11}(t)]$, $n\mathbb{E}[\nabla f(Z_t^\theta) I_{12,j}(t)]$ and $n\mathbb{E}[\nabla f(Z_t^\theta) \sum_{j=1}^m I_{2,j}(t)]$ converge to the second, third and fourth term in (9.3), respectively. For example, let us consider $n\mathbb{E}[\nabla f(Z_t^\theta) \sum_{j=1}^m I_{2,j}(t)]$. We have shown in Theorem 8.1 that

$$
n I_{2,j}(t) \to \frac{T}{2} \Lambda_t \int_0^t \Gamma_s (\nabla \sigma^j b)(X_s)\, dB_s^j
$$

in L^p for any $p \ge 1$. So it follows from the Hölder inequality that

$$
\left| \mathbb{E}\left[n \nabla f(Z_t^\theta) I_{2,j}(t) - \nabla f(X_t) \frac{T}{2} \Lambda_t \int_0^t \Gamma_s (\nabla \sigma^j b)(X_s)\, dB_s^j \right] \right| \to 0
$$

as $n \to \infty$. The other two terms can be studied in similar way. This completes the proof of the theorem. □

REMARK 9.1. Theorem 9.1 may be used to construct a Richard extrapolation scheme with error bound $o(n^{-1})$.

10. Rate of convergence for the Euler scheme. In this section, we apply our approach based on Malliavin calculus developed in Section 4 to study the rate of convergence of the naive Euler scheme defined in (1.2). Our first result is the rate of the strong convergence of the naive Euler scheme. As we will see, the weak rate of convergence and the rate of strong convergence are the same for the naive Euler scheme. We still use X^n to represent the naive Euler scheme (1.2). This will not cause confusion since we will only deal with this scheme in this section.

THEOREM 10.1. *Let X and X^n be the processes defined in* (1.1) *and* (1.2), *respectively. Suppose that $b \in C_b^1(\mathbb{R}^d;\mathbb{R}^d)$ and $\sigma \in C_b^2(\mathbb{R}^d;\mathbb{R}^{d\times m})$. Then for each $p \geq 1$, we have*

$$n^{2H-1} \sup_{t\in[0,T]} \mathbb{E}\big(|X_t - X_t^n|^p\big)^{1/p} \leq C.$$

If we assume $b \in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$ and $\sigma \in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$, then as n tends to infinity,

$$n^{2H-1}(X_t - X_t^n) \to \frac{T^{2H-1}}{2} \sum_{j=1}^m \int_0^t \Lambda_t \Gamma_s (\nabla\sigma^j \sigma^j)(X_s)\,ds,$$

where Λ is the solution to linear equation (4.16) *and $\Gamma_t = \Lambda_t^{-1}$, and the convergence holds in L^p for all $p \geq 1$.*

PROOF. We let $Y_t = X_t - X_t^n$, $t \in [0,T]$. Then as in the proof of Theorem 4.1, we can derive the decomposition of Y_t

$$\begin{aligned}
Y_t &= \Lambda_t^n \int_0^t \Gamma_s^n b_3(s)\Bigg[b\big(X_{\eta(s)}^n\big)\big(s-\eta(s)\big) + \sum_{l=1}^m \sigma^l\big(X_{\eta(s)}^n\big)\big(B_s^l - B_{\eta(s)}^l\big)\Bigg]ds \\
&\quad + \sum_{j=1}^m \int_0^t \Lambda_t^n \Gamma_s^n b_2^j(s)\big(s-\eta(s)\big)\,dB_s^j \\
&\quad + \sum_{i,j=1}^m \int_0^t \Lambda_t^n \Gamma_s^n \sigma_2^{j,i}(s)\big(B_s^i - B_{\eta(s)}^i\big)\,dB_s^j \\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t),
\end{aligned}$$

where Λ^n, Γ^n, $b_2^j(s)$, $\sigma_2^{j,i}(s)$ and $b_3(s)$ are the same terms as those defined in the proof of Theorem 4.1 with the scheme X^n replaced by the classical Euler scheme (1.2).

It is clear that $\|I_1(t)\|_p \leq Cn^{-1}$. On the other hand, estimates (A.4) and (A.5) of Lemma A.3 imply that $\|I_2(t)\|_p \leq Cn^{-1}$ and $\|I_3(t)\|_p \leq Cn^{-1}$. Finally, as in the proof of (A.15) in Lemma A.5 we obtain $\|I_4(t)\|_p \leq Cn^{1-2H}$. This completes the proof of the first part of the theorem.

