

## ASYMPTOTIC DISTRIBUTION OF COMPLEX ZEROS OF RANDOM ANALYTIC FUNCTIONS

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Let  $\xi_0, \xi_1, \dots$  be independent identically distributed complex-valued random variables such that  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ . We consider random analytic functions of the form

$$\mathbf{G}_n(z) = \sum_{k=0}^{\infty} \xi_k f_{k,n} z^k,$$

where  $f_{k,n}$  are deterministic complex coefficients. Let  $\mu_n$  be the random measure counting the complex zeros of  $\mathbf{G}_n$  according to their multiplicities. Assuming essentially that  $-\frac{1}{n} \log f_{[tn],n} \rightarrow u(t)$  as  $n \rightarrow \infty$ , where  $u(t)$  is some function, we show that the measure  $\frac{1}{n} \mu_n$  converges in probability to some deterministic measure  $\mu$  which is characterized in terms of the Legendre–Fenchel transform of  $u$ . The limiting measure  $\mu$  does not depend on the distribution of the  $\xi_k$ 's. This result is applied to several ensembles of random analytic functions including the ensembles corresponding to the three two-dimensional geometries of constant curvature. As another application, we prove a random polynomial analogue of the circular law for random matrices.

### 1. Introduction.

1.1. *Statement of the problem.* Let  $\xi_0, \xi_1, \dots$  be nondegenerate independent identically distributed (i.i.d.) random variables with complex values. The simplest ensemble of random polynomials are the *Kac polynomials* defined as

$$\mathbf{K}_n(z) = \sum_{k=0}^n \xi_k z^k.$$

The distribution of zeros of Kac polynomials has been much studied; see [1, 10, 14–16, 28, 31, 36]. It is known that under a very mild moment assumption, the complex zeros of  $\mathbf{K}_n$  cluster asymptotically near the unit circle  $\mathbb{T} = \{|z| = 1\}$  and that the distribution of zeros is asymptotically uniform with regard to the argument. To make this precise, we need to introduce some notation. Let  $G$  be an analytic function in some domain  $D \subset \mathbb{C}$ . Assuming that  $G$  does not vanish identically,

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we consider a measure  $\mu_G$  counting the complex zeros of  $G$  according to their multiplicities:

$$\mu_G = \sum_{z \in D: G(z)=0} n_G(z)\delta(z).$$

Here,  $n_G(z)$  is the multiplicity of the zero at  $z$  and  $\delta(z)$  is the unit point mass at  $z$ . If  $G$  vanishes identically, we put  $\mu_G = 0$ . Then, Ibragimov and Zaporozhets [16] proved that the following two conditions are equivalent:

- (1) With probability 1, the sequence of measures  $\frac{1}{n}\mu_{\mathbf{K}_n}$  converges as  $n \rightarrow \infty$  weakly to the uniform probability distribution on  $\mathbb{T}$ .
- (2)  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ .

Along with the Kac polynomials, many other remarkable ensembles of random polynomials (or, more generally, random power series) appeared in the literature. These ensembles are usually characterized by invariance properties with respect to certain groups of transformations and have the general form

$$\mathbf{G}_n(z) = \sum_{k=0}^{\infty} \xi_k f_{k,n} z^k,$$

where  $\xi_0, \xi_1, \dots$  are i.i.d. complex-valued random variables and  $f_{k,n}$  are complex deterministic coefficients. The aim of the present work is to study the distribution of zeros of  $\mathbf{G}_n$  asymptotically as  $n \rightarrow \infty$ . We will show that under certain assumptions on the coefficients  $f_{k,n}$ , the random measure  $\frac{1}{n}\mu_{\mathbf{G}_n}$  converges, as  $n \rightarrow \infty$ , to some limiting deterministic measure  $\mu$ . The limiting measure  $\mu$  does not depend on the distribution of the random variables  $\xi_k$ ; see Figure 1. Results of this

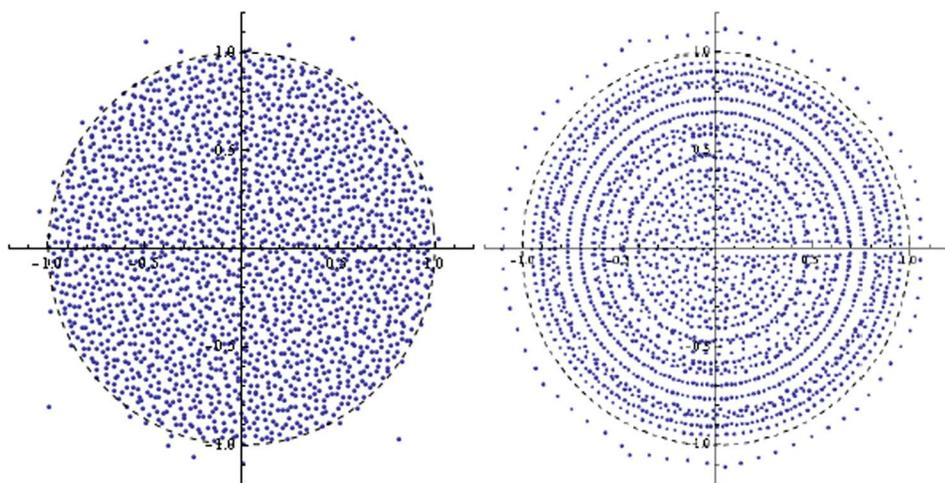


FIG. 1. Zeros of the Weyl random polynomial  $\mathbf{W}_n(z) = \sum_{k=0}^n \xi_k \frac{z^k}{\sqrt{k!}}$  of degree  $n = 2000$ . The zeros were divided by  $\sqrt{n}$ . Left: Complex normal coefficients. Right: Coefficients are positive with  $\mathbb{P}[\log \xi_k > t] = t^{-4}$  for  $t > 1$ . In both cases, the limiting distribution of zeros is uniform on the unit disk.

type are known in the context of random matrices; see, for example, [35]. However, the literature on random polynomials and random analytic functions usually concentrates on the Gaussian case, since in this case explicit calculations are possible; see, for example, [2, 4, 6, 8, 10, 13, 28–30, 32, 33]. The only ensemble of random polynomials for which the independence of the limiting distribution of zeros on the distribution of the coefficients is well understood is the Kac ensemble; see [1, 15, 16, 36]. In the context of random polynomials, there were many results on the universal character of local correlations between close zeros [3, 19, 29, 30]. In this work, we focus on the global distribution of zeros.

The paper is organized as follows. In Sections 2.1–2.4, we state our results for a number of concrete ensembles of random analytic functions. These results are special cases of the general Theorem 2.8 whose statement, due to its technicality, is postponed to Section 2.5. Proofs are given in Sections 3 and 4.

1.2. *Notation.* Let  $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$  be the open disk with radius  $r > 0$  centered at the origin. Let  $\mathbb{D} = \mathbb{D}_1$  be the unit disk. Put  $\mathbb{D}_\infty = \mathbb{C}$ . Denote by  $\lambda$  the Lebesgue measure on  $\mathbb{C}$ . A Borel measure  $\mu$  on a locally compact metric space  $X$  is called *locally finite* (l.f.) if  $\mu(A) < \infty$  for every compact set  $A \subset X$ . A sequence  $\mu_n$  of l.f. measures on  $X$  converges *vaguely* to a l.f. measure  $\mu$  if for every continuous, compactly supported function  $\varphi : X \rightarrow \mathbb{R}$ ,

$$(1) \quad \lim_{n \rightarrow \infty} \int_X \varphi(z) \mu_n(dz) = \int_X \varphi(z) \mu(dz).$$

If  $\mu_n$  and  $\mu$  are probability measures, the vague convergence is equivalent to the more familiar weak convergence for which (1) is required to hold for all continuous, bounded functions  $\varphi$ ; see Lemma 4.20 in [17]. Let  $\mathcal{M}(X)$  be the space of all l.f. measures on  $X$  endowed with the vague topology. Note that  $\mathcal{M}(X)$  is a Polish space; see Theorem A2.3 in [17]. A *random measure* on  $X$  is a random element defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and taking values in  $\mathcal{M}(X)$ . The a.s. convergence and convergence in probability of random measures are defined as the convergence of the corresponding  $\mathcal{M}(X)$ -valued random elements. An equivalent definition: a sequence of random measures  $\mu_n$  converges to a random measure  $\mu$  in probability (resp., a.s.), if (1) holds in probability (resp., a.s.) for every continuous, compactly supported function  $\varphi : X \rightarrow \mathbb{R}$ .

