Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2013, Vol. 49, No. 3, 781–816
DOI: 10.1214/12-AIHP493

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Hierarchical pinning model in correlated random environment

Quentin Berger and Fabio Lucio Toninelli

Université de Lyon and CNRS Ecole Normale Supérieure de Lyon, Laboratoire de Physique, 46, allée d'Italie, 69364 Lyon Cedex 07, France.
E-mail: quentin.berger@ens-lyon.fr; fabio-lucio.toninelli@ens-lyon.fr

Received 7 November 2011; revised 12 April 2012; accepted 18 April 2012

Abstract. We consider the hierarchical disordered pinning model studied in (*J. Statist. Phys.* **66** (1992) 1189–1213), which exhibits a localization/delocalization phase transition. In the case where the disorder is i.i.d. (independent and identically distributed), the question of relevance/irrelevance of disorder (i.e. whether disorder changes or not the critical properties with respect to the homogeneous case) is by now mathematically rather well understood (*Probab. Theory Related Fields* **148** (2010) 159–175, *Pure Appl. Math.* **63** (2010) 233–265). Here we consider the case where randomness is spatially correlated and correlations respect the hierarchical structure of the model; in the non-hierarchical model our choice would correspond to a power-law decay of correlations.

In terms of the critical exponent of the homogeneous model and of the correlation decay exponent, we identify three regions. In the first one (non-summable correlations) the phase transition disappears. In the second one (correlations decaying fast enough) the system behaves essentially like in the i.i.d. setting and the relevance/irrelevance criterion is not modified. Finally, there is a region where the presence of correlations changes the critical properties of the annealed system.

Résumé. Nous considérons le modèle hiérarchique d'accrochage sur une ligne de défaut inhomogène étudié dans (*J. Statist. Phys.* **66** (1992) 1189–1213), qui possède une transition de phase de localisation/délocalisation. Dans le cas où le désordre est i.i.d. (indépendant et identiquement distribué), la question de pertinence/non pertinence du désordre (i.e. de savoir si le désordre change ou non les propriétés critiques du système par rapport au cas homogène) est maintenant bien comprise d'un point de vue mathématique (*Probab. Theory Related Fields* **148** (2010) 159–175, *Pure Appl. Math.* **63** (2010) 233–265). Nous considérons ici le cas où le désordre est corrélé spatialement, et où les corrélations respectent la structure hiérarchique du modèle; dans le cadre non-hiérarchique, notre choix correspondrait à une décroissance en loi de puissance des corrélations.

En termes d'exposant critique du modèle homogène et d'exposant de décroissance des corrélations, nous identifions trois régions. Dans la première (corrélations non sommables), la transition de phase disparaît. Dans la deuxième (corrélations décroissant suffisamment vite), le système se comporte essentiellement comme dans le cas i.i.d., et le critère de pertinence/non pertinence du désordre n'est pas modifié. Enfin, il existe une région où le présence de corrélations change les propriétés critiques du système annealed.

MSC: 82B44; 82D60; 60K37

Keywords: Pinning models; Polymer; Disordered models; Harris criterion; Critical phenomena; Correlation

1. Introduction

A fundamental problem in the study of disordered systems is to understand to what extent quenched (i.e. frozen) randomness modifies the critical properties of a homogeneous (i.e. non-disordered) system. Basically, the first question is whether the transition survives in presence of disorder that locally randomizes the thermodynamic parameter which measures the distance from the critical point (e.g. for a ferromagnet $T - T_c$ can be randomized by adding a random component to the couplings J_{ij}). If yes, then one can ask whether the critical exponents are modified. The celebrated Harris criterion [18] states that disorder is irrelevant (i.e. a sufficiently weak disorder does not change the critical exponents) if $d\nu > 2$, where d is the space dimension and ν is the correlation length critical exponent of the homogeneous model, while it is relevant if $d\nu < 2$. The case $d\nu = 2$ is called marginal and deciding between relevance and irrelevance is a very model-dependent question.

Despite much effort, the Harris criterion is still far from having a mathematical justification. In the last few years, the *disordered pinning model* [11,13] emerged as a case where the disorder relevance question can be attacked from a rigorous point of view. This is a class of one-dimensional ($d = 1$) models, based on an underlying renewal process with power-law inter-arrival distribution; the model lives in a random environment, such that the occurrence of a renewal at step n is modified with respect to the law of the renewal by a factor $\exp(\epsilon_n)$, where ϵ_n is a sequence of i.i.d. random variables: if $\epsilon_n > 0$ (resp. $\epsilon_n < 0$) there is an energetic gain (resp. penalization) in having the renewal at n. The pinning model exhibits a localization/delocalization phase transition when the average $h := \mathbb{E}\epsilon_n$ is varied, and in the non-disordered case ($\beta^2 := \mathrm{Var}(\epsilon_n) = 0$) the critical point h_c and the critical exponent ν can be computed exactly (ν depends only on the tail exponent of the renewal inter-arrival law). Thanks to a series of recent works, the Harris criterion has been put on mathematical grounds on this case: it is now proven that, for β small, ν does not change if it is larger than 2 [1,19,23] and it does change as soon as $\beta \neq 0$ if $\nu < 2$ [17]. For the pinning model, the relevance/irrelevance question can be also asked in the following sense [9]: is the critical point of the disordered model (quenched critical point) equal to the critical point of the *annealed model*, where the partition function is replaced by its disorder average? It turns out that for β small the difference of the two critical points is zero if $\nu > 2$ [1,23], while it behaves like $\beta^{2/(2-\nu)}$ if $\nu < 2$ [2,8]. In the marginal case $\nu = 2$, relevance of disorder has also been shown, though in the weaker sense that the difference between quenched and annealed critical points is non-zero (it is essentially of order $\exp(-c/\beta^2)$, as argued in [9] and proven in [15,16]). Recently, a variational approach to the relevance/irrelevance question, based on a large deviation principle, has been proposed in [5].

Let us also add that, for the pinning model, the correlation length exponent ν should coincide with the exponent governing the vanishing of the free energy at the critical point: $\mathtt{F}(h, \beta) \simeq (h - h_c(\beta))^\nu$ (this is proven in special situations, e.g. [12,22], but it should be a rather general fact). In the rest of this work, ν will actually denote the free energy critical exponent.

It is widely expected, on general grounds, that correlations in the environment may change qualitatively the Harris criterion: in the case of a d-dimensional system where the correlation between the random potentials at i and j decays as $|i - j|^{-\xi}$, Weinrib and Halperin [24] predict that the Harris criterion is unchanged if $\xi > d$ (summable correlations), while for $\xi < d$ the condition for disorder irrelevance should be $\xi\nu > 2$.

The study of the random pinning model with correlated disorder is still in a rudimentary form. In [21] a case with finite-range correlations was studied, and no modification of the Harris criterion was found. On the other extreme, in the pinning model of [3] not only correlations decay in a power-law way, but potentials are so strongly correlated that in a system of length N there are typically regions of size N^b, for some $b > 0$, where the ϵ_n take the same value. In this case, the authors of [3] are able to compute the critical point and to give sharp estimates on the critical behavior for $\beta > 0$. In particular, they find that an arbitrarily small amount of disorder *does* change the critical exponent, irrespective of the value of the non-disordered critical exponent ν.

Hierarchical models on diamond lattices, homogeneous or disordered [4,6,7], are a powerful tool in the study of the critical behavior of statistical mechanics models, especially because real-space renormalization group transformations à la Migdal–Kadanoff are exact in this case. In this spirit, in the present work we consider the hierarchical version of the pinning model introduced in the i.i.d. setting in [9] and later studied in [14,15]. The idea is to study a polymer on a diamond hierarchical lattice, interacting with a one-dimensional defect line where the potentials ϵ_n are placed (cf. [9], Sec. 4.2, and [14], Sec. 1.2, for more details on the relation with the non-hierarchical pinning model). Thanks to the diamond structure, the partition function for a system of size 2^n turns out to be expressed by a simple recursive relation in terms of the partition functions of two systems of size 2^{n-1}, cf. (2.1). At this point one can (as we will in the following) forget about the polymer interpretation and just retain the recursion. As in the non-hierarchical case, the system exhibits a localization/delocalization phase transition witnessed by the vanishing of the free energy when h is smaller than a certain threshold value $h_c(\beta)$.

We consider the case where disorder is Gaussian and its correlation structure respects the hierarchical structure of the model: the correlation between the potential at i and j is given by $\kappa^{d(i,j)}$, where $0 < \kappa < 1$ and $d(i, j)$ is the tree distance between i and j on a binary tree. The Weinrib–Halperin criterion in this context would say that disorder is irrelevant if and only if $\nu \log_2(1/\max(\kappa, 1/2)) > 2$ which for $\kappa = 0$ (no correlations) reduces to $\nu > 2$ as for the i.i.d. case. In terms of a parameter $B \in (1, 2)$ which defines the geometry of the diamond lattice, the criterion would read equivalently (cf. (2.21))

$$\text{irrelevance} \Longleftrightarrow \max(\kappa, 1/2) < B^2/4. \tag{1.1}$$

A closer inspection of the model, however, shows easily that *the phase transition does not survive* for $\kappa > 1/2$ (cf. Section 4). When instead correlations are summable (which corresponds to $\kappa < 1/2$) we find, in agreement with (1.1), irrelevance if $B > \sqrt{2}$ (see Theorem 3.3 and Proposition 6.1). As for $B \leq \sqrt{2}$, again we find agreement with the Weinrib–Halperin criterion: disorder is relevant (see Proposition 3.5) and if in addition $\kappa < B^2/4$, the model behaves like in the i.i.d. case as far as the difference between quenched and annealed critical points is concerned, see Theorem 3.3. The crucial step (and the one which requires the most technical work) in proving Theorem 3.3 (and Proposition 6.1) is to show that for $\kappa < \min(1/2, B^2/4)$ the Gibbs measure of the annealed system near the annealed critical point is close (in a suitable sense) to the Gibbs measure of the homogeneous system near its critical point (cf. Theorem 3.1 and Proposition 3.2). This requires some work, in particular because the annealed critical point is not known explicitly for $\kappa \neq 0$. Once this is done, the proof of disorder relevance/irrelevance according to $B \lesseqgtr \sqrt{2}$ can be obtained generalizing the ideas that were developed for the i.i.d. model.

Finally, the region $B^2/4 < \kappa < 1/2$, $B < \sqrt{2}$ reserves somewhat of a surprise: while we are not able to capture sharply the behavior of the annealed model and of the difference between quenched and annealed critical points (as we do for $\kappa < \min(1/2, B^2/4)$), see Theorem 3.1, Proposition 3.2 and Theorem 3.3), we can prove that the annealed model has a different critical behavior than the homogeneous model with the same value of B. In particular, the contact fraction at the annealed critical point scales qualitatively differently (as a function of the system size) than for the homogeneous model, see Equation (5.33). In view of Theorem 3.1 mentioned above, this means that if we fix $B < \sqrt{2}$ and we increase κ starting from 0, at $\kappa = B^2/4$ the annealed system has a "phase transition" where its critical properties change. As we discuss in Section 4, this suggests that, while for $\kappa < B^2/4$ the annealed free energy near the annealed critical point $h_c^{\mathrm{a}}(\beta)$ has a singularity of type $(h - h_c^{\mathrm{a}}(\beta))^\nu$ and $\nu = \log_2 / \log(2/B)$, for $B^2/4 < \kappa < 1/2$ the annealed free energy should vanish as $h \searrow h_c^{\mathrm{a}}(\beta)$ with a larger exponent.