Applying the integration by parts to $I_4(t)$ yields

$$\begin{aligned}
&\int_0^t \Lambda_t^n \Gamma_s^n \sigma_2^{j,i}(s)\big(B_s^i - B_{\eta(s)}^i\big)\,dB_s^j \\
&\quad = \int_0^t \Lambda_t^n \Gamma_s^n \sigma_2^{j,i}(s)\big(B_s^i - B_{\eta(s)}^i\big)\delta B_s^j \\
&\qquad + \alpha_H \int_0^t \int_0^t D_r^j\big[\Lambda_t^n \Gamma_s^n \sigma_2^{j,i}(s)\big]\big(B_s^i - B_{\eta(s)}^i\big)\mu(ds\,dr)
\end{aligned}$$

$$+\delta_{ij}\alpha_H\int_0^t\int_0^t \Lambda_t^n\Gamma_s^n\sigma_2^{j,i}(s)\mathbf{1}_{[\eta(s),s]}(r)\mu(ds\,dr)$$
$$=: A_n^1(t)+A_n^2(t)+A_n^3(t).$$

From (A.8) we have $\|A_n^1(t)\|_p\le C\gamma_n^{-1}$. Applying (A.5) with F_u replaced by

$$\int_0^t D_r^j[\Lambda_t^n\Gamma_s^n\sigma_2^{j,i}(s)]|r-u|^{2H-2}\,dr$$

we obtain $\|A_n^2(t)\|_p\le Cn^{-1}$. So it suffices to identify the limit of $n^{2H-1}A_n^3(t)$ in L^p. It follows from Lemma 4.1 and Remark 4.1 that

$$\|\Lambda_t^n\Gamma_s^n\sigma_2^{j,j}(s)-\Lambda_t\Gamma_s(\nabla\sigma^j\sigma^j)(X_s)\|_p\le Cn^{1-2\beta}.$$

Therefore, $n^{2H-1}A_n^3(t)$, and the quantity

$$n^{2H-1}\int_0^t\int_0^t \Lambda_t\Gamma_s(\nabla\sigma^j\sigma^j)(X_s)\mathbf{1}_{[\eta(s),s]}(r)|r-s|^{2H-2}\,ds\,dr$$

converges to the same value in L^p. The theorem now follows by noticing that

$$n^{2H-1}\int_0^t\int_0^t \Lambda_t\Gamma_s(\nabla\sigma^j\sigma^j)(X_s)\mathbf{1}_{[\eta(s),s]}(r)|r-s|^{2H-2}\,ds\,dr$$
$$=n^{2H-1}\int_0^t \Lambda_t\Gamma_s(\nabla\sigma^j\sigma^j)(X_s)\frac{(s-\eta(s))^{2H-1}}{2H-1}\,ds$$
$$\to\frac{T^{2H-1}}{2\alpha_H}\int_0^t \Lambda_t\Gamma_s(\nabla\sigma^j\sigma^j)(X_s)\,ds,$$

in L^p for all $p\ge 1$. □

As a consequence of the above theorem, we can deduce the following result.

COROLLARY 10.1. *Let X and X^n be the processes defined in* (1.1) *and* (1.2), *respectively. Suppose that $b\in C_b^3(\mathbb{R}^d;\mathbb{R}^d)$, $\sigma\in C_b^4(\mathbb{R}^d;\mathbb{R}^{d\times m})$ and $f\in C_b^2(\mathbb{R}^d)$. Let Λ be defined in* (4.16). *Then we have the following L^p-convergence as $n\to\infty$ for all $p\ge 1$:*

$$n^{2H-1}[f(X_t^n)-f(X_t)]\to\frac{T^{2H-1}}{2}\sum_{j=1}^m\int_0^t\nabla f(X_t)\Lambda_t\Gamma_s(\nabla\sigma^j\sigma^j)(X_s)\,ds.$$

PROOF. We can write

$$n^{2H-1}[f(X_t^n)-f(X_t)]=n^{2H-1}\left(\int_0^1\nabla f(Z_t^\theta)\,d\theta\right)(X_t^n-X_t),$$

where we denote $Z_t^\theta=\theta X_t+(1-\theta)X_t^n$, $t\in[0,T]$. Then the result follows from Theorem 10.1, the convergence of X_t^n to X_t and the assumption on f. □

The above corollary implies the following weak approximation result:

$$\lim_{n\to\infty} n^{2H-1}\{\mathbb{E}[f(X_t)]-\mathbb{E}[f(X_t^n)]\}$$
$$=\frac{T^{2H-1}}{2}\sum_{j=1}^m\int_0^t \mathbb{E}[\nabla f(X_t)\Lambda_t\Gamma_s(\nabla\sigma^j\sigma^j)(X_s)]\,ds.$$

APPENDIX

A.1. Estimates of a Young integral. In this section, we give an estimate on the pathwise integral using fractional calculus.