## 2. Statement of results.

2.1. *The three invariant ensembles.* Let  $\xi_0, \xi_1, \dots$  be i.i.d. random variables. Unless stated otherwise, they take values in  $\mathbb{C}$ , are nondegenerate, and satisfy the condition  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ . Fix a parameter  $\alpha > 0$ . We start by considering the

following three ensembles of random analytic functions (see, e.g., [13, 33]):

$$\mathbf{F}_n(z) = \begin{cases} \sum_{k=0}^n \xi_k \left( \frac{n(n-1)\cdots(n-k+1)}{k!} \right)^\alpha z^k & \text{(elliptic, } n \in \mathbb{N}, z \in \mathbb{C}), \\ \sum_{k=0}^\infty \xi_k \left( \frac{n^k}{k!} \right)^\alpha z^k & \text{(flat, } n > 0, z \in \mathbb{C}), \\ \sum_{k=0}^\infty \xi_k \left( \frac{n(n+1)\cdots(n+k-1)}{k!} \right)^\alpha z^k & \text{(hyperbolic, } n > 0, z \in \mathbb{D}). \end{cases}$$

Note that in the elliptic case  $\mathbf{F}_n$  is a random polynomial of degree  $n$ , in the flat case it is a random entire function, whereas in the hyperbolic case it is a random analytic function defined on the unit disk  $\mathbb{D}$ . The a.s. convergence of the series in the latter two cases follows from Lemma 4.4 below. In the particular case when  $\alpha = 1/2$  and  $\xi_k$  are complex standard Gaussian with density  $z \mapsto \pi^{-1} \exp\{-|z|^2\}$  on  $\mathbb{C}$ , the zero sets of these analytic functions possess remarkable invariance properties relating them to the three geometries of constant curvature; see [13, 33]. In this special case, the expected number of zeros of  $\mathbf{F}_n$  in a Borel set  $B$  can be computed exactly [13, 33]:

$$\mathbb{E}[\mu_{\mathbf{F}_n}(B)] = \begin{cases} \frac{n}{\pi} \int_B (1 + |z|^2)^{-2} \lambda(dz) & \text{(elliptic case, } B \subset \mathbb{C}), \\ \frac{n}{\pi} \lambda(B) & \text{(flat case, } B \subset \mathbb{C}), \\ \frac{n}{\pi} \int_B (1 - |z|^2)^{-2} \lambda(dz) & \text{(hyperbolic case, } B \subset \mathbb{D}). \end{cases}$$

In the next theorem, we compute the asymptotic distribution of zeros of  $\mathbf{F}_n$  for more general  $\xi_k$ 's.

**THEOREM 2.1.** *Let  $\xi_0, \xi_1, \dots$  be nondegenerate i.i.d. random variables such that  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ . As  $n \rightarrow \infty$ , the sequence of random measures  $\frac{1}{n} \mu_{\mathbf{F}_n}$  converges in probability to the deterministic measure having a density  $\rho_\alpha$  with respect to the Lebesgue measure, where*

$$\rho_\alpha(z) = \begin{cases} \frac{1}{2\pi\alpha} |z|^{(1/\alpha)-2} (1 + |z|^{1/\alpha})^{-2} & \text{(elliptic case, } z \in \mathbb{C}), \\ \frac{1}{2\pi\alpha} |z|^{(1/\alpha)-2} & \text{(flat case, } z \in \mathbb{C}), \\ \frac{1}{2\pi\alpha} |z|^{(1/\alpha)-2} (1 - |z|^{1/\alpha})^{-2} & \text{(hyperbolic case, } z \in \mathbb{D}). \end{cases}$$

2.2. *Littlewood–Offord random polynomials.* Next, we consider an ensemble of random polynomials which was introduced by Littlewood and Offord [21, 22]. It is related to the flat model. First, we give some motivation. Let  $\xi_0, \xi_1, \dots$  be non-degenerate i.i.d. random variables. Given a sequence  $w_0, w_1, \dots \in \mathbb{C} \setminus \{0\}$  consider a random polynomial  $\mathbf{W}_n$  defined by

$$(2) \quad \mathbf{W}_n(z) = \sum_{k=0}^n \xi_k w_k z^k.$$

For  $w_k = 1$ , we recover the Kac polynomials, for which the zeros concentrate near the unit circle. The next result shows that the structure of the zeros does not differ essentially from the Kac case if the sequence  $w_k$  grows or decays not too fast.

**THEOREM 2.2.** *Let  $\xi_0, \xi_1, \dots$  be nondegenerate i.i.d. random variables such that  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ . If  $\lim_{k \rightarrow \infty} \frac{1}{k} \log |w_k| = w$  for some constant  $w \in \mathbb{R}$ , then the sequence of random measures  $\frac{1}{n} \mu_{\mathbf{W}_n}$  converges in probability to the uniform probability distribution on the circle of radius  $e^{-w}$  centered at the origin.*

We would like to construct examples where there is no concentration near a circle. Let us make the following assumption on the sequence  $w_k$ :

$$(3) \quad \log |w_k| = -\alpha(k \log k - k) - \beta k + o(k), \quad k \rightarrow \infty,$$

where  $\alpha > 0$  and  $\beta \in \mathbb{R}$  are parameters. Particular cases are polynomials of the form

$$\begin{aligned} \mathbf{W}_n^{(1)}(z) &= \sum_{k=0}^n \frac{\xi_k}{(k!)^\alpha} z^k, \\ \mathbf{W}_n^{(2)}(z) &= \sum_{k=0}^n \frac{\xi_k}{k^{\alpha k}} z^k, \\ \mathbf{W}_n^{(3)}(z) &= \sum_{k=0}^n \frac{\xi_k}{\Gamma(\alpha k + 1)} z^k. \end{aligned}$$

The family  $\mathbf{W}_n^{(1)}$  has been studied by Littlewood and Offord [21, 22] in one of the earliest works on random polynomials. They were interested in the number of real zeros. In the next theorem, we describe the limiting distribution of complex zeros of  $\mathbf{W}_n$ . Let  $\mu_n$  be the measure counting the points of the form  $e^{-\beta} n^{-\alpha} z$ , where  $z$  is a zero of  $\mathbf{W}_n$ . That is, for every Borel set  $B \subset \mathbb{C}$ ,

$$(4) \quad \mu_n(B) = \mu_{\mathbf{W}_n}(e^\beta n^\alpha B).$$

**THEOREM 2.3.** *Let  $\xi_0, \xi_1, \dots$  be nondegenerate i.i.d. random variables such that  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ . Let  $w_0, w_1, \dots$  be a complex sequence satisfying (3).*

With probability 1, the sequence of random measures  $\frac{1}{n}\mu_n$  converges to the deterministic probability measure having the density

$$(5) \quad z \mapsto \frac{1}{2\pi\alpha} |z|^{(1/\alpha)-2} \mathbb{1}_{z \in \mathbb{D}}$$

with respect to the Lebesgue measure on  $\mathbb{C}$ .

For the so-called *Weyl random polynomials*  $\mathbf{W}_n(z) = \sum_{k=0}^n \xi_k \frac{z^k}{\sqrt{k!}}$  having  $\alpha = 1/2$  and  $\beta = 0$ , the limiting distribution is uniform on  $\mathbb{D}$ ; see Figure 1. This result can be seen as an analogue of the famous circular law for the distribution of eigenvalues of non-Hermitian random matrices with i.i.d. entries [5, 35]. Forrester and Honner [9] stated the circular law for Weyl polynomials and discussed differences and similarities between the matrix and the polynomial cases; see also [18].

Under a minor additional assumption on the coefficients  $w_k$  we can prove that the logarithmic moment condition is not only sufficient, but also necessary for the a.s. convergence of the empirical distribution of zeros. It is easy to check that the additional assumption is satisfied for  $\mathbf{W}_n = \mathbf{W}_n^{(i)}$  with  $i = 1, 2, 3$ .

**THEOREM 2.4.** *Let  $\xi_0, \xi_1, \dots$  be nondegenerate i.i.d. random variables. Let  $w_0, w_1, \dots$  be a complex sequence satisfying (3) and such that for some  $C > 0$ ,*

$$(6) \quad |w_{n-k}/w_n| < C e^{\beta k} n^{\alpha k} \quad \text{for all } n \in \mathbb{N}, k \leq n.$$

Let  $\mu_n$  be as in (4). Then, the following are equivalent:

- (1) With probability 1, the sequence of random measures  $\frac{1}{n}\mu_n$  converges to the probability measure with density (5).
- (2)  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ .

It should be stressed that in all our results we assume that the random variables  $\xi_k$  are nondegenerate (i.e., not a.s. constant). To see that this assumption is essential, consider the deterministic polynomials

$$(7) \quad s_n(z) = \sum_{k=0}^n \frac{z^k}{k!}.$$

A classical result of Szegő [34] states that the zeros of  $s_n(nz)$  cluster asymptotically (as  $n \rightarrow \infty$ ) along the curve  $\{|ze^{1-z}| = 1\} \cap \mathbb{D}$ ; see Figure 2 (left). This behavior is manifestly different from the distribution with density  $1/(2\pi|z|)$  on  $\mathbb{D}$  we have obtained in Theorem 2.3 for the same polynomial with randomized coefficients; see Figure 2 (right).

