The Annals of Applied Probability
2015, Vol. 25, No. 3, 1383–1419
DOI: 10.1214/14-AAP1026

UTILITY MAXIMIZATION WITH ADDICTIVE CONSUMPTION HABIT FORMATION IN INCOMPLETE SEMIMARTINGALE MARKETS[1]

BY XIANG YU

University of Michigan

This paper studies the continuous time utility maximization problem on consumption with addictive habit formation in incomplete semimartingale markets. Introducing the set of auxiliary state processes and the modified dual space, we embed our original problem into a time-separable utility maximization problem with a shadow random endowment on the product space $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \overline{\mathbb{P}})$. Existence and uniqueness of the optimal solution are established using convex duality approach, where the primal value function is defined on two variables, that is, the initial wealth and the initial standard of living. We also provide sufficient conditions on the stochastic discounting processes and on the utility function for the well-posedness of the original optimization problem. Under the same assumptions, classical proofs in the approach of convex duality analysis can be modified when the auxiliary dual process is not necessarily integrable.

1. Introduction. During the past decades, the time separable von Neumann–Morgenstern preferences on consumption have been observed to be inconsistent with some empirical evidences. For instance, the well-known magnitude of the equity premium (Mehra and Prescott [21]) cannot be reconciled with the preference $\mathbb{E}[\int_0^T U(t, c_t)\,dt]$ where the instantaneous utility function U is only derived from the consumption rate process. As an alternative modeling tool, *linear addictive* habit formation preference has attracted a lot of attention and has been actively investigated in recent years. This new preference is defined by $\mathbb{E}[\int_0^T U(t, c_t - Z(c)_t)\,dt]$, where $U:[0,T]\times(0,\infty)\to\mathbb{R}$ and the additional accumulative process $Z(c)_t$, called the *habit formation* or *the standard of living* process, describes the consumption history impact. To be more precise, $Z(c)_t$ is the solution of the following recursive equation:

$$dZ(c)_t = \big(\delta_t c_t - \alpha_t Z(c)_t\big)\,dt,$$
$$Z(c)_0 = z,$$

Received July 2012; revised November 2013.

[1]This work is part of the author's Ph.D. dissertation at the University of Texas at Austin and is partially supported by NSF Grant under the award number DMS-09-08441.

MSC2010 subject classifications. Primary 91G10, 91B42; secondary 91G80.

Key words and phrases. Time nonseparable utility maximization, consumption habit formation, auxiliary processes, convex duality, incomplete markets.

where the discounting factors α_t and δ_t are assumed to be nonnegative optional processes, and the given real number $z \geq 0$ is called the *initial habit* or the *initial standard of living*. Moreover, the consumption habits are assumed to be *addictive* in the sense that $c_t \geq Z(c)_t$ for all time $t \in [0, T]$. Compared to the time separable case, a small drop in consumption may cause large fluctuation in consumption net of the subsistence level due to the standard of living constraint. The habit formation preference can possibly explain sizable excess returns on risky assets in equilibrium models even for moderate values of the degree of risk aversion. Based on this, a vast literature recommends this time nonseparable preference as the new economic paradigm. We refer readers to, for instance, Constantinides [4], Samuelson [25] and Campbell and Cochrane [3].

The study of habit formation in modern economics dates back to Hicks [12] in 1965 and Ryder and Heal [24] in 1973. More recently, there have been some important contributions in complete Itô processes markets; see Detemple and Zapatero [9, 10], Schroder and Skiadas [27], Munk [22], Detemple and Karatzas [8] and Englezos and Karatzas [11]. Several pioneering work have derived the explicit feedback form of the optimal policies under different assumptions and market models. However, in the words by Englezos and Karatzas [11], "The existence of an optimal portfolio/consumption pair in an incomplete market is an open question . . . , and new methodologies are needed to handle the problem." Therefore, in this paper, we are interested in the general incomplete semimartingale framework and aim to prove the existence and uniqueness of the optimal solution to this path dependent optimization problem.

The convex duality approach plays an important role in the treatment of utility maximization problems in incomplete markets. To list a very small subset of the existing literature, we refer to Karatzas et al. [15], Kramkov and Schachermayer [18, 19], Cvitanic, Schachermayer and Wang [5], Karatzas and Žitković [16], Hugonnier and Kramkov [14] and Žitković [29, 30].

Typically, the critical step to build conjugate duality is to define the dual space as a proper extension of space $\mathcal{M}$, which is the set of equivalent local martingale measure density processes. The first natural choice is the bipolar set of the space $\mathcal{M}$, which is the smallest convex, closed and solid set containing the set $\mathcal{M}$. Kramkov and Schachermayer [18, 19] and Žitković [29], proved that this bipolar set can be characterized as the solid hull of the set $\mathcal{Y}(y)$, which is defined as the set of supermartingale deflators

$$\mathcal{Y}(y) = \big\{Y | Y_0 = y, Y_t > 0, t \in [0, T] \text{ and } XY = (X_t Y_t)_{0 \leq t \leq T} \\ \text{is a supermartingale for each } X \in \mathcal{X}(x)\big\}.$$

Here $\mathcal{X}(x)$ denotes the set of accumulated gains/losses processes under some admissible portfolios with initial endowment less than or equal to x. However, according to the definition of habit formation process $Z(c)_t$, if we derive the naive dual problem using the Legendre–Fenchel transform and the first order condition, we arrive at

$$\inf_{y>0, Y\in\mathcal{Y}(y)} \mathbb{E}\biggl[\int_0^T V\biggl(Y_t + \delta_t \mathbb{E}\biggl[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)\,dv} Y_s\,ds\Big|\mathcal{F}_t\biggr]\biggr)\,dt\biggr]$$
$$- z\mathbb{E}\biggl[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)\,dv} Y_t\,dt\biggr].$$

The first mathematical difficulty is the extra integral $\mathbb{E}[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)\,dv} Y_t\,dt]$, from which we can see that $\mathcal{Y}(y)$ is not the appropriate space to show the existence of the optimal dual solution. However, it still reminds us to invoke the general treatment of random endowment developed by Cvitanic, Schachermayer and Wang [5], Karatzas and Žitković [16] and Žitković [30]. They proposed another extension of the set $\mathcal{M}$, which is now considered as the set of equivalent local martingale measures, to the set $\mathcal{D}$ of bounded finitely additive measures. Nevertheless, their approach is inadequate to deal with the first term of the dual problem, when the conditional integral part $\mathbb{E}[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)\,dv} Y_s\,ds|\mathcal{F}_t]$ in the conjugate function V is taken into account.

In order to avoid the complexity of the path-dependence, we propose the transform from the consumption rate process c_t to the auxiliary process $\tilde{c}_t = c_t - Z(c)_t$, so that the primal utility maximization problem becomes time separable with respect to the process $\tilde{c}_t$. This substitution idea from c_t to $\tilde{c}_t$ appeared first in the market isomorphism result for complete markets by Schroder and Skiadas [27]. And meanwhile, for each equivalent local martingale measure density process $Y \in \mathcal{M}$, we define the auxiliary dual process Γ_t exactly by

$$\Gamma_t \triangleq Y_t + \delta_t \mathbb{E}\biggl[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)\,dv} Y_s\,ds\Big|\mathcal{F}_t\biggr] \qquad \forall t \in [0, T].$$

The dual problem can therefore be formulated in terms of auxiliary process Γ_t instead of Y_t so that the path dependence of Y_t can also be hidden in the definition of process Γ_t.

By introducing the process $\tilde{w}_t = e^{\int_0^t(-\alpha_v)\,dv}$, one can shift the integral $\mathbb{E}[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)\,dv} Y_t\,dt]$ to the integral $\mathbb{E}[\int_0^T \tilde{w}_t\Gamma_t\,dt]$ with respect to its auxiliary process Γ_t. With the aid of this equality, we can treat the extra exogenous random term $\tilde{w}_t$ as the shadow random endowment density process and define the dual functional on the properly modified space of Γ_t instead of Y_t. By enlarging the effective domain of values for x and z, the original utility maximization problem with habit formation can be embedded into the framework of Hugonnier and Kramkov [14] as an abstract time-separable utility maximization problem on the product space.

On the other hand, we are facing some troubles in applying the classical duality results since the auxiliary process Γ_t is not integrable. For instance, to show the existence of the dual optimizer, the trick of applying the de la Vallée–Poussin theorem in the proof of Lemma 3.2 in Kramkov and Schachermayer [18] does not work. And the argument of contradiction in the proof of Lemma 1 in Kramkov and Schachermayer [19] using the subsequence splitting lemma will also fail by observing that constants may not be contained in the corresponding space. Therefore, we impose the additional sufficient conditions on habit formation discounting factors α_t and δ_t; see Assumption 3.1 to guarantee the well-posedness of the primal optimization problem. We also ask for reasonable asymptotic elasticity conditions on utility functions U both at $x \to 0$ and $x \to \infty$ for the validity of several key assertions of our main results to hold true.

The rest of this paper is organized as follows: Section 2 introduces the financial market and consumption habit formation process. In Section 3, we define the auxiliary process space $\overline{\mathcal{A}}(x, z)$, the enlarged space $\widetilde{\mathcal{A}}(x, z)$ and the auxiliary dual space $\widetilde{\mathcal{M}}$. The original path-dependent utility maximization problem is embedded into an abstract time separable optimization problem with the shadow random endowment. Section 4 is devoted to the formulation of the two-dimensional dual problem over the properly enlarged dual space $\widetilde{\mathcal{Y}}(y, r)$ such that the shadow random endowment part can be hidden, and our main results are stated at the end. Section 5 contains detailed proofs of our main theorems.

2. Market model.

2.1. *The financial market model.* We consider a financial market with $d \in \mathbb{N}$ risky assets modeled by a d-dimensional semimartingale

$$S = (S_t^{(1)}, \dots, S_t^{(d)})_{t\in[0,T]} \tag{2.1}$$

on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, where the filtration $\mathbb{F}$ satisfies the usual conditions, and the maturity time is given by T. To simplify the notation, we take $\mathcal{F} = \mathcal{F}_T$. We make the standard assumption that there exists one riskless bond $S_t^{(0)} \equiv 1$, $\forall t \in [0, T]$, which amounts to considering $S_t^{(0)}$ as the numéraire asset.

The portfolio process $H = (H_t^{(1)}, \dots, H_t^{(d)})_{t\in[0,T]}$ is a predictable S-integrable process representing the number of shares of each risky asset held by the investor at time $t \in [0, T]$. The accumulated gains/losses process of the investor under his trading strategy H by time t is given by

$$X_t^H = (H \cdot S)_t = \sum_{k=1}^{d} \int_0^t H_u^{(k)}\, dS_u^{(k)}, \qquad t \in [0, T]. \tag{2.2}$$

2.2. *Admissible portfolios and consumption habit formation.* The portfolio process $(H_t)_{t\in[0,T]}$ is called *admissible* if there exists a constant bound $a \in \mathbb{R}$ such that $X_t^H \ge a$, a.s. for all $t \in [0, T]$.

Now, given the initial wealth $x > 0$, the agent will also choose an intermediate consumption plan during the whole investment horizon, and we denote the consumption rate process by c_t. The resulting self-financing *wealth process* $(W_t^{x,H,c})_{t\in[0,T]}$ is given by

$$W_t^{x,H,c} \triangleq x + (H\cdot S)_t - \int_0^t c_s\,ds, \qquad t\in[0,T]. \tag{2.3}$$

Besides of the wealth process, as we defined in the Introduction, the associated *consumption habit formation* process $Z(c)_t$ is given equivalently by the following exponentially weighted average of agent's past consumption integral and the initial habit:

$$Z(c)_t = ze^{-\int_0^t \alpha_v\,dv} + \int_0^t \delta_s e^{-\int_s^t \alpha_v\,dv} c_s\,ds. \tag{2.4}$$

Here discounting factors α_t and δ_t measure, respectively, the persistence of the initial habits level and the intensity of consumption history. In this paper, we shall be mostly interested in the general case when discounting factors α_t and δ_t are stochastic processes which are allowed to be unbounded. However, for technical reasons, we will assume that $\int_0^t(\delta_u - \alpha_u)\,du < \infty$ a.s. for each $t\in[0,T]$.

Throughout this paper, we make the assumption that the consumption habit is addictive, that is, $c_t \geq Z(c)_t$, $\forall t\in[0,T]$, which is to say, the investor's current consumption rate shall never fall below *the standard of living* process.

A consumption process $(c_t)_{t\in[0,T]}$ is defined to be (x,z)-*financeable* if there exists an admissible portfolio process $(H_t)_{t\in[0,T]}$ such that $W_t^{x,H,c}\geq 0$, a.s. for $\forall t\in[0,T]$ and the addictive habit formation constraint $c_t \geq Z(c)_t$, a.s. for $\forall t\in[0,T]$ holds. The class of all (x,z)-financeable consumption rate processes will be denoted by $\mathcal{A}(x,z)$, for $x>0$, $z\geq 0$.

2.3. *Absence of arbitrage.* A probability measure $\mathbb{Q}$ is called the *equivalent local martingale measure* if it is equivalent to $\mathbb{P}$ and if X_t^H is a local martingale under $\mathbb{Q}$. We denote by $\mathcal{M}$ the family of equivalent local martingale measures, and in order to rule out the arbitrage opportunities in the market, we assume that

$$\mathcal{M}\neq\varnothing. \tag{2.5}$$

See Delbaen and Schachermayer [6] and [7] for comprehensive discussions on the topic of no arbitrage.

Define the RCLL process $Y^{\mathbb{Q}}$ by

$$Y_t^{\mathbb{Q}} = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\middle|\mathcal{F}_t\right]$$

for the $\mathbb{Q}\in\mathcal{M}$. Then $Y^{\mathbb{Q}}$ is called an equivalent local martingale measure density and with a slight abuse of notation, we denote $\mathcal{M}$ also as the set of all equivalent local martingale density processes.

The celebrated optional decomposition theorem (see Kramkov [20]) enables us to characterize the (x,z)-financeable consumption process by the following budget constraint condition:

PROPOSITION 2.1. *The process* $(c_t)_{t\in[0,T]}$ *is* (x,z)*-financeable if and only if* $c_t \geq Z(c)_t$, $\forall t \in [0,T]$ *and*

$$\mathbb{E}\left[\int_0^T c_t Y_t\,dt\right] \leq x \qquad \forall Y \in \mathcal{M}. \tag{2.6}$$

2.4. *The utility function.* The individual investor's preference is represented by a utility function $U:[0,T]\times(0,\infty)\to\mathbb{R}$, such that, for every $x>0$, $U(\cdot,x)$ is continuous on $[0,T]$, and for every $t\in[0,T]$, the function $U(t,\cdot)$ is strictly concave, strictly increasing, continuously differentiable and satisfies the Inada conditions,

$$U'(t,0) \triangleq \lim_{x\to 0} U'(t,x) = \infty, \qquad U'(t,\infty) \triangleq \lim_{x\to\infty} U'(t,x) = 0, \tag{2.7}$$

where $U'(t,x) \triangleq \frac{\partial}{\partial x}U(t,x)$. For each $t\in[0,T]$, we extend the definition of the utility function by $U(t,x)=-\infty$ for all $x<0$, which is equivalent to the addictive habit formation constraint $c_t \geq Z(c)_t$ when the utility function is defined on the difference between the consumption rate process c_t and the habit formation process $Z(c)_t$.