LEMMA A.1. *Let $z=\{z_t, t\in[0,T]\}$ be a Hölder continuous function with index $\beta\in(0,1)$. Suppose that $f:\mathbb{R}^{l+m}\to\mathbb{R}$ is continuously differentiable. We denote by $\nabla_x f$ the l-dimensional vector with coordinates $\frac{\partial f}{\partial x_i}$, $i=1,\dots,l$, and by $\nabla_y f$ the m-dimensional vector with coordinates $\frac{\partial f}{\partial x_{l+i}}$, $i=1,\dots,m$. Consider processes $x=\{x_t, t\in[0,T]\}$ and $y=\{y_t, t\in[0,T]\}$ with dimensions l and m, respectively, such that $\|x\|_{0,T,\beta'}$ and $\|y\|_{0,T,\beta',n}$ are finite for each $n\geq 1$, where $\beta'\in(0,1)$ is such that $\beta'+\beta>1$. Then we have the following estimates*:

(i) *for any $s,t\in[0,T]$ such that $s\leq t$ and $s=\eta(s)$, we have*

$$\left|\int_s^t f(x_r,y_{\eta(r)})\,dz_r\right|\leq K_1\sup_{r\in[s,t]}|f(x_r,y_{\eta(r)})|\|z\|_\beta(t-s)^\beta$$
$$+K_2\sup_{r_1,r_2\in[s,t]}|\nabla_x f(x_{r_1},y_{\eta(r_2)})|\|x\|_{s,t,\beta'}\|z\|_\beta(t-s)^{\beta+\beta'}$$
$$+K_3\sup_{r_1,r_2\in[s,t]}|\nabla_y f(x_{r_1},y_{r_2})|\|y\|_{s,t,\beta',n}\|z\|_\beta(t-s)^{\beta+\beta'},$$

where the K_i, $i=1,2,3$, are constants depending on β and β';

(ii) *if the function f only depends on the first l variables, then the above estimate holds for all $0\leq s\leq t\leq T$.*

PROOF. Take α such that $\beta'>\alpha>1-\beta$. Let $s,t\in[0,T]$ be such that $s=\eta(s)$ and $s\leq t$. Applying the fractional integration by parts formula in Proposition 2.1, we obtain

$$\left|\int_s^t f(x_r,y_{\eta(r)})\,dz_r\right|\leq\int_s^t|D_{s+}^\alpha f(x_r,y_{\eta(r)})||D_{t-}^{1-\alpha}(z_r-z_t)|\,dr. \tag{A.1}$$

By the definition of fractional differentiation in (2.3) and taking into account that $\alpha+\beta-1>0$, we can show that

$$|D_{t-}^{1-\alpha}(z_r-z_t)|\leq K_0\|z\|_\beta(t-r)^{\alpha+\beta-1},\qquad s\leq r\leq t, \tag{A.2}$$

where $K_0 = \frac{\beta}{(\beta+\alpha-1)\Gamma(\alpha)}$. On the other hand, using (2.2) we obtain

$$
\begin{aligned}
&\bigl|D_{s+}^{\alpha} f(x_r, y_{\eta(r)})\bigr| \\
&\quad \le \frac{1}{\Gamma(1-\alpha)}\biggl[\frac{|f(x_r, y_{\eta(r)})|}{(r-s)^{\alpha}} + \alpha \int_s^r \frac{|f(x_r, y_{\eta(r)}) - f(x_u, y_{\eta(u)})|}{(r-u)^{\alpha+1}}\,du\biggr] \\
&\quad \le \frac{1}{\Gamma(1-\alpha)} \\
&\qquad \times \biggl[\sup_{r\in[s,t]} \bigl|f(x_r, y_{\eta(r)})\bigr|(r-s)^{-\alpha} \\
&\qquad\quad + \alpha \sup_{r_1,r_2\in[s,t]} \bigl|\nabla_x f(x_{r_1}, y_{\eta(r_2)})\bigr| \|x\|_{s,t,\beta'} \int_s^r (r-u)^{\beta'-\alpha-1}\,du \\
&\qquad\quad + \alpha \sup_{r_1,r_2\in[s,t]} \bigl|\nabla_y f(x_{r_1}, y_{r_2})\bigr| \|y\|_{s,t,\beta',n} \int_s^r \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}}\,du\biggr].
\end{aligned}
\tag{A.3}
$$