Let us conclude by discussing how our results would presumably read for the correlated, non-hierarchical disordered pinning model. If the disorder is Gaussian and correlations decay as $|i - j|^{-\xi}$, then we should get the same results as for the hierarchical model, provided that $\log_2(1/\kappa) = \xi$. In particular, if $\xi < 1$ (non-summable correlations) there is no phase transition (the proof of Theorem 4.1 can actually be easily adapted), and the annealed system, well defined if $\xi > 1$, would have a critical behavior different from the homogeneous one if $1 < \xi < 2/\nu$ with ν the free energy critical exponent of the homogeneous pinning model. As a side remark, let us recall that Dyson [10] used a *hierarchical* ferromagnetic Ising model (which, at least formally, resembles very much our annealed pinning model, cf. (3.2)) plus the Griffiths correlation inequalities, to derive criteria for existence of a ferromagnetic phase transition for a *non-hierarchical*, one-dimensional Ising ferromagnet with couplings decaying as $J_{i-j} \sim |i - j|^{-\xi}$. We stress that, in contrast, in our case there are no available correlation inequalities which would allow to infer directly results on the non-hierarchical pinning model starting from the hierarchical one.

Let us now give an overview of the organization of the paper:

- In Section 2 we define the model and give preliminary results, in particular on the homogeneous case, and we state our main results in Section 3;
- In Section 4 we discuss the case $\kappa > 1/2$, showing that the phase transition does not survive;
- In Section 5, we study in detail the annealed model, giving first some preliminary tools (Section 5.1), then looking at the case $\kappa < 1/2 \wedge B^2/4$ and proving Theorem 3.1 and Proposition 3.2 (Section 5.2), and finally focusing on the case $B^2/4 < \kappa < 1/2$ (Section 5.3);
- In Section 6 we prove disorder irrelevance for $\kappa < 1/2$, $B > \sqrt{2}$, and in Section 7 we prove disorder relevance for $\kappa < 1/2 \wedge B^2/4$, $B \leq \sqrt{2}$.

2. Model and preliminaries

2.1. *The hierarchical pinning model with hierarchically correlated disorder*

Let $1 < B < 2$. We consider the following iteration

$$Z_{n+1}^{(i)} = \frac{Z_n^{(2i-1)} Z_n^{(2i)} + B - 1}{B} \tag{2.1}$$

for $n \in \mathbb{N} \cup \{0\}$ and $i \in \mathbb{N}$. We study the case in which the initial condition is random and given by $Z_0^{(i)} = \mathrm{e}^{\beta\omega_i + h}$, with $h \in \mathbb{R}$, $\beta \geq 0$ and where $\omega := \{\omega_i\}_{i\in\mathbb{N}}$ is a sequence of centered Gaussian variables, whose law is denoted by $\mathbb{P}$. One defines the law $\mathbb{P}$ thanks to the correlations matrix K and note $\kappa_{ij} := \mathbb{E}[\omega_i\omega_j]$. We interpret $Z_n^{(i)}$ as the partition function on the ith block of size 2^n.

In view of the recursive definition of the partition function, we make the very natural choice of restricting to a correlation structure of hierarchical type. For $p \in \mathbb{N} \cup \{0\}$ and $k \in \mathbb{N}$, let

$$I_{k,p} := \{(k-1)2^p + 1, \dots, k2^p\} \tag{2.2}$$

be the kth block of size 2^p. We define the hierarchical distance $d(\cdot,\cdot)$ on $\mathbb{N}$ by establishing that $d(i,j) = p$ if i, j are contained in the same block of size 2^p but not in the same block of size 2^{p-1}. In other words, $d(i,j)$ is just the tree distance between i and j, if $\mathbb{N}$ is seen as the set of the leaves of an infinite binary tree.

We assume that κ_{ij} depends only on $d(i,j)$ and for $d(i,j) = p$ we write $\kappa_{ij} =: \kappa_p$ with $\kappa_0 = 1$, $\kappa_p \geq 0$ for every p. Actually, we make the explicit choice

$$\kappa_p = \kappa^p \quad \text{for some } 0 < \kappa < 1/2. \tag{2.3}$$

We will see in Section 4 that the reason why we exclude the case $\kappa \geq 1/2$ is that the model becomes less interesting (there is no phase transition for the quenched model and the annealed model is not well defined). For $\kappa = 0$, one recovers the model with i.i.d. disorder. We moreover stress that if $\kappa > 0$ such a Gaussian sequence is not ergodic, actually its law is not even translation invariant. It has however the following property: if we define $\omega_{I_{k,p}} := (\omega_{(k-1)2^p+1}, \dots, \omega_{k2^p})$ (with $I_{k,p} := \{(k-1)2^p + 1, \dots, k2^p\}$) we have that for all $p \geq 0$, the variables $(\omega_{I_{k,p}})_{k\in\mathbb{N}}$ have the same law.

It is standard that such a Gaussian law actually exists. An explicit construction can be obtained as follows. Let $\mathcal{I} = \{I_{k,p}, p \geq 0, k \in \mathbb{N}\}$ and let $\{\widehat{\omega}_I\}_{I\in\mathcal{I}}$ be a family of i.i.d. standard Gaussian $\mathcal{N}(0,1)$ variables, and note its law $\widehat{\mathbb{P}}$. Then one has the following equality in law:

$$\omega_i := \sum_{I\in\mathcal{I}; i\in I} \widehat{\kappa}_I \widehat{\omega}_I, \tag{2.4}$$

where $\widehat{\kappa}_{I_{k,p}} := \widehat{\kappa}_p := \sqrt{\kappa^p - \kappa^{p+1}}$ (just check that the Gaussian family thus constructed has the correct correlation structure; the sum in the r.h.s. of (2.4) is well defined since $\sum_p \widehat{\kappa}_p^2 = 1 < \infty$).

We point out that all our results can be easily extended to the case where $\kappa := \lim_{p\to\infty} |\kappa_p|^{1/p}$ exists and is in $(0, 1/2)$.

The *quenched* free energy of the model is defined by

$$\mathtt{F}(\beta, h) := \lim_{n\to\infty} \frac{1}{2^n} \log Z_{n,h}^{\omega} \overset{\mathbb{P}\text{-a.s.}}{=} \lim_{n\to\infty} \frac{1}{2^n} \mathbb{E}\big[\log Z_{n,h}^{\omega}\big], \tag{2.5}$$

where $Z_{n,h}^{\omega}$ denotes $Z_n^{(1)}$ (it is helpful to indicate explicitly the dependence on h and on ω, the dependence on β being implicit to get simpler notations). The above definition is justified by the following theorem:

Theorem 2.1. *The limit in* (2.5) *exists* $\mathbb{P}$*-almost surely and in* $L^1(\mathrm{d}\mathbb{P})$, *is almost surely constant and non-negative. The function* $\mathtt{F}$ *is convex, and* $\mathtt{F}(\beta,\cdot)$ *is non-decreasing. These properties are inherited from*

$$\mathtt{F}_n(\beta,h) := \frac{1}{2^n}\mathbb{E}\big[\log Z^{\omega}_{n,h}\big]. \tag{2.6}$$

$\mathtt{F}_n(\beta,h)$ *converges exponentially fast to* $\mathtt{F}(\beta,h)$, *and more precisely one has for all* $n\geq 1$

$$\mathtt{F}_n(\beta,h) - \frac{1}{2^n}\log B \leq \mathtt{F}(\beta,h) \leq \mathtt{F}_n(\beta,h) + \frac{1}{2^n}\log\left(\frac{B^2+B-1}{B(B-1)}\right). \tag{2.7}$$

We define also the *annealed* partition function $Z^{\mathrm{a}}_{n,h} := \mathbb{E}[Z^{\omega}_{n,h}]$, and the *annealed* free energy:

$$\mathtt{F}^{\mathrm{a}}(\beta,h) := \lim_{n\to\infty}\frac{1}{2^n}\log\mathbb{E}\big[Z^{\omega}_{n,h}\big]. \tag{2.8}$$

Proposition 2.2. *The limit in* (2.8) *exists, is non-negative and finite. The function* $\mathtt{F}^{\mathrm{a}}$ *is convex and* $\mathtt{F}^{\mathrm{a}}(\beta,\cdot)$ *is non-decreasing. These properties are inherited from*

$$\mathtt{F}^{\mathrm{a}}_n(\beta,h) := \frac{1}{2^n}\log\mathbb{E}\big[Z^{\omega}_{n,h}\big]. \tag{2.9}$$

$\mathtt{F}^{\mathrm{a}}_n(\beta,h)$ *converges exponentially fast to* $\mathtt{F}^{\mathrm{a}}(\beta,h)$, *and more precisely one has for all* $n\geq 1$

$$\mathtt{F}^{\mathrm{a}}_n(\beta,h) - \frac{1}{2^n}\log B \leq \mathtt{F}^{\mathrm{a}}(\beta,h) \leq \mathtt{F}^{\mathrm{a}}_n(\beta,h) + \mathrm{O}\big((2\kappa)^n\big). \tag{2.10}$$

Note that the error terms in the upper bounds in (2.7)–(2.10) are not of the same order.

Finiteness of the annealed free energy would fail if the correlations where not summable, i.e. if $\sum_j \kappa_{ij}=\infty$, which would be the case for $\kappa\geq 1/2$.

The proof of Theorem 2.1 uses the classical convergence of $\mathtt{F}_n(\beta,h)$, plus a concentration inequality of $2^{-n}\log Z^{\beta,h}_{n,h}$ around its mean (see the proof below). The fact that $\mathtt{F}(\beta,h)<\infty$ is a trivial consequence of $Z^{\omega}_{n,h}\leq \exp(\sum_{i=1}^{2^n}(\beta|\omega_i|+h))$. The proof of Proposition 2.2 is postponed to Section 5.1.

Proof of Theorem 2.1. The convergence of the average $\mathtt{F}_n(\beta,h)$ is classical, and the proof is similar to the one of Theorem 1.1 in [14]. Then, for any $\varepsilon>0$, we define

$$\Psi_{n,\varepsilon} := \mathbb{P}\big(\big|\log Z^{\omega,\beta}_{n,h} - \mathbb{E}\log Z^{\omega,\beta}_{n,h}\big| \geq \varepsilon 2^n\big),$$

and we show that $\sum_{n\in\mathbb{N}}\Psi_{n,\varepsilon}<\infty$, so that we are able to use Borel–Cantelli lemma to prove the almost-sure convergence in (2.5).

Let us briefly estimate $\Psi_{n,\varepsilon}$, recalling the following inequality for functions of a Gaussian vector $\widehat{\omega}\in\mathbb{R}^m$ of *i.i.d.* standard $\mathcal{N}(0,1)$ with law $\widehat{\mathbb{P}}$. If $f:\mathbb{R}^m\to\mathbb{R}$ is a function with Lipschitz norm L, then $\widehat{\mathbb{P}}(|f(\widehat{\omega})-\widehat{\mathbb{E}}[f(\widehat{\omega})]|\geq\varepsilon)\leq 2\mathrm{e}^{-t^2/(4L^2)}$.

We adapt the construction (2.4) for the (finite) vector $\omega^{(n)} := \{\omega_i\}_{1\leq i\leq 2^n}$. Set $\mathcal{I}^{(n)} := \{I_{k,p}, p\in\{0,\dots,n\}, k\in\{1,\dots,2^{n-p}\}\}$ (the set of blocks in a system of size 2^n), with $I_{k,p}$ defined in (2.2), and let $\widehat{\omega}^{(n)} = \{\widehat{\omega}_I\}_{I\in\mathcal{I}^{(n)}}$ be a Gaussian vector of *i.i.d.* standard $\mathcal{N}(0,1)$ variables. Then, one has the following equality in law

$$\omega_i = \sum_{I\in\mathcal{I}^{(n)}, I\ni i}\widehat{\kappa}_I\widehat{\omega}_I \quad \text{for all } i\in\big\{1,\dots,2^n\big\}, \tag{2.11}$$

where $\widehat{\kappa}_{I_{k,p}} = \widehat{\kappa}_p$, with $\widehat{\kappa}_n = \sqrt{\kappa^n}$ and $\widehat{\kappa}_p = \sqrt{\kappa^p - \kappa^{p+1}}$ for $p<n$ (one only has to check that the vector constructed as above has the right correlation structure).