According to these assumptions, the inverse $I(t,\cdot):\mathbb{R}_+\to\mathbb{R}_+$ of the function $U'(t,\cdot)$ exists for every $t\in[0,T]$, and is continuous and strictly decreasing. The convex conjugate of the utility function is defined by

$$V(t,y) \triangleq \sup_{x>0}\{U(t,x)-xy\}, \qquad y>0.$$

Following the asymptotic elasticity condition on utility functions coined by Kramkov and Schachermayer [18] (see also Karatzas and Žitković [16]), we make assumptions on the asymptotic behavior of U at both $x=0$ and $x=\infty$ for future purposes.

ASSUMPTION 2.1. The utility function U satisfies the reasonable asymptotic elasticity condition both at $x=\infty$ and $x=0$, that is,

$$\mathrm{AE}_\infty[U] = \limsup_{x\to\infty}\left(\sup_{t\in[0,T]}\frac{xU'(t,x)}{U(t,x)}\right) < 1 \tag{2.8}$$

and

$$\mathrm{AE}_0[U] = \limsup_{x\to 0}\left(\sup_{t\in[0,T]}\frac{xU'(t,x)}{|U(t,x)|}\right) < \infty. \tag{2.9}$$

Moreover, in order to get some inequalities uniformly in time t, we shall assume

$$\lim_{x\to\infty}\Big(\inf_{t\in[0,T]} U(t,x)\Big) > 0 \tag{2.10}$$

and

$$\lim_{x\to 0}\Big(\sup_{t\in[0,T]} U(t,x)\Big) < 0. \tag{2.11}$$

REMARK 1. Many well-known utility functions satisfy reasonable asymptotic elasticity conditions (2.8) and (2.9), for example, the discounted log utility function $U(t,x) = e^{-\beta t}\log(x)$ or discounted power utility function $U(t,x) = e^{-\beta t}\frac{x^p}{p}$ ($p < 1$ and $p \neq 0$), for some constant $\beta > 0$. However, it is also easy to check that the utility function $U(t,x) = -e^{1/x}$ does not satisfy condition (2.9), and the utility function $U(t,x) = \frac{x}{\log x}$ does not satisfy the condition (2.8).

REMARK 2. If the utility function satisfies the lower bound assumption $\inf_{t\in[0,T]} U(t,0) > -\infty$, then condition (2.9) is automatically verified, and if the utility function satisfies the upper bound assumption $\sup_{t\in[0,T]} U(t,\infty) < \infty$, condition (2.8) holds true.

Next, some technical results give the equivalent characterizations of reasonable asymptotic elasticity conditions (2.8) and (2.9). The proof is based on the fact that $-V$ is a concave function and on similar arguments in Lemma 6.3 of Kramkov and Schachermayer [18]; see also Proposition 3.7 of Karatzas and Žitković [16].

LEMMA 2.1. *Let $U(t,x)$ be a utility function satisfying Assumption 2.1. We have* $\mathrm{AE}_0[U] < \infty$ *if and only if* $\mathrm{AE}_\infty[V] < 1$, *where we define*

$$\mathrm{AE}_\infty[V] = \limsup_{y\to\infty}\Big(\sup_{t\in[0,T]} \frac{yV'(t,y)}{V(t,y)}\Big) < 1. \tag{2.12}$$

In each of the subsequent assertions, the infimum of $\gamma_1 > 0$, for which these assertions hold true, equals the reasonable asymptotic elasticity $\mathrm{AE}_\infty[U]$, *and the infimum of $\gamma_2 > 0$ equals the reasonable asymptotic elasticity* $\mathrm{AE}_\infty[V]$.

(i) *There are $x_0 > 0$ and $y_0 > 0$ for all $t \in [0,T]$ s.t.*

$$U(t,\lambda x) < \lambda^{\gamma_1} U(t,x) \qquad \text{for } \lambda > 1,\ x \geq x_0;$$

$$V(t,\lambda y) > \lambda^{\gamma_2} V(t,y) \qquad \text{for } \lambda > 1,\ y \geq y_0.$$

(ii) *There are $x_0 > 0$ and $y_0 > 0$ for all $t \in [0,T]$ s.t.*

$$U'(t,x) < \gamma_1 \frac{U(t,x)}{x} \qquad \text{for } x \geq x_0;$$

$$V'(t,y) > \gamma_2 \frac{V(t,y)}{y} \qquad \text{for } y \geq y_0.$$

(iii) *There are* $x_0 > 0$ *and* $y_0 > 0$ *for all* $t \in [0, T]$ *s.t.*

$$V(t, \mu y) < \mu^{-\gamma_1/(1-\gamma_1)} V(t, y) \qquad \text{for } 0 < \mu < 1, 0 < y \le y_0;$$
$$U(t, \mu x) > \mu^{-\gamma_2/(1-\gamma_2)} U(t, x) \qquad \text{for } 0 < \mu < 1, 0 < x \le x_0.$$

(iv) *There are* $x_0 > 0$ *and* $y_0 > 0$ *for all* $t \in [0, T]$ *s.t.*

$$-V'(t, y) < \left(\frac{\gamma_1}{1-\gamma_1}\right)\frac{V(t, y)}{y} \qquad \text{for } 0 < y \le y_0;$$
$$-U'(t, x) > \left(\frac{\gamma_2}{1-\gamma_2}\right)\frac{U(t, x)}{x} \qquad \text{for } 0 < x \le x_0.$$

3. A new characterization of financeable consumption processes.

3.1. *Functional set up.* In the spirit of Bouchard and Pham [1] who treated the wealth dependent problem (see also Žitković [30] on consumption and endowment with stochastic clock), we denote $\mathcal{O}$ as σ-algebra of optional sets relative to the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, and let $d\overline{\mathbb{P}} = dt \times d\mathbb{P}$ be the measure on the product space $(\Omega \times [0, T], \mathcal{O})$ defined as

$$\overline{\mathbb{P}}[A] = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \mathbf{1}_A(t, \omega)\, dt\right] \qquad \text{for } A \in \mathcal{O}. \tag{3.1}$$

We denote by $\mathbb{L}^0(\Omega \times [0, T], \mathcal{O}, \overline{\mathbb{P}})$ ($\mathbb{L}^0$ for short) the set of all random variables on the product space $\Omega \times [0, T]$ with respect to the optional σ-algebra $\mathcal{O}$ endowed with the topology of convergence in measure $\overline{\mathbb{P}}$. And from now on, we shall identify the optional stochastic process $(Y_t)_{t\in[0,T]}$ with the random variable $Y \in \mathbb{L}^0(\Omega \times [0, T], \mathcal{O}, \overline{\mathbb{P}})$. We also define the positive orthant $\mathbb{L}^0_+(\Omega \times [0, T], \mathcal{O}, \overline{\mathbb{P}})$ ($\mathbb{L}^0_+$ for short) as the set of $Y = Y(t, \omega) \in \mathbb{L}^0$ such that

$$Y \ge 0, \qquad \overline{\mathbb{P}}\text{-a.s.}$$

Endow $\mathbb{L}^0_+$ with the bilinear form valued in $[0, \infty]$ as

$$\langle X, Y\rangle = \mathbb{E}\left[\int_0^T X_t Y_t\, dt\right] \qquad \text{for all } X, Y \in \mathbb{L}^0_+.$$

3.2. *Path-dependence reduction by auxiliary processes.* At this point, we are able to define the set of all (x, z)-financeable consumption rate processes as a set of random variables on the product space $(\Omega \times [0, T], \mathcal{O}, \overline{\mathbb{P}})$, and Proposition 2.1 states that

$$\mathcal{A}(x, z) = \{c \in \mathbb{L}^0_+ : c_t \ge Z(c)_t,\ \forall t \in [0, T] \text{ and } \langle c, Y\rangle \le x,\ \forall Y \in \mathcal{M}\}. \tag{3.2}$$

However, the family $\mathcal{A}(x,z)$ may be empty for some values $x>0$, $z\geq 0$. We shall restrict ourselves to the *effective domain* $\overline{\mathcal{H}}$ which is defined as the union of the *interior* of set such that $\mathcal{A}(x,z)$ is not empty, and the boundary $\{x>0, z=0\}$

$$\overline{\mathcal{H}} \triangleq \int \{(x,z)\in(0,\infty)\times[0,\infty):\mathcal{A}(x,z)\neq\varnothing\} \cup (0,\infty)\times\{0\}. \tag{3.3}$$

From the definition, $\overline{\mathcal{H}}$ includes the special case of zero initial habit, that is, $z=0$.

Before we state the next result, we shall first impose some additional conditions on the discounting factors α_t and δ_t, which are essential for the well-posedness of the primal utility optimization problem:

ASSUMPTION 3.1. We assume that nonnegative optional processes α_t and δ_t satisfy

$$\sup_{Y\in\mathcal{M}} \mathbb{E}\left[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)\,dv} Y_t\,dt\right]<\infty, \tag{3.4}$$

and there exists a constant $\bar{x}>0$ such that

$$\mathbb{E}\left[\int_0^T U^-\big(t,\bar{x}e^{-\int_0^t\alpha_v\,dv}\big)\,dt\right]<\infty. \tag{3.5}$$

REMARK 3. If stochastic discounting processes α_t and δ_t are assumed to be bounded, conditions (3.4) and (3.5) will be satisfied. Condition (3.4) is the well-known super-hedging property of the random variable $\int_0^T e^{\int_0^t(\delta_v-\alpha_v)\,dv}\,dt$ in the original market.

LEMMA 3.1. *Under condition* (3.4), *the effective domain $\overline{\mathcal{H}}$ can be rewritten explicitly as*

$$\overline{\mathcal{H}}=\left\{(x,z)\in(0,\infty)\times[0,\infty):x>z\sup_{Y\in\mathcal{M}}\mathbb{E}\left[\int_0^T e^{\int_0^t(\delta_v-\alpha_v)\,dv}Y_t\,dt\right]\right\}. \tag{3.6}$$

The proof of the lemma is straightforward, and we refer to Yu [28] for details.

By choosing $(x,z)\in\overline{\mathcal{H}}$, we can now define the preliminary version of our *primal utility maximization problem* as

$$u(x,z)\triangleq\sup_{c\in\mathcal{A}(x,z)}\mathbb{E}\left[\int_0^T U\big(t,c_t-Z(c)_t\big)\,dt\right],\qquad (x,z)\in\overline{\mathcal{H}}. \tag{3.7}$$

Now, for fixed $(x,z)\in\overline{\mathcal{H}}$, and each (x,z)-financeable consumption rate process, we want to generalize the Market Isomorphism idea by Schroder and Skiadas [27] in order to reduce the path-dependence structure. By introducing the auxiliary process $\tilde{c}_t=c_t-Z(c)_t$, the auxiliary set of $\mathcal{A}(x,z)$ is given as

$$\overline{\mathcal{A}}(x,z)\triangleq\big\{\tilde{c}\in\mathbb{L}_+^0:\tilde{c}_t=c_t-Z(c)_t,\ \forall t\in[0,T],c\in\mathcal{A}(x,z)\big\}. \tag{3.8}$$

It is straightforward to verify the following lemma.

LEMMA 3.2. *For each fixed $(x, z) \in \overline{\mathcal{H}}$, there is one-to-one correspondence between sets $\mathcal{A}(x, z)$ and $\overline{\mathcal{A}}(x, z)$, and hence we have $\overline{\mathcal{A}}(x, z) \neq \varnothing$ for $(x, z) \in \overline{\mathcal{H}}$.*

Let us turn our attention to the set $\mathcal{M}$ of equivalent local martingale measure densities, and for each $Y \in \mathcal{M}$, the auxiliary optional process with respect to Y_t is defined as

$$\Gamma_t \triangleq Y_t + \delta_t \mathbb{E}\left[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)\, dv} Y_s \, ds \Big| \mathcal{F}_t\right] \qquad \forall t \in [0, T]. \tag{3.9}$$

Let us denote the set of all these auxiliary optional processes by

$$\begin{aligned} \widetilde{\mathcal{M}} = \biggl\{ \Gamma \in \mathbb{L}^0_+ : \Gamma_t = Y_t + \delta_t \mathbb{E}\left[\int_t^T e^{\int_t^s (\delta_v - \alpha_v)\, dv} Y_s \, ds \Big| \mathcal{F}_t\right], \\ \forall t \in [0, T], Y \in \mathcal{M} \biggr\}. \end{aligned} \tag{3.10}$$

We remark here that although stochastic discounting processes δ_t and α_t are unbounded, under condition (3.4), the auxiliary dual process Γ is well defined in $\mathbb{L}^0_+$, but it is not necessarily in $\mathbb{L}^1$.

A direct application of the Fubini–Tonelli theorem induces the key equalities below; for the detailed proof, we refer to Proposition 2.3.3 of Yu [28].

PROPOSITION 3.1. *Under condition (3.4), for each nonnegative optional process c_t such that $c_t \geq Z(c)_t$ with $Z(c)_t$ defined by (2.4) for fixed initial standard of living $z \geq 0$ and the nonnegative optional process Y_t, we have the following equalities with respect to their corresponding auxiliary processes $\tilde{c}_t = c_t - Z(c)_t$ and Γ_t which is defined by (3.9), that*

$$\begin{aligned} \langle c, Y \rangle &= \langle \tilde{c}, \Gamma \rangle + z \langle w, Y \rangle \\ &= \langle \tilde{c}, \Gamma \rangle + z \langle \tilde{w}, \Gamma \rangle. \end{aligned} \tag{3.11}$$

Here we define random processes $w, \tilde{w} \in \mathbb{L}^0_+$ by

$$w_t \triangleq e^{\int_0^t (\delta_v - \alpha_v)\, dv} \quad \textit{and} \quad \tilde{w}_t \triangleq e^{\int_0^t (-\alpha_v)\, dv} \qquad \textit{for all } t \in [0, T]. \tag{3.12}$$

Based on Propositions 2.1 and 3.1, under conditions (3.4) and (3.5), clearly we will have the alternative budget constraint characterization of the consumption rate process c_t as:

PROPOSITION 3.2. *For any given pair $(x, z) \in \overline{\mathcal{H}}$, the consumption rate process c is (x, z)-financeable if and only if $c_t \geq Z(c)_t$, $\forall t \in [0, T]$ and*

$$\langle c - Z(c), \Gamma \rangle \leq x - z \langle \tilde{w}, \Gamma \rangle \qquad \textit{for all } \Gamma \in \widetilde{\mathcal{M}}.$$

Proposition 3.2 provides us the alternative definition of set $\overline{\mathcal{A}}(x, z)$ for $(x, z) \in \overline{\mathcal{H}}$ by

$$\overline{\mathcal{A}}(x, z) = \{\tilde{c} \in \mathbb{L}^0_+ : \langle \tilde{c}, \Gamma \rangle \leq x - z \langle \tilde{w}, \Gamma \rangle,\ \forall \Gamma \in \widetilde{\mathcal{M}}\}. \tag{3.13}$$

3.3. *Embedding into an abstract utility maximization problem with the shadow random endowment*. In order to apply the convex duality approach for the random endowment, we need to enlarge the domain of the set $\overline{\mathcal{H}}$ to $\mathcal{H}$ and also enlarge the corresponding auxiliary set $\overline{\mathcal{A}}(x,z)$ to $\widetilde{\mathcal{A}}(x,z)$,

$$\widetilde{\mathcal{A}}(x,z) \triangleq \{\tilde{c} \in \mathbb{L}_+^0 : \langle \tilde{c}, \Gamma\rangle \le x - z\langle \tilde{w}, \Gamma\rangle,\ \forall \Gamma \in \widetilde{\mathcal{M}}\}, \tag{3.14}$$

where now $(x,z) \in \mathbb{R}^2$ and is restricted in the enlarged domain $\mathcal{H}$,

$$\mathcal{H} \triangleq \operatorname{int}\{(x,z) \in \mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z) \neq \varnothing\}.$$

Under condition (3.4) and Proposition 3.1, it is easy to verify the equivalent characterization of $\mathcal{H}$ by the following:

LEMMA 3.3.

$$\begin{aligned} \mathcal{H} &= \{(x,z) \in \mathbb{R}^2 : x > z\langle \tilde{w}, \Gamma\rangle, \textit{ for all } \Gamma \in \widetilde{\mathcal{M}}\} \\ &= \{(x,z) \in \mathbb{R}^2 : x > \bar{p}z, z \ge 0\} \cup \{(x,z) \in \mathbb{R}^2 : x > \underline{p}z, z < 0\}. \end{aligned} \tag{3.15}$$

Here

$$\bar{p} \triangleq \sup_{Y \in \mathcal{M}} \langle w, Y\rangle = \sup_{\Gamma \in \widetilde{\mathcal{M}}} \langle \tilde{w}, \Gamma\rangle, \tag{3.16}$$

and

$$\underline{p} \triangleq \inf_{Y \in \mathcal{M}} \langle w, Y\rangle = \inf_{\Gamma \in \widetilde{\mathcal{M}}} \langle \tilde{w}, \Gamma\rangle, \tag{3.17}$$

where $\bar{p}, \underline{p} < \infty$, *and* $\mathcal{H}$ *is a well-defined convex cone in* $\mathbb{R}^2$. *Moreover,*

$$\begin{aligned} \operatorname{cl}\mathcal{H} &= \{(x,z) \in \mathbb{R}^2 : \widetilde{\mathcal{A}}(x,z) \neq \varnothing\} \\ &= \{(x,z) \in \mathbb{R}^2 : x \ge z\langle \tilde{w}, \Gamma\rangle, \textit{ for all } \Gamma \in \widetilde{\mathcal{M}}\}, \end{aligned} \tag{3.18}$$

where $\operatorname{cl}\mathcal{H}$ *denotes the closure of the set* $\mathcal{H}$ *in* $\mathbb{R}^2$.