Inequalities (A.1), (A.3) and (A.2) together imply

$$
\begin{aligned}
&\biggl|\int_s^t f(x_r, y_{\eta(r)})\,dz_r\biggr| \\
&\quad \le \frac{1}{\Gamma(1-\alpha)} \int_s^t \biggl[\sup_{r\in[s,t]} \bigl|f(x_r, y_{\eta(r)})\bigr|(r-s)^{-\alpha} \\
&\qquad\qquad + \alpha \sup_{r_1,r_2\in[s,t]} \bigl|\nabla_x f(x_{r_1}, y_{\eta(r_2)})\bigr| \|x\|_{s,t,\beta'} \\
&\qquad\qquad \times \int_s^r (r-u)^{\beta'-\alpha-1}\,du \\
&\qquad\qquad + \alpha \sup_{r_1,r_2\in[s,t]} \bigl|\nabla_y f(x_{r_1}, y_{r_2})\bigr| \|y\|_{s,t,\beta',n} \\
&\qquad\qquad\qquad \times \int_s^r \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}}\,du\biggr] \\
&\qquad\qquad \times K_0 \|z\|_{\beta} (t-r)^{\alpha+\beta-1}\,dr \\
&\quad \le K_1 \sup_{r\in[s,t]} \bigl|f(x_r, y_{\eta(r)})\bigr| \|z\|_{\beta} (t-s)^{\beta} \\
&\qquad + K_2 \sup_{r_1,r_2\in[s,t]} \bigl|\nabla_x f(x_{r_1}, y_{\eta(r_2)})\bigr| \|x\|_{s,t,\beta'} \|z\|_{\beta} (t-s)^{\beta+\beta'} \\
&\qquad + K_3 \sup_{r_1,r_2\in[s,t]} \bigl|\nabla_y f(x_{r_1}, y_{r_2})\bigr| \|y\|_{s,t,\beta',n} \|z\|_{\beta} (t-s)^{\beta+\beta'},
\end{aligned}
$$

where $K_1 = K_0 \frac{\Gamma(\alpha+\beta)}{\Gamma(\beta+1)}$, $K_2 = K_0 \frac{\alpha\Gamma(\alpha+\beta)\Gamma(\beta'-\alpha+1)}{\Gamma(1-\alpha)\Gamma(\beta+\beta'+1)(\beta'-\alpha)}$, $K_3 = K_0 K_4 \frac{\alpha}{\Gamma(1-\alpha)}$ and K_4 is the constant in Lemma A.2. This completes the proof. □

LEMMA A.2. *Let β, β' and α be such that $\beta' > \alpha > 1-\beta$. Then for any $s, t \in [0, T]$ such that $s < t$, $s = \eta(s)$, there exists a constant K_4 depending on α, β and T, such that*

$$\int_s^t (t-r)^{\alpha+\beta-1} \int_s^r \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}}\, du\, dr \le K_4 (t-s)^{\beta+\beta'}.$$

PROOF. Without loss of generality, we let $T = 1$. Note that when $\eta(s) = s < t \le \eta(s) + \frac{1}{n}$, the double integral equals zero. In the following we will assume $t > \eta(s) + \frac{1}{n}$.

We first write

$$\begin{aligned}
&\int_s^t (t-r)^{\alpha+\beta-1} \int_s^r \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}}\, du\, dr \\
&\quad = \int_{\eta(s)+1/n}^t (t-r)^{\alpha+\beta-1} \int_{\eta(s)}^{\eta(r)} \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}}\, du\, dr \\
&\quad = \int_{\eta(s)+1/n}^t (t-r)^{\alpha+\beta-1} \left(\int_{\eta(r)-1/n}^{\eta(r)} + \int_{\eta(s)}^{\eta(r)-1/n} \right) \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}}\, du\, dr \\
&\quad := J_1 + J_2.
\end{aligned}$$