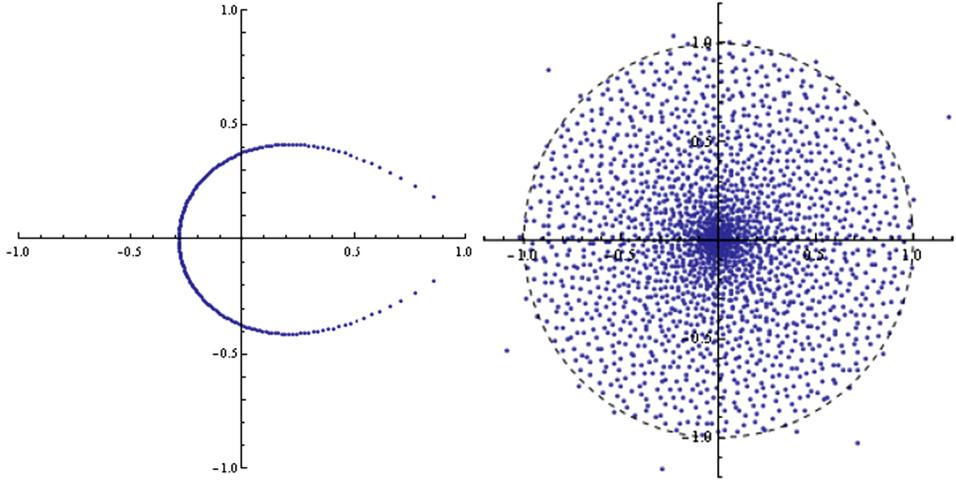


FIG. 2. *Left:* Zeros of the Szegő polynomial  $s_n(z) = \sum_{k=0}^n \frac{z^k}{k!}$  of degree  $n = 200$ . *Right:* Zeros of the Littlewood–Offord random polynomial  $W_n(z) = \sum_{k=0}^n \xi_k \frac{z^k}{k!}$  of degree  $n = 2000$  with complex normal coefficients. In both cases, the zeros were divided by  $n$ .

2.3. *Littlewood–Offord random entire function.* Next we discuss a random entire function which also was introduced by Littlewood and Offord [23, 24]. Their aim was to describe the properties of a “typical” entire function of a given order  $1/\alpha$ . Given a complex sequence  $w_0, w_1, \dots$  satisfying (3) consider a random entire function

$$(8) \quad \mathbf{W}(z) = \sum_{k=0}^{\infty} \xi_k w_k z^k.$$

Examples are given by

$$\mathbf{W}^{(1)}(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{(k!)^\alpha} z^k,$$

$$\mathbf{W}^{(2)}(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{k^{\alpha k}} z^k,$$

$$\mathbf{W}^{(3)}(z) = \sum_{k=0}^{\infty} \frac{\xi_k}{\Gamma(\alpha k + 1)} z^k.$$

The first function is essentially the flat model considered above, namely  $\mathbf{W}^{(1)}(n^\alpha z) = \mathbf{F}_n(z)$ . For  $\alpha = 1$ , it is a randomized version of the Taylor series for the exponential. The last function is a randomized version of the Mittag–Leffler function. Our aim is to describe the density of zeros of  $\mathbf{W}$  on the global scale. Let  $\mu_n$  be the measure counting the points of the form  $e^{-\beta} n^{-\alpha} z$ , where  $z$  is a zero

of  $\mathbf{W}$ . That is, for every Borel set  $B \subset \mathbb{C}$ ,

$$(9) \quad \mu_n(B) = \mu_{\mathbf{W}}(e^{\beta} n^{\alpha} B).$$

We have the following strengthening of the flat case of Theorem 2.1.

**THEOREM 2.5.** *Let  $\xi_0, \xi_1, \dots$  be nondegenerate i.i.d. random variables such that  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ . Let  $w_0, w_1, \dots$  be a complex sequence satisfying (3). With probability 1, the random measure  $\frac{1}{n} \mu_n$  converges to the deterministic measure having the density*

$$(10) \quad z \mapsto \frac{1}{2\pi\alpha} |z|^{(1/\alpha)-2}$$

with respect to the Lebesgue measure on  $\mathbb{C}$ .

As a corollary, we obtain a law of large numbers for the number of zeros of  $\mathbf{W}$ .

**COROLLARY 2.6.** *Let  $N(r) = \mu_{\mathbf{W}}(\mathbb{D}_r)$  be the number of zeros of  $\mathbf{W}$  in the disk  $\mathbb{D}_r$ . Under the assumptions of Theorem 2.5,*

$$N(r) = e^{-\beta/\alpha} r^{1/\alpha} (1 + o(1)) \quad \text{a.s. as } r \rightarrow \infty.$$

In the case  $\alpha = 1/2$  the limiting measure in Theorem 2.5 has constant density  $1/\pi$ . The difference between the limiting densities in Theorems 2.3 and 2.5 is that in the latter case there is no restriction to the unit disk. It has been pointed out by the unknown referee that in the special case of the Bernoulli-distributed  $\xi_k$ 's Theorem 2.5 can be deduced from the results of Littlewood and Offord [23, 24] using the Levin–Pfluger theory ([20], Chapter 3). Our proof is simpler than the proof of Littlewood and Offord [23, 24]. For a related work, see also [25, 26].

Let us again stress the importance of the nondegeneracy assumption. The exponential function  $e^z$  has no complex zeros, whereas the zeros of its randomized version  $\sum_{k=0}^{\infty} \xi_k \frac{z^k}{k!}$  have the global-scale density  $1/(2\pi|z|)$  on  $\mathbb{C}$ . For the absolute values of the zeros, the limiting density is constant and equal to 1 on  $(0, \infty)$ .

**2.4. Randomized theta function.** Given a parameter  $\alpha \in (0, 1) \cup (1, \infty)$  we consider a random analytic function

$$\mathbf{H}_n(z) = \begin{cases} \sum_{k=0}^{\infty} \xi_k e^{n^{1-\alpha} k^{\alpha}} z^k & (\text{case } \alpha < 1, z \in \mathbb{D}), \\ \sum_{k=0}^{\infty} \xi_k e^{-n^{1-\alpha} k^{\alpha}} z^k & (\text{case } \alpha > 1, z \in \mathbb{C}). \end{cases}$$

**THEOREM 2.7.** *Let  $\xi_0, \xi_1, \dots$  be nondegenerate i.i.d. random variables such that  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ . As  $n \rightarrow \infty$ , the sequence of random measures  $\frac{1}{n} \mu_{\mathbf{H}_n}$  converges in probability to the deterministic measure having the density*

$$z \mapsto \frac{1}{2\pi\alpha|1-\alpha|} \frac{1}{|z|^2} \left| \frac{\log|z|}{\alpha} \right|^{(2-\alpha)/(\alpha-1)}$$

*with respect to the Lebesgue measure on  $\mathbb{C}$ . The density is restricted to  $\mathbb{D}$  in the case  $\alpha < 1$  and to  $\mathbb{C} \setminus \mathbb{D}$  in the case  $\alpha > 1$ .*

As the parameter  $\alpha$  crosses the value 1, the zeros of  $\mathbf{H}_n$  jump from the unit disk  $\mathbb{D}$  to its complement  $\mathbb{C} \setminus \mathbb{D}$ . Note that the case  $\alpha = 1$  corresponds formally to Kac polynomials for which the zeros are on the boundary of  $\mathbb{D}$ . The special case  $\alpha = 2$  corresponds to the randomized theta function

$$(11) \quad \mathbf{H}_n(z) = \sum_{k=0}^{\infty} \xi_k e^{-k^2/n} z^k.$$

The limiting distribution of zeros has the density  $\frac{1}{4\pi|z|^2}$  on  $\mathbb{C} \setminus \mathbb{D}$ . One can also take the sum in (11) over  $k \in \mathbb{Z}$  in which case the zeros fill the whole complex plane with the same density.

A similar model, namely the polynomials  $\mathbf{Q}_n(z) = \sum_{k=0}^n \xi_k e^{-k\alpha} z^k$ , where  $\alpha > 1$ , has been considered by Schehr and Majumdar [27]. Assuming that  $\xi_k$  are real-valued they showed that almost all zeros of  $\mathbf{Q}_n$  become real if  $\alpha > 2$ . In our model, the distribution of the arguments of the zeros remains uniform for every  $\alpha$ .