We will now define the *auxiliary primal utility maximization problem* based on the auxiliary domain $\widetilde{\mathcal{A}}(x,z)$ as

$$\tilde{u}(x,z) \triangleq \sup_{\tilde{c} \in \widetilde{\mathcal{A}}(x,z)} \mathbb{E}\left[\int_0^T U(t, \tilde{c}_t)\,dt\right], \qquad (x,z) \in \mathcal{H}. \tag{3.19}$$

By definitions of $\overline{\mathcal{A}}(x,z)$ for $(x,z) \in \overline{\mathcal{H}}$ and $\widetilde{\mathcal{A}}(x,z)$ for $(x,z) \in \mathcal{H}$, we successfully embedded our original utility maximization problem (3.7) with consumption habit formation into the auxiliary abstract utility maximization problem (3.19) without habit formation, but with the shadow random endowment. More precisely, the following equivalence can be guaranteed that for any $(x,z) \in \overline{\mathcal{H}} \subset \mathcal{H}$,

$$\overline{\mathcal{A}}(x,z) = \widetilde{\mathcal{A}}(x,z), \tag{3.20}$$

and the two value functions coincide,

$$u(x,z) = \tilde{u}(x,z). \tag{3.21}$$

In addition, we have that c_t^* is the optimal solution for $u(x,z)$ if and only if $\tilde{c}_t^* = c_t^* - Z(c^*)_t \geq 0$, and for all $t \in [0,T]$ is the optimal solution for $\tilde{u}(x,z)$, when $(x,z) \in \overline{\mathcal{H}}$.

4. The dual optimization problem and main results. Inspired by the idea in Hugonnier and Kramkov [14] for optimal investment with random endowment, we concentrate now on the construction of the dual problem by first introducing the set $\mathcal{R}$,

$$\mathcal{R} \triangleq ri\{(y,r) \in \mathbb{R}^2 : xy - zr \geq 0, \text{ for all } (x,z) \in \mathcal{H}\}. \tag{4.1}$$

Let us make the following assumption on stochastic discounting processes α_t and δ_t.

ASSUMPTION 4.1. The random variable defined by

$$\mathcal{E} \triangleq \int_0^T w_t\,dt = \int_0^T e^{\int_0^t (\delta_v - \alpha_v)\,dv}\,dt \tag{4.2}$$

is not replicable under our original financial market; that is, there is no constant K such that

$$\mathbb{E}^{\mathbb{Q}}[\mathcal{E}] = K \qquad \text{for any } \mathbb{Q} \in \mathcal{M}.$$

REMARK 4. Under Assumption 4.1, the existence of market isomorphism by Schroder and Skiadas [27] may no longer hold. Our work can generally extend their conclusions and provide the existence and uniqueness of the optimal solution in incomplete markets using convex analysis.

LEMMA 4.1. *Under Assumption 4.1, we know that $\mathcal{R}$ is an open convex cone in $\mathbb{R}^2$ and can be rewritten as*

$$\mathcal{R} = \{(y,r) \in \mathbb{R}^2 : y > 0 \text{ and } \underline{p}y < r < \bar{p}y\}, \tag{4.3}$$

where $\bar{p}$ and $\underline{p}$ are defined by (3.16) *and* (3.17), *and $\bar{p} < \underline{p}$.*

For an arbitrary pair $(y,r) \in \mathcal{R}$, we denote by $\widetilde{\mathcal{Y}}(y,r)$ the set of nonnegative processes as a proper extension of the auxiliary set $\widetilde{\mathcal{M}}$ in the way that

$$\widetilde{\mathcal{Y}}(y,r) \triangleq \{\Gamma \in \mathbb{L}_+^0 : \langle \tilde{c}, \Gamma \rangle \leq xy - zr, \text{ for all } \tilde{c} \in \widetilde{\mathcal{A}}(x,z) \text{ and } (x,z) \in \mathcal{H}\}. \tag{4.4}$$

The *auxiliary dual utility maximization problem* to (3.19) can be now defined by

$$\tilde{v}(y,r) \triangleq \inf_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T V(t,\Gamma_t)\,dt\right], \qquad (y,r) \in \mathcal{R}. \tag{4.5}$$

The following theorems constitute our main results, and we provide their proofs through a number of auxiliary results in the next section.

THEOREM 4.1. *Given Assumptions 3.1 and 4.1, assume also that conditions* (2.5), (2.7), (2.9), (*i.e.*, $\mathrm{AE}_0[U] < \infty$), (2.10) *and* (2.11) *hold true. Moreover, assume that*

$$\tilde{u}(x,z) < \infty \qquad \text{for some } (x,z) \in \mathcal{H}, \tag{4.6}$$

then we have:

(i) *The function $\tilde{u}$ is $(-\infty,\infty)$-valued on $\mathcal{H}$ and $\tilde{v}(y,r)$ is $(-\infty,\infty]$-valued on $\mathcal{R}$. For each $(y,r) \in \mathcal{R}$ there exists a constant $s = s(y,r) > 0$ such that $\tilde{v}(sy,sr) < \infty$, and the conjugate duality of value functions $\tilde{u}$ and $\tilde{v}$ holds*

$$\tilde{u}(x,z) = \inf_{(y,r)\in\mathcal{R}} \{\tilde{v}(y,r) + xy - zr\}, \qquad (x,z) \in \mathcal{H},$$

$$\tilde{v}(y,r) = \sup_{(x,z)\in\mathcal{H}} \{\tilde{u}(x,z) - xy + zr\}, \qquad (y,r) \in \mathcal{R}.$$

(ii) *The solution $\Gamma^*(y,r)$ to the optimization problem* (4.5) *exists and is unique* (*in the sense of $=$ under $\overline{\mathbb{P}}$ in $\mathbb{L}^0_+$*) *for all $(y,r) \in \mathcal{R}$ such that $\tilde{v}(y,r) < \infty$.*

THEOREM 4.2. *In addition to assumptions of Theorem 4.1, we also assume that condition* (2.8) *holds*, (*i.e.*, $\mathrm{AE}_\infty[U] < 1$). *Then we also have*:

(i) *The value function $\tilde{v}(y,r)$ is $(-\infty,\infty)$-valued on $(y,r) \in \mathcal{R}$, and $\tilde{v}$ is continuously differentiable on $\mathcal{L}$.*

(ii) *The solution $\tilde{c}^*(x,z)$ to optimization problem* (3.19) *exists and is unique* (*in the sense of $=$ under $\overline{\mathbb{P}}$ in $\mathbb{L}^0_+$*) *for any $(x,z) \in \mathcal{H}$, and there exists a representation of the optimal solution such that $\tilde{c}^*_t(x,z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$.*

(iii) *The superdifferential of $\tilde{u}$ maps $\mathcal{H}$ into $\mathcal{R}$, that is,*

$$\partial\tilde{u}(x,z) \subset \mathcal{R}, \qquad (x,z) \in \mathcal{H}. \tag{4.7}$$

Moreover, if $(y,r) \in \partial\tilde{u}(x,z)$, then there exists a representation of the optimal solution such that $\Gamma^*_t(y,r) > 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$ and $\tilde{c}^*(x,z)$ and $\Gamma^*(y,r)$ are related by

$$\begin{aligned} &\Gamma^*_t(y,r) = U'\bigl(t, \tilde{c}^*_t(x,z)\bigr) \quad \text{or} \\ &\tilde{c}^*_t(x,z) = I\bigl(t, \Gamma^*_t(y,r)\bigr), \qquad t \in [0,T], \\ &\bigl\langle \Gamma^*(y,r), \tilde{c}^*(x,z)\bigr\rangle = xy - zr. \end{aligned} \tag{4.8}$$

(iv) *If we restrict the choice of initial wealth x and initial standard of living z such that $(x,z) \in \overline{\mathcal{H}} \subset \mathcal{H}$, the solution $c^*_t(x,z)$ to our primal utility optimization problem* (3.7) *exists and is unique, moreover,*

$$\tilde{c}^*_t(x,z) = c^*_t(x,z) - Z(c^*)_t(x,z), \qquad t \in [0,T]. \tag{4.9}$$

5. Proofs of main results.

5.1. *The proof of Theorem 4.1.* The following proposition will serve as the key step to build Bipolar relationships:

PROPOSITION 5.1. *Assume that all conditions of Theorem 4.1 hold true. The families* $(\widetilde{\mathcal{A}}(x,z))_{(x,z)\in\mathcal{H}}$ *and* $(\widetilde{\mathcal{Y}}(y,r))_{(y,r)\in\mathcal{R}}$ *have the following properties*:

(i) *For any* $(x,z)\in\mathcal{H}$, *the set* $\widetilde{\mathcal{A}}(x,z)$ *contains a strictly positive random variable on the product space. A nonnegative random variable* $\tilde{c}$ *belongs to* $\widetilde{\mathcal{A}}(x,z)$ *if and only if*

$$\langle \tilde{c}, \Gamma\rangle \le xy - zr \qquad \text{for all } (y,r)\in\mathcal{R} \text{ and } \Gamma\in\widetilde{\mathcal{Y}}(y,r). \tag{5.1}$$

(ii) *For any* $(y,r)\in\mathcal{R}$, *the set* $\widetilde{\mathcal{Y}}(y,r)$ *contains a strictly positive random variable on the product space. A nonnegative random variable* Γ *belongs to* $\widetilde{\mathcal{Y}}(y,r)$ *if and only if*

$$\langle \tilde{c}, \Gamma\rangle \le xy - zr \qquad \text{for all } (x,z)\in\mathcal{H} \text{ and } \tilde{c}\in\widetilde{\mathcal{A}}(x,z). \tag{5.2}$$

In order to prove Proposition 5.1, for any $p>0$, we denote by $\mathcal{M}(p)$ the subset of $\mathcal{M}$ that consists of densities $Y\in\mathcal{M}$ such that $\langle w, Y\rangle = p$. For any density process $Y\in\mathcal{M}(p)$, define the auxiliary set as

$$\begin{aligned}\widetilde{\mathcal{M}}(p) \triangleq \biggl\{\Gamma\in\mathbb{L}^0_+ : \Gamma_t = Y_t + \delta_t\mathbb{E}\biggl[\int_t^T e^{\int_t^s(\delta_v-\alpha_v)\,dv} Y_s\,ds\Bigm|\mathcal{F}_t\biggr],\\ \forall t\in[0,T], Y\in\mathcal{M}(p)\biggr\}.\end{aligned} \tag{5.3}$$

We have $\langle \tilde{w}, \Gamma\rangle = \langle w, Y\rangle = p$.

Define $\mathcal{P}$ as the open interval $\mathcal{P} = (\underline{p}, \bar{p})$, where $\underline{p}, \bar{p}$ are given in (3.16) and (3.17). We have the following result.

LEMMA 5.1. *Assume that conditions of Proposition 5.1 hold true, and let* $p>0$. *Then the set* $\widetilde{\mathcal{M}}(p)$ *is not empty if and only if* $p\in\mathcal{P}=(\underline{p},\bar{p})$, *where* $\underline{p}, \bar{p}$ *are defined in* (3.16) *and* (3.17). *In particular,*

$$\bigcup_{p\in\mathcal{P}} \widetilde{\mathcal{M}}(p) = \widetilde{\mathcal{M}}, \tag{5.4}$$

where the set $\widetilde{\mathcal{M}}$ *is defined by* (3.10).

PROOF. The proof reduces to verifying that $\mathcal{P}=\mathcal{P}'$, where

$$\mathcal{P}' \triangleq \{p>0 : \widetilde{\mathcal{M}}(p)\neq\varnothing\}.$$

Similar to the proof of Lemma 8 of Hugonnier and Kramkov [14], one direction inclusion that $\mathcal{P}\subseteq\mathcal{P}'$ is obvious.

For the inverse, let $p \in \mathcal{P}'$, $(x, z) \in \operatorname{cl}\mathcal{H}$, $\Gamma \in \widetilde{\mathcal{M}}(p)$, and we claim that there exists a $\tilde{c} \in \widetilde{\mathcal{A}}(x, z)$ such that

$$\overline{\mathbb{P}}[\tilde{c} > 0] > 0,$$

which implies that

$$0 < \langle \tilde{c}, \Gamma \rangle \leq x - zp.$$

As (x, z) is an arbitrary element of $\operatorname{cl}\mathcal{H}$, it follows that $p \in \mathcal{P}$.

As for the above claim, according to Theorem 2.11 of Schachermayer [26], condition (4.1) guarantees that for all $Y \in \mathcal{M}$,

$$\underline{p} < \langle w, Y \rangle < \bar{p},$$

which is

$$\underline{p} < \langle \tilde{w}, \Gamma \rangle < \bar{p},$$

for all the $\Gamma \in \widetilde{\mathcal{M}}$. By the definition of $\operatorname{cl}\mathcal{H}$ in Lemma 3.3, for any $(x, z) \in \operatorname{cl}\mathcal{H}$, we have

$$x - z\langle \tilde{w}, \Gamma \rangle > 0,$$

for all $\Gamma \in \widetilde{\mathcal{M}}$, and the claim holds by the definition of $\widetilde{\mathcal{A}}(x, z)$. $\square$

LEMMA 5.2. *Assume that the conditions of Proposition 5.1 hold true, let $p \in \mathcal{P} = (\underline{p}, \bar{p})$ and we have $\widetilde{\mathcal{M}}(p) \subseteq \widetilde{\mathcal{Y}}(1, p)$.*

PROOF. The conclusion can be directly derived in light of definitions of sets $\widetilde{\mathcal{A}}(x, z)$ and $\widetilde{\mathcal{Y}}(1, p)$. $\square$

According to the definition of $\widetilde{A}(x, z)$, similar to the proof of Lemma 10 of Hugonnier and Kramkov [14], it is straightforward to show the following result:

LEMMA 5.3. *Assume that conditions of Proposition 5.1 hold true. For any $(x, z) \in \mathcal{H}$, a nonnegative random variable $\tilde{c}$ belongs to $\widetilde{\mathcal{A}}(x, z)$ if and only if*

$$\langle \tilde{c}, \Gamma \rangle \leq x - zp \qquad \text{for all } p \in \mathcal{P} \text{ and } \Gamma \in \widetilde{\mathcal{M}}(p). \tag{5.5}$$

PROOF OF PROPOSITION 5.1. For the validity of assertion (i), given any $(x, z) \in \mathcal{H}$, there exists a $\lambda > 0$ such that $(x - \lambda, z) \in \mathcal{H}$ since $\mathcal{H}$ is an open set.