From this construction, we set $f(\widehat{\omega}^{(n)}) := 2^{-n} \log Z^{\omega,\beta}_{N,h}$, and we remark that f is a function of Lipschitz norm L, with L^2 being bounded from above by

$$\sup_{\widehat{\omega}^{(n)}} \big\| \nabla f\big(\widehat{\omega}^{(n)}\big)\big\|^2 := \sup_{\widehat{\omega}^{(n)}} \sum_{I\in\mathcal{I}^{(n)}} \left|\frac{\partial f}{\partial \widehat{\omega}_I}\right|^2$$
$$= \sup_{\omega} \sum_{p=0}^{n} \sum_{k=1}^{2^{n-p}} \frac{\beta^2}{2^{2n}} \mathbf{E}^{\omega,\beta}_{n,h}\Bigg[\widehat{\kappa}_p \sum_{i\in I_{k,p}} \delta_i\Bigg]^2 \le \frac{\beta^2}{2^n} \sum_{p=0}^{n} 2^p \kappa^p. \tag{2.12}$$

Therefore, the inequality mentioned above leads to $\Psi_{n,\varepsilon} \le 2\mathrm{e}^{-cst.\varepsilon^2\beta^{-2}(2\wedge 1/\kappa)^n}$, which is summable for all $\varepsilon > 0$ if $\kappa < 1$. □

We can compare the *quenched* and *annealed* free energies, with the Jensen inequality:

$$\mathtt{F}(\beta,h) = \lim_{n\to\infty} \frac{1}{2^n}\mathbb{E}\big[\log Z^{\omega}_{n,h}\big] \le \lim_{n\to\infty}\frac{1}{2^n}\log\mathbb{E}\big[Z^{\omega}_{n,h}\big] = \mathtt{F}^{\mathrm{a}}(\beta,h). \tag{2.13}$$

The properties of $\mathtt{F}^{\mathrm{a}}$ are well known in the non-correlated case, since in this case the annealed model is just the hierarchical homogeneous pinning model (see the Section 2.3). We also have the existence of critical points for both *quenched* and *annealed* models, thanks to the convexity and the monotonicity of the free energies with respect to h:

Proposition 2.3 (Critical points). *Let $\beta > 0$ being fixed. There exist critical values $h^{\mathrm{a}}_c(\beta), h_c(\beta)$ such that*:

- $\mathtt{F}^{\mathrm{a}}(\beta,h) = 0$ *if* $h \le h^{\mathrm{a}}_c(\beta)$ *and* $\mathtt{F}^{\mathrm{a}}(\beta,h) > 0$ *if* $h > h^{\mathrm{a}}_c(\beta)$,
- $\mathtt{F}(\beta,h) = 0$ *if* $h \le h_c(\beta)$ *and* $\mathtt{F}(\beta,h) > 0$ *if* $h > h_c(\beta)$.

One has $-c_\kappa\beta^2 \le h^{\mathrm{a}}_c(\beta) \le h_c(\beta) \le 0$ for some constant $c_\kappa < \infty$.

The inequality $h^{\mathrm{a}}_c(\beta) \le h_c(\beta)$ is a direct consequence of (2.13). The fact that $h^{\mathrm{a}}_c(\beta) \ge -c_\kappa\beta^2$ is discussed after (3.2). The bound $h_c(\beta) \le 0$ follows from $\mathtt{F}(\beta,h) \ge \mathtt{F}(0,h)$, which is proven in [11], Prop. 5.1 (the proof is given there for the i.i.d. disorder model but it works identically for the correlated case, since it simply requires that $\mathbb{E}(\omega_i) = 0$).

In the sequel, we often write h^{a}_c instead of $h^{\mathrm{a}}_c(\beta)$ for brevity.

2.2. *Galton–Watson interpretation and polymer measure*

Let us take $1 < B < 2$, and set $\mathbf{P}_n$ the law of a Galton–Watson tree $\mathcal{T}_n$ of depth $n+1$, where the offspring distribution concentrates on 0 with probability $\frac{B-1}{B}$ and on 2 with probability $\frac{1}{B}$. Thus, the mean offspring size is $2/B > 1$, and the Galton–Watson process is supercritical. We then have a random binary tree with a random subset of descendants and we define the set $\mathcal{R}_n \subset \{1,\dots,2^n\}$ of individuals that are present at the nth generation (which are the leaves of $\mathcal{T}_n$).

Recall the definition (2.2) of $I_{k,p}$, the kth block of size 2^p, and of the hierarchical (tree) distance $d(\cdot,\cdot)$ introduced in Section 2.1.

One has the useful following proposition.

Proposition 2.4 ([15], Proposition 4.1). *For any $n \ge 0$ and given a subset $I \subset \{1,\dots,2^n\}$, one defines $\mathcal{T}^{(n)}_I$ to be the subtree of the standard binary tree of depth $n+1$, obtained by deleting all the edges, except those which link leaves $i \in I$ to the root. We note $v(n,I)$ the number of nodes of $\mathcal{T}^{(n)}_I$, with the convention that leaves are not counted as nodes, while the root is. Then one has*

$$\mathbf{E}_n[\delta_I] = B^{-v(n,I)}, \tag{2.14}$$

where $\delta_I := \prod_{i\in I}\delta_i$ and where $\delta_i = 1$ if the individual i is present at generation n (i.e. if $i \in \mathcal{R}_n$), and $\delta_i = 0$ otherwise. In particular $\mathbf{E}_n[\delta_i] = B^{-n}$ for every $i \in \{1,\dots,2^n\}$.

Using the recursive structure of the Galton–Watson tree $\mathcal{T}_n$, one can rewrite the partition function as

$$Z_n^{(i)} = \mathbf{E}_n\left[\exp\left(\sum_{k=1}^{2^n}(\beta\omega_{2^n(i-1)+k} + h)\delta_k\right)\right], \tag{2.15}$$

since it satisfies the iteration (2.1) and the correct initial condition $Z_0^{(i)} = \exp(\beta\omega_i + h)$. It is convenient to define

$$H_{n,h}^{\omega,(i)} = \sum_{k\in I_{i,n}}(\beta\omega_k + h)\delta_k \tag{2.16}$$

as the Hamiltonian on the ith block of size 2^n (we also write $H_{n,h}^{\omega}$ for $H_{n,h}^{\omega,(1)}$ if there is no ambiguity). This allows to introduce the polymer measure

$$\frac{\mathrm{d}\mathbf{P}_{n,h}^{\omega}}{\mathrm{d}\mathbf{P}_n} := \frac{1}{Z_{n,h}^{\omega}}\exp\left(H_{n,h}^{\omega}\right). \tag{2.17}$$

Remark 2.5. *As in the pinning model* [11], *the critical point $h_c(\beta)$ marks the transition from a delocalized to a localized regime. We observe that thanks to the convexity of the free energy, for a fixed β*

$$\partial_h \mathtt{F}(\beta, h) = \lim_{n\to\infty}\mathbf{E}_{n,h}^{\omega}\left[\frac{1}{2^n}\sum_{k=1}^{2^n}\delta_k\right], \tag{2.18}$$

almost surely in ω, for every h such that $\mathtt{F}$ is differentiable at h. This is the so-called average "contact fraction" under the measure $\mathbf{P}_{n,h}^{\omega}$. If $h < h_c(\beta)$, $\mathtt{F}(\beta, h) = 0$ and the density of contact goes to 0: *we are in the delocalized regime. On the other hand, if $h > h_c(\beta)$, we have $\mathtt{F}(\beta, h) > 0$, and there is a positive density of contacts*: *this is the localized regime.*

Such a remark applies also naturally to the annealed model.

2.3. *Critical behavior of the pure model*

It is convenient to set

$$S_n^{(i)} = \sum_{k\in I_{i,n}}\delta_k \tag{2.19}$$

to be the number of contact points on the block $I_{i,n}$, and write $S_n = S_n^{(1)}$ if there is no ambiguity. We then have of course $S_n^{(i)} = S_{n-1}^{(2i-1)} + S_{n-1}^{(2i)}$.

The pure model is the model in which $\beta = 0$: its partition function is $Z_{n,h}^{\mathrm{pure}} = \mathbf{E}_n[\exp(hS_n)]$ and we let $\mathtt{F}(h)$ denote its free energy. It is well known that the pure model exhibits a phase transition at the critical point $h_c(\beta = 0) = 0$:

Theorem 2.6 ([14], Theorem 1.2). *For every $B \in (1, 2)$, there exist two constants $c_0 := c_0(B) > 0$ and $c_0' := c_0'(B) > 0$ such that for all $0 \le h \le 1$, we have*

$$c_0 h^{\nu} \le \mathtt{F}(h) \le c_0' h^{\nu} \tag{2.20}$$

with

$$\nu = \frac{\log 2}{\log(2/B)} > 1. \tag{2.21}$$

The exponent ν is called the pure critical exponent. Note that ν is an increasing function of B, and that we have $\nu = 2$ for $B = B_c := \sqrt{2}$. We give other useful estimates on the pure model in Appendix A.

3. Main results

In this section we frequently write h_c^a instead of $h_c^{\mathrm{a}}(\beta)$.

It turns out that the effect of correlations is extremely different according to whether $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ or not. In the former case, our first result says that, the correlations decaying fast enough, the critical properties of the annealed model are very close to those of the pure one.

First, let us write down more explicitly what $Z_{n,h}^{\mathrm{a}} = \mathbb{E}[Z_{n,h}^{\omega}]$ is. Note that the Gaussian structure of the disorder is very helpful, to be able to give an explicit formula for the annealed partition function, only in terms of two points correlations. The computation gives

$$Z_{n,h}^{\mathrm{a}} = \mathbf{E}_n\left[\exp\left(\left(\frac{\beta^2}{2}+h\right)\sum_{k=1}^{2^n}\delta_k + \beta^2/2\sum_{p=1}^{n}\kappa_p \sum_{\substack{1\le i,j\le 2^n\\ d(i,j)=p}}\delta_i\delta_j\right)\right] =: \mathbf{E}_n\left[\mathrm{e}^{H_{n,h}^{\mathrm{a}}}\right]. \tag{3.1}$$

One easily realizes that

$$H_{n,h}^{\mathrm{a}} = h\sum_{k=1}^{2^n}\delta_k + \frac{\beta^2}{2}\sum_{i,j=1}^{2^n}\kappa_{ij}\delta_i\delta_j = \left(\frac{\beta^2}{2}+h\right)S_n + \beta^2\sum_{p=1}^{n}\kappa_p\sum_{i=1}^{2^{n-p}} S_{p-1}^{(2i-1)}S_{p-1}^{(2i)}. \tag{3.2}$$

In particular note that

$$\left(h+\beta^2/2\right)\sum_{k=1}^{2^n}\delta_k \le H_{n,h}^{\mathrm{a}} \le \left(h+c_\kappa\beta^2\right)\sum_{k=1}^{2^n}\delta_k := \left(h+\frac{\beta^2}{2}\sum_{p\ge 0}2^{p-1}\kappa^p\right)\sum_{k=1}^{2^n}\delta_k,$$

which together with the fact that $h_c(\beta=0)=0$, implies $-c_\kappa\beta^2 \le h_c^{\mathrm{a}}(\beta) \le -\beta^2/2$.