**2.5. The general result.** We are going to state a theorem which contains all examples considered above as special cases. Let  $\xi_0, \xi_1, \dots$  be nondegenerate i.i.d. complex-valued random variables such that  $\mathbb{E} \log(1 + |\xi_0|) < \infty$ . Consider a random Taylor series

$$(12) \quad \mathbf{G}_n(z) = \sum_{k=0}^{\infty} \xi_k f_{k,n} z^k,$$

where  $f_{k,n} \in \mathbb{C}$  are deterministic coefficients. Essentially, we will assume that for some function  $u(t)$  the coefficients  $f_{k,n}$  satisfy

$$|f_{k,n}| = e^{-nu(k/n)+o(n)}, \quad n \rightarrow \infty.$$

Here is a precise statement. We assume that there is a function  $f : [0, \infty) \rightarrow [0, \infty)$  and a number  $T_0 \in (0, \infty]$  such that

- (A1)  $f(t) > 0$  for  $t < T_0$  and  $f(t) = 0$  for  $t > T_0$ .
- (A2)  $f$  is continuous on  $[0, T_0)$ , and, in the case  $T_0 < +\infty$ , left continuous at  $T_0$ .
- (A3)  $\lim_{n \rightarrow \infty} \sup_{k \in [0, An]} ||f_{k,n}|^{1/n} - f(\frac{k}{n})| = 0$  for every  $A > 0$ .

(A4)  $R_0 := \liminf_{t \rightarrow \infty} f(t)^{-1/t} \in (0, \infty]$ ,  $\liminf_{k \rightarrow \infty} |f_{k,n}|^{-1/k} \geq R_0$  for every fixed  $n \in \mathbb{N}$  and additionally,  $\liminf_{n,k/n \rightarrow \infty} |f_{k,n}|^{-1/k} \geq R_0$ .

It will be shown later that condition (A4) ensures that the series (12) defining  $\mathbf{G}_n$  converges with probability 1 on the disk  $\mathbb{D}_{R_0}$ . Let  $I : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$  be the Legendre–Fenchel transform of the function  $u(t) = -\log f(t)$ , where  $\log 0 = -\infty$ . That is,

$$(13) \quad I(s) = \sup_{t \geq 0} (st - u(t)) = \sup_{t \geq 0} (st + \log f(t)).$$

Note that  $I$  is a convex function,  $I(s)$  is finite for  $s < \log R_0$  and  $I(s) = +\infty$  for  $s > \log R_0$ . Recall that  $\mu_{\mathbf{G}_n}$  is the measure assigning to each zero of  $\mathbf{G}_n$  a weight equal to its multiplicity.

**THEOREM 2.8.** *Under the above assumptions, the sequence of random measures  $\frac{1}{n}\mu_{\mathbf{G}_n}$  converges in probability to some deterministic locally finite measure  $\mu$  on the disk  $\mathbb{D}_{R_0}$ . The measure  $\mu$  is rotationally invariant and is characterized by*

$$(14) \quad \mu(\mathbb{D}_r) = I'(\log r), \quad r \in (0, R_0).$$

By convention,  $I'$  is the left derivative of  $I$ . Since  $I$  is convex, the left derivative exists everywhere on  $(-\infty, \log R_0)$  and is a nondecreasing, left-continuous function. Since the supremum in (13) is taken over  $t \geq 0$ , we have  $\lim_{s \rightarrow -\infty} I'(s) = 0$ . Hence,  $\mu$  has no atom at zero. If  $I'$  is absolutely continuous on some interval  $(\log r_1, \log r_2)$ , then the density of  $\mu$  on the annulus  $r_1 < |z| < r_2$  with respect to the Lebesgue measure on  $\mathbb{C}$  is

$$(15) \quad \rho(z) = \frac{I''(\log |z|)}{2\pi |z|^2}.$$

It is possible to give a characterization of the measure  $\mu$  without referring to the Legendre–Fenchel transform. The radial part of  $\mu$  is a measure  $\bar{\mu}$  on  $(0, \infty)$  defined by  $\bar{\mu}((0, r)) = \mu(\mathbb{D}_r)$ . Suppose first that  $u$  is convex on  $(0, T_0)$  (which is the case in all our examples). Then,  $\bar{\mu}$  is the image of the Lebesgue measure on  $(0, \infty)$  under the mapping  $t \mapsto e^{u'(t)}$ , where  $u'$  is the left derivative of  $u$ . This follows from the fact that  $(u')^\leftarrow = I'$  and  $(I')^\leftarrow = u'$  by the Legendre–Fenchel duality, where  $\varphi^\leftarrow(t) = \inf\{s \in \mathbb{R} : \varphi(s) \geq t\}$  is the generalized left-continuous inverse of a nondecreasing function  $\varphi$ . In particular, the support of  $\mu$  is contained in the annulus

$$\{e^{\lim_{t \downarrow 0} u'(t)} \leq |z| \leq e^{\lim_{t \uparrow T_0} u'(t)}\}$$

and is equal to this annulus if  $u'$  has no jumps. In general, any jump of  $u'$  (or, by duality, any constancy interval of  $I'$ ) corresponds to a missing annulus in the support of  $\mu$ . Also, any jump of  $I'$  (or, by duality, any constancy interval of  $u'$ ) corresponds to a circle with positive  $\mu$ -measure. More precisely, if  $I'$  has a jump

at  $s$  (or, by duality,  $u'$  takes the value  $s$  on an interval of positive length), then  $\mu$  assigns a positive weight (equal to the size of the jump) to the circle of radius  $e^s$  centered at the origin. In the case when  $u$  is nonconvex, we can apply the same considerations after replacing  $u$  by its convex hull.

One may ask what measures  $\mu$  may appear as limits in Theorem 2.8. Clearly,  $\mu$  has to be rotationally invariant, with no atom at 0. The next theorem shows that there are no further essential restrictions.

**THEOREM 2.9.** *Let  $\mu$  be a rotationally invariant measure on  $\mathbb{C}$  such that*

- (1)  $\mu(\mathbb{C} \setminus \mathbb{D}_{R_0}) = 0$ , where  $R_0 := \sup\{r > 0 : \mu(\mathbb{D}_r) < \infty\} \in (0, \infty]$ .
- (2)  $\int_0^R \mu(\mathbb{D}_r)r^{-1} dr < \infty$  for some (hence, every)  $R < R_0$ .

*Then, there is a random Taylor series  $\mathbf{G}_n$  of the form (12) with convergence radius a.s.  $R_0$  such that  $\frac{1}{n}\mu_{\mathbf{G}_n}$  converges in probability to  $\mu$  on the disk  $\mathbb{D}_{R_0}$ .*

**EXAMPLE 2.10.** Consider a random polynomial

$$(16) \quad \mathbf{G}_n(z) = \sum_{k=0}^n \xi_k z^k + 2^n \sum_{k=n+1}^{2n} \xi_k \left(\frac{z}{2}\right)^k + \left(\frac{9}{2}\right)^n \sum_{k=2n+1}^{3n} \xi_k \left(\frac{z}{3}\right)^k.$$

We can apply Theorem 2.8 with

$$u(t) = \begin{cases} 0, & t \in [0, 1], \\ (\log 2)(t - 1), & t \in [1, 2], \\ (\log 3)t - \log \frac{9}{2}, & t \in [2, 3], \\ +\infty, & t \geq 3, \end{cases}$$

$$I(s) = \begin{cases} 0, & s \leq 0, \\ s, & s \in [0, \log 2], \\ 2s - \log 2, & s \in [\log 2, \log 3], \\ 3s - \log 6, & t \geq \log 3. \end{cases}$$

The function  $u'$  has three constancy intervals of length 1 where it takes values 0,  $\log 2$ ,  $\log 3$ . Dually, the function  $I'$  has three jumps of size 1 at 0,  $\log 2$ ,  $\log 3$  and is locally constant outside these points. It follows that the limiting distribution of the zeros of  $\mathbf{G}_n$  is the sum of uniform probability distributions on three concentric circles with radii 1, 2, 3.

**REMARK 2.11.** Suppose that  $\mathbf{G}_n$  satisfies the assumptions of Theorem 2.8. Then, so does the derivative  $\mathbf{G}'_n$  (and, moreover,  $f$  is the same in both cases). Thus, the derivative of any fixed order of  $\mathbf{G}_n$  has the same limiting distribution of zeros as  $\mathbf{G}_n$ . Similarly, for every complex sequence  $c_n$  such that  $\limsup_{n \rightarrow \infty} \frac{1}{n} \log |c_n| \leq f(0)$ , the function  $\mathbf{G}_n(z) - c_n$  satisfies the assumptions. Hence, the limiting distribution of the solutions of the equation  $\mathbf{G}_n(z) = c_n$  is the same as for the zeros of  $\mathbf{G}_n$ .

**3. Proofs: Special cases.** We are going to prove the results of Section 1. We will verify the assumptions of Section 2.5 and apply Theorem 2.8. Recall the notation  $u(t) = -\log f(t)$ .