Let $\tilde{c} \in \widetilde{\mathcal{A}}(x - \lambda, z)$, for any $\Gamma \in \widetilde{\mathcal{M}}$, and $\tilde{w}_t = e^{-\int_0^t \alpha_v \, dv} > 0$ for all $t \in [0, T]$, we have

$$\langle \tilde{c}, \Gamma \rangle \leq x - \lambda - z\langle \tilde{w}, \Gamma \rangle. \tag{5.6}$$

By condition (3.4) and Proposition 3.1, we define $\rho_t \triangleq \frac{\lambda}{\bar{p}}\tilde{w}_t > 0$ for all $t \in [0,T]$, and then for all $\Gamma \in \widetilde{\mathcal{M}}$, we obtain

$$\begin{aligned}\langle \rho, \Gamma\rangle &\le \langle \tilde{c} + \rho, \Gamma\rangle \le x - \lambda - z\langle \tilde{w}, \Gamma\rangle + \frac{\lambda}{\bar{p}}\langle \tilde{w}, \Gamma\rangle \\ &\le x - \lambda - z\langle \tilde{w}, \Gamma\rangle + \lambda \le x - z\langle \tilde{w}, \Gamma\rangle.\end{aligned}$$

Hence, the existence of a strictly positive element $\rho_t \in \widetilde{\mathcal{A}}(x,z)$ follows by the definition of $\widetilde{\mathcal{A}}(x,z)$.

If (5.1) holds for some $\tilde{c} \in \mathbb{L}^0_+$, the density process $\Gamma \in \widetilde{\mathcal{M}}(p)$ belongs to $\widetilde{\mathcal{Y}}(1,p)$ for all $p \in \mathcal{P}$ by Lemma 5.2, and hence (5.5) holds. Lemma 5.3 then implies that $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$. Conversely, suppose $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$, the definition of set $\widetilde{\mathcal{Y}}(y,r)$, $(y,r) \in \mathcal{R}$ implies (5.1), and we complete the proof of assertion (i).

For the proof of the assertion (ii), first we have

$$k\widetilde{\mathcal{Y}}(y,r) = \widetilde{\mathcal{Y}}(ky,kr) \qquad \text{for all } k > 0, (y,r) \in \mathcal{R}.$$

Therefore, it is enough to consider $(y,r) = (1,p)$ for some $p \in \mathcal{P}$. Lemma 5.2 implies $\Gamma \in \widetilde{\mathcal{M}}(p) \subseteq \widetilde{\mathcal{Y}}(1,p)$, and the existence of strictly positive $Y \in \mathcal{M}(p)$ takes care of the existence $\Gamma \in \widetilde{\mathcal{M}}(p)$ and $\Gamma > 0$ $\overline{\mathbb{P}}$-a.s.

The second part is a direct consequence of the definition of $\widetilde{\mathcal{Y}}(y,r)$. □

For the proof of Theorem 4.1, we will also need the following lemmas:

LEMMA 5.4. *Under assumptions of Theorem 4.1, the value function $\tilde{u}$ is $(-\infty,\infty)$-valued on $\mathcal{H}$.*

PROOF. First, by Lemma 2.1, the condition $\mathrm{AE}_0[U] < \infty$ implies that for any positive constant $s > 0$, there exist $s_1 > 0$ and $s_2 > 0$ such that for all $t \in [0,T]$,

$$U(t, x/s) \ge s_1 U(t,x) + s_2, \qquad x > 0. \tag{5.7}$$

According to condition (3.5) and the proof of Proposition 5.1, for each fixed pair $(x,z) \in \mathcal{H}$, there exists $\lambda = \lambda(x,z) > 0$ such that $\frac{\lambda}{\bar{p}}\tilde{w}_t \in \widetilde{\mathcal{A}}(x,z)$, and therefore we deduce that $\bar{x}\tilde{w}_t \in \widetilde{\mathcal{A}}(\frac{\bar{x}\bar{p}}{\lambda}x, \frac{\bar{x}\bar{p}}{\lambda}z)$, and

$$\begin{aligned}\tilde{u}\left(\frac{\bar{x}\bar{p}}{\lambda}x, \frac{\bar{x}\bar{p}}{\lambda}z\right) &= \sup_{\tilde{c} \in \widetilde{\mathcal{A}}(((\bar{x}\bar{p})/\lambda)x, ((\bar{x}\bar{p})/\lambda)z)} \mathbb{E}\left[\int_0^T U(t,\tilde{c}_t)\,dt\right] \\ &\ge \mathbb{E}\left[\int_0^T U(t,\bar{x}\tilde{w}_t)\,dt\right] > -\infty.\end{aligned}$$

Hence, for any $(x,z) \in \mathcal{H}$, there exists a constant $s(x,z) > 0$ such that $\tilde{u}(sx,sz) > -\infty$, with $s(x,z) = \frac{\bar{x}\bar{p}}{\lambda}$.

For any constant $s > 0$,

$$\widetilde{\mathcal{A}}(x,z) = \widetilde{\mathcal{A}}(sx,sz)/s,$$

which implies that $\tilde{u}(x,z) > -\infty$ if $\tilde{u}(sx, sz) > -\infty$ holds for a constant $s = s(x,z) > 0$. By the result above, we can conclude that $\tilde{u}(x,z) > -\infty$ in the whole domain $\mathcal{H}$.

Now, since the set $\mathcal{H}$ is open, and $\tilde{u}(x,z) < \infty$ for some $(x,z) \in \mathcal{H}$ by the condition (4.6), we deduce that $\tilde{u}$ is finitely valued on $\mathcal{H}$ by the concavity of $\tilde{u}$ on $\mathcal{H}$. And the proof is complete. $\square$

Before we state the next lemma, let us introduce the definition given by Žitković [31].

DEFINITION 5.1. A convex subset C of a topological vector space X is said to be *convexly compact* if for any nonempty set A and any family $\{F_a\}_{a\in A}$ of closed, convex subsets of C, the condition

$$\forall D \in \mathrm{Fin}(A), \qquad \bigcap_{a\in D} F_a \neq \varnothing \quad \Longrightarrow \quad \bigcap_{a\in A} F_a \neq \varnothing,$$

where the set $\mathrm{Fin}(A)$ consists of all nonempty finite subsets of A for an arbitrary nonempty set A.

Žitković [31] furthermore derived an easy characterization on the space of nonnegative, measurable functions; see Theorem 3.1 of Žitković [31]. We modify his result to fit into our framework.

PROPOSITION 5.2. *A closed and convex subset C of $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \overline{\mathbb{P}})$ is convexly compact if and only if it is bounded in the finite measure $\overline{\mathbb{P}}$.*

Based on the above proposition, we have the following lemma on the convexly compactness of sets $\widetilde{\mathcal{A}}(x,z)$ and $\widetilde{\mathcal{Y}}(y,r)$:

LEMMA 5.5. *For each pair $(x,z) \in \mathcal{H}$ and $(y,r) \in \mathcal{R}$, the sets $\widetilde{\mathcal{A}}(x,z)$ and $\widetilde{\mathcal{Y}}(y,r)$ are convex, solid and closed in the topology of convergence in measure $\overline{\mathbb{P}}$. Moreover, they are both bounded in $\mathbb{L}^0_+(\Omega \times [0,T], \mathcal{O}, \overline{\mathbb{P}})$; hence they are both convexly compact.*

PROOF. For $(y,r) \in \mathcal{R}$, we define auxiliary sets as

$$\begin{aligned} \mathfrak{H}(y,r) &\triangleq \{(x,z) \in \mathcal{H} : xy - zr \leq 1\}, \\ \mathfrak{A}(k) &\triangleq \bigcup_{(x,z)\in k\mathfrak{H}(y,r)} \widetilde{\mathcal{A}}(x,z), \end{aligned} \tag{5.8}$$

and denote by $\widetilde{\mathfrak{A}}(k)$ the closure of $\mathfrak{A}(k)$ with respect to convergence in measure $\overline{\mathbb{P}}$.

From Proposition 5.1, it follows that

$$\Gamma \in \widetilde{\mathcal{Y}}(y,r) \quad \Longleftrightarrow \quad \langle \tilde{c}, \Gamma\rangle \leq 1 \qquad \forall \tilde{c} \in \widetilde{\mathfrak{A}}(1).$$

Hence, sets $\widetilde{\mathcal{Y}}(y,r)$ and $\widetilde{\mathfrak{A}}(1)$ satisfy

$$\widetilde{\mathcal{Y}}(y,r) = \widetilde{\mathfrak{A}}(1)^{\circ}.$$

At the same time, by its definition, $\widetilde{\mathfrak{A}}(1)$ itself is closed, convex and solid. The bipolar theorem in Brannath and Schachermayer [2] asserts that $\widetilde{\mathfrak{A}}(1) = \widetilde{\mathfrak{A}}(1)^{\circ\circ}$, and hence we have the following Bipolar relationship:

$$\begin{aligned} \widetilde{\mathfrak{A}}(1) &= \widetilde{\mathcal{Y}}(y,r)^{\circ}, \\ \widetilde{\mathcal{Y}}(y,r) &= \widetilde{\mathfrak{A}}(1)^{\circ}. \end{aligned} \tag{5.9}$$

The Bipolar theorem on $\mathbb{L}^0_+$ implies that $\mathcal{Y}(y,r)$ is convex, solid and closed under the convergence in measure $\overline{\mathbb{P}}$.

Similarly, for $(x,z) \in \mathcal{H}$, we define the set

$$\begin{aligned} \mathfrak{R}(x,z) &\triangleq \{(y,r) \in \mathcal{R} : xy - zr \le 1\}, \\ \mathfrak{Y}(k) &\triangleq \bigcup_{(y,r)\in k\mathfrak{R}(x,z)} \widetilde{\mathcal{Y}}(y,r), \end{aligned} \tag{5.10}$$

and denote by $\widetilde{\mathfrak{Y}}(k)$ the closure of $\mathfrak{Y}(k)$ with respect to convergence in measure $\overline{\mathbb{P}}$.

Again, Proposition 5.1 implies

$$\tilde{c} \in \widetilde{\mathcal{A}}(x,z) \quad \Longleftrightarrow \quad \langle \tilde{c}, \Gamma \rangle \le 1 \qquad \forall \Gamma \in \widetilde{\mathfrak{Y}},$$

and the Bipolar relationship

$$\begin{aligned} \widetilde{\mathfrak{Y}}(1) &= \widetilde{\mathcal{A}}(x,z)^{\circ}, \\ \widetilde{\mathcal{A}}(x,z) &= \widetilde{\mathfrak{Y}}(1)^{\circ}. \end{aligned} \tag{5.11}$$

Hence, $\widetilde{\mathcal{A}}(x,z)$ is also convex, solid and closed under convergence in measure $\overline{\mathbb{P}}$.

Thanks to the existence of strictly positive $\Gamma \in \widetilde{\mathcal{M}}(p)$ which is also in $\widetilde{\mathcal{Y}}(1,p)$, the set $\widetilde{\mathcal{A}}(x,z)$ is therefore bounded in measure $\overline{\mathbb{P}}$ by Proposition 5.1 part (i).

Similarly, as in the proof of Proposition 5.1, there exists $\lambda = \lambda(x,z)$ such that $\rho_t > 0$ for all $t \in [0,T]$ and $\rho_t = \frac{\lambda}{p}\tilde{w}_t \in \widetilde{\mathcal{A}}(x,z)$. Due to Proposition 5.1 part (ii), the set $\widetilde{\mathcal{Y}}(y,r)$ is also bounded in measure $\overline{\mathbb{P}}$. Therefore both of them are convexly compact in $\mathbb{L}^0_+$. $\square$

Contrary to the existing literature, we cannot mimic the classical proof of the existence of the dual optimizer due to the lack of integrability of the dual process $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$ for $(y,r) \in \mathcal{R}$. Conditions $\mathrm{AE}_0[U] < \infty$ and $\mathbb{E}[\int_0^T U(t, \bar{x}\tilde{w}_t)\,dt] > -\infty$ are critical to prove lemmas below.

LEMMA 5.6. *Under assumptions of Theorem 4.1, for each fixed $(y,r)\in\mathcal{R}$, we have*

$$\sup_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\left[\int_0^T V^-(t,\Gamma_t)\,dt\right]<\infty.$$

PROOF. Condition (3.5) admits the existence of $\bar{x}\tilde{w}_t\in\mathbb{L}_+^0$ such that $\mathbb{E}[\int_0^T U(t,\bar{x}\tilde{w}_t)\,dt]>-\infty$. Moreover, by the proof of Proposition 5.1, for each fixed $(y,r)\in\mathcal{R}$, we can find a pair $(x,z)\in\mathfrak{H}(y,r)$ and there exists a constant $\lambda(x,z)>0$ such that $\tilde{w}\in\widetilde{\mathfrak{A}}(\frac{\bar{p}}{\lambda})$, where $\bar{p}$ is defined by (3.16). Taking into account the inequality $U(t,x)\le V(t,y)+xy$, for any $\Gamma\in\widetilde{\mathcal{Y}}(y,r)$ and $y_0(t)\triangleq\inf\{y>0:V(t,y)<0\}$, we have

$$\begin{aligned}\mathbb{E}\left[\int_0^T V^-(t,\Gamma_t)\,dt\right]&\le-\mathbb{E}\left[\int_0^T V\big(t,\Gamma_t\mathbf{1}_{\{\Gamma_t\ge y_0(t)\}}+y_0(t)\mathbf{1}_{\{\Gamma_t<y_0(t)\}}\big)\,dt\right]\\&\le-\mathbb{E}\left[\int_0^T U(t,\bar{x}\tilde{w}_t)\,dt\right]+\bar{x}\mathbb{E}\left[\int_0^T\tilde{w}_t\Gamma_t\,dt\right]\\&\quad+\bar{x}\mathbb{E}\left[\int_0^T\tilde{w}_t\big(y_0(t)-\Gamma_t\big)\mathbf{1}_{\{\Gamma_t<y_0(t)\}}\,dt\right]\\&\le-\mathbb{E}\left[\int_0^T U(t,\bar{x}\tilde{w}_t)\,dt\right]+\bar{x}\frac{\bar{p}}{\lambda}+\bar{x}\int_0^T y_0(t)\,dt.\end{aligned}$$

The last term is finitely valued and independent of the initial choice of Γ since $\tilde{w}_t\triangleq e^{\int_0^t(-\alpha_v)\,dv}\le1$ for $t\in[0,T]$ and $\sup_{t\in[0,T]}y_0(t)<\infty$ by condition (2.11). Thus the conclusion holds true. $\square$

LEMMA 5.7. *Under assumptions of Theorem 4.1, for any $(y,r)\in\mathcal{R}$, $\{V^-(\cdot,\Gamma_\cdot)\}_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}$ is uniformly integrable.*

PROOF. By Lemma 2.1, the condition $\mathrm{AE}_0[U]<\infty$ is equivalent to the following assertion:

$$\exists y_0>0\text{ and }\mu\in(1,2),\ \forall y\ge y_0,\qquad V(t,2y)\ge\mu V(t,y).\tag{5.12}$$

Let $y_0>0$ and $\mu\in(1,2)$ be constants in the above inequality (5.12). Taking $\gamma=\log_2\mu\in(0,1)$, we define the auxiliary function $\widetilde{V}(t,y):[0,\infty)\times(0,\infty)\to\mathbb{R}$ by

$$\widetilde{V}(t,y)\triangleq\begin{cases}-\dfrac{2y_0}{\gamma}V'(t,2y_0)-V(t,y), & y\ge2y_0,\\[2mm] -V(t,2y_0)-\dfrac{2y_0}{\gamma}V'(t,2y_0)\left(\dfrac{y}{2y_0}\right)^{\gamma}, & y<2y_0.\end{cases}\tag{5.13}$$

For each fixed $t > 0$, $\widetilde{V}(t, y)$ is a nonnegative, concave and nondecreasing function which agrees with $-V(t, y)$ up to a constant for large enough values of y and satisfies

$$\widetilde{V}(t, 2y) \leq \mu \widetilde{V}(t, y) \qquad \text{for all } y > 0. \tag{5.14}$$

Lemma 5.6 asserts that

$$\sup_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T V^-(t, \Gamma_t)\, dt\right] < \infty,$$

and hence in light of the fact that V^- and $\widetilde{V}$ differ only by a constant in a neighborhood of ∞, we will get

$$\sup_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T \widetilde{V}(t, \Gamma_t)\, dt\right] < \infty. \tag{5.15}$$

The validity of uniform integrability of the sequence $(V^-(\cdot, \Gamma^n_\cdot))_{n \geq 1}$ for $\Gamma^n \in \widetilde{\mathcal{Y}}(y, r)$, is therefore equivalent to the uniform integrability of $(\widetilde{V}(\cdot, \Gamma^n_\cdot))_{n \geq 1}$.