On one hand, notice that in the term J_2 we always have $r - u > \frac{1}{n}$, and thus $\eta(r) - \eta(u) \le r - u + \frac{1}{n} \le 2(r-u)$. Therefore,

$$\begin{aligned}
J_2 &\le \int_{\eta(s)+1/n}^t (t-r)^{\alpha+\beta-1} \int_{\eta(s)}^{\eta(r)-1/n} \frac{2^{\beta'}(r-u)^{\beta'}}{(r-u)^{\alpha+1}}\, du\, dr \\
&\le K(t-s)^{\beta+\beta'}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
J_1 &= \int_{\eta(s)+1/n}^t (t-r)^{\alpha+\beta-1} \int_{\eta(r)-1/n}^{\eta(r)} \frac{|\eta(r)-\eta(u)|^{\beta'}}{(r-u)^{\alpha+1}}\, du\, dr \\
&\le K n^{-\beta'} (t-s)^{\alpha+\beta-1} \int_{\eta(s)+1/n}^t \left[\frac{1}{(r-\eta(r))^\alpha} - \frac{1}{(r-\eta(r)+1/n)^\alpha} \right] dr \\
&\le K n^{-\beta'} (t-s)^{\alpha+\beta-1} \int_{\eta(s)+1/n}^t \frac{1}{(r-\eta(r))^\alpha}\, dr \\
&\le K n^{-\beta'} (t-s)^{\alpha+\beta-1} \frac{(\eta(t)+1/n) - (\eta(s)+1/n)}{1/n} n^{\alpha-1} \\
&\le K(t-s)^{\beta+\beta'}.
\end{aligned}$$

The lemma is now proved. $\square$

A.2. Estimates for some special Young and Skorohod integrals. In this section we derive estimates for some specific Young and Skorohod integrals. We fix $n \in \mathbb{N}$ and consider the uniform partition on $[0, T]$.

LEMMA A.3. *Let $B = \{B_t, t \in [0, T]\}$ be a one-dimensional fBm with Hurst parameter $H > \frac{1}{2}$. Fix $\nu \geq 0$ and $p \geq \frac{1}{H}$. Let $F = \{F_t, t \in [0, T]\}$ be a stochastic process whose trajectories are Hölder continuous of order $\gamma > 1 - H$ and such that $F_t \in \mathbb{D}^{1,q}$, $t \in [0, T]$, for some $q > p$. For any $\rho > 1$ we set*

$$F_{1,\rho} = \sup_{s,t \in [0,T]} \big(\|F_t\|_\rho \vee \|D_s F_t\|_\rho \big).$$

Then there exists a constant C (independent of F) such that the following inequalities hold for all $0 \leq s < t \leq T$:

$$\left\| \int_s^t F_u \big(u - \eta(u)\big)^\nu \, dB_u \right\|_p \leq C n^{-\nu} (t-s)^H F_{1,p}, \tag{A.4}$$

$$\left\| \int_s^t F_u (B_u - B_{\eta(u)}) \, du \right\|_p \leq C n^{-1} (t-s)^H F_{1,q}. \tag{A.5}$$

PROOF OF (A.4). Applying (2.9) we can decompose the Young integral as the sum of a Skorohod integral plus a complementary term,

$$\begin{aligned} &\int_s^t F_u \big(u - \eta(u)\big)^\nu \, dB_u \\ &\qquad = \int_s^t F_u \big(u - \eta(u)\big)^\nu \delta B_u \\ &\qquad\quad + \alpha_H \int_s^t \int_0^T \big(u - \eta(u)\big)^\nu D_r F_u |r - u|^{2H-2} \, dr \, du. \end{aligned} \tag{A.6}$$

It follows from (2.11) that the L^p-norm of the first integral of the right-hand side of (A.6) is bounded by $C n^{-\nu} (t-s)^H F_{1,p}$. On the other hand, from Minkowski's inequality it follows that the L^p-norm of the second integral is less than or equal to $C n^{-\nu} (t-s) F_{1,p}$. These estimates imply (A.4) because $(t-s) \leq (t-s)^H T^{1-H}$. □

PROOF OF (A.5). If $t - s \leq \frac{1}{n}$, we can write

$$\begin{aligned} \left\| \int_s^t F_u (B_u - B_{\eta(u)}) \, du \right\|_p &\leq \int_s^t \big\| F_u (B_u - B_{\eta(u)}) \big\|_p \, du \\ &\leq C \sup_{t \in [0,T]} \|F_t\|_q n^{-H} (t-s) \\ &\leq C \sup_{t \in [0,T]} \|F_t\|_q n^{-1} (t-s)^H, \end{aligned}$$

where the first inequality follows from Minkowski's inequality and the second one from Hölder's inequality. Suppose that $t - s \geq \frac{1}{n}$. Applying Fubini's theorem for the Young integral, we obtain

$$\int_s^t F_u(B_u - B_{\eta(u)})\, du = \int_{\eta(s)}^t \left(\int_v^{\varepsilon(v)} \mathbf{1}_{[s,t]}(u) F_u\, du \right) dB_v.$$