**PROOF OF THEOREM 2.2.** We can assume that  $w = 0$  since otherwise we can consider the polynomial  $\mathbf{W}_n(e^{-w}z)$ . It follows from  $\lim_{k \rightarrow \infty} \frac{1}{k} \log |w_k| = 0$  that assumptions (A1)–(A4) of Section 2.5 are fulfilled with  $T_0 = 1$ ,  $R_0 = +\infty$  and

$$f(t) = \begin{cases} 1, & t \in [0, 1], \\ 0, & t > 1, \end{cases} \quad u(t) = \begin{cases} 0, & t \in [0, 1], \\ +\infty, & t > 1. \end{cases}$$

The Legendre–Fenchel transform of  $u$  is given by  $I(r) = \max(0, r)$ . It follows from (14) that  $\mu$  is the uniform probability measure on  $\mathbb{T}$ .  $\square$

**REMARK 3.1.** Under a slightly more restrictive assumption  $\mathbb{E} \log |\xi_0| < \infty$ , Theorem 2.2 can be deduced from the result of Hughes and Nikeghbali [14] (which is partially based on the Erdős–Turan inequality). This method, however, requires a subexponential growth of the coefficients and therefore fails in all other examples we consider here.

**PROOF OF THEOREM 2.1.** By the Stirling formula,  $\log n! = n \log n - n + o(n)$  as  $n \rightarrow \infty$ . It follows that assumption (A3) holds with

$$u(t) = \begin{cases} \alpha(t \log t + (1 - t) \log(1 - t)), & 0 \leq t \leq 1, \text{ elliptic case,} \\ \alpha(t \log t - t), & t \geq 0, \text{ flat case,} \\ \alpha(t \log t - (1 + t) \log(1 + t)), & t \geq 0, \text{ hyperbolic case.} \end{cases}$$

In the elliptic case,  $u(t) = +\infty$  for  $t > 1$ . The Legendre–Fenchel transform of  $u$  is given by

$$I(s) = \begin{cases} \alpha \log(1 + e^{s/\alpha}), & s \in \mathbb{R}, \text{ elliptic case,} \\ \alpha e^{s/\alpha}, & s \in \mathbb{R}, \text{ flat case,} \\ -\alpha \log(1 - e^{s/\alpha}), & s < 0, \text{ hyperbolic case.} \end{cases}$$

In the hyperbolic case,  $I(s) = +\infty$  for  $s \geq 0$ . We have  $R_0 = 1$  in the hyperbolic case and  $R_0 = +\infty$  in the remaining two cases. The proof is completed by applying Theorem 2.8.  $\square$

**PROOF OF THEOREM 2.3.** We are going to apply Theorem 2.8 to the polynomial  $\mathbf{G}_n(z) = \mathbf{W}_n(e^\beta n^\alpha z)$ . We have  $f_{k,n} = e^{\beta k + \alpha k \log n} w_k$ , for  $0 \leq k \leq n$ . Equation (3) implies that assumption (A3) is satisfied with

$$u(t) = \begin{cases} \alpha(t \log t - t), & t \in [0, 1], \\ +\infty, & t > 1. \end{cases}$$

The Legendre–Fenchel transform of  $u$  is given by

$$I(s) = \begin{cases} \alpha e^{s/\alpha}, & s \leq 0, \\ \alpha + s, & s \geq 0. \end{cases}$$

Applying Theorem 2.8, we obtain that  $\frac{1}{n}\mu_{\mathbf{G}_n}$  converges in probability to the required limit. A.s. convergence will be demonstrated in Section 4.6 below.  $\square$

PROOF OF THEOREM 2.5. We apply Theorem 2.8 to  $\mathbf{G}_n(z) = \mathbf{W}(e^\beta n^\alpha z)$ . We have  $u(t) = \alpha(t \log t - t)$  for all  $t \geq 0$ . Hence,  $I(s) = \alpha e^{s/\alpha}$  for all  $s \in \mathbb{R}$ . We can apply Theorem 2.8 to prove convergence in probability. A.s. convergence will be demonstrated in Section 4.7 below.  $\square$

PROOF OF THEOREM 2.7. Put  $\sigma = +1$  in the case  $\alpha > 1$  and  $\sigma = -1$  in the case  $\alpha < 1$ . We have  $u(t) = \sigma t^\alpha$  for  $t \geq 0$ . It follows that

$$I(r) = \begin{cases} \sigma(\alpha - 1) \left(\frac{\sigma r}{\alpha}\right)^{\alpha/(\alpha-1)}, & \sigma r \geq 0, \\ +\infty, & \sigma r < 0. \end{cases}$$

We can apply Theorem 2.8.  $\square$

#### 4. Proofs: General results.

4.1. *Method of proof of Theorem 2.8.* We use the notation and the assumptions of Section 2.5. We denote the probability space on which the random variables  $\xi_0, \xi_1, \dots$  are defined by  $(\Omega, \mathcal{F}, \mathbb{P})$ . We will write  $\mu_n = \mu_{\mathbf{G}_n}$  for the measure counting the zeros of  $\mathbf{G}_n$ . To stress the randomness of the object under consideration we will sometimes write  $\mathbf{G}_n(z; \omega)$  and  $\mu_n(\omega)$  instead of  $\mathbf{G}_n(z)$  and  $\mu_n$ . Here,  $\omega \in \Omega$ . The starting point of the proof of Theorem 2.8 is the formula

$$(17) \quad \mu_n(\omega) = \frac{1}{2\pi} \Delta \log |\mathbf{G}_n(z; \omega)|$$

for every fixed  $\omega \in \Omega$  for which  $\mathbf{G}_n(z; \omega)$  does not vanish identically. Here,  $\Delta$  denotes the Laplace operator in the complex  $z$ -plane. The Laplace operator should always be understood as an operator acting on  $\mathcal{D}'(\mathbb{D}_{R_0})$ , the space of generalized functions on the disk  $\mathbb{D}_{R_0}$ ; see, for example, Chapter II of [11]. Equation (17) follows from the formula  $\frac{1}{2\pi} \Delta \log |z - z_0| = \delta(z_0)$ , for every  $z_0 \in \mathbb{C}$ ; see Example 4.1.10 in [11]. First, we will compute the limiting logarithmic potential in (17).

THEOREM 4.1. *Under the assumptions of Section 2.5, for every  $z \in \mathbb{D}_{R_0} \setminus \{0\}$ ,*

$$(18) \quad p_n(z) := \frac{1}{n} \log |\mathbf{G}_n(z)| \xrightarrow[n \rightarrow \infty]{P} I(\log |z|).$$

We will prove Theorem 4.1 in Sections 4.2, 4.3, 4.4 below. Theorem 4.1 follows from equations (22) and (27) below. Moreover, it follows from (22) that  $\limsup_{n \rightarrow \infty} p_n(z) \leq I(\log |z|)$  a.s. Unfortunately, we were unable to prove that  $\liminf_{n \rightarrow \infty} p_n(z) \geq I(\log |z|)$  a.s. Instead, we have the following slightly weaker statement.

PROPOSITION 4.2. *Let  $l_1, l_2, \dots$  be an increasing sequence of natural numbers such that  $l_k \geq k^3$  for all  $k \in \mathbb{N}$ . Under the assumptions of Section 2.5 we have, for every  $z \in \mathbb{D}_{R_0} \setminus \{0\}$ ,*

$$(19) \quad p_{l_k}(z) = \frac{1}{l_k} \log |\mathbf{G}_{l_k}(z)| \xrightarrow[k \rightarrow \infty]{\text{a.s.}} I(\log |z|).$$

Proposition 4.2 follows from equations (22) and (27) by noting that  $\sum_{k=1}^\infty k^{-3/2} < \infty$  and applying the Borel–Cantelli lemma. The next proposition allows us to pass from convergence of potentials to convergence of measures. We will prove it Section 4.5. Recall that  $\mu_n$  counts the zeros of  $\mathbf{G}_n$ .