To this end, we argue by contradiction. Suppose this sequence is not uniformly integrable. Then by Rosenthal's subsequence splitting lemma, we can find a subsequence $(f^n)_{n \geq 1}$, a constant $\varepsilon > 0$ and a disjoint sequence $(A^n)_{n \geq 1}$ of $(\Omega \times [0, T], \mathcal{O})$ with

$$A^n \in \mathcal{O}, \qquad A^i \cap A^j = \varnothing \qquad \text{if } i \neq j,$$

such that

$$\mathbb{E}\left[\int_0^T \widetilde{V}(t, f_t^n) \mathbf{1}_{A^n}\, dt\right] \geq \varepsilon \qquad \text{for } n \geq 1.$$

Define the sequence of random variables $(h^n)_{n \geq 1}$

$$h_t^n = \sum_{k=1}^n f_t^k \mathbf{1}_{A^k}.$$

For any $\tilde{c} \in \widetilde{\mathfrak{A}}(1)$, we get

$$\langle \tilde{c}, h^n \rangle \leq \sum_{k=1}^n \langle \tilde{c}, f^k \rangle \leq n.$$

Hence $\frac{h^n}{n} \in \widetilde{\mathcal{Y}}(y, r)$.

One the other hand,

$$\mathbb{E}\left[\int_0^T \widetilde{V}(t, h_t^n)\, dt\right] \geq \sum_{k=1}^n \mathbb{E}\left[\int_0^T \widetilde{V}(t, f_t^k) \mathbf{1}_{A^k}\, dt\right] \geq \varepsilon n.$$

Therefore by taking $n = 2^m$, via iteration, it produces

$$\mu^m \sup_{\Gamma_t \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T \widetilde{V}(t, \Gamma_t)\, dt\right] \geq \mu^m \mathbb{E}\left[\int_0^T \widetilde{V}\left(t, \frac{h_t^{2^m}}{2^m}\right) dt\right]$$

$$\geq \mathbb{E}\left[\int_0^T \widetilde{V}\big(t, h_t^{2^m}\big)\, dt\right] \geq 2^m \varepsilon.$$

Since $\mu \in (1, 2)$, this contradicts (5.15) for m large enough, and therefore the conclusion holds true. □

Following the classical analysis, Lemma 5.7 together with Fatou's lemma will deduce the existence of the dual optimizer.

LEMMA 5.8. *For any pair $(y, r) \in \mathcal{R}$ such that $\tilde{v}(y, r) < \infty$, the optimal solution Γ^* to the optimization problem* (4.5) *exists and is unique*.

For the proof of conjugate duality between value functions $\tilde{u}(x, z)$ and $\tilde{v}(y, r)$, similar to Lemma 11 of Hugonnier and Kramkov [14], we have the following result:

LEMMA 5.9. *If $\mathcal{G} \subseteq \mathbb{L}_+^0$ is convex and contains a strictly positive random variable*, *then*

$$\sup_{g \in \mathcal{G}} \mathbb{E}\left[\int_0^T U(t, x g_t)\, dt\right] = \sup_{g \in \mathrm{cl}\,\mathcal{G}} \mathbb{E}\left[\int_0^T U(t, x g_t)\, dt\right], \qquad x > 0,$$

where $\mathrm{cl}\,\mathcal{G}$ *denotes the closure of* $\mathcal{G}$ *with respect to convergence in measure* $\overline{\mathbb{P}}$.

LEMMA 5.10. *For $\tilde{w}_t \triangleq e^{\int_0^t (-\alpha_v)\, dv}$*, *we have the following result*:

$$\mathbb{E}\left[\int_0^T V^-\big(t, U'(t, \tilde{w}_t)\big)\, dt\right] < \infty. \tag{5.16}$$

PROOF. Similar to the proof of Lemma 5.6, we have $\mathbb{E}[\int_0^T U(t, \bar{x}\tilde{w}_t)\, dt] > -\infty$, and by the inequality $U(t, x) < V(t, y) + xy$, for any $y_0(t) \triangleq \inf\{y > 0 : V(t, y) < 0\}$, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T V^-\big(t, U'(t, \tilde{w}_t)\big)\, dt\right] \\
&\quad \leq -\mathbb{E}\left[\int_0^T V\big(t, U'(t, \tilde{w}_t)\mathbf{1}_{\{U'(t,\tilde{w}_t) \geq y_0(t)\}} + y_0(t)\mathbf{1}_{\{U'(t,\tilde{w}_t) < y_0(t)\}}\big)\, dt\right] \\
&\quad \leq -\mathbb{E}\left[\int_0^T U(t, \bar{x}\tilde{w}_t)\, dt\right] + \bar{x}\mathbb{E}\left[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)\, dt\right] \qquad (5.17)
\end{aligned}$$

$$+\bar{x}\mathbb{E}\biggl[\int_0^T \tilde{w}_t\bigl(y_0(t)-U'(t,\tilde{w}_t)\bigr)\mathbf{1}_{\{U'(t,\tilde{w}_t)<y_0(t)\}}\,dt\biggr]$$

$$\le -\mathbb{E}\biggl[\int_0^T U(t,\bar{x}\tilde{w}_t)\,dt\biggr]+\bar{x}\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\,dt\biggr]+\bar{x}\int_0^T y_0(t)\,dt.$$

We already know the first term and the third term are bounded, as for the second term, we have two different cases:

Case 1. If $\bar{x}\le 1$, the second term can be rewritten as

$$\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\,dt\biggr]=\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\mathbf{1}_{\{\tilde{w}_t\le x_0\}}\,dt\biggr]+\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\mathbf{1}_{\{\tilde{w}_t> x_0\}}\,dt\biggr],$$

where x_0 is the uniform constant in Lemma 2.1 such that for all $t\in[0,T]$,

$$xU'(t,x)<\biggl(\frac{\gamma}{1-\gamma}\biggr)\bigl(-U(t,x)\bigr)\qquad \text{for } 0<x\le x_0. \tag{5.18}$$

Again, by the fact that $\tilde{w}_t\le 1$ for $t\in[0,T]$, it follows that

$$\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\mathbf{1}_{\{\tilde{w}_t> x_0\}}\,dt\biggr]<\infty,$$

and we also have

$$\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\mathbf{1}_{\{\tilde{w}_t\le x_0\}}\,dt\biggr]\le -\biggl(\frac{\gamma}{1-\gamma}\biggr)\mathbb{E}\biggl[\int_0^T U(t,\tilde{w}_t)\mathbf{1}_{\{\tilde{w}_t\le x_0\}}\,dt\biggr]$$

$$\le \biggl(\frac{\gamma}{1-\gamma}\biggr)\mathbb{E}\biggl[\int_0^T U^-(t,\bar{x}\tilde{w}_t)\,dt\biggr]<\infty,$$

by using inequality (5.18), the increasing property of $U(t,x)$ with respect to x and condition (3.5).

Case 2. If $\bar{x}>1$, the second term can be rewritten as

$$\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\,dt\biggr]=\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\mathbf{1}_{\{\bar{x}\tilde{w}_t\le x_0\}}\,dt\biggr]+\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t,\tilde{w}_t)\mathbf{1}_{\{\bar{x}\tilde{w}_t> x_0\}}\,dt\biggr],$$

where x_0 is the uniform constant in Lemma 2.1 such that for all $t\in[0,T]$, inequality (5.18) holds and moreover,

$$U\biggl(t,\frac{1}{\bar{x}}x\biggr)>\biggl(\frac{1}{\bar{x}}\biggr)^{-\gamma/(1-\gamma)}U(t,x)\qquad \text{for } 0<x\le x_0, \tag{5.19}$$

holds for all $t\in[0,T]$.

Again, the second term is bounded since $\bar{x}\tilde{w}_t \le \bar{x}$ for $t \in [0, T]$, and for the first term, we have

$$\begin{aligned}\mathbb{E}\biggl[\int_0^T \tilde{w}_t U'(t, \tilde{w}_t)\mathbf{1}_{\{\bar{x}\tilde{w}_t \le x_0\}}\,dt\biggr] &\le -\biggl(\frac{\gamma}{1-\gamma}\biggr)\mathbb{E}\biggl[\int_0^T U(t, \tilde{w}_t)\mathbf{1}_{\{\bar{x}\tilde{w}_t \le x_0\}}\,dt\biggr]\\ &\le \biggl(\frac{\gamma}{1-\gamma}\biggr)\biggl(\frac{1}{\bar{x}}\biggr)^{-\gamma/(1-\gamma)}\mathbb{E}\biggl[\int_0^T U^-(t, \bar{x}\tilde{w}_t)\,dt\biggr]\\ &< \infty\end{aligned}$$

by inequalities (5.18) and (5.19) and condition (3.5).

Hence the second term in (5.17) is finite, and therefore result (5.16) holds true. □

We emphasize that we have to revise the classical Minimax theorem based on $\mathbb{L}^1$ to derive the important conjugate duality relationship. The following Minimax theorem by Kauppila [17] can serve as a substitute tool on the space $\mathbb{L}^0_+$ for convexly compact sets.

THEOREM 5.1 (Minimax theorem). *Let A be a nonempty convex subset of a topological space, and B a nonempty, closed, convex and convexly compact subset of a topological vector space. Let $H : A \times B \to \mathbb{R}$ be convex on A, and concave and upper-semicontinuous on B. Then*

$$\sup_B \inf_A H = \inf_A \sup_B H.$$

See Theorem A.1 in Appendix A of Kauppila [17].

LEMMA 5.11. *Under assumptions of Theorem 4.1, the conjugate duality results hold*

$$\begin{aligned}\tilde{u}(x,z) &= \inf_{(y,r)\in\mathcal{R}}\{\tilde{v}(y,r) + xy - zr\}, \qquad (x,z)\in\mathcal{H},\\ \tilde{v}(y,r) &= \sup_{(x,z)\in\mathcal{H}}\{\tilde{u}(x,z) - xy + zr\}, \qquad (y,r)\in\mathcal{R}.\end{aligned} \tag{5.20}$$

PROOF. For $n > 0$, we define $\mathcal{S}_n$ as a subset in $\mathbb{L}^0_+(\Omega \times [0, T], \mathcal{O}, \overline{\mathbb{P}})$ by

$$\mathcal{S}_n = \{\tilde{c} \in \mathbb{L}^0_+ : 0 \le \tilde{c} \le n\tilde{w}\}.$$

It is clear that sets $\mathcal{S}_n$ are closed, convex and bounded in probability and hence convexly compact in $\mathbb{L}^0_+$.

It is easy to verify that the functional

$$\tilde{c} \mapsto \mathbb{E}\biggl[\int_0^T \bigl(U(t, \tilde{c}_t) - \tilde{c}_t\Gamma_t\bigr)\,dt\biggr]$$

is upper-semicontinuous on $\mathcal{S}_n$ under convergence in measure $\overline{\mathbb{P}}$, for all $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$ and $(y,r) \in \mathcal{R}$.

Lemma 5.5 implies that $\widetilde{\mathcal{Y}}(y,r)$ is a closed convex subset of $\mathbb{L}^0_+$. We can use the above Minimax Theorem 5.1 to get the following equality: for fixed n,

$$\begin{aligned}&\sup_{\tilde{c}\in\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\biggl[\int_0^T \bigl(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\bigr)\,dt\biggr]\\ &\qquad=\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\sup_{\tilde{c}\in\mathcal{S}_n}\mathbb{E}\biggl[\int_0^T \bigl(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\bigr)\,dt\biggr].\end{aligned}$$

By the bipolar relationship (5.9) and the definition, we have

$$\bigcup_{(x,z)\in\mathcal{H}}\widetilde{\mathcal{A}}(x,z)=\bigcup_{k>0}\widetilde{\mathfrak{A}}(k). \tag{5.21}$$

To continue the proof, we define the auxiliary set

$$\mathfrak{A}'(k)\triangleq\Bigl\{\tilde{c}\in\widetilde{\mathfrak{A}}(k):\sup_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\langle\tilde{c},\Gamma\rangle=k\Bigr\},$$

and clearly, it follows that

$$\bigcup_{k>0}\widetilde{\mathfrak{A}}(k)=\bigcup_{(x,z)\in\mathcal{H}}\widetilde{\mathcal{A}}(x,z)=\bigcup_{k>0}\mathfrak{A}'(k). \tag{5.22}$$

We first show that

$$\begin{aligned}&\lim_{n\to\infty}\sup_{\tilde{c}\in\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\biggl[\int_0^T \bigl(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\bigr)\,dt\biggr]\\ &\qquad=\sup_{k>0}\sup_{\tilde{c}\in\mathfrak{A}'(k)}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\biggl[\int_0^T \bigl(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\bigr)\,dt\biggr].\end{aligned} \tag{5.23}$$

The direction of inequality "$\geq$" holds by

$$\begin{aligned}&\lim_{n\to\infty}\sup_{\tilde{c}\in\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\biggl[\int_0^T \bigl(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\bigr)\,dt\biggr]\\ &\qquad\geq\lim_{n\to\infty}\sup_{\tilde{c}\in\mathfrak{A}'(k)\cap\mathcal{S}_n}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\biggl[\int_0^T \bigl(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\bigr)\,dt\biggr]\\ &\qquad=\sup_{\tilde{c}\in\mathfrak{A}'(k)}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\mathbb{E}\biggl[\int_0^T \bigl(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\bigr)\,dt\biggr]\qquad\forall k>0.\end{aligned}$$

The other direction "$\leq$" is obvious since for any $(x,z)\in\mathcal{H}$, we have $n\tilde{w}\in\mathfrak{A}'(n\bar{p})$, and hence $\mathcal{S}_n\subset\mathfrak{A}'(n\bar{p})$.