We also use the notation $H_n^{\mathrm{a},(k)}$ for the "annealed Hamiltonian" on the kth block of size 2^n

$$H_{n,h}^{\mathrm{a},(k)} = h\sum_{l\in I_{k,n}}\delta_l + \frac{\beta^2}{2}\sum_{i,j\in I_{k,n}}\kappa_{ij}\delta_i\delta_j$$

and the following relation holds:

$$H_{n+1,h}^{\mathrm{a}} = H_{n,h}^{\mathrm{a},(1)} + H_{n,h}^{\mathrm{a},(2)} + \beta^2\kappa_{n+1}S_n^{(1)}S_n^{(2)}. \tag{3.3}$$

If we set $h = h_c^{\mathrm{a}} + u$, so that the phase transition is at $u=0$, one has

$$Z_{n,h}^{\mathrm{a}} = \mathbf{E}_n\left[\exp(uS_n)\mathrm{e}^{H_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}}\right] = Z_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}\mathbf{E}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}\left[\exp(uS_n)\right], \tag{3.4}$$

where

$$\frac{\mathrm{d}\mathbf{P}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}}{\mathrm{d}\mathbf{P}_n} := \frac{1}{Z_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}}\exp\left(H_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}\right). \tag{3.5}$$

The measure $\mathbf{P}_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}$ is the annealed polymer measure at the critical point h_c^{a}.

We can finally formulate our first result:

Theorem 3.1. *Let $\kappa < \frac{B^2}{4}\wedge\frac{1}{2}$. There exist some $\beta_0>0$ and constants $c_1, c_2 > 0$ such that for every $\beta\le\beta_0$ and $u\in[0,1]$, one has*

$$-c_2\beta^2\left(\frac{4\kappa}{B^2}\right)^n + \mathbf{E}_n\left[\exp\left(\mathrm{e}^{-c_1\beta^2}uS_n\right)\right] \le \mathbf{E}_n\left[\exp(uS_n)\mathrm{e}^{H_{n,h_c^{\mathrm{a}}}^{\mathrm{a}}}\right] \le \mathbf{E}_n\left[\exp\left(\mathrm{e}^{c_1\beta^2}uS_n\right)\right] \tag{3.6}$$

so that, for any $u \in [0,1]$,

$$\mathtt{F}\big(e^{-c_1\beta^2}u\big) \le \mathtt{F}^a\big(\beta, h_c^a + u\big) \le \mathtt{F}\big(e^{c_1\beta^2}u\big). \tag{3.7}$$

Theorem 3.1 is saying that the critical behavior of the annealed free energy around h_c^a is the same as that of the pure model around $h = 0$ (in particular, same critical exponent ν).

The essential tool is to prove that the measures $\mathbf{P}_n$ and $\mathbf{P}^a_{n,h_c^a}$ are close. This is the contents of the following proposition:

Proposition 3.2. *If $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$, then there exist some $\beta_0 > 0$ and a constant $c_1 > 0$ such that, for every $\beta \le \beta_0$, for any non-empty subset I of $\{1, \dots, 2^n\}$ one has*

$$\big(e^{-c_1\beta^2}\big)^{|I|}\mathbf{E}_n[\delta_I] \le \mathbf{E}_n\big[\delta_I e^{H^a_{n,h_c^a}}\big] \le \big(e^{c_1\beta^2}\big)^{|I|}\mathbf{E}_n[\delta_I], \tag{3.8}$$

where $\delta_I := \prod_{i\in I}\delta_i$. The case $I = \varnothing$ is dealt with by Lemma 5.1 below, that says that the partition function at the critical point approaches 1 exponentially fast:

$$e^{-c_2\beta^2(4\kappa/B^2)^n} \le Z^a_{n,h_c^a} \le 1. \tag{3.9}$$

Observe that (3.9) says that if $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ the partition function of the annealed model at h_c^a is very close to that of the pure model at its critical point $h = 0$ (which equals identically 1). We will see in Theorem 3.6 that (3.8) fails, even for $\beta > 0$ small, if $\kappa > \frac{B^2}{4} \wedge \frac{1}{2}$.

With the crucial Proposition 3.2 in hand, it is not hard to prove that for $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ the Harris criterion for disorder relevance is not modified by the presence of disorder correlations:

Theorem 3.3. *Let $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$. Recall the value of the pure critical exponent $\nu = \log 2/\log(2/B)$.*

- *If $1 < B \le B_c = \sqrt{2}$, then disorder is relevant: the quenched and annealed critical points differ for every $\beta > 0$, and:*
  - *if $B < B_c$, there exist a constant $c_3 > 0$ such that for every $0 \le \beta \le 1$*

$$(c_3)^{-1}\beta^{2/(2-\nu)} \le h_c(\beta) - h_c^a(\beta) \le c_3\beta^{2/(2-\nu)}; \tag{3.10}$$

  - *if $B = B_c$, there exist a constant $c_4 > 0$ and some $\beta_0 > 0$ such that for every $0 \le \beta \le \beta_0$*

$$\exp\left(-\frac{c_4}{\beta^4}\right) \le h_c(\beta) - h_c^a(\beta) \le \exp\left(-\frac{c_4^{-1}}{\beta^{2/3}}\right). \tag{3.11}$$

- *If $B_c < B < 2$, then disorder is irrelevant: there exists some $\beta_0 > 0$ such that $h_c(\beta) = h_c^a(\beta)$ for any $0 < \beta \le \beta_0$. More precisely, for every $\eta > 0$ and choosing $u > 0$ sufficiently small, $\mathtt{F}(\beta, h_c^a(\beta) + u) \ge (1-\eta)\mathtt{F}^a(\beta, h_c^a(\beta) + u)$.*

With some extra effort one can presumably improve the upper bound (3.11) to $e^{-c_2^{-1}/\beta^2}$ and the lower bound to $\exp(-c_2(\epsilon)/\beta^{2+\epsilon})$ for every $\epsilon > 0$, as is known for the uncorrelated case $\kappa = 0$ [15,16]. We will not pursue this line.

Remark 3.4. It is important to note that Theorems 3.1 and 3.3 do not require the knowledge of the value of h_c^a (in general there is no hope to compute it exactly). This makes the analysis of the quenched model considerably more challenging than in the i.i.d. disorder case $\kappa = 0$, where it is immediate to see that $h_c^a(\beta) = -\beta^2/2$.

We mentioned in the Introduction that for the i.i.d. model one can prove that, when the free-energy critical exponent ν of the homogeneous model is smaller than 2, such exponent is modified by an arbitrarily small amount of disorder (more precisely, the result is that the exponent is at least 2 as soon as $\beta > 0$). The same holds for the model with correlated disorder:

Proposition 3.5. *If $\kappa < 1/2$, for every $B \in (1,2)$ there exists a constant $c(B) < \infty$ such that for all $\beta > 0$ and $h \in \mathbb{R}$, we have*

$$\mathtt{F}(\beta,h) \le \frac{c(B)}{\beta^2}\big(h - h_c(\beta)\big)_+^2. \tag{3.12}$$

We restrict to $\kappa < 1/2$ since otherwise there is no phase transition.

We do not give here the proof of this proposition since, thanks to summability of the correlations, it is very similar to the one for the i.i.d. hierarchical model [20].

In the case $1/2 > \kappa \ge B^2/4$ correlations have a much more dramatic effect on critical properties and in particular we expect them to change the value of the annealed critical exponent from the value $\nu = \log 2/\log(2/B)$ to a larger one. Partial results in this direction are collected in the following theorem, which shows that (some) critical properties of the annealed model differ from those of the homogenous one.

Theorem 3.6. *Let $B^2/4 < \kappa < 1/2$ and $\beta > 0$. In contrast with (3.9), the partition function at the critical point does not converge to 1. Rather, one has*

$$\prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}} \le \frac{1}{\beta\sqrt{\kappa}}\left(\frac{B}{2\sqrt{\kappa}}\right)^n. \tag{3.13}$$

Also, the average number of individuals at generation n at the critical point satisfies

$$\mathbf{E}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}[S_n] = \mathbf{E}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\left[\sum_{i=1}^{2^n}\delta_i\right] \le \frac{c(B)}{\beta}\frac{1}{\kappa^{(n+1)/2}}. \tag{3.14}$$

When proving Theorem 3.6 we will actually prove that the mth moment of S_n under $\mathbf{P}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ is at most of order $\kappa^{-mn/2}$. Therefore, with high probability S_n is much smaller than $(2/B)^n$, which would be the order of magnitude of S_n for $\kappa < B^2/4 \wedge 1/2$, as can be deduced from Propositions 3.2 and 2.4.

In other words, if we fix $B < \sqrt{2}$ and we let κ grow but tuning h so that we are always at the annealed critical point, there is a phase transition in the behavior of the finite-volume contact fraction when crossing the value $\kappa = B^2/4$, cf. also Fig. 1.

4. The case $\kappa > 1/2$

Restricting to the event where all the δ_n are equal to 1 and using Proposition 2.4, one sees that

$$Z^{\mathrm{a}}_{n,h} \ge \left(\frac{1}{B}\right)^{2^n}\exp\left(\left(\big(h+\beta^2/2\big) + \beta^2/2\sum_{p=1}^{n}\kappa_p 2^{p-1}\right)2^n\right). \tag{4.1}$$

Thus, we see that $\mathtt{F}^{\mathrm{a}}(\beta,h) = \infty$ unless

$$K_\infty := \sum_{p=0}^{\infty}\kappa_p 2^p < +\infty. \tag{4.2}$$

For $\kappa > 1/2$, not only the annealed free energy is ill-defined. One can also prove that the quenched free energy is strictly positive for every value of $h \in \mathbb{R}$: the quenched system does not have a localization/delocalization phase transition.

Theorem 4.1. *If $\kappa > 1/2$, then $\mathtt{F}(\beta,h) > 0$ for every $\beta > 0$, $h \in \mathbb{R}$, so that $h_c(\beta) = -\infty$. There exists some constant $c_5 > 0$ such that for all $h \le -1$ and $\beta > 0$*

$$\mathtt{F}(\beta,h) \ge \exp\big(-c_5|h|\big(|h|/\beta^2\big)^{\log 2/\log(2\kappa)}\big). \tag{4.3}$$

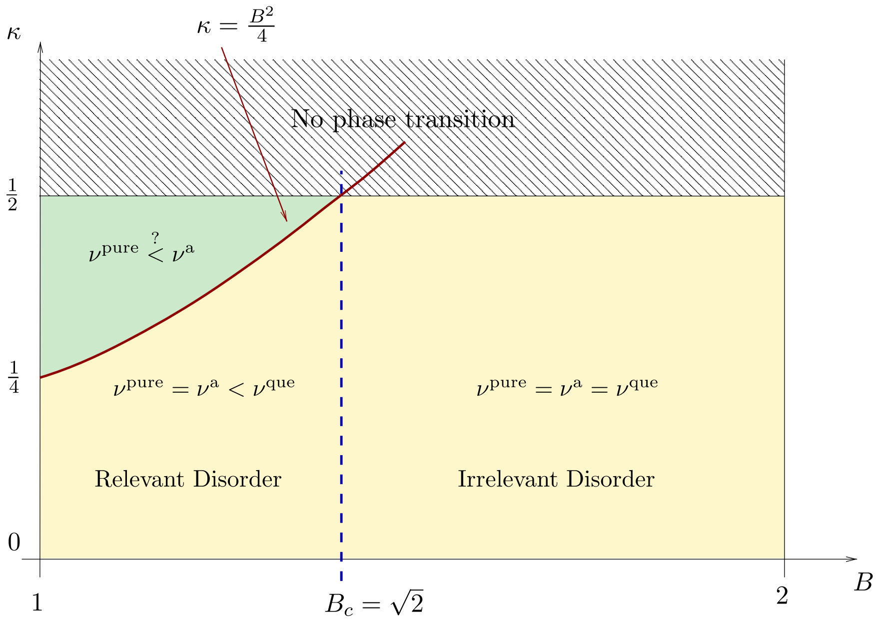

Fig. 1. Overview of the qualitative behavior of the model. One takes $\kappa < 1/2$, otherwise neither annealed nor quenched model have any phase transition. For $\kappa < 1/2 \wedge B^2/4$ the annealed model exhibits the same critical behavior as the pure one, and so the critical exponent is $\nu^{\mathrm{a}} = \nu = \log 2/\log(2/B)$. Moreover, the measures $\mathbf{P}_n$ and $\mathbf{P}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$ are similar (in the sense of Proposition 3.2) and the criterion relevance/irrelevance of disorder is the same as in the i.i.d. disorder case: disorder is irrelevant for $B > B_c := \sqrt{2}$, marginally relevant at $B = B_c$ and relevant for $B < B_c$ (cf. Theorem 3.3). The region above the parabola $\kappa = B^2/4$ remains to be understood, but partial results (Theorem 3.6) suggest that the critical behavior of the annealed model is different from the one of the pure model, in particular the annealed critical exponent should be larger. Note that disorder is proven to be relevant for all $B < B_c$, $\kappa < 1/2$ through the "smoothing result" of Proposition 3.5, showing that the quenched critical exponent is strictly larger than the pure one.