PROPOSITION 4.3. *Let  $l_1, l_2, \dots$  be any increasing sequence of natural numbers. Assume that for Lebesgue-a.e.  $z \in \mathbb{D}_{R_0}$  equation (19) holds. Then,*

$$(20) \quad \frac{1}{l_k} \mu_{l_k} \xrightarrow[k \rightarrow \infty]{\text{a.s.}} \frac{1}{2\pi} \Delta I(\log |z|).$$

With these results, we are in position to prove Theorem 2.8. We need to show that  $\frac{1}{n} \mu_n$  converges to  $\mu$  in probability, as a sequence of  $\mathcal{M}(\mathbb{D}_{R_0})$ -valued random variables. A sequence of random variables with values in a metric space converges in probability to some limit if and only if every subsequence of these random variables contains a subsubsequence which converges a.s. to the same limit; see, for example, Lemma 3.2 in [17]. Let a subsequence  $\frac{1}{n_1} \mu_{n_1}, \frac{1}{n_2} \mu_{n_2}, \dots$ , where  $n_1 < n_2 < \dots$ , be given. Write  $l_k = n_{k^3}$ , so that  $\{l_k\}$  is a subsequence of  $\{n_k\}$  and  $l_k \geq k^3$ . It follows from Propositions 4.2 and 4.3 that (20) holds. So, the random measure  $\frac{1}{n} \mu_n$  converges in probability to  $\frac{1}{2\pi} \Delta I(\log |z|)$ . It remains to observe that the generalized function  $\frac{1}{2\pi} \Delta I(\log |z|)$  is equal to the measure  $\mu$  given in (14). This follows from the fact that the radial part of  $\Delta$  in polar coordinates is given by  $\frac{1}{r} \frac{d}{dr} r \frac{d}{dr}$ . This gives the desired result.

4.2. *The logarithmic moment condition.* The next well-known lemma states that i.i.d. random variables grow subexponentially with probability 1 if and only if their logarithmic moment is finite.

LEMMA 4.4. *Let  $\xi_0, \xi_1, \dots$  be i.i.d. random variables. Fix  $\varepsilon > 0$ . Then,*

$$(21) \quad S := \sup_{k=0,1,\dots} \frac{|\xi_k|}{e^{\varepsilon k}} < +\infty \text{ a.s.} \iff \mathbb{E} \log(1 + |\xi_0|) < \infty.$$

PROOF. For every nonnegative random variable  $X$  we have

$$\sum_{k=1}^\infty \mathbb{P}[X \geq k] \leq \mathbb{E}X \leq \sum_{k=0}^\infty \mathbb{P}[X \geq k].$$

With  $X = \frac{1}{\varepsilon} \log(1 + |\xi_0|)$  it follows that  $\mathbb{E} \log(1 + |\xi_0|) < \infty$  if and only if  $\sum_{k=1}^{\infty} \mathbb{P}[|\xi_0| \geq e^{\varepsilon k} - 1] < \infty$  for some (equivalently, every)  $\varepsilon > 0$ . The proof is completed by applying the Borel–Cantelli lemma.  $\square$

Note in passing that Lemma 4.4 and condition (A4) imply that for every  $n \in \mathbb{N}$  the series (12) converges with probability 1 on  $\mathbb{D}_{R_0}$ .

4.3. *Upper bound in Theorem 4.1.* Fix an  $\varepsilon > 0$ . All constants which we will introduce below depend only on  $\varepsilon$ . Let us agree that all inequalities will hold uniformly over  $z \in \mathbb{D}_{e^{-2\varepsilon} R_0} \setminus \{0\}$  if  $R_0 < \infty$  and over  $z \in \mathbb{D}_{1/\varepsilon} \setminus \{0\}$  if  $R_0 = \infty$ . We will show that there exists an a.s. finite random variable  $M = M(\varepsilon)$  such that for all sufficiently large  $n$ ,

$$(22) \quad |\mathbf{G}_n(z)| \leq M e^{n(I(\log |z|) + 3\varepsilon)}.$$

First, we estimate the tail of the Taylor series (12) defining  $\mathbf{G}_n$ . By assumption (A4) there is  $A > \max(0, -\log f(0))$  such that for all  $n \geq A$  and all  $k \geq An$ ,

$$|f_{k,n}| < (|z|e^{2\varepsilon})^{-k}.$$

Lemma 4.4 implies that there exist a.s. finite random variables  $S, M'$  such that for all  $n \geq A$ ,

$$(23) \quad \left| \sum_{k \geq An} \xi_k f_{k,n} z^k \right| \leq S \sum_{k \geq An} e^{\varepsilon k} |f_{k,n}| |z|^k \leq S \sum_{k \geq An} e^{-\varepsilon k} \leq M' e^{-An}.$$

We now consider the initial part of the Taylor series (12) defining  $\mathbf{G}_n$ . Take some  $\delta > 0$ . By assumption (A3), there is  $N$  such that for all  $n > N$  and all  $k \leq An$ ,

$$(24) \quad |f_{k,n}| < \left( f\left(\frac{k}{n}\right) + \delta \right)^n.$$

It follows from (13) that for all  $t \geq 0$ ,

$$(25) \quad t \log |z| + \log f(t) \leq I(\log |z|).$$

Using (24), (25) and Lemma 4.4 with  $\varepsilon/A$  instead of  $\varepsilon$  we obtain that there is an a.s. finite random variable  $M''$  such that for all sufficiently large  $n$ ,

$$(26) \quad \begin{aligned} \left| \sum_{0 \leq k < An} \xi_k f_{k,n} z^k \right| &\leq M'' \sum_{0 \leq k < An} e^{(\varepsilon k)/A} \left( f\left(\frac{k}{n}\right) + \delta \right)^n |z|^k \\ &\leq M'' e^{\varepsilon n} \sum_{0 \leq k < An} \left( e^{(k/n) \log |z| + \log f(k/n)} + \delta |z|^{k/n} \right)^n \\ &\leq M'' e^{2\varepsilon n} \left( e^{I(\log |z|)} + \delta \max(1, |z|^A) \right)^n \\ &\leq M'' e^{n(I(\log |z|) + 3\varepsilon)}, \end{aligned}$$

where the last inequality holds if  $\delta = \delta(\varepsilon)$  is sufficiently small. Combining (23) and (26) and noting that  $-A < \log f(0) \leq I(\log |z|)$  by (25), we obtain that (22) holds with  $M = M' + M''$  for sufficiently large  $n$ . By enlarging  $M$ , if necessary, we can achieve that it holds for all  $n \geq A$ .

4.4. *Lower bound in Theorem 4.1.* Fix  $\varepsilon > 0$  and  $z \in \mathbb{D}_{R_0} \setminus \{0\}$ . We are going to show that

$$(27) \quad \mathbb{P}[|\mathbf{G}_n(z)| < e^{n(I(\log |z|) - 4\varepsilon)}] = O\left(\frac{1}{\sqrt{n}}\right), \quad n \rightarrow \infty.$$

We will use the Kolmogorov–Rogozin inequality in a multidimensional form which can be found in [7]. Given a  $d$ -dimensional random vector  $X$  define its concentration function by

$$(28) \quad Q(X; r) = \sup_{x \in \mathbb{R}^d} \mathbb{P}[X \in \mathbb{D}_r(x)], \quad r > 0,$$

where  $\mathbb{D}_r(x)$  is a  $d$ -dimensional ball of radius  $r$  centered at  $x$ . An easy consequence of (28) is that for all independent random vectors  $X, Y$  and all  $r, a > 0$ ,

$$(29) \quad Q(X + Y; r) \leq Q(X; r), \quad Q(aX; r) = Q(X; r/a).$$

The next result follows from Corollary 1 on page 304 of [7].

**THEOREM 4.5 (Kolmogorov–Rogozin inequality).** *There is a constant  $C_d$  depending only on  $d$  such that for all independent (not necessarily identically distributed) random  $d$ -dimensional vectors  $X_1, \dots, X_n$  and for all  $r > 0$ , we have*

$$Q(X_1 + \dots + X_n; r) \leq C_d \cdot \left(\sum_{k=1}^n (1 - Q(X_k; r))\right)^{-1/2}.$$

The idea of our proof of (27) is to use the Kolmogorov–Rogozin inequality to show that the probability of very strong cancellation among the terms of the series (12) defining  $\mathbf{G}_n$  is small. First, we have to single out those terms of  $\mathbf{G}_n$  in which  $|f_{k,n}z^k|$  is large enough. By definition of  $I$ , see (13), there is  $t_0 \in [0, T_0]$  such that  $t_0 \log |z| + \log f(t_0) > I(\log |z|) - \varepsilon$ . Moreover, by assumption (A2), we can find a closed interval  $J$  of length  $|J| > 0$  containing  $t_0$  such that

$$f(t)|z|^t > e^{I(\log |z|) - 2\varepsilon}, \quad t \in J.$$

Define a set  $\mathcal{J}_n = \{k \in \mathbb{N}_0 : k/n \in J\}$ . By assumption (A3) there is  $N$  such that for all  $n > N$  and all  $k \in \mathcal{J}_n$ ,

$$|f_{k,n}||z|^k > e^{n(I(\log |z|) - 3\varepsilon)}.$$

Let  $n > N$ . For  $k \in \mathbb{N}_0$  define

$$a_{k,n} = e^{-n(I(\log |z|) - 3\varepsilon)} f_{k,n}z^k.$$

Note that  $|a_{k,n}| > 1$  for  $k \in \mathcal{J}_n$ . Define

$$\mathbf{G}_{n,1} = \sum_{k \in \mathcal{J}_n} a_{k,n} \xi_k, \quad \mathbf{G}_{n,2} = \sum_{k \notin \mathcal{J}_n} a_{k,n} \xi_k.$$