To continue the proof, we need to prepare finiteness results as below.

From definitions in Lemma 5.5 and by Lemma 5.9, it is easy to see that

$$\begin{aligned}\sup_{\tilde{c}\in\widetilde{\mathfrak{A}}(k)} \mathbb{E}\left[\int_0^T U(t,\tilde{c}_t)\,dt\right] &= \sup_{\tilde{c}\in\mathfrak{A}(k)} \mathbb{E}\left[\int_0^T U(t,\tilde{c}_t)\,dt\right] \\ &= \sup_{(x,z)\in k\mathfrak{H}(y,r)} \tilde{u}(x,z), \qquad k>0,\end{aligned} \tag{5.24}$$

and we claim that

$$\sup_{(x,z)\in k\mathfrak{H}(y,r)} \tilde{u}(x,z) < \infty, \qquad k>0. \tag{5.25}$$

Since the set $\mathcal{R}$ is open, and the set $\mathfrak{H}(y,r)$ is bounded, (5.25) follows from the concavity of $\tilde{u}$ and $\tilde{u}(x,z)<\infty$ for all $(x,z)\in\mathcal{H}$.

Now, by (5.22), (5.25) and the definition of domain $\mathcal{H}$, we have further equalities:

$$\begin{aligned}&\sup_{k>0}\sup_{\tilde{c}\in\mathfrak{A}'(k)}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T \big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\big)\,dt\right] \\ &\quad= \sup_{k>0}\left\{\sup_{\tilde{c}\in\mathfrak{A}'(k)} \mathbb{E}\left[\int_0^T U(t,\tilde{c}_t)\,dt\right]-k\right\} \\ &\quad= \sup_{k>0}\left\{\sup_{\tilde{c}\in\widetilde{\mathfrak{A}}(k)} \mathbb{E}\left[\int_0^T U(t,\tilde{c}_t)\,dt\right]-k\right\} \\ &\quad= \sup_{k>0}\left\{\sup_{(x,z)\in k\mathfrak{H}(y,r)} \tilde{u}(x,z)-k\right\} = \sup_{(x,z)\in\mathcal{H}}\{\tilde{u}(x,z)-xy+zr\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)}\sup_{\tilde{c}\in\mathcal{S}_n} \mathbb{E}\left[\int_0^T \big(U(t,\tilde{c}_t)-\tilde{c}_t\Gamma_t\big)\,dt\right] \\ &\qquad= \inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T V^n(t,\Gamma_t,\omega)\,dt\right] \triangleq \tilde{v}^n(y,r),\end{aligned}$$

where we define $V^n(t,y,\omega)$ according to the definition of set $\mathcal{S}_n$ as

$$V^n(t,y,\omega) = \sup_{0<x\le n\tilde{w}} \big[U(t,x)-xy\big].$$

Consequently, it is sufficient to show that

$$\lim_{n\to\infty}\tilde{v}^n(y,r) = \lim_{n\to\infty}\inf_{\Gamma\in\widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T V^n(t,\Gamma_t,\omega)\,dt\right] = \tilde{v}(y,r), \qquad (y,r)\in\mathcal{R}.$$

Evidently, $\tilde{v}^n(y,r)\le\tilde{v}(y,r)$, for $n\ge1$. Let $(\Gamma^n)_{n\ge1}$ be a sequence in $\widetilde{\mathcal{Y}}(y,r)$ such that

$$\lim_{n\to\infty}\mathbb{E}\left[\int_0^T V^n\big(t,\Gamma_t^n,\omega\big)\,dt\right] = \lim_{n\to\infty}\tilde{v}^n(y,r).$$

There exists a sequence $h^n \in \operatorname{conv}(\Gamma^n, \Gamma^{n+1}, \dots)$, $n \geq 1$, converging almost surely to a random variable Γ. Then $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$ is verified because the set $\widetilde{\mathcal{Y}}(y,r)$ is closed under convergence in finite measure $\overline{\mathbb{P}}$.

We claim that the sequence of processes $(V^n(\cdot, h^n_\cdot, \omega)^-)$, $n \geq 1$ is uniformly integrable. In fact, we can rewrite

$$(V^n(t, h^n_t, \omega))^- = (V^n(t, h^n_t, \omega))^- \mathbf{1}_{\{h^n_t \leq U'(t, \tilde{w}_t)\}} + (V^n(t, h^n_t, \omega))^- \mathbf{1}_{\{h^n_t > U'(t, \tilde{w}_t)\}}, \tag{5.26}$$

since $V^n(t,y,\omega) = V(t,y)$ for $y \geq U'(t, \tilde{w}_t) \geq U'(t, n\tilde{w}_t)$ by definition. The proof of Lemma 5.7 gives the uniform integrability of the sequence of processes $(V^n(\cdot, h^n_\cdot, \omega))^- \mathbf{1}_{\{h^n_\cdot > U'(\cdot, \tilde{w}_\cdot)\}}$, $n \geq 1$.

On the other hand, by the monotonicity of $(V^n)^-$, for all $n > 1$,

$$\begin{aligned}(V^n(t, h^n_t, \omega))^- \mathbf{1}_{\{h^n_t \leq U'(t, \tilde{w}_t)\}} &\leq (V^1(t, h^n_t, \omega))^- \mathbf{1}_{\{h^n_t \leq U'(t, \tilde{w}_t)\}} \\ &\leq (V(t, U'(t, \tilde{w}_t)))^-.\end{aligned} \tag{5.27}$$

By Lemma 5.10, the right-hand side is integrable in the product space, and hence the sequence $(V^n(\cdot, h^n_\cdot, \omega))^- \mathbf{1}_{\{h^n_\cdot \leq U'(\cdot, \tilde{w}_\cdot)\}}$, $n \geq 1$ is also uniformly integrable. Thus our claim holds true. Moreover, we have the following inequalities:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\left[\int_0^T V^n(t, \Gamma^n_t, \omega)\,dt\right] &\geq \liminf_{n\to\infty} \mathbb{E}\left[\int_0^T V^n(t, h^n_t, \omega)\,dt\right] \\ &\geq \mathbb{E}\left[\int_0^T V(t, \Gamma_t)\,dt\right] \geq \tilde{v}(y,r),\end{aligned}$$

which prove

$$\tilde{v}(y,r) = \sup_{(x,z)\in\mathcal{H}} \{\tilde{u}(x,z) - xy + zr\}. \tag{5.28}$$

Equality (5.20) is a direct consequence of equality (5.28) and the properties of convex conjugation; see Corollary 12.2.2 and Theorem 12.2 in Rockafellar [23]. □

PROOF OF THEOREM 4.1. It is now sufficient to show that the conjugate value function $\tilde{v}$ is $(-\infty, \infty]$-valued on $\mathcal{R}$.

The Legendre–Fenchel transform gives that

$$U(t,x) \leq V(t,y) + xy.$$

By integration, it is easy to see for any $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$ and $\Gamma \in \widetilde{\mathcal{Y}}(y,r)$,

$$\mathbb{E}\left[\int_0^T U(t, \tilde{c}_t)\,dt\right] \leq \mathbb{E}\left[\int_0^T V(t, \Gamma_t)\,dt\right] + \mathbb{E}\left[\int_0^T \tilde{c}_t \Gamma_t\,dt\right].$$

Proposition 5.1 deduces that

$$\tilde{u}(x,z) \le \tilde{v}(y,r) + xy - zr.$$

Hence for all $(y,r) \in \mathcal{R}$, we have $\tilde{v}(y,r) > -\infty$ by Lemma 5.4.

On the other hand, thanks to conjugate duality (5.20) and Bipolar relationship (5.9), we can follow proofs of Lemmas 5.5 and 5.11 and obtain that for each fixed $(y,r) \in \mathcal{R}$,

$$\sup_{(x,z)\in k\mathfrak{H}(y,r)} \tilde{u}(x,z) = \inf_{s>0}\{\tilde{v}(sy,sr) + ks\}.$$

The finiteness result (5.25) for all $k > 0$ in the proof of Lemma 5.11 guarantees the existence of a constant $s(y,r) > 0$ such that $\tilde{v}(sy,sr) < \infty$. $\square$

5.2. *The proof of Theorem* 4.2. Let us move on to the proof of Theorem 4.2 and some further lemmas and auxiliary results are needed.

LEMMA 5.12. *Under assumptions of Theorem* 4.2, *we have* $\tilde{v}(y,r)$ *is* $(-\infty,\infty)$-*valued on* $\mathcal{R}$.

PROOF. Similar to the proof of Lemma 5.4, under the additional condition (2.8), we can show that $\tilde{v}(y,r) < \infty$ if $\tilde{v}(sy,sr) < \infty$ for a constant $s = s(y,r) > 0$. However, it has been shown that Theorem 4.1 admits the existence of $s = s(y,r) > 0$. $\square$

Notice that we cannot mimic proofs of Lemmas 5.6, 5.7 and 5.8 to obtain the existence of the optimal solution to the problem (3.19). In fact, our arguments for the dual problem depend on the existence of a bounded process $\tilde{w} \in \widetilde{\mathfrak{A}}(\frac{\bar{p}}{\lambda})$, which is missing in the dual space. To this end, we resort to another auxiliary optimization problem and take advantage of the Bipolar results built in Lemma 5.5.

LEMMA 5.13. *Define the auxiliary optimization problem to the auxiliary dual utility minimization problem* (4.5) *as*

$$\hat{v}(k) = \inf_{\Gamma\in\widetilde{\mathfrak{Y}}(k)} \mathbb{E}\left[\int_0^T V(t,\Gamma_t)\,dt\right], \tag{5.29}$$

where $\widetilde{\mathfrak{Y}}(k)$ *is defined in Lemma* 5.5 *as the bipolar set of* $\widetilde{\mathcal{A}}(x,z)$ *on the product space for any* $(x,z) \in \mathcal{H}$.

Then, for all $k > 0$, *under hypothesis of Theorem* 4.2, *the value function* $\hat{v}(k) < \infty$ *for all* $k > 0$, *and the optimal solution* $\widehat{\Gamma}(k)$ *exists and is unique and* $\widehat{\Gamma}_t(k) > 0$ *for all* $t \in [0,T]$. *Moreover, for each* $k > 0$, *and any* $\Gamma \in \widetilde{\mathfrak{Y}}(k)$, *we have*

$$\mathbb{E}\left[\int_0^T (\Gamma_t - \widehat{\Gamma}_t(k)) I(t,\widehat{\Gamma}_t(k))\,dt\right] \le 0.$$

PROOF. According to the definition in Lemma 5.5 and by Lemma 5.12, it is easy to see

$$\hat{v}(k) = \inf_{\Gamma \in \widetilde{\mathfrak{Y}}(k)} \mathbb{E}\left[\int_0^T V(t, \Gamma_t)\, dt\right] \leq \inf_{\Gamma \in \mathfrak{Y}(k)} \mathbb{E}\left[\int_0^T V(t, \Gamma_t)\, dt\right]$$
$$= \inf_{(y,r) \in k\mathfrak{R}(x,z)} \tilde{v}(y, r) < \infty, \qquad k > 0.$$

Taking into account the Bipolar relationship (5.11), we have that $\widetilde{\mathfrak{Y}}(k)$ is convexly compact in $\mathbb{L}^0_+$, and the existence and uniqueness of optimal solution $\widehat{\Gamma}(k)$ will follow the similar proof of Theorem 4.1.

For $k > 0$, $\epsilon \in (0, 1)$, we define $\Gamma^\epsilon_t = (1 - \epsilon)\widehat{\Gamma}_t(k) + \epsilon\Gamma_t$, and for all $t \in [0, T]$, the optimality of $\widehat{\Gamma}(k)$ implies

$$\begin{aligned} 0 &\leq \frac{1}{\epsilon}\mathbb{E}\left[\int_0^T \big(V(t, \Gamma^\epsilon_t) - V(t, \widehat{\Gamma}_t(k))\big)\, dt\right] \\ &\leq \frac{1}{\epsilon}\mathbb{E}\left[\int_0^T \big(\widehat{\Gamma}_t(k) - \Gamma^\epsilon_t\big) I(t, \Gamma^\epsilon_t)\, dt\right] \\ &= \mathbb{E}\left[\int_0^T \big(\widehat{\Gamma}_t(k) - \Gamma_t\big) I(t, \Gamma^\epsilon_t)\, dt\right]. \end{aligned} \tag{5.30}$$

We claim that the family $\{((\Gamma_t - \widehat{\Gamma}_t(k))I(t, \Gamma^\epsilon_t))^-, \epsilon \in (0, 1)\}$ is uniformly integrable with respect to $\overline{\mathbb{P}}$. Observe that

$$\big((\Gamma_t - \widehat{\Gamma}_t(k))I(t, \Gamma^\epsilon_t)\big)^- \leq \widehat{\Gamma}_t(k)I(t, \Gamma^\epsilon_t) \leq \widehat{\Gamma}_t(k)I\big(t, (1 - \epsilon)\widehat{\Gamma}_t(k)\big) \qquad \forall t \in [0, T].$$

For fixed $\epsilon_0 < 1$ and $\epsilon < \epsilon_0$, we deduce that for each $t \in [0, T]$,

$$\begin{aligned} \big|\widehat{\Gamma}_t(k)I\big(t, (1 - \epsilon)\widehat{\Gamma}_t(k)\big)\big| &\leq \big|\widehat{\Gamma}_t(k)I\big(t, (1 - \epsilon)\widehat{\Gamma}_t(k)\big)\big|\mathbf{1}_{\{\widehat{\Gamma}_t(k) \leq y_1\}} \\ &\quad + \big|\widehat{\Gamma}_t(k)I\big(t, (1 - \epsilon)\widehat{\Gamma}_t(k)\big)\big|\mathbf{1}_{\{\widehat{\Gamma}_t(k) \geq y_2/(1-\epsilon_0)\}} \\ &\quad + \big|\widehat{\Gamma}_t(k)I\big(t, (1 - \epsilon)\widehat{\Gamma}_t(k)\big)\big|\mathbf{1}_{\{y_1 < \widehat{\Gamma}_t(k) < y_2/(1-\epsilon_0)\}}. \end{aligned}$$

By Lemma 2.1, reasonable asymptotic elasticity conditions $\mathrm{AE}_0[U] < \infty$ and $\mathrm{AE}_\infty[U] < 1$ imply that for fixed $\mu > 0$, there exist constants $C_1 > 0$, $C_2 > 0$, $y_1 > 0$ and $y_2 > 0$ such that

$$\begin{aligned} -V'(t, \mu y) &< C_1\frac{V(t, y)}{y} \qquad \text{for } 0 < y \leq y_1, \\ -V'(t, y) &< C_2\frac{-V(t, y)}{y} \qquad \text{for } y_2 \leq y. \end{aligned} \tag{5.31}$$

Hence, the first term is dominated by

$$\big|\widehat{\Gamma}_t(k)I\big(t, (1 - \epsilon)\widehat{\Gamma}_t(k)\big)\big|\mathbf{1}_{\{\widehat{\Gamma}_t(k) \leq y_1\}} \leq \frac{1}{1 - \epsilon_0}C_1\big|V\big(t, \widehat{\Gamma}_t(k)\big)\big|,$$

and the second term is dominated by

$$\begin{aligned}
&\big|\widehat{\Gamma}_t(k)I\big(t,(1-\epsilon)\widehat{\Gamma}_t(k)\big)\big|\mathbf{1}_{\{\widehat{\Gamma}_t(k)\geq y_2/(1-\epsilon_0)\}}\\
&\quad\leq \frac{-1}{1-\epsilon_0}C_2V\big(t,(1-\epsilon)\widehat{\Gamma}_t(k)\big)\mathbf{1}_{\{\widehat{\Gamma}_t(k)\geq y_2/(1-\epsilon_0)\}}\\
&\quad\leq \frac{1}{1-\epsilon_0}C_2\big|V\big(t,\widehat{\Gamma}_t(k)\big)\big|.
\end{aligned}$$

These two terms are both in $\mathbb{L}^1$ by the finiteness of $\hat{v}(k)$. On the other hand, the third remaining term $|\widehat{\Gamma}_t(k)I(t,(1-\epsilon)\widehat{\Gamma}_t(k))|\mathbf{1}_{\{y_1<\widehat{\Gamma}_t(k)<y_2/(1-\epsilon_0)\}}$ is dominated by $k\widehat{\Gamma}_t(k)\mathbf{1}_{\{y_1<\widehat{\Gamma}_t(k)<y_2/(1-\epsilon_0)\}}$ for a constant $k>0$, and it is obviously integrable as well.