The proof of $h_c(\beta) = -\infty$ can be presumably extended to the case $\kappa = 1/2$. To avoid technicalities, we do not develop this case here.

Proof. In this proof (and in the sequel), we do not keep track of the constants $c, C, \ldots,$ and therefore they can change from line to line.

The idea is to lower bound the partition function by choosing a suitable localization strategy for the polymer to adopt, and to compute the contribution to the free energy of this strategy. This is inspired by what is done in [11], Chapter 6, to bound the critical point of the random copolymer model. More precisely one gives a definition of a "good block", supposed to be favorable to localization in that the ω_i are sufficiently positive, and analyses the contribution of the strategy of aiming only at the good blocks. For $\kappa > 1/2$ (non-summable correlations), it is a lot easier to find such large block (see Lemma 4.2 to be compared with the independent case). In this sense the behavior of the system is qualitatively different from the $\kappa < 1/2$ case.

Clearly it is sufficient to prove the claim for h negative and large enough in absolute value. Let us fix some $l \in \mathbb{N}$ (to be optimized later), take $n > l$ and let $\mathcal{I} \subset \{1, \ldots, 2^{n-l}\}$, which is supposed to denote the set of indices corresponding to "good blocks" of size 2^l. Then for any fixed ω, targeting only the blocks in $\mathcal{I}$ gives (a similar inequality was proven in [20], see Eq. (3.7))

$$Z^{\omega}_{n,h} \geq \left(\frac{B-1}{B^2}\right)^{v(n-l,\mathcal{I})} \prod_{k\in\mathcal{I}} Z^{\omega,(k)}_{l,h}, \tag{4.4}$$

where $v(n-l, \mathcal{I}_n)$ is the number of nodes in the subtree $\mathcal{T}^{(n-l)}_{\mathcal{I}}$ defined in Proposition 2.4 and $Z^{\omega,(i)}_{l,h}$ is the partition function on $I_{k,l}$, the ith block of size 2^l, cf. (2.2). The term $(\frac{B-1}{B^2})^{v(n-l,\mathcal{I})}$ is a lower bound on the probability that the node $1 \leq i \leq 2^{n-l}$ at generation $n-l$ has at least one descendant at level $n-l+1$ if and only if $i \in \mathcal{I}$ (see Fig. 2).

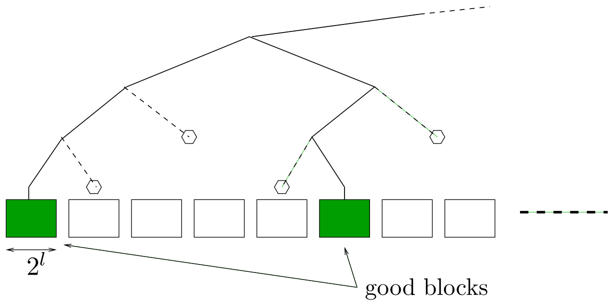

Fig. 2. The strategy of aiming exactly at the good (colored) blocks is represented above. One first places the subtree $\mathcal{T}_{\mathcal{I}}^{(n-l)}$, which is present with probability $(1/B)^{v(n-l,\mathcal{I})}$, and then forces all the leaves that do not lead to any good block (the hexagons in the figure) not to have any children, which happens with probability larger than $((B-1)/B)^{v(n-l,\mathcal{I})}$. The maximal amount of nodes that such a tree can contain is reached when all the good blocks are all equally distant one from another, and is thus bounded as in (4.5).

It was shown in [20], Eq. (3.9), that

$$v(n,\mathcal{I}) \le |\mathcal{I}|\big(2+n-l-\lfloor \log_2|\mathcal{I}|\rfloor\big) \tag{4.5}$$

so that

$$\frac{1}{2^n}\log Z_{n,h}^{\omega} \ge \frac{1}{2^n}\sum_{k\in\mathcal{I}}\log Z_{l,h}^{\omega,(k)} - \log\left(\frac{B^2}{B-1}\right)\frac{|\mathcal{I}|}{2^n}\big(2+n-l-\lfloor \log_2|\mathcal{I}|\rfloor\big). \tag{4.6}$$

Let us fix h negative with $|h|$ large and take $l=l(h)\in\mathbb{N}$ to be chosen later. Define then

$$\mathcal{A}_l^{(k)} := \big\{\text{for all } i\in I_{k,l},\ \text{ one has } \beta\omega_i+h\ge|h|\big\}, \tag{4.7}$$

and

$$\mathcal{I}(\omega)=\mathcal{I}_n(\omega) := \big\{1\le k\le 2^{n-l} : \mathcal{A}_l^{(k)} \text{ is verified}\big\}. \tag{4.8}$$

One notices that for all $k\in\mathcal{I}_n$ one has $Z_{l,h}^{\omega,(k)}\ge Z_{l,|h|}^{\mathrm{pure}}$, so that one gets from (4.6)

$$\frac{1}{2^n}\log Z_{n,h}^{\omega} \ge \frac{|\mathcal{I}_n|}{2^{n-l}}\frac{1}{2^l}\log Z_{l,|h|}^{\mathrm{pure}} - \log\left(\frac{B^2}{B-1}\right)\frac{|\mathcal{I}_n|}{2^n}\big(2+n-l-\lfloor \log_2|\mathcal{I}_n|\rfloor\big). \tag{4.9}$$

We also note $p_l := \mathbb{P}(\mathcal{A}_l^{(1)})$, so that one has $\mathbb{E}|\mathcal{I}_n| = 2^{n-l}p_l$. Then, taking the expectation in (4.9), one has

$$\frac{1}{2^n}\mathbb{E}\log Z_{n,h}^{\omega} \ge p_l\frac{1}{2^l}\log Z_{l,|h|}^{\mathrm{pure}} - c(B)p_l 2^{-l}(2-\log_2 p_l), \tag{4.10}$$

where we applied Jensen's inequality to $\mathbb{E}[|\mathcal{I}_n|\log_2|\mathcal{I}_n|]$, since $x\mapsto x\log x$ is convex. Then, letting n go to infinity, and provided that l is large enough so that $2^{-l}\log Z_{l,|h|}^{\mathrm{pure}}\ge\frac{1}{2}\mathtt{F}(|h|)$, one has $\mathbb{P}$-a.s.

$$\mathtt{F}(\beta,h)\ge p_l\mathtt{F}\big(|h|\big)/2 - c(B)2^{-l}p_l(2-\log_2 p_l) \ge p_l\big(c|h| - c'2^{-l}(2-\log_2 p_l)\big), \tag{4.11}$$

where we used that for $|h|\ge1$ one has $\mathtt{F}(|h|)\ge \mathit{const}\times|h|$.

It then remains to estimate the probability p_l.

Lemma 4.2. *If $\kappa>1/2$, there exist two constants $c,C>0$ such that for every $l\in\mathbb{N}$ and $A\ge C\sqrt{l}$ one has*

$$\mathbb{P}\big(\forall i\in\big\{1,\dots,2^l\big\},\omega_i\ge A\big)\ge c^{-1}\exp\big(-cA^2(1/\kappa)^l\big). \tag{4.12}$$

From this lemma, and choosing l such that $\sqrt{l} \le 2|h|/(C\beta)$, one gets that

$$p_l = \mathbb{P}\big(\forall i \in \{1,\dots,2^l\}, \omega_i \ge 2|h|/\beta\big) \ge c^{-1}\exp\big(-c\kappa^{-l}h^2/\beta^2\big). \tag{4.13}$$

Then in view of (4.11) one chooses $l = \log(\bar{C}|h|/\beta^2)/\log(2\kappa)$ (this is compatible with $\sqrt{l} \le 2|h|/(C\beta)$ if $|h|$ is large enough) so that $c|h| - c'2^{-l}(2 - \log_2 p_l) \ge c|h|/2 \ge c/2$ provided that $\bar{C}$ is large enough. And (4.11) finally gives with this choice of l

$$\mathtt{F}(\beta,h) \ge const \times \exp\big(-c\kappa^{-l}h^2/\beta^2\big) \ge const \times \exp\big(-c'|h|\big(|h|/\beta^2\big)^{\log 2/\log(2\kappa)}\big). \tag{4.14}$$

□

Proof of Lemma 4.2. First of all, note $\mathcal{A} = \{\forall i \in \{1,\dots,2^l\}, \omega_i \ge A\}$. We consider the measure $\bar{\mathbb{P}}$ on $\{\omega_1,\dots,\omega_{2^l}\}$ which is absolutely continuous with respect to $\mathbb{P}$, and consists in translating the ω_i's of $2A$, without changing the correlation matrix K. Then one uses the inequality

$$\mathbb{P}(\mathcal{A}) \ge \bar{\mathbb{P}}(\mathcal{A})\exp\big(-\bar{\mathbb{P}}(\mathcal{A})^{-1}\big(\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) + \mathrm{e}^{-1}\big)\big), \tag{4.15}$$

with $\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P})$ the relative entropy of $\bar{\mathbb{P}}$ w.r.t. $\mathbb{P}$. Note that $\bar{\mathbb{P}}(\mathcal{A}) = \mathbb{P}(\min_{i=1,\dots,2^l}\omega_i \ge -A) = \mathbb{P}(\max_{i=1,\dots,2^l}\omega_i \le A)$, so that from the Claim 4.3 below, and using that $A \ge C\sqrt{l}$, one has $\bar{\mathbb{P}}(\mathcal{A}) \ge 1/2$.

Claim 4.3. *Let $\{\omega_i\}_{i\in\{1,\dots,2^l\}}$ be a centered Gaussian vector of law $\mathbb{P}$, with covariance matrix K such that all $\kappa_{ij} \ge 0$ and $\kappa_{ii} = 1$. There exists a constant $C > 0$ such that*

$$\mathbb{P}\Big(\max_{i=1,\dots,2^l}\omega_i \le C\sqrt{l}\Big) \ge 1/2. \tag{4.16}$$

It follows from the classical Slepian's lemma that if $\{\widehat{\omega}_i\}_{i\in\{1,\dots,2^l\}}$ is a vector of i.i.d. standard Gaussian variables (whose law is denoted $\widehat{\mathbb{P}}$), then one has

$$\mathbb{E}\Big[\max_{i=1,\dots,2^l}\omega_i\Big] \le \widehat{\mathbb{E}}\Big[\max_{i=1,\dots,2^l}\widehat{\omega}_i\Big] \le c\sqrt{l}, \tag{4.17}$$

where the second inequality is classical. Thus one gets

$$\mathbb{P}\Big(\max_{i=1,\dots,2^l}\omega_i \ge 2c\sqrt{l}\Big) \le \frac{1}{2c\sqrt{l}}\mathbb{E}\Big[\max_{i=1,\dots,2^l}\omega_i\Big] \le 1/2. \tag{4.18}$$

One is thus left with estimating the relative entropy $\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P})$ in (4.15). A straightforward Gaussian computation gives

$$\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) = 2A^2\big\langle K^{-1}\mathbf{1},\mathbf{1}\big\rangle,$$

where $\mathbf{1}$ is the vector whose 2^l elements are all equal to 1. From Lemma B.1 one sees that $\mathbf{1}$ is an eigenvector of K, with eigenvalue $\lambda := \kappa_0 + \sum_{p=1}^{l}2^{p-1}\kappa_p \ge const \times (2\kappa)^l$, so that $\mathrm{H}(\bar{\mathbb{P}}|\mathbb{P}) \le cA^2(1/\kappa)^l$, which combined with (4.15) gives the right bound. □

5. Study of the annealed model

Let us remark first of all that since $\kappa_n \ge 0$, thanks to (3.3) one has $H^{\mathrm{a}}_{n+1,h} \ge H^{\mathrm{a},(1)}_{n,h} + H^{\mathrm{a},(2)}_{n,h}$, and therefore

$$Z^{\mathrm{a}}_{n+1,h} \ge \frac{(Z^{\mathrm{a}}_{n,h})^2 + B - 1}{B}. \tag{5.1}$$

From this one deduces that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \le 1$. Indeed, the map $x \mapsto (x^2 + (B-1))/B$ has an unstable fixed point at 1, and $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} > 1$ would imply that $\mathtt{F}^{\mathrm{a}}(\beta, h^{\mathrm{a}}_c) > 0$.