By considering real and imaginary parts, we can view the complex random variables  $a_{k,n} \xi_k$  as two-dimensional random vectors. Using (29), we arrive at

$$(30) \quad \mathbb{P}[|\mathbf{G}_n(z)| < e^{n(I(\log |z|) - 4\epsilon)}] \leq Q(\mathbf{G}_{n,1} + \mathbf{G}_{n,2}; e^{-\epsilon n}) \leq Q(\mathbf{G}_{n,1}; e^{-\epsilon n}).$$

By Theorem 4.5, there is an absolute constant  $C$  such that for all  $r > 0$ ,

$$\begin{aligned} Q(\mathbf{G}_{n,1}; r) &\leq C \cdot \left( \sum_{k \in \mathcal{J}_n} (1 - Q(a_{k,n} \xi_k; r)) \right)^{-1/2} \\ &\leq C \cdot \left( \sum_{k \in \mathcal{J}_n} (1 - Q(\xi_k; r)) \right)^{-1/2}. \end{aligned}$$

Here, the second inequality follows from the fact that  $|a_{k,n}| > 1$  for  $k \in \mathcal{J}_n$ . Now, since the random variable  $\xi_0$  is supposed to be nondegenerate, we can choose  $r > 0$  so small that  $Q(\xi_0; r) < 1$ . Note that this is the only place in the proof of Theorem 2.8 where we use the randomness of the  $\xi_k$ 's in a nonobvious way. The rest of the proof is valid for any deterministic sequence  $\xi_0, \xi_1, \dots$  such that  $|\xi_n| = O(e^{\delta n})$  for every  $\delta > 0$ . If  $n$  is sufficiently large, then  $e^{-\epsilon n} \leq r$  and hence,

$$(31) \quad Q(\mathbf{G}_{n,1}; e^{-\epsilon n}) \leq Q(\mathbf{G}_{n,1}; r) \leq C_1 |\mathcal{J}_n|^{-1/2} \leq C_2 n^{-1/2}.$$

In the last inequality, we have used that the number of elements of  $\mathcal{J}_n$  is larger than  $(|J|/2)n$  for large  $n$ . Taking (30) and (31) together completes the proof of (27).

4.5. *Proof of Proposition 4.3.* Define a set  $A \subset \mathbb{D}_{R_0} \times \Omega$ , measurable with respect to the product of the Borel  $\sigma$ -algebra on  $\mathbb{D}_{R_0}$  and  $\mathcal{F}$ , by

$$A = \left\{ (z, \omega) : \lim_{k \rightarrow \infty} p_{l_k}(z; \omega) = I(\log |z|) \right\}.$$

We know from assumption (19) that for Lebesgue-a.e.  $z \in \mathbb{D}_{R_0}$  it holds that  $\int_{\Omega} \mathbb{1}_{(z, \omega) \notin A} \mathbb{P}(d\omega) = 0$ . By Fubini's theorem, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ , it holds that  $\int_{\mathbb{D}_{R_0}} \mathbb{1}_{(z, \omega) \notin A} \lambda(dz) = 0$ . Hence, there is a measurable set  $E_1 \subset \Omega$  with  $\mathbb{P}[E_1] = 0$  such that for every  $\omega \notin E_1$ ,

$$(32) \quad \lim_{k \rightarrow \infty} p_{l_k}(z; \omega) = I(\log |z|), \quad \text{for Lebesgue-a.e. } z \in \mathbb{D}_{R_0}.$$

Let  $k(\omega) = \min\{k \in \mathbb{N}_0 : \xi_k(\omega) \neq 0\}$ ,  $\omega \in \Omega$ . Since the  $\xi_k$ 's are assumed to be nondegenerate, the set  $E_0 = \{\omega \in \Omega : k(\omega) = \infty\}$  satisfies  $\mathbb{P}[E_0] = 0$ . By conditions (A3) and (A1), after ignoring finitely many values of  $n$ , we can assume that  $f_{k,n} \neq 0$  for  $0 \leq k \leq T_0 n/2$ . Define  $n(\omega) = 2k(\omega)/T_0$ . For  $\omega \notin E_0$  and  $n > n(\omega)$  the function  $\mathbf{G}_n$  does not vanish identically. For every fixed  $\omega \notin E_0$  and  $n > n(\omega)$

the function  $p_n(z; \omega) = \frac{1}{n} \log |\mathbf{G}_n(z; \omega)|$  is subharmonic, as a function of  $z$ ; see Example 4.1.10 in [11]. Also, it follows from (22) that there is a measurable set  $E_2 \subset \Omega$  with  $\mathbb{P}[E_2] = 0$  such that for every  $\omega \notin E_2$ , the family of functions  $\mathcal{P}_\omega = \{z \mapsto p_{l_k}(z; \omega) : k \in \mathbb{N}\}$ , is uniformly bounded above on every compact subset of  $\mathbb{D}_{R_0}$ . Let  $E = E_0 \cup E_1 \cup E_2$ , so that  $\mathbb{P}[E] = 0$ . Fix  $\omega \notin E$ . By Theorem 4.1.9 of [11], the family  $\mathcal{P}_\omega$  is either precompact in  $\mathcal{D}'(\mathbb{D}_{R_0})$ , the space of generalized functions on the disk  $\mathbb{D}_{R_0}$ , or contains a subsequence converging to  $-\infty$  uniformly on compact subsets of  $\mathbb{D}_{R_0}$ . The latter possibility is excluded by (32). Thus, the family  $\mathcal{P}_\omega$  is precompact in  $\mathcal{D}'(\mathbb{D}_{R_0})$ . Any subsequential limit of  $\mathcal{P}_\omega$  must coincide with the function  $I(\log |z|)$  by (32) and Proposition 16.1.2 in [12]. It follows that for every fixed  $\omega \notin E$ ,

$$(33) \quad p_{l_k}(z; \omega) \xrightarrow[k \rightarrow \infty]{} I(\log |z|) \quad \text{in } \mathcal{D}'(\mathbb{D}_{R_0}).$$

Since the Laplace operator is continuous on  $\mathcal{D}'(\mathbb{D}_{R_0})$ , we may apply it to the both sides of (33). Recalling (17) we obtain that for every  $\omega \notin E$ ,

$$\frac{1}{l_k} \mu_{l_k}(\omega) = \frac{1}{2\pi} \Delta p_{l_k}(z; \omega) \xrightarrow[k \rightarrow \infty]{} \frac{1}{2\pi} \Delta I(\log |z|) \quad \text{in } \mathcal{D}'(\mathbb{D}_{R_0}).$$

A sequence of locally finite measures converges in  $\mathcal{D}'(\mathbb{D}_{R_0})$  if and only if it converges vaguely. This completes the proof of (20).

4.6. *Proof of the a.s. convergence in Theorem 2.3.* Recall that convergence in probability has already been established in Section 3. To prove the a.s. convergence we first extract a subsequence to which we can apply the Borel–Cantelli lemma. Given  $n \in \mathbb{N}$  we can find a unique  $j_n \in \mathbb{N}$  such that  $j_n^3 \leq n < (j_n + 1)^3$ . Write  $m_n = j_n^3$  and  $\mathbf{G}_n(z) = \mathbf{W}_n(e^{\beta} m_n^\alpha z)$ . Note that  $\lim_{n \rightarrow \infty} m_n/n = 1$ . Thus, it suffices to show that  $\frac{1}{n} \mu_{\mathbf{G}_n}$  converges a.s. to the measure with density (5). As a first step, we will prove the a.s. convergence of the corresponding potentials. Fix  $z \in \mathbb{D} \setminus \{0\}$ . We will prove that

$$(34) \quad p_n(z) = \frac{1}{n} \log |\mathbf{G}_n(z)| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \alpha |z|^{1/\alpha}.$$

Note that  $\mathbf{G}_n$  satisfies all assumptions of Section 2.5. It follows from Proposition 4.2 applied to the subsequence  $l_j = j^3$  that

$$(35) \quad \frac{1}{m_n} \log |\mathbf{G}_{m_n}(z)| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \alpha |z|^{1/\alpha}.$$

Let now  $n \in \mathbb{N}$  be a sufficiently large number not of the form  $j^3$ . We have, by Lemma 4.4 and (3),

$$\begin{aligned} |\mathbf{G}_n(z) - \mathbf{G}_{m_n}(z)| &= \left| \sum_{k=m_n+1}^n \xi_k w_k e^{\beta k} m_n^{\alpha k} z^k \right| \\ &\leq S e^{2\epsilon n} \sum_{k=m_n+1}^n e^{-\alpha(k \log k - k)} n^{\alpha k} |z|^k. \end{aligned}$$

The function  $x \mapsto -\alpha(x \log x - x) + \alpha x \log n$  defined for  $x > 0$  attains its maximum, which is equal to  $\alpha n$ , at  $x = n$ . Recall that  $|z| < 1$ . Since  $m_n > (1 - \varepsilon)n$  and  $\varepsilon n S < e^{\varepsilon n}$  if  $n$  is sufficiently large, we have the estimate

$$|\mathbf{G}_n(z) - \mathbf{G}_{m_n}(z)| \leq e^{3\varepsilon n} e^{\alpha n} |z|^{(1-\varepsilon)n}.$$

Since  $\alpha + \log |z| < \alpha |z|^{1/\alpha}$ , we have, if  $\varepsilon > 0$  is small enough,

$$(36) \quad |\mathbf{G}_n(z) - \mathbf{G}_{m_n}(z)| \leq e^{(1-\varepsilon)n(\alpha |z|^{1/\alpha} - 2\varepsilon)} \leq e^{m_n(\alpha |z|^{1/\alpha} - 2\varepsilon)}.$$

Bringing (35) and (36) together we obtain (34).