We can let $\epsilon\to 0$ and apply dominated convergence theorem and Fatou's lemma to obtain the stated inequality.

To show that the optimal solution $\widehat{\Gamma}_t(k)>0$ for all $t\in[0,T]$, we can choose an element $\Gamma\in\widetilde{\mathfrak{Y}}(k)$ and $\Gamma_t>0$ for all $t\in[0,T]$. Inequality (5.30) can be rewritten as

$$\begin{aligned}
0\geq{}&\mathbb{E}\bigg[\int_0^T(\Gamma_t-\widehat{\Gamma}_t(k))I\big(t,\Gamma_t^{\epsilon}\big)\mathbf{1}_{\{\widehat{\Gamma}_t>0\}}\,dt\bigg]\\
&+\mathbb{E}\bigg[\int_0^T(\Gamma_t-\widehat{\Gamma}_t(k))I\big(t,\Gamma_t^{\epsilon}\big)\mathbf{1}_{\{\widehat{\Gamma}_t=0\}}\,dt\bigg].
\end{aligned}$$

Now suppose $\overline{\mathbb{P}}\{\widehat{\Gamma}_t(k)=0\}>0$. By the uniform integrability of $\{((\Gamma_t-\widehat{\Gamma}_t(k))\times I(t,\Gamma_t^{\epsilon}))^-,\epsilon\in(0,1)\}$, the second term of (5.32) goes to ∞ as ϵ converges to 0 since $I(t,0)=\infty$, and $\Gamma_t>0$ for all $t\in[0,T]$. Then we obtain the contradiction and hence the conclusion holds. $\square$

LEMMA 5.14. *Under the assumptions of Theorem 4.2, the auxiliary dual value function $\hat{v}(k)$ is continuously differentiable on $(0,\infty)$, and*

$$-k\hat{v}'(k)=\mathbb{E}\bigg[\int_0^T\widehat{\Gamma}_t(k)I\big(t,\widehat{\Gamma}_t(k)\big)\,dt\bigg].$$

PROOF. In order to show $\hat{v}(k)$ is continuously differentiable, by the convex property, it is enough to justify that its derivative exists on $(0,\infty)$. Now fix $k>0$, and define the function

$$h(s)\triangleq\mathbb{E}\bigg[\int_0^T V\Big(t,\frac{s}{k}\widehat{\Gamma}_t(k)\Big)\,dt\bigg].$$

This function is convex, and by optimality of $\widehat{\Gamma}(k)$ of problem (5.29), we have $h(s)\geq\hat{v}(s)$ for all $s>0$ and $h(k)=\hat{v}(k)$. Again, convexity implies that

$$\Delta^-h(k)\leq\Delta^-\hat{v}(k)\leq\Delta^+\hat{v}(k)\leq\Delta^+h(k),$$

where Δ^+ and Δ^- denote right and left derivatives, respectively. Now

$$\begin{aligned}\Delta^+ h(k) &= \lim_{\epsilon\to 0}\frac{h(k+\epsilon)-h(k)}{\epsilon}\\ &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\mathbb{E}\left[\int_0^T \left(V\left(t,\frac{k+\epsilon}{k}\widehat{\Gamma}_t(k)\right)-V\big(t,\widehat{\Gamma}_t(k)\big)\right)dt\right]\\ &\le \liminf_{\epsilon\to 0}\left(-\frac{1}{k\epsilon}\right)\mathbb{E}\left[\int_0^T \epsilon\widehat{\Gamma}_t(k)I\left(t,\frac{k+\epsilon}{k}\widehat{\Gamma}_t(k)\right)dt\right]\\ &= -\frac{1}{k}\mathbb{E}\left[\int_0^T \widehat{\Gamma}_t(k)I\big(t,\widehat{\Gamma}_t(k)\big)\,dt\right]\end{aligned}$$

by monotone convergence theorem. Similarly, we get

$$\Delta^- h(k) \ge \limsup_{\epsilon\to 0}\mathbb{E}\left[-\int_0^T \widehat{\Gamma}_t(k)I\left(t,\frac{k-\epsilon}{k}\widehat{\Gamma}_t(k)\right)dt\right].$$

We can follow the same reasoning as in Lemma 5.13 to show the family $\{(\widehat{\Gamma}_t(k)I(t,\frac{k-\epsilon}{k}\widehat{\Gamma}_t(k))),\epsilon\in(0,1)\}$ is uniformly integrable. Dominated convergence theorem and Fatou's lemma deduce

$$\Delta^- h(k) \ge -\frac{1}{k}\mathbb{E}\left[\int_0^T \widehat{\Gamma}_t(k)I\big(t,\widehat{\Gamma}_t(k)\big)\,dt\right],$$

which completes the proof. $\square$

LEMMA 5.15. *The auxiliary dual value function $\hat{v}(\cdot)$ has the asymptotic property*

$$-\hat{v}'(0)=\infty,\qquad -\hat{v}'(\infty)=0. \tag{5.32}$$

PROOF. We first show $-\hat{v}'(0)=\infty$, and to this end, we claim that

$$\hat{v}(0+) \ge \int_0^T V(t,0+)\,dt. \tag{5.33}$$

First, for any $k>0$, it follows by definition that

$$\begin{aligned}\hat{v}(k) &= \mathbb{E}\left[\int_0^T V\big(t,\widehat{\Gamma}_t(k)\big)\,dt\right]\\ &= \mathbb{E}\left[\int_0^T V^+\big(t,\widehat{\Gamma}_t(k)\big)\,dt\right]-\mathbb{E}\left[\int_0^T V^-\big(t,\widehat{\Gamma}_t(k)\big)\,dt\right].\end{aligned}$$

Recall that $\widetilde{\mathfrak{Y}}(k)=k\widetilde{\mathfrak{Y}}(1)$, and thus $\widehat{\Gamma}_t(k)=k\widehat{\Gamma}_t(1)$. Now by Fatou's lemma, first, we have

$$\lim_{k\to 0}\mathbb{E}\left[\int_0^T V^+\big(t,\widehat{\Gamma}_t(k)\big)\,dt\right] \ge \mathbb{E}\left[\int_0^T V^+(t,0+)\,dt\right]. \tag{5.34}$$

On the other hand, similar to the proof of Lemma 5.6, we can show that

$$\mathbb{E}\left[\int_0^T V^-\big(t, \widehat{\Gamma}_t(1)\big)\,dt\right] < \infty,$$

and therefore, by the monotonicity of function $V^-(t,\cdot)$ and the dominated convergence theorem, it follows that

$$\lim_{k\to 0}\mathbb{E}\left[\int_0^T V^-\big(t, \widehat{\Gamma}_t(k)\big)\,dt\right] = \mathbb{E}\left[\int_0^T V^-(t, 0+)\,dt\right],$$

which together with (5.34) imply that (5.33) holds true.

Therefore, if $\int_0^T V(t,0+)\,dt = \infty$, we have $\hat{v}(0+) = \infty$, and by convexity, it follows that $\hat{v}'(0+) = -\infty$.

In the case $\int_0^T V(t,0+)\,dt < \infty$, it is easy to see that

$$-\hat{v}(0+) \geq \lim_{k\to 0}\frac{\hat{v}(0) - \hat{v}(k)}{k} \geq \lim_{k\to 0}\frac{\int_0^T V(t,0+)\,dt - \mathbb{E}[\int_0^T V(t,\widehat{\Gamma}_t(k))\,dt]}{k},$$

and hence we have

$$\begin{aligned}
-\hat{v}(0+) &\geq \lim_{k\to 0}\frac{\mathbb{E}[\int_0^T V(t,0+)\,dt] - \mathbb{E}[\int_0^T V(t,\widehat{\Gamma}_t(k))\,dt]}{k}\\
&\geq \lim_{k\to 0}\mathbb{E}\left[\int_0^T \widehat{\Gamma}_t(1) I\big(t, k\widehat{\Gamma}_t(1)\big)\,dt\right] = \infty
\end{aligned}$$

by the monotone convergence theorem.

As for $-\hat{v}'(\infty) = 0$, since the function $-\hat{v}$ is concave and increasing, there is a finite positive limit

$$-\hat{v}'(\infty) \triangleq \lim_{k\to\infty} -\hat{v}'(y).$$

By the definition of Legendre–Fenchel transform, for any $y > 0$,

$$-V(t,y) \leq -U(t,x) + xy \qquad \text{for all } x > 0,$$

and then for any $\epsilon > 0$, we always have

$$\begin{aligned}
0 &\leq -\hat{v}'(\infty)\\
&= \lim_{k\to\infty}\frac{-\hat{v}(k)}{k} = \lim_{k\to\infty}\frac{\mathbb{E}[\int_0^T -V(t,\widehat{\Gamma}_t(k))\,dt]}{k}\\
&\leq \lim_{k\to\infty}\frac{\mathbb{E}[\int_0^T -U(t,\epsilon\tilde{w}_t)\,dt]}{k} + \lim_{k\to\infty}\frac{\langle \epsilon\tilde{w}, \widehat{\Gamma}(k)\rangle}{k}.
\end{aligned}$$

Now, recall that for each fixed $(x,z) \in \mathcal{H}$, there exists a constant $\lambda(x,z) > 0$ such that $\tilde{w}_t \in \widetilde{\mathcal{A}}(\frac{\bar{p}}{\lambda}x, \frac{\bar{p}}{\lambda}z)$, and by the definition of $\widetilde{\mathfrak{Y}}(k)$, we can see the second term above satisfies

$$\lim_{k\to\infty}\frac{\langle \epsilon\tilde{w}, \widehat{\Gamma}(k)\rangle}{k} \leq \lim_{k\to\infty}\frac{\epsilon(\bar{p}/\lambda)k}{k} = \epsilon\frac{\bar{p}}{\lambda}.$$

As for the first term, we claim that $\mathbb{E}[\int_0^T -U(t,\epsilon\tilde{w}_t)\,dt] < \infty$ for each fixed ϵ small enough. Without loss of generality, it is enough to consider that $\epsilon < \bar{x}$, and we can apply Lemma 2.1 again. Since there exists a constant x_0 such that for all $t \in [0,T]$,

$$U\left(t, \frac{\epsilon}{\bar{x}}x\right) > \left(\frac{\epsilon}{\bar{x}}\right)^{-\gamma/(1-\gamma)} U(t,x) \qquad \text{for } 0 < x \le x_0,$$

we will have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T -U(t,\epsilon\tilde{w}_t)\,dt\right] \\
&\quad = \mathbb{E}\left[\int_0^T -U(t,\epsilon\tilde{w}_t)\mathbf{1}_{\{\bar{x}\tilde{w}_t > x_0\}}\,dt\right] + \mathbb{E}\left[\int_0^T -U(t,\epsilon\tilde{w}_t)\mathbf{1}_{\{\bar{x}\tilde{w}_t \le x_0\}}\,dt\right] \\
&\quad \le \mathbb{E}\left[\int_0^T -U(t,\epsilon\tilde{w}_t)\mathbf{1}_{\{\bar{x}\tilde{w}_t > x_0\}}\,dt\right] + \left(\frac{\epsilon}{\bar{x}}\right)^{-\gamma/(1-\gamma)} \mathbb{E}\left[\int_0^T -U(t,\bar{x}\tilde{w}_t)\,dt\right] \\
&\quad < \infty
\end{aligned}$$

by the fact that $\tilde{w}_t \le 1$ for $t \in [0,T]$ and condition (3.5).

Hence, in conclusion,

$$0 \le -\hat{v}'(\infty) = \lim_{k\to\infty} \frac{-\hat{v}(k)}{k} \le \epsilon\frac{\bar{p}}{\lambda},$$

and consequently, we have $-\hat{v}'(\infty) = 0$ by letting ϵ go to 0. □

LEMMA 5.16. *Under assumptions of Theorem 4.2, for any $(x,z) \in \mathcal{H}$, suppose k satisfies $1 = -\hat{v}'(k)$ where $\hat{v}(k)$ is the value function of the auxiliary dual optimization problem (5.29). Then $\tilde{c}_t^*(x,z) \triangleq I(t,\widehat{\Gamma}_t(k))$ is the unique (in the sense of $=$ under $\overline{\mathbb{P}}$ in $\mathbb{L}_+^0$) optimal solution to problem (3.19). Moreover we have $\tilde{c}_t^*(x,z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0,T]$.*

PROOF. Lemma 5.14 asserts that

$$\langle \tilde{c}^*(x,z), \widehat{\Gamma}(k)\rangle = -k\hat{v}'(k) = k.$$

And for any $\Gamma \in \widetilde{\mathfrak{Y}}(k)$, by Lemma 5.13, we have

$$\langle \tilde{c}^*(x,z), \Gamma(k)\rangle \le \langle \tilde{c}^*(x,z), \widehat{\Gamma}(k)\rangle = k.$$

Hence, we first get $\tilde{c}_t^*(x,z) \in \widetilde{\mathcal{A}}(x,z)$ by the Bipolar relationship (5.11).
Now, for any $\tilde{c} \in \widetilde{\mathcal{A}}(x,z)$, we have

$$\begin{aligned}
&\langle \tilde{c}, \widehat{\Gamma}(k)\rangle \le k, \\
&U(t,\tilde{c}_t) \le V(t,\widehat{\Gamma}_t(k)) + \tilde{c}_t\widehat{\Gamma}_t(k) \qquad \forall t \in [0,T].
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T U(t, \tilde{c}_t)\, dt\right] \\
&\quad \le \hat{v}(k) + k = \mathbb{E}\left[\int_0^T \big(V\big(t, \widehat{\Gamma}_t(k)\big) + \widehat{\Gamma}_t(k) I\big(t, \widehat{\Gamma}_t(k)\big)\big)\, dt\right] \qquad (5.35)\\
&\quad = \mathbb{E}\left[\int_0^T U\big(t, I\big(\widehat{\Gamma}_t(k)\big)\big)\, dt\right] = \mathbb{E}\left[\int_0^T U\big(t, \tilde{c}_t^*\big)\, dt\right],
\end{aligned}$$

which infers the optimality of $\tilde{c}^*$. The uniqueness of the optimal solution follows from the strict concavity of the function U.