Applying (A.4) with $\nu = 0$ we obtain

$$\begin{aligned} \left\| \int_{\eta(s)}^t \left(\int_v^{\varepsilon(v)} \mathbf{1}_{[s,t]}(u) F_u\, du \right) dB_v \right\|_p &\leq C\big(t - \eta(s)\big)^H n^{-1} F_{1,p} \\ &\leq C(t-s)^H n^{-1} F_{1,p}. \end{aligned} \tag{A.7}$$

This completes the proof of (A.5). □

LEMMA A.4. *Let $B = \{B_t, t \in [0,T]\}$ be an m-dimensional fBm with Hurst parameter $H > \frac{1}{2}$. Fix $p \geq \frac{1}{H}$. Let $F = \{F_t, t \in [0,T]\}$ be a stochastic process such that $F_t \in \mathbb{D}^{2,q}$, $t \in [0,T]$, for some $q > p$. For any $\rho > 1$ we set*

$$F_{2,\rho} = \sup_{r,s,t \in [0,T]} \big(\|F_t\|_\rho \vee \|D_s F_t\|_\rho \vee \|D_r D_s F_t\|_\rho \big).$$

Set also

$$F_* = \sup_{r,s,t \in [0,T]} \big(|F_t| \vee |D_s F_t| \vee |D_r D_s F_t| \big).$$

Then there exists a constant C (independent of F) such that the following holds for all $0 \leq s < t \leq T$, $i, j = 1, \dots, m$:

$$\left\| \sum_{k=\lfloor ns/T \rfloor}^{\lfloor nt/T \rfloor} F_{t_k} \int_{t_k \vee s}^{t_{k+1} \wedge t} \int_{t_k}^u \delta B_v^i \delta B_u^j \right\|_p \leq C \gamma_n^{-1} (t-s)^{1/2} \|F_*\|_q, \tag{A.8}$$

$$\left\| \sum_{k=\lfloor ns/T \rfloor}^{\lfloor nt/T \rfloor} F_{t_k} \int_{t_k \vee s}^{t_{k+1} \wedge t} \int_{t_k}^u \delta B_v^i \delta B_u^j \right\|_p \leq C n^{-H} (t-s)^H F_{2,q}. \tag{A.9}$$

PROOF. Using (2.8), we can write

$$\begin{aligned} &\sum_{k=\lfloor ns/T \rfloor}^{\lfloor nt/T \rfloor} F_{t_k} \int_{t_k \vee s}^{t_{k+1} \wedge t} \int_{t_k}^u \delta B_v^i \delta B_u^j \\ &\quad = \int_s^t F_{\eta(u)} \big(B_u^i - B_{\eta(u)}^i\big) \delta B_u^j \\ &\qquad + \alpha_H \int_s^t \int_0^T D_r^j F_{\eta(u)} \big(B_u^i - B_{\eta(u)}^i\big) \mu(dr\, du). \end{aligned} \tag{A.10}$$

Applying (A.5) to the second integral of the right-hand side of (A.10) with F_u replaced by $\int_0^T D_r^j F_{\eta(u)}|r-u|^{2H-2}\,dr$ (notice that here we do not need the Hölder continuity of the integrand for the Young integral to be well defined) yields

$$\begin{aligned}
&\left\|\int_s^t \int_0^T D_r^j F_{\eta(u)}\big(B_u^i - B_{\eta(u)}^i\big)\mu(dr\,du)\right\|_p \\
&\quad \le Cn^{-1}(t-s)^H F_{2,q} \sup_{u\in[0,T]} \int_0^T |r-u|^{2H-2}\,dr \qquad \text{(A.11)}\\
&\quad \le Cn^{-1}(t-s)^H F_{2,q}.
\end{aligned}$$

This implies both estimates (A.8) and (A.9).