We are ready to complete the proof. It follows from (34) and Proposition 4.3 that the restriction of  $\frac{1}{n}\mu_{\mathbf{G}_n}$  to  $\mathbb{D}$  converges a.s. to a measure  $\mu$  with density (5), as a sequence of random elements with values in  $\mathcal{M}(\mathbb{D})$ . To prove that the a.s. convergence holds in the sense of  $\mathcal{M}(\mathbb{C})$ -valued elements, we need to show that  $\lim_{n \rightarrow \infty} \frac{1}{n}\mu_{\mathbf{G}_n}(\mathbb{C} \setminus \mathbb{D}) = 0$  a.s., or, equivalently, that  $\liminf_{n \rightarrow \infty} \frac{1}{n}\mu_{\mathbf{G}_n}(\mathbb{D}) = 1$  a.s. Let  $f: \mathbb{C} \rightarrow [0, 1]$  be a continuous function with support in  $\mathbb{D}$ . Then, since  $\nu \mapsto \int f d\nu$  defines a continuous functional on  $\mathcal{M}(\mathbb{D})$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n}\mu_{\mathbf{G}_n}(\mathbb{D}) \geq \liminf_{n \rightarrow \infty} \frac{1}{n} \int_{\mathbb{C}} f d\mu_{\mathbf{G}_n} = \int_{\mathbb{C}} f d\mu \quad \text{a.s.}$$

The supremum of the right-hand side over all admissible  $f$  is equal to 1 since  $\mu(\mathbb{D}) = 1$ . This proves the claim.

4.7. *Proof of the a.s. convergence in Theorem 2.5.* Let  $m_n$  be defined in the same way as in the previous proof. Write  $\mathbf{G}_n(z) = \mathbf{W}(e^\beta m_n^\alpha z)$ . Note that  $\mathbf{G}_n$  satisfies the assumptions of Section 2.5 with  $I(s) = \alpha e^{s/\alpha}$ . By Proposition 4.2, for all  $z \in \mathbb{C} \setminus \{0\}$ ,

$$(37) \quad p_n(z) = \frac{1}{n} \log |\mathbf{G}_n(z)| \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \alpha |z|^{1/\alpha}.$$

Then, it follows from Proposition 4.3 that  $\frac{1}{n}\mu_{\mathbf{G}_n}$  converges a.s. to the measure with density (10).

4.8. *Proof of Theorem 2.4.* We prove only the implication (1)  $\Rightarrow$  (2) since the converse implication has been established in Theorem 2.3. Let  $\mathbf{W}_n(z) = \sum_{k=0}^n \xi_k w_k z^k$ , where  $w_k$  is a sequence satisfying (3) and (6). Assume that  $\mathbb{E} \log(1 + |\xi_0|) = \infty$ . Fix  $\varepsilon > 0$ . We will show that with probability 1 there exist infinitely many  $n$ 's such that all zeros of  $\mathbf{W}_n(e^\beta n^\alpha z)$  are located in the disk  $\mathbb{D}_{2\varepsilon}$ . This implies that  $\frac{1}{n}\mu_n$  does not converge a.s. to the measure with density (5). We use an idea of [16]. By Lemma 4.4,  $\limsup_{n \rightarrow \infty} |\xi_n|^{1/n} = +\infty$ . Hence, with probability 1 there exist infinitely many  $n$ 's such that

$$(38) \quad \begin{aligned} |\xi_n|^{1/n} &> \max_{k=1, \dots, n-1} |\xi_{n-k}|^{1/(n-k)}, \\ |\xi_n|^{1/n} &> \max \left\{ \frac{3C+1}{\varepsilon}, \frac{1}{e^{\alpha\varepsilon}} \right\}. \end{aligned}$$

Let  $n$  be such that (38) holds. By (6) and (38), we have for every  $z \in \mathbb{C}$  and  $k < n$ ,

$$\begin{aligned} |w_{n-k}\xi_{n-k}(e^\beta n^\alpha z)^{n-k}| &\leq C|w_n|e^{\beta k}n^{\alpha k}|\xi_n|^{(n-k)/n}|e^\beta n^\alpha z|^{n-k} \\ &= C|w_n\xi_n(e^\beta n^\alpha z)^n|(|\xi_n|^{1/n}|z|)^{-k}. \end{aligned}$$

For every  $z$  such that  $|z| > \varepsilon$ , we obtain

$$\begin{aligned} \left| \sum_{k=1}^{n-1} w_{n-k}\xi_{n-k}(e^\beta n^\alpha z)^{n-k} \right| &\leq C|w_n\xi_n(e^\beta n^\alpha z)^n| \cdot \left( \sum_{k=1}^{n-1} \frac{1}{(3C+1)^k} \right) \\ &< \frac{1}{3}|w_n\xi_n(e^\beta n^\alpha z)^n|. \end{aligned}$$

By (3) and (38), the right-hand side of this inequality goes to  $+\infty$  as  $n \rightarrow \infty$ . In particular, for sufficiently large  $n$ , it is larger than  $|\xi_0 w_0|$ . It follows that for  $|z| > \varepsilon$ , the term of degree  $n$  in the polynomial  $\mathbf{W}_n(e^\beta n^\alpha z)$  is larger, in the sense of absolute value, than the sum of all other terms. Hence, the polynomial  $\mathbf{W}_n(e^\beta n^\alpha z)$  has no zeros outside the disk  $\mathbb{D}_{2\varepsilon}$ .

4.9. *Proof of Theorem 2.9.* Start with a measure  $\mu$  satisfying the assumptions of Theorem 2.9. Define a function  $I$  by  $I(s) = \int_{-\infty}^s \mu(\mathbb{D}_{e^r}) dr$  for  $s < \log R_0$ . The integral is finite by the second assumption of the theorem. Clearly,  $I$  is nondecreasing, continuous and convex on  $(-\infty, \log R_0)$ . For  $s > \log R_0$  let  $I(s) = +\infty$ . Define  $I(\log R_0)$  by left continuity. Let now  $u$  be defined as the Legendre–Fenchel transform of  $I$ :

$$u(t) = \sup_{s \in \mathbb{R}} (st - I(s)).$$

We claim that the random analytic function  $\mathbf{G}_n(z) = \sum_{k=0}^\infty \xi_k f_{k,n} z^k$  with  $f_{k,n} = e^{-nu(k/n)}$  satisfies assumptions (A1)–(A4) of Theorem 2.8 with  $f = e^{-u}$ . By the Legendre–Fenchel duality, the function  $u$  possesses the following properties. First, it is convex and lower-semicontinuous. Second, it is finite on the interval  $[0, T_0)$ , where  $T_0 = \limsup_{t \rightarrow +\infty} I(t)/t$  satisfies  $T_0 \in (0, +\infty]$ . This holds since  $I$  is nondecreasing and  $\lim_{s \rightarrow -\infty} I(s) = 0$  by construction. Third,  $u(t) = +\infty$  for  $t > T_0$  and  $t < 0$ . This verifies assumption (A1). Fourth, formula (13) holds and  $\lim_{t \rightarrow +\infty} u(t)/t = \log R_0$ . This, together with Lemma 4.4, shows that the convergence radius of  $\mathbf{G}_n$  is  $R_0$  a.s. and verifies assumption (A4). Finally,  $u$  is continuous on  $[0, T_0)$  (since it is convex and finite there), and, in the case  $T_0 < +\infty$ , the function  $u$  is left continuous at  $T_0$  (follows from the lower-semicontinuity of  $u$ ). This verifies assumption (A2). Assumption (A3) holds trivially with  $f = e^{-u}$ .

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