5.1. *An auxiliary partition function, proof of Proposition* 2.2

It is very convenient for the following to introduce a modified partition function, both for the quenched case and for the annealed one, defining

$$\bar{Z}^{\omega}_{n,h} = \mathbf{E}_n\bigl[\exp\bigl(H^{\omega}_{n,h} + \theta\beta^2\kappa_n(S_n)^2\bigr)\bigr], \quad \text{with } \theta := \frac{\kappa}{2(1-2\kappa)} \tag{5.2}$$

and

$$\bar{Z}^{\mathrm{a}}_{n,h} = \mathbb{E}\bigl[\bar{Z}^{\omega}_{n,h}\bigr] = \mathbf{E}_n\bigl[\exp\bigl(\bar{H}^{\mathrm{a}}_{n,h}\bigr)\bigr], \tag{5.3}$$

with

$$\bar{H}^{\mathrm{a}}_{n,h} = H^{\mathrm{a}}_{n,h} + \theta\beta^2\kappa_n(S_n)^2. \tag{5.4}$$

Note that θ vanishes for $\kappa \to 0$ (no need of the auxiliary partition function for the non-correlated model) and that it diverges for $\kappa \to 1/2$, where the annealed model is not well-defined.

We also naturally define $\bar{\mathtt{F}}^{\mathrm{a}}(\beta,h) := \lim_{n\to\infty} 2^{-n}\log \bar{Z}^{\mathrm{a}}_{n,h}$ (the existence of the limit will be shown in the course of the proof of Proposition 2.2) and, using $\delta_k \le 1$, one gets that $Z^{\mathrm{a}}_{n,h} \le \bar{Z}^{\mathrm{a}}_{n,h} \le \mathrm{e}^{\theta\beta^2(4\kappa)^n} Z^{\mathrm{a}}_{n,h}$, so that $\bar{\mathtt{F}}^{\mathrm{a}}(\beta,h) = \mathtt{F}^{\mathrm{a}}(\beta,h)$ (recall we chose $\kappa < 1/2$). Similarly, if $\bar{\mathtt{F}}(\beta,h) := \lim_{n\to\infty} 2^{-n}\log \bar{Z}^{\omega}_{n,h}$ then $\bar{\mathtt{F}}(\beta,h) = \mathtt{F}(\beta,h)$.

Then, from (3.3), one gets that (recall $\kappa_n = \kappa^n$ and (2.19))

$$\begin{aligned}
\bar{H}^{\mathrm{a}}_{n+1,h} &\le H^{\mathrm{a},(1)}_{n,h} + H^{\mathrm{a},(2)}_{n,h} + \frac{\beta^2}{2}\kappa^{n+1}\bigl(S^{(1)}_n\bigr)^2 + \frac{\beta^2}{2}\kappa^{n+1}\bigl(S^{(2)}_n\bigr)^2 \\
&\quad + 2\theta\beta^2\kappa^{n+1}\bigl(S^{(1)}_n\bigr)^2 + 2\theta\beta^2\kappa^{n+1}\bigl(S^{(1)}_n\bigr)^2 \\
&= H^{\mathrm{a},(1)}_{n,h} + \theta\beta^2\kappa^n\bigl(S^{(1)}_n\bigr)^2 + H^{\mathrm{a},(2)}_{n,h} + \theta\beta^2\kappa^n\bigl(S^{(2)}_n\bigr)^2 = \bar{H}^{\mathrm{a},(1)}_{n,h} + \bar{H}^{\mathrm{a},(2)}_{n,h},
\end{aligned} \tag{5.5}$$

where we used the self-explanatory notation $\bar{H}^{\mathrm{a},(i)}_{n,h}$ for the auxiliary Hamiltonian in the block $I_{i,n}$. We used the bounds $ab \le 1/2(a^2+b^2)$ and $(a+b)^2 \le 2(a^2+b^2)$ and then the definition of θ.

This gives in particular that

$$\bar{Z}^{\mathrm{a}}_{n+1,h} \le \frac{(\bar{Z}^{\mathrm{a}}_{n,h})^2 + B - 1}{B}, \tag{5.6}$$

from which one deduces that $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \ge 1$ for all $n \in \mathbb{N}$. Indeed, otherwise, for some $n_0 \in \mathbb{N}$ one has $\bar{Z}^{\mathrm{a}}_{n_0,h^{\mathrm{a}}_c} < 1$, and then one can find some $h > h^{\mathrm{a}}_c$ such that $\bar{Z}^{\mathrm{a}}_{n_0,h} \le 1$, which combined with (5.6) gives that $\bar{Z}^{\mathrm{a}}_{n,h} \le 1$ for all $n \ge n_0$. Therefore one would have $\mathtt{F}^{\mathrm{a}}(\beta,h) = \bar{\mathtt{F}}^{\mathrm{a}}(\beta,h) = 0$, which is a contradiction with the definition of h^{a}_c.

Proof of Proposition 2.2. One has from (5.1)

$$\frac{Z^{\mathrm{a}}_{n+1,h}}{B} \ge \left(\frac{Z^{\mathrm{a}}_{n,h}}{B}\right)^2, \tag{5.7}$$

and from (5.6) and the fact that $\bar{Z}^{\mathrm{a}}_{n,h} \ge (B-1)/B$

$$K_B\bar{Z}^{\mathrm{a}}_{n+1,h} \le \bigl(K_B\bar{Z}^{\mathrm{a}}_{n,h}\bigr)^2 \quad \text{with} \quad K_B = \frac{B^2+B-1}{B(B-1)}. \tag{5.8}$$

Therefore, the sequence $\{2^{-n}\log(Z^{\mathrm{a}}_{n,h}/B)\}_{n\ge1}$ and $\{2^{-n}\log(K_B\bar{Z}^{\mathrm{a}}_{n,h})\}_{n\ge1}$ are non-decreasing and non-increasing respectively, so that both converge to a limit, $\mathtt{F}^{\mathrm{a}}(\beta,h)$ and $\bar{\mathtt{F}}^{\mathrm{a}}(\beta,h)$ respectively, but we have already remarked earlier in this section that $\mathtt{F}^{\mathrm{a}}(\beta,h)=\bar{\mathtt{F}}^{\mathrm{a}}(\beta,h)$. One finally has

$$\begin{aligned}&\mathtt{F}^{\mathrm{a}}(\beta,h)\geq \mathtt{F}^{\mathrm{a}}_n(\beta,h)-2^{-n}\log B,\\&\mathtt{F}^{\mathrm{a}}(\beta,h)=\bar{\mathtt{F}}^{\mathrm{a}}(\beta,h)\leq \bar{\mathtt{F}}^{\mathrm{a}}_n(\beta,h)+2^{-n}\log K_B,\end{aligned}\tag{5.9}$$

so that since $\bar{\mathtt{F}}^{\mathrm{a}}_n(\beta,h)\leq \mathtt{F}^{\mathrm{a}}_n(\beta,h)+\theta\beta^2(2\kappa)^n$, one gets the desired result. □

5.2. *Proof of Theorem 3.1 and Proposition 3.2*

The really crucial point is to prove that, provided that $\kappa<\frac{B^2}{4}\wedge\frac{1}{2}$, the annealed partition function (and the auxiliary one $\bar{Z}^{\mathrm{a}}_{n,h}$) at the annealed critical point converges exponentially fast to 1.

Lemma 5.1. *If $\kappa<\frac{B^2}{4}\wedge\frac{1}{2}$ then there exist some constant $c_2>0$ and some $\beta_0>0$ such that for any $n\geq 0$ and every $\beta\leq\beta_0$, one has*

$$\begin{aligned}&\exp\bigl(-c_2\beta^2\bigl(4\kappa/B^2\bigr)^n\bigr)\leq Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\leq 1,\\&1\leq \bar{Z}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\leq \exp\bigl(c_2\beta^2\bigl(4\kappa/B^2\bigr)^n\bigr).\end{aligned}$$

Proof of Theorem 3.1 given Lemma 5.1 and Proposition 3.2 . We expand $\exp(uS_n)$, to get

$$\mathbf{E}_n\bigl[\exp(uS_n)e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\bigr]=\sum_{k=0}^{\infty}\frac{u^k}{k!}\mathbf{E}_n\bigl[(S_n)^k e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\bigr].\tag{5.10}$$

Thanks to Proposition 3.2, we have that for any $k\geq 1$

$$\bigl(e^{-c_1\beta^2}\bigr)^k\mathbf{E}_n\bigl[(S_n)^k\bigr]\leq \mathbf{E}_n\bigl[(S_n)^k e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\bigr]\leq \bigl(e^{c_1\beta^2}\bigr)^k\mathbf{E}_n\bigl[(S_n)^k\bigr],\tag{5.11}$$

and with (5.10) we have then

$$\mathbf{E}_n\bigl[\exp(uS_n)e^{H_{n,h_c^{\mathrm{a}}}}\bigr]\leq Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}+\mathbf{E}_n\Biggl[\sum_{k=1}^{\infty}\frac{(ue^{c_1\beta^2})^k}{k!}(S_n)^k\Biggr]\leq \mathbf{E}_n\bigl[\exp\bigl(e^{c_1\beta^2}uS_n\bigr)\bigr],\tag{5.12}$$

where we used that $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\leq 1$. We naturally get the other inequality in the same way

$$\mathbf{E}_n\bigl[\exp(uS_n)e^{H_{n,h_c^{\mathrm{a}}}}\bigr]\geq \mathbf{E}_n\bigl[\exp\bigl(e^{-c_1\beta^2}uS_n\bigr)\bigr]-c_2\beta^2\left(\frac{4\kappa}{B^2}\right)^n,\tag{5.13}$$

where we used Lemma 5.1 to get that $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\geq 1-c_2\beta^2(4\kappa/B^2)^n$. □

Remark 5.2. *Using the same type of expansion, Proposition 3.2 gives more general results: for example, one can get*

$$\begin{aligned}&\mathbf{E}_n\bigl[\exp\bigl(e^{-pc_1\beta^2}u(S_n)^p\bigr)\bigr]-c_2\beta^2\left(\frac{4\kappa}{B^2}\right)^n\leq \mathbf{E}_n\bigl[e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\exp\bigl(u(S_n)^p\bigr)\bigr],\\&\mathbf{E}_n\bigl[e^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\exp\bigl(u(S_n)^p\bigr)\bigr]\leq \mathbf{E}_n\bigl[\exp\bigl(e^{pc_1\beta^2}u(S_n)^p\bigr)\bigr].\end{aligned}\tag{5.14}$$

In the sequel, we refer to this remark to avoid repeating this kind of computation.

Before proving Proposition 3.2 and Lemma 5.1, we prove the following result, valid for any $\kappa<1/2$. Given $I\subset\{1,\dots,2^n\}$ we say that I is *complete* if $2i-1\in I$ for some $i\in\mathbb{N}$ if and only if $2i\in I$.