Applying (2.8) to the first summand on the right-hand side of (A.10) yields

$$\begin{aligned}
&\int_s^t F_u\big(B_u^i - B_{\eta(u)}^i\big)\delta B_u^j \\
&\quad = \int_s^t \int_{\eta(u)}^u F_u \delta B_v^i \delta B_u^j + \alpha_H \int_s^t \left\{\int_0^T \int_{\eta(u)}^u D_v^i F_u \mu(dr dv)\right\}\delta B_u^j. \qquad \text{(A.12)}
\end{aligned}$$

Now we apply (2.11) to the second term of the right-hand side of (A.12), and we obtain

$$\begin{aligned}
&\left\|\int_s^t \left\{\int_0^T \int_{\eta(u)}^u D_v^i F_u \mu(dr\,dv)\right\}\delta B_u^j\right\|_p \\
&\quad \le C F_{2,p} \left\|\mathbf{1}_{[s,t]}(u)\int_0^T \int_{\eta(u)}^u \mu(dr\,dv)\right\|_{L^{1/H}([0,T])} \qquad \text{(A.13)}\\
&\quad \le C F_{2,p} n^{-1}(t-s)^H.
\end{aligned}$$

Again, this inequality implies both estimates (A.8) and (A.9).

It remains to estimate the term $I_{s,t} := \int_s^t \int_{\eta(u)}^u F_u \delta B_v^i \delta B_u^j$. It follows from (2.11) that

$$\begin{aligned}
\|I_{s,t}\|_p &\le C F_{2,p} \big\|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v)\big\|_{L^{1/H}([0,T]^2)} \\
&\le C F_{2,p} n^{-H}(t-s)^H,
\end{aligned}$$

which completes the proof of (A.9).

To derive (A.8) we need a more accurate estimate.

Meyer's inequality implies that

$$\begin{aligned}
\|I_{s,t}\|_p &\le C\big[\big\|\|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v)F_u\|_{\mathcal{H}^{\otimes 2}}\big\|_p \\
&\qquad + \big\|\|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v)D_r F_u\|_{\mathcal{H}^{\otimes 3}}\big\|_p \\
&\qquad + \big\|\|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v)D_{r'}D_r F_u\|_{\mathcal{H}^{\otimes 4}}\big\|_p\big] \\
&\le C\|F_*\|_p \big\|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v)\big\|_{\mathcal{H}^{\otimes 2}}.
\end{aligned}$$

Therefore, to complete the proof, it suffices to show that

$$\begin{aligned}
&\|\mathbf{1}_{[s,t]}(u)\mathbf{1}_{[\eta(u),u]}(v)\|^2_{\mathcal{H}^{\otimes 2}} \\
&\quad = \alpha_H^2 \int_s^t \int_s^t \int_{\eta(u')}^{u'} \int_{\eta(u)}^{u} \mu(dv\,dv')\mu(du\,du') \qquad \text{(A.14)} \\
&\quad \le (t-s)\gamma_n^{-2}.
\end{aligned}$$

In the case $t - s \geq \frac{1}{n}$,

$$\begin{aligned}
&\int_s^t \int_s^t \int_{\eta(u')}^{u'} \int_{\eta(u)}^{u} \mu(dv\,dv'\,du\,du') \\
&\quad \le \sum_{k,k'=\lfloor ns/T\rfloor}^{\lfloor nt/T\rfloor} \int_{t_{k'}}^{t_{k'+1}} \int_{t_k}^{t_{k+1}} \int_{t_{k'}}^{u'} \int_{t_k}^{u} \mu(dv\,dv'\,du\,du') \\
&\quad \le \sum_{k=\lfloor ns/T\rfloor}^{\lfloor nt/T\rfloor} \sum_{p=1-n}^{n-1} \int_{t_{k+p}}^{t_{k+p+1}} \int_{t_k}^{t_{k+1}} \int_{t_{k+p}}^{u'} \int_{t_k}^{u} \mu(dv\,dv'\,du\,du') \\
&\quad = n^{-4H} \sum_{k=\lfloor ns/T\rfloor}^{\lfloor nt/T\rfloor} \sum_{p=1-n}^{n-1} Q(p) \\
&\quad \le C(t-s)\gamma_n^{-2},
\end{aligned}$$

where we recall that $Q(p)$ is defined in Section 2.4, and inequality (A.14) follows.