Under assumptions of Theorem 4.2, for any pair $(x, z) \in \mathcal{H}$, because $\widetilde{\mathfrak{Y}}(k)$ is convexly compact and $\widehat{\Gamma}_t(k)$ is bounded in probability, we have the optimal solution $\tilde{c}_t^*(x, z) > 0$, $\mathbb{P}$-a.s. for all $t \in [0, T]$ since $\widehat{\Gamma}_t(k)$ is bounded in probability if and only if $\widehat{\Gamma}_t(k)$ is finite $\overline{\mathbb{P}}$-a.s. and by definition, we know $I(t, x) > 0$ for $x < \infty$. $\square$

Let $(x, z) \in \operatorname{cl}\mathcal{H}$, the proof of Lemma 5.1 shows that there exists $\tilde{c} \in \widetilde{\mathcal{A}}(x, z)$ such that $\overline{\mathbb{P}}[\tilde{c} > 0] > 0$. Similar to the proof of Lemma 12 of Hugonnier and Kramkov [14], we will have:

LEMMA 5.17. *Assume that conditions of Proposition 5.1 hold, and let $(y^n, r^n) \in \mathcal{R}$ and $\Gamma^n \in \widetilde{\mathcal{Y}}(y^n, r^n)$, $n \ge 1$, converge to $(y, r) \in \mathbb{R}^2$ and $\Gamma \in \mathbb{L}^0_+$, respectively. If Γ is a strictly positive random variable, we have $(y, r) \in \mathcal{R}$ and $\Gamma \in \widetilde{\mathcal{Y}}(y, r)$.*

The next lemma is the last result we need to prepare to proceed to the proof of Theorem 4.2:

LEMMA 5.18. *Under Assumption 4.1, we have*

$$\overline{\mathbb{P}}\big[\tilde{c}^*(x_1, z_1) \ne \tilde{c}^*(x_2, z_2)\big] > 0,$$

for two different points $(x_i, z_i) \in \mathcal{H}$, $i = 1, 2$.

PROOF. Assume that there exist two distinct pairs (x_1, z_1) and (x_2, z_2) in $\mathcal{H}$ and

$$\overline{\mathbb{P}}\big[\tilde{c}^*(x_1, z_1) \ne \tilde{c}^*(x_2, z_2)\big] = 0.$$

The definition of set $\widetilde{A}(x_1, z_1)$ implies that

$$\big\langle \tilde{c}^*(x_2, z_2), \Gamma\big\rangle \le x_1 - z_1 \langle \tilde{w}, \Gamma\rangle \qquad \forall \Gamma \in \widetilde{\mathcal{M}}.$$

However, we also know that $\tilde{c}^*(x_2, z_2) \in \widetilde{A}(x_2, z_2)$, which deduces that

$$x_2 - z_2 \langle \tilde{w}, \Gamma\rangle \le x_1 - z_1 \langle \tilde{w}, \Gamma\rangle \qquad \forall \Gamma \in \widetilde{\mathcal{M}}.$$

On the other hand, by symmetry and replacing $\tilde{c}^*(x_2, z_2)$ by $\tilde{c}^*(x_1, z_1)$, we can conclude that

$$x_1 - z_1\langle\tilde{w}, \Gamma\rangle \leq x_2 - z_2\langle\tilde{w}, \Gamma\rangle \qquad \forall \Gamma \in \widetilde{\mathcal{M}}.$$

Therefore we must have

$$x_1 - z_1\langle\tilde{w}, \Gamma\rangle = x_2 - z_2\langle\tilde{w}, \Gamma\rangle \qquad \forall \Gamma \in \widetilde{\mathcal{M}},$$

which is a contradiction to Assumption 4.1 since we can obtain a constant $K = \frac{x_1 - x_2}{z_1 - z_2}$ and

$$\mathbb{E}^{\mathbb{Q}}[\mathcal{E}] = \langle w, Y\rangle = \langle\tilde{w}, \Gamma\rangle = \frac{x_1 - x_2}{z_1 - z_2} \qquad \forall \mathbb{Q} \in \mathcal{M}. \qquad \square$$

PROOF OF THEOREM 4.2. We first show that the dual value function $\tilde{v}(y, z)$ is continuously differentiable on $\mathcal{R}$. Theorems 4.1.1 and 4.1.2 in Hiriart-Urruty and Lemaréchal [13] give the equivalence between the above statement and the fact that the value function $\tilde{u}(x, z)$ is strictly concave on $\mathcal{H}$, since U is a strictly concave function. Showing that the value function is strictly concave is equivalent to showing that for any two distinct points $(x_i, z_i) \in \mathcal{H}$, $i = 1, 2$, the optimal consumption policies are different, that is,

$$\overline{\mathbb{P}}\big[\tilde{c}^*(x_1, z_1) \neq \tilde{c}^*(x_2, z_2)\big] > 0,$$

which is the consequence of Lemma 5.18.

To continue the remaining part, it amounts to show that assertion (ii) holds. Recall that $\widehat{\Gamma}(k)$ is the optimal solution of the auxiliary dual problem (5.29) such that

$$\widehat{\Gamma}_t(k) = U'\big(t, \tilde{c}_t^*(x, z)\big) \qquad \forall t \in [0, T], k = \big\langle\tilde{c}^*(x, z), \widehat{\Gamma}(k)\big\rangle.$$

As $\widetilde{\mathfrak{Y}}(k)$ is closed under convergence in measure $\overline{\mathbb{P}}$, there exists a sequence $(y^n, r^n) \in k\mathfrak{R}(x, z)$ such that $\Gamma^n \in \widetilde{\mathcal{Y}}(y^n, r^n)$, and Γ^n converges to $\widehat{\Gamma}(k)$ $\overline{\mathcal{P}}$-a.s. by passing to a subsequence if necessary. Since the set $k\mathfrak{R}(x, z)$ is bounded, there exists a further subsequence (y^n, r^n) converging to $(y, r) \in \mathbb{R}^2$. By passing to this further subsequence, as we have shown $\overline{\mathbb{P}}[\widehat{\Gamma}(k) > 0] = 1$, it follows that $(y, r) \in k\mathfrak{R}(x, z)$ such that $\widehat{\Gamma}(k) \in \widetilde{\mathcal{Y}}(y, r)$ due to Lemma 5.17. Moreover, for this pair $(y, r) \in \mathcal{R}$, by Fatou's lemma and Proposition 5.1, we have the equality that

$$xy - zr = k = \big\langle\tilde{c}^*(x, z), \widehat{\Gamma}(k)\big\rangle. \tag{5.36}$$

The corresponding optimizer $\Gamma_t^*(y, r)$ of (4.5) then verifies

$$\Gamma_t^*(y, r) = \widehat{\Gamma}_t(k) = U'\big(t, \tilde{c}^*(x, z)\big), \qquad t \in [0, T]. \tag{5.37}$$

To see this, on one hand, we have $\widehat{\Gamma}(k) \in \widetilde{\mathcal{Y}}(y, r)$, hence

$$\mathbb{E}\left[\int_0^T V\big(t, \Gamma_t^*(y, r)\big)\, dt\right] = \inf_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T V\big(t, \Gamma_t(y, r)\big)\, dt\right]$$
$$\le \mathbb{E}\left[\int_0^T V\big(t, \widehat{\Gamma}_t(k)\big)\, dt\right].$$

On the other hand, we have

$$\mathbb{E}\left[\int_0^T V\big(t, \widehat{\Gamma}_t(y, r)\big)\, dt\right] = \inf_{\Gamma \in \widetilde{\mathfrak{Y}}(k)} \mathbb{E}\left[\int_0^T V\big(t, \Gamma_t(y, r)\big)\, dt\right]$$
$$\le \inf_{\Gamma \in \widetilde{\mathcal{Y}}(y,r)} \mathbb{E}\left[\int_0^T V\big(t, \Gamma_t(y, r)\big)\, dt\right]$$
$$= \mathbb{E}\left[\int_0^T V\big(t, \Gamma_t^*(y, r)\big)\, dt\right].$$

By the equality

$$U\big(t, \tilde{c}_t^*(x, z)\big) = V\big(t, \widehat{\Gamma}_t(k)\big) + \tilde{c}_t^*(x, z)\widehat{\Gamma}_t(k), \qquad t \in [0, T],$$

we can conclude $(y, r) \in \partial\tilde{u}(x, z)$ by Theorem 23.5 of Rockafellar [23], since

$$\tilde{u}(x, z) = \tilde{v}(y, z) + xy - zr. \tag{5.38}$$

In particular, it implies that

$$\partial\tilde{u}(x, z) \cap \mathcal{R} \neq \varnothing. \tag{5.39}$$

Similar to the proof of Theorem 2 in Hugonnier and Kramkov [14], it is easy to show that

$$\partial\tilde{u}(x, z) \subset \mathcal{R}.$$

For any $(y, r) \in \partial\tilde{u}(x, z)$, there exists a sequence $(y^n, r^n) \in \partial\tilde{u}(x, z) \cap \mathcal{R}$ converging to (y, r) by (5.39) and the fact that $\partial\tilde{u}(x, z)$ is closed and convex. Since $U'(\cdot, \tilde{c}_\cdot^*(x, z))$ is strictly positive and $U'(\cdot, \tilde{c}_\cdot^*(x, z)) \in \widetilde{\mathcal{Y}}(y, r)$, Lemma 5.17 infers that $(y, r) \in \mathcal{R}$.

Conversely, for any $(y, r) \in \partial\tilde{u}(x, z)$, we have

$$\mathbb{E}\left[\int_0^T \big|V\big(t, \Gamma_t^*(y, r)\big) + \tilde{c}_t^*(x, z)\Gamma_t^*(y, r) - U\big(t, \tilde{c}_t^*(x, z)\big)\big|\, dt\right]$$
$$= \mathbb{E}\left[\left(\int_0^T V\big(t, \Gamma_t^*(y, r)\big) + \tilde{c}_t^*(x, z)\Gamma_t^*(y, r) - U\big(t, \tilde{c}_t^*(x, z)\big)\, dt\right)\right]$$
$$\le \tilde{v}(y, r) + xy - zr - \tilde{u}(x, z) = 0,$$

which infers (5.36) and (5.37). $\square$

Acknowledgement. I sincerely thank my advisor Mihai Sîrbu for numerous helpful discussions on the topic of this paper as well as his generous support and kind encouragement during my research.

REFERENCES

[1] BOUCHARD, B. and PHAM, H. (2004). Wealth-path dependent utility maximization in incomplete markets. *Finance Stoch*. **8** 579–603. MR2212119

[2] BRANNATH, W. and SCHACHERMAYER, W. (1999). A bipolar theorem for $L^0_+(\Omega, \mathcal{F}, \mathbf{P})$. In *Séminaire de Probabilités, XXXIII. Lecture Notes in Math*. **1709** 349–354. Springer, Berlin. MR1768009

[3] CAMPBELL, J. Y. and COCHRANE, J. H. (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. *J. Polit. Econ*. **107** 205–251.

[4] CONSTANTINIDES, G. M. (1988). Habit formation: A resolution of the equity premium puzzle. Working paper series. Center for Research in Security Prices, Graduate School of Business, Univ. Chicago, Chicago, IL.

[5] CVITANIĆ, J., SCHACHERMAYER, W. and WANG, H. (2001). Utility maximization in incomplete markets with random endowment. *Finance Stoch*. **5** 259–272. MR1841719

[6] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann*. **300** 463–520. MR1304434

[7] DELBAEN, F. and SCHACHERMAYER, W. (1998). The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann*. **312** 215–250. MR1671792

[8] DETEMPLE, J. B. and KARATZAS, I. (2003). Non-addictive habits: Optimal consumption-portfolio policies. *J. Econom. Theory* **113** 265–285. MR2021304

[9] DETEMPLE, J. B. and ZAPATERO, F. (1991). Asset prices in an exchange economy with habit formation. *Econometrica* **59** 1633–1657.

[10] DETEMPLE, J. B. and ZAPATERO, F. (1992). Optimal consumption-portfolio policies with habit formation. *Math. Finance* **2** 251–274.

[11] ENGLEZOS, N. and KARATZAS, I. (2009). Utility maximization with habit formation: Dynamic programming and stochastic PDEs. *SIAM J. Control Optim*. **48** 481–520. MR2486081

[12] HICKS, J. (1965). *Capital and Growth*. Oxford Univ. Press, New York.

[13] HIRIART-URRUTY, J.-B. and LEMARÉCHAL, C. (2001). *Fundamentals of Convex Analysis*. Springer, Berlin. MR1865628

[14] HUGONNIER, J. and KRAMKOV, D. (2004). Optimal investment with random endowments in incomplete markets. *Ann. Appl. Probab*. **14** 845–864. MR2052905

[15] KARATZAS, I., LEHOCZKY, J. P., SHREVE, S. E. and XU, G.-L. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM J. Control Optim*. **29** 702–730. MR1089152

[16] KARATZAS, I. and ŽITKOVIĆ, G. (2003). Optimal consumption from investment and random endowment in incomplete semimartingale markets. *Ann. Probab*. **31** 1821–1858. MR2016601

[17] KAUPPILA, H. (2010). *Convex Duality in Singular Control—Optimal Consumption Choice with Intertemporal Substitution and Optimal Investment in Incomplete Markets*. ProQuest LLC, Ann Arbor, MI. Ph.D. thesis—Columbia Univ. MR2733498

[18] KRAMKOV, D. and SCHACHERMAYER, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Ann. Appl. Probab*. **9** 904–950. MR1722287

[19] KRAMKOV, D. and SCHACHERMAYER, W. (2003). Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab*. **13** 1504–1516. MR2023886

[20] KRAMKOV, D. O. (1996). Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probab. Theory Related Fields* **105** 459–479. MR1402653

[21] MEHRA, R. and PRESCOTT, E. C. (1985). The equity premium: A puzzle. *J. Monet. Econ.* **15** 145–161.

[22] MUNK, C. (2008). Portfolio and consumption choice with stochastic investment opportunities and habit formation in preferences. *J. Econom. Dynam. Control* **32** 3560–3589. MR2464350

[23] ROCKAFELLAR, R. T. (1970). *Convex Analysis*. Princeton Univ. Press, Princeton, NJ. MR0274683

[24] RYDER, H. E. and HEAL, G. M. (1973). Optimal growth with intertemporally dependent preferences. *Rev. Econom. Stud.* **40** 1–33.

[25] SAMUELSON, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *Rev. Econ. Stat.* **51** 239–246.

[26] SCHACHERMAYER, W. (2004). *Portfolio Optimization in Incomplete Financial Markets*. Scuola Normale Superiore, Classe di Scienze, Pisa. MR2144570

[27] SCHRODER, M. and SKIADAS, C. (2002). An isomorphism between asset pricing models with and without linear habit formation. *Rev. Financ. Stud.* **15** 1189–1221.

[28] YU, X. (2012). Utility maximization with consumption habit formation in incomplete markets. Ph.D. thesis, Univ. Texas at Austin.

[29] ŽITKOVIĆ, G. (2002). A filtered version of the bipolar theorem of Brannath and Schachermayer. *J. Theoret. Probab.* **15** 41–61. MR1883202

[30] ŽITKOVIĆ, G. (2005). Utility maximization with a stochastic clock and an unbounded random endowment. *Ann. Appl. Probab.* **15** 748–777. MR2114989

[31] ŽITKOVIĆ, G. (2010). Convex compactness and its applications. *Math. Financ. Econ.* **3** 1–12. MR2651515

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MICHIGAN
530 CHURCH STREET
ANN ARBOR, MICHIGAN 48109
USA
E-MAIL: xymath@umich.edu