Lemma 5.3. *For every $n \ge 1$, and any non-empty and complete subset I of $\{1, \dots, 2^n\}$, one has*

$$\left(\prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}}\right)^{|I|} \mathbf{E}_n[\delta_I] \le \mathbf{E}_n\left[\delta_I \mathrm{e}^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\right] \tag{5.15}$$

$$\le \mathbf{E}_n\left[\delta_I \mathrm{e}^{\bar{H}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\right] \le \left(\prod_{p=0}^{n-1} \bar{Z}^{\mathrm{a}}_{p,h_c^{\mathrm{a}}}\right)^{|I|} \mathbf{E}_n[\delta_I]. \tag{5.16}$$

Note that if $I = \varnothing$, these inequalities are false, since $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \le 1 \le \bar{Z}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}$.

Proof of Lemma 5.3. As the two bounds rely on a similar argument, that is $H^{\mathrm{a}}_{n+1,h_c^{\mathrm{a}}} \ge H^{\mathrm{a},(1)}_{n,h_c^{\mathrm{a}}} + H^{\mathrm{a},(2)}_{n,h_c^{\mathrm{a}}}$ in one case, and $\bar{H}^{\mathrm{a}}_{n+1,h_c^{\mathrm{a}}} \le \bar{H}^{\mathrm{a},(1)}_{n,h_c^{\mathrm{a}}} + \bar{H}^{\mathrm{a},(2)}_{n,h_c^{\mathrm{a}}}$ in the other case, we focus only on the lower bound.

We prove it by iteration, the case $n = 1$ being trivial (the only non-empty complete subset is $I = \{1, 2\}$ and the inequalities can be checked by hand). Now assume that the assumption is true for some $n \ge 1$ and take I a non-empty complete subset of $\{1, \dots, 2^{n+1}\}$. We decompose I into two subsets $I_1 = I \cap [1, 2^n]$ and $I_2 = I \cap [2^n + 1, 2^{n+1}]$ and we define $\widetilde{I}_2$ to be the subset obtained by shifting I_2 to the left by 2^n. It is easy to realize that both I_1 and $\widetilde{I}_2$ are complete subsets of $\{1, \dots, 2^n\}$ and one has $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]$.

Now, using that $H^{\mathrm{a}}_{n+1,h_c^{\mathrm{a}}} \ge H^{\mathrm{a},(1)}_{n,h_c^{\mathrm{a}}} + H^{\mathrm{a},(2)}_{n,h_c^{\mathrm{a}}}$, one has

$$\mathbf{E}_{n+1}\left[\delta_I \mathrm{e}^{H^{\mathrm{a}}_{n+1,h_c^{\mathrm{a}}}}\right] \ge \frac{1}{B}\mathbf{E}_n\left[\delta_{I_1} \mathrm{e}^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\right]\mathbf{E}_n\left[\delta_{\widetilde{I}_2} \mathrm{e}^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\right] \tag{5.17}$$

and two cases can occur.

(1) $\widetilde{I}_2 = \varnothing$, $|I_1| = |I|$ (or $I_1 = \varnothing$, $|\widetilde{I}_2| = |I|$). Then, (5.17) plus the induction step gives

$$\mathbf{E}_{n+1}\left[\delta_I \mathrm{e}^{H^{\mathrm{a}}_{n+1,h_c^{\mathrm{a}}}}\right] \ge \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}] Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \left(\prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}}\right)^{|I|}. \tag{5.18}$$

Since $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \le 1$, one has $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \ge (Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}})^{|I|}$, and obtains the claim at level $n + 1$.

(2) $I_1, I_2 \ne \varnothing$. In this case, from (5.17), the recurrence assumption directly gives

$$\mathbf{E}_{n+1}\left[\delta_I \mathrm{e}^{H^{\mathrm{a}}_{n+1,h_c^{\mathrm{a}}}}\right] \ge \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}] \left(\prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}}\right)^{|I_1|+|I_2|}. \tag{5.19}$$

This gives the result at level $n + 1$, using that $|I| = |I_1| + |I_2|$, and bounding again $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \le 1$. □

Proof of Proposition 3.2. Given $I \subset \{1, \dots, 2^n\}$, let I' be the smallest complete subset of $\{1, \dots, 2^n\}$ that contains I, and note that $|I'| \le 2|I|$. Note that

$$\mathbf{E}_n\left[\delta_I \exp\left(H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\right)\right] = \mathbf{E}_n\left[\delta_{I'} \exp\left(H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}\right)\right], \qquad \mathbf{E}_n[\delta_I] = \mathbf{E}_n[\delta_{I'}],$$

simply because of the offspring distribution of the Galton–Watson tree: if the individual $2i - 1$ is present at generation n, so is the individual $2i$. This immediately implies that the statement of Lemma 5.3 holds for every I (not necessarily complete), if $|I|$ is replaced by $2|I|$.

Then, Lemmas 5.3 and 5.1 imply Proposition 3.2 with $c_1 = 2c_2 \sum_{p=0}^{\infty} (\frac{4\kappa}{B^2})^n = 2c_2 \frac{B^2}{B^2 - 4\kappa}$. □

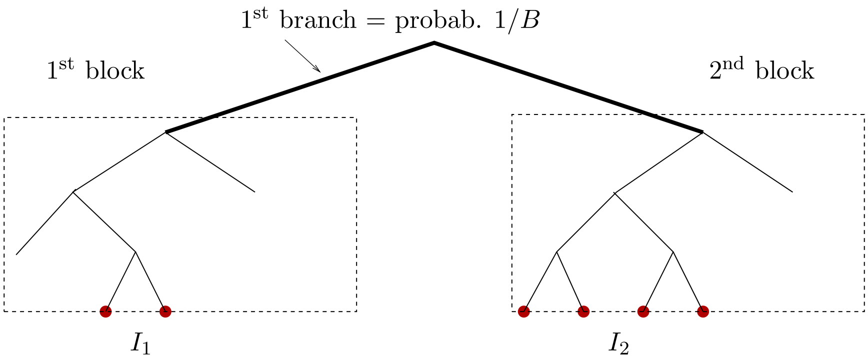

Fig. 3. Decomposition of a non-empty complete set I into two subsets I_1 and I_2. If I is non empty, the first generation must be non-empty (this has probability $1/B$). Conditionally on this, the occurrence of I_1 and I_2 are independent events.

Proof of Lemma 5.1. One would like to use a result analogue to Proposition 3.2 to bound $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} = \mathbf{E}_n[\mathrm{e}^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}} \times \exp(\theta\kappa^n(S_n)^2)]$. So we first prove a weaker upper bound. The proof relies strongly on the pure model estimates presented in Appendix A, which show that the term $\theta\kappa_n(S_n)^2$ in $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}$ has little effect if $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$.

Take $\varphi := (2\kappa) \vee \frac{4\kappa}{B^2} < 1$ and C the constant c associated to $A = 1$ in Corollary A.4, and fix some $\beta \le \beta_0$, with $\beta_0 := (\prod_{p=0}^{\infty} \mathrm{e}^{C(p+2)\varphi^p})^{-2} \le 1$. We prove iteratively on n that for all subsets I of $\{1,\dots,2^n\}$ one has

$$\mathbf{E}_n\big[\delta_I \mathrm{e}^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\big] \le (x_n)^{|I|}\mathbf{E}_n[\delta_I], \quad \text{with } x_n := \prod_{p=0}^{n} \mathrm{e}^{C(p+1)\beta\varphi^p}. \tag{5.20}$$

Note that with our choice of β_0 one has $(x_n)^2 \le \beta_0^{-1}$ for all $n \ge 0$.

The case $n = 0$ is trivial (just use that $h^{\mathrm{a}}_c \le -\beta^2/2$, as discussed after (3.2)). Now assume that (5.20) is true for some $n \ge 0$ and take I a subset of $\{1,\dots,2^{n+1}\}$.

If $I = \varnothing$, then we simply use that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \le 1$. If $I \ne \varnothing$ decompose it as in the proof of Lemma 5.3 into two subsets I_1, I_2 and let $\widetilde{I}_2$ be obtained by translating I_2 to the left by 2^n, so that $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]$ (see Fig. 3). Then, from the iteration (3.3) on $H^{\mathrm{a}}_{n,h}$ one has

$$H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c} \le H^{\mathrm{a},(1)}_{n,h^{\mathrm{a}}_c} + \frac{\beta^2}{2}\kappa^{n+1}\big(S^{(1)}_n\big)^2 + H^{\mathrm{a},(1)}_{n,h^{\mathrm{a}}_c} + \frac{\beta^2}{2}\kappa^{n+1}\big(S^{(2)}_n\big)^2, \tag{5.21}$$

so that one gets

$$\begin{aligned}\mathbf{E}_{n+1}\big[\delta_I \mathrm{e}^{H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c}}\big] \le \frac{1}{B}\mathbf{E}_n\left[\delta_{I_1}\mathrm{e}^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}} \exp\left(\frac{\beta^2}{2}\kappa^{n+1}(S_n)^2\right)\right] \\ \times \mathbf{E}_n\left[\delta_{\widetilde{I}_2}\mathrm{e}^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}} \exp\left(\frac{\beta^2}{2}\kappa^{n+1}(S_n)^2\right)\right].\end{aligned} \tag{5.22}$$

Now one can use the inductive assumption to estimate each part of (5.22). Expanding the exponential term and recalling that $\beta_0(x_n)^2 \le 1$, one has for instance

$$\begin{aligned}\mathbf{E}_n\left[\delta_{I_1}\mathrm{e}^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}} \exp\left(\frac{\beta^2}{2}\kappa^{n+1}(S_n)^2\right)\right] &= \sum_{k=0}^{\infty} \frac{(\beta^2\kappa^{n+1}/2)^k}{k!}\mathbf{E}_n\big[\delta_{I_1}\mathrm{e}^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}(S_n)^{2k}\big] \\ &\le \sum_{k=0}^{\infty}(x_n)^{|I_1|+2k}\frac{(\beta^2\kappa^{n+1}/2)^k}{k!}\mathbf{E}_n\big[\delta_{I_1}(S_n)^{2k}\big]\end{aligned} \tag{5.23}$$

$$\le (x_n)^{|I_1|}\mathbf{E}_n\big[\delta_{I_1}\mathrm{e}^{(x_n)^2(\beta^2/2)\kappa^{n+1}(S_n)^2}\big]$$
$$\le (x_n)^{|I_1|}\mathbf{E}_n\big[\delta_{I_1}\mathrm{e}^{(\beta\kappa/2)\kappa^n(S_n)^2}\big].$$

We now use Corollary A.4 to get that

$$\mathbf{E}_n\big[\delta_{I_1}\mathrm{e}^{(\beta\kappa/2)\kappa^n(S_n)^2}\big] \le \exp\left(C\frac{\beta\kappa}{2}\varphi^{-1}\varphi^{n+1}\right)^{n|I_1|+1}\mathbf{E}_n[\delta_{I_1}]. \tag{5.24}$$

Combining this with (5.22) and (5.23) and the definition of $\varphi \ge 2\kappa$ one gets

$$\mathbf{E}_{n+1}\big[\delta_I \mathrm{e}^{H^{\mathrm{a}}_{n+1,h^{\mathrm{a}}_c}}\big] \le (x_n)^{|I|}\big(\mathrm{e}^{C(\beta/4)\varphi^{n+1}}\big)^{n|I|+2}\frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]. \tag{5.25}$$

Using that $n|I|+2 \le (n+2)|I|$ (because $I \ne \varnothing$) and the definition of $x_{n+1} = x_n\mathrm{e}^{C(n+2)\beta\varphi^{n+1}}$, one gets equation (5.20) at level $n+1$.