In the case $t - s \leq \frac{1}{n}$, we have the raw estimate

$$\int_s^t \int_s^t \int_{\eta(u')}^{u'} \int_{\eta(u)}^{u} \mu(dr\,dr'\,du\,du') \le \frac{1}{n^{2H}} \int_s^t \int_s^t \mu(du\,du') = \frac{1}{n^{2H}}(t-s)^{2H}$$

and

$$n^{-2H}(t-s)^{2H} \le (t-s)\gamma_n^{-2}.$$

So (A.14) is also true for this case. The proof of the lemma is now complete. $\square$

LEMMA A.5. *Let $B = \{B_t, t \in [0,T]\}$ be a one-dimensional fBm with Hurst parameter $H > \frac{1}{2}$. Suppose that $F = \{F_t, t \in [0,T\}$, $G = \{G_t, t \in [0,T]\}$ are processes that are Hölder continuous of order $\beta \in (\frac{1}{2}, H)$. Then there exists a constant C (not depending on F or G) such that for all $0 \le s < t \le T$, $\nu \ge 0$,*

$$\begin{aligned}
&\left|\int_s^t F_u(G_u - G_{\eta(u)})(B_u - B_{\eta(u)})\,dB_u\right| \qquad \text{(A.15)} \\
&\quad \le C(\|F\|_\infty + \|F\|_\beta)\|G\|_\beta \|B\|_\beta^2 n^{1-3\beta}(t-s)^\beta
\end{aligned}$$

and

$$\left|\int_s^t F_u(G_u - G_{\eta(u)})\big(u-\eta(u)\big)^\nu \, dB_u\right| \tag{A.16}$$
$$\le C\big(\|F\|_\infty + \|F\|_\beta\big)\|G\|_\beta \|B\|_\beta n^{1-2\beta-\nu}(t-s)^\beta.$$

PROOF OF (A.15). We assume first that $s,t \in [t_k, t_{k+1}]$ for some $k = 0,1,\dots,n-1$. By Lemma A.1(ii),

$$\begin{aligned}
&\left|\int_s^t F_u(G_u - G_{\eta(u)})(B_u - B_{\eta(u)})\, dB_u\right| \\
&\quad \le K_1 \sup_{u\in[s,t]} \big|F_u(G_u - G_{t_k})(B_u - B_{t_k})\big| \|B\|_\beta (t-s)^\beta \\
&\qquad + K_2 \sup_{u\in[s,t]} \big[\big|F_u(G_u - G_{t_k})\big| \|B\|_\beta^2 (t-s)^{2\beta} \\
&\qquad\qquad + \big|F_u(B_u - B_{t_k})\big| \|G\|_\beta \|B\|_\beta (t-s)^{2\beta} \\
&\qquad\qquad + \big|(G_u - G_{t_k})(B_u - B_{t_k})\big| \|F\|_\beta \|B\|_\beta (t-s)^{2\beta}\big] \\
&\quad \le C\kappa_\beta(F,G) n^{-2\beta}(t-s)^\beta,
\end{aligned} \tag{A.17}$$

where $\kappa_\beta(F,G) = (\|F\|_\infty + \|F\|_\beta)\|G\|_\beta \|B\|_\beta^2$. In the general case, we can write

$$\begin{aligned}
&\left|\int_s^t F_u(G_u - G_{\eta(u)})(B_u - B_{\eta(u)})\, dB_u\right| \\
&\quad = \left|\left(\int_s^{\varepsilon(s)} + \sum_{k=\lfloor ns/T\rfloor+1}^{\lfloor nt/T\rfloor} \int_{t_k}^{t_{k+1}} + \int_{\eta(t)}^t\right) F_u(G_u - G_{\eta(u)})(B_u - B_{\eta(u)})\, dB_u\right| \\
&\quad \le C\kappa_\beta(F,G) n^{-2\beta}\left[\big(\varepsilon(s)-s\big)^\beta + \big(t-\eta(t)\big)^\beta + \sum_{k=\lfloor ns/T\rfloor+1}^{\lfloor nt/T\rfloor} (T/n)^\beta\right] \\
&\quad \le C\kappa_\beta(F,G) n^{-2\beta}\big[\big(\varepsilon(s)-s\big)^\beta + \big(t-\eta(t)\big)^\beta + \big(\eta(t)-\varepsilon(s)\big)n^{1-\beta}\big] \\
&\quad \le C\kappa_\beta(F,G) n^{1-3\beta}(t-s)^\beta,
\end{aligned}$$

where the first inequality follows from (A.17). □

PROOF OF (A.16). This estimate can be proved by following the lines of the proof of (A.15) and noticing the fact that $(u-\eta(u))^\nu$ has finite ν-Hölder seminorm on (t_k, t_{k+1}) for each $k = 1,\dots,n-1$. □

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