We have performed a first crucial step: there exist some $\beta_0 > 0$ and a constant $x := \lim_{n\to\infty} x_n$, such that for every $n \in \mathbb{N}$ and every $\beta \le \beta_0$ one has

$$\mathbf{E}_n\big[\delta_I \mathrm{e}^{H^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\big] \le x^{|I|}\mathbf{E}_n[\delta_I] \quad \text{for every } I \subset \big\{1,\dots,2^n\big\}. \tag{5.26}$$

Then using the idea of Remark 5.2, one has from the definition of $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}$ (and expanding the exponential term)

$$\begin{aligned}\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} &= \mathbf{E}_n\big[\mathrm{e}^{H_{n,h^{\mathrm{a}}_c}}\big] + \sum_{k=1}^{\infty}\frac{(\theta\beta^2\kappa^n)^k}{k!}\mathbf{E}_n\big[\mathrm{e}^{H_{n,h^{\mathrm{a}}_c}}(S_n)^{2k}\big]\\ &\le Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} + \mathbf{E}_n\big[\exp\big(x^2\theta\beta^2\kappa^n(S_n)^2\big)-1\big]\\ &\le Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} + \exp\big(c\beta^2\big(4\kappa/B^2\big)^n\big)-1,\end{aligned} \tag{5.27}$$

where we used (5.26) for the first inequality and Theorem A.3 for the second one. Then using that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \le 1$, one has the desired upper bound for $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}$. On the other hand, with $\bar{Z}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \ge 1$ one gets that $Z^{\mathrm{a}}_{n,h^{\mathrm{a}}_c} \ge 1 - c'\beta^2(4\kappa/B^2)^n$, which concludes the proof. □

Remark 5.4. *Adapting the proof of Proposition 3.2 to the auxiliary partition function $\bar{Z}^{\omega}_{n,h}$, one gets under the same hypothesis that there exists a constant c'_1 such that for any non-empty subset I of $\{1,\dots,2^n\}$ one has*

$$\big(\mathrm{e}^{-c'_1\beta^2}\big)^{|I|}\mathbf{E}_n[\delta_I] \le \mathbf{E}_n\big[\delta_I\mathrm{e}^{\bar{H}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\big] \le \big(\mathrm{e}^{c'_1\beta^2}\big)^{|I|}\mathbf{E}_n[\delta_I]. \tag{5.28}$$

This implies, together with Lemma 5.1, an analog of Theorem 3.1: there exist some $\beta_0 > 0$ and constants $c'_1, c'_2 > 0$ such that for every $\beta \le \beta_0$ and $u \in [0,1]$, one has

$$\mathbf{E}_n\big[\exp\big(\mathrm{e}^{-c'_1\beta^2}uS_n\big)\big] \le \mathbf{E}_n\big[\exp(uS_n)\mathrm{e}^{\bar{H}^{\mathrm{a}}_{n,h^{\mathrm{a}}_c}}\big] \le \mathbf{E}_n\big[\exp\big(\mathrm{e}^{c'_1\beta^2}uS_n\big)\big] + c'_2\beta^2\left(\frac{4\kappa}{B^2}\right)^n. \tag{5.29}$$

5.3. *The case $B^2/4 < \kappa < 1/2$*

Proof of Theorem 3.6. Using the identity (3.3), one has for all $n \in \mathbb{N}$ and $h \in \mathbb{R}$

$$Z^{\mathrm{a}}_{n+1,h} = \frac{1}{B}\mathbf{E}_n^{\otimes 2}\big[\mathrm{e}^{H^{\mathrm{a},(1)}_{n,h}}\mathrm{e}^{H^{\mathrm{a},(2)}_{n,h}}\exp\big(\beta^2\kappa^{n+1}S_n^{(1)}S_n^{(2)}\big)\big] + \frac{B-1}{B} \tag{5.30}$$

$$= \frac{1}{B}\sum_{m=0}^{\infty}\frac{(\beta^2\kappa^{n+1})^m}{m!}\mathbf{E}_n\big[\mathrm{e}^{H^{\mathrm{a}}_{n,h}}(S_n)^m\big]^2 + \frac{B-1}{B}. \tag{5.31}$$

If one takes $h = h_c^{\mathrm{a}}$ and uses the bound $Z^{\mathrm{a}}_{n+1,h_c^{\mathrm{a}}} \le 1$, one gets

$$\sum_{m=0}^{\infty} \frac{(\beta^2 \kappa^{n+1})^m}{m!} \mathbf{E}_n\big[\mathrm{e}^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}} (S_n)^m\big]^2 \le 1, \tag{5.32}$$

so that bounding each term of the sum by 1, one gets that for all $m \ge 0$

$$\mathbf{E}_n\big[\mathrm{e}^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}} (S_n)^m\big] \le \sqrt{m!}\left(\frac{1}{\beta}\left(\frac{1}{\sqrt{\kappa}}\right)^{n+1}\right)^m. \tag{5.33}$$

For $m = 1$ (using $Z^{\mathrm{a}}_{n,h} \ge (B-1)/B$) we obtain (3.14), but also an estimate for all the moments of S_n.

Using Lemma 5.3 one has

$$\left(\frac{2}{B}\right)^n \prod_{p=0}^{n-1} Z^{\mathrm{a}}_{p,h_c^{\mathrm{a}}} \le \mathbf{E}_n\big[\mathrm{e}^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}} S_n\big] \le \frac{1}{\beta}\left(\frac{1}{\sqrt{\kappa}}\right)^{n+1}, \tag{5.34}$$

which implies (3.13). $\square$

Another observation is that, writing $h = h_c^{\mathrm{a}} + u$, one gets from (5.33) that

$$\mathbf{E}_n\big[\mathrm{e}^{H^{\mathrm{a}}_{n,h}}\big] = \mathbf{E}_n\big[\mathrm{e}^{u S_n} \mathrm{e}^{H^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}}\big] \le \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}}\left(\frac{u}{\beta}\left(\frac{1}{\sqrt{\kappa}}\right)^{n+1}\right)^m. \tag{5.35}$$

Thus if $u \le (\sqrt{\kappa})^n$, one has that $Z^{\mathrm{a}}_{n,h_c^{\mathrm{a}}+u} = \mathbf{E}_n[\mathrm{e}^{H^{\mathrm{a}}_{n,h}}]$ does not grow with n. This is in contrast with the pure model where

$$Z^{\mathrm{pure}}_{n,u} = \mathbf{E}_n\big[\exp(u S_n)\big] \ge \exp\big(u \mathbf{E}_n(S_n)\big) = \exp\big(u(2/B)^n\big)$$

which diverges with n if $u = (\sqrt{\kappa})^n$ (recall we are considering $\kappa > B^2/4$).

All these facts lead us to conjecture that the phase transition of the annealed model for $B^2/4 < \kappa < 1/2$ is smoother than that of the pure model.

6. Disorder irrelevance

To prove disorder irrelevance for $B > B_c$ and the upper bounds on the difference between quenched and annealed critical points in Theorem 3.3, we use the following proposition.

Proposition 6.1. *Let $\kappa < (B^2/4 \wedge 1/2)$. If $B > B_c$, there exists a $\beta_0 > 0$ such that for $\beta \le \beta_0$ and for every $\eta \in (0,1)$ one can find $\varepsilon > 0$ such that for all $u \in (0, \varepsilon)$*

$$\mathtt{F}\big(\beta, h_c^{\mathrm{a}} + u\big) \ge (1-\eta)\mathtt{F}^{\mathrm{a}}\big(\beta, h_c^{\mathrm{a}} + u\big). \tag{6.1}$$

If $B < B_c$, then for every $\eta \in (0,1)$ one can find constants $c, \beta_0, \epsilon > 0$ such that if $\beta \le \beta_0$, for all $u \in (c\beta^{2/(2-\nu)}, \epsilon(\eta))$

$$\mathtt{F}\big(\beta, h_c^{\mathrm{a}} + u\big) \ge (1-\eta)\mathtt{F}^{\mathrm{a}}\big(\beta, h_c^{\mathrm{a}} + u\big) \tag{6.2}$$

with ν as in (2.21).

If $B = B_c$, then for every $\eta \in (0,1)$ one can find $\beta_0 > 0$ and a constant $c > 0$ such that if $\beta \le \beta_0$, for all $u \in (c\exp(-c\beta^{-2/3}), 1)$

$$\mathtt{F}\big(\beta, h_c^{\mathrm{a}} + u\big) \ge (1-\eta)\mathtt{F}^{\mathrm{a}}\big(\beta, h_c^{\mathrm{a}} + u\big). \tag{6.3}$$

Proof. This is based on the study of the variance $\mathcal{V}_n := \mathbb{E}[(\bar{Z}^\omega_{n,h})^2] - \mathbb{E}[\bar{Z}^\omega_{n,h}]^2$.

Fix some $B \in (1,2)$. One has

$$\mathbb{E}\big[\big(\bar{Z}^\omega_{n,h}\big)^2\big] = \mathbf{E}_n^{\otimes 2}\Bigg[\exp\Bigg(\bar{H}^{\mathrm{a}}_{n,h}(\delta) + \bar{H}^{\mathrm{a}}_{n,h}\big(\delta'\big) + \beta^2 \sum_{i,j=1}^{2^n} \kappa_{ij}\delta_i\delta'_j\Bigg)\Bigg] \tag{6.4}$$

with δ and δ' two independent copies of the same Galton–Watson process. We also have $\mathbb{E}[\bar{Z}^\omega_{n,h}]^2 = \mathbf{E}_n^{\otimes 2}[\exp(\bar{H}^{\mathrm{a}}_{n,h}(\delta) + \bar{H}^{\mathrm{a}}_{n,h}(\delta'))]$. To simplify notations, we write $h = h_c^{\mathrm{a}} + u$ and we define

$$D_n := \sum_{i,j=1}^{2^n} \kappa_{ij}\delta_i\delta'_j. \tag{6.5}$$

Then,

$$\begin{aligned}\mathcal{V}_n &= \mathbf{E}_n^{\otimes 2}\big[\mathrm{e}^{uS_n}\mathrm{e}^{uS'_n}\big(\mathrm{e}^{\beta^2 D_n} - 1\big)\mathrm{e}^{\bar{H}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}(\delta)}\mathrm{e}^{\bar{H}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}(\delta')}\big] \\ &\le \widetilde{\mathcal{V}}_n := \mathbf{E}_n^{\otimes 2}\big[\mathrm{e}^{CuS_n}\mathrm{e}^{CuS'_n}\big(\mathrm{e}^{C\beta^2 D_n} - 1\big)\big],\end{aligned} \tag{6.6}$$

where we expanded the exponential and used Remark 5.2 and Eq. (5.28).

Using the Cauchy–Schwarz inequality in (6.6),

$$\widetilde{\mathcal{V}}_n \le \mathbf{E}_n\big[\mathrm{e}^{2CuS_n}\big]\sqrt{\mathbf{E}_n^{\otimes 2}\big[\big(\mathrm{e}^{C\beta^2 D_n} - 1\big)^2\big]} \le \mathbf{E}_n\big[\mathrm{e}^{2CuS_n}\big]\sqrt{\mathbf{E}_n^{\otimes 2}\big[\mathrm{e}^{2C\beta^2 D_n} - 1\big]}. \tag{6.7}$$

We define $Q_n := \mathcal{V}_n/\mathbb{E}[\bar{Z}^\omega_{n,h}]^2 \le \mathcal{V}_n$ (recall that $h \ge h_c^{\mathrm{a}}$ and that $\mathbb{E}\bar{Z}^\omega_{n,h_c^{\mathrm{a}}} \ge 1$). Then one also uses Lemma A.1 to get that $\mathbf{E}_n[\mathrm{e}^{2CuS_n}] \le c\exp(c2^n u^\nu)$. Therefore, one has

$$Q_n \le c\exp\big(c2^n u^\nu\big)\sqrt{\mathbf{E}_n^{\otimes 2}\big[\mathrm{e}^{2C\beta^2 D_n} - 1\big]}. \tag{6.8}$$

Defining

$$n_1 = n_1(u) := \log(1/u)/\log(2/B) = \nu\log(1/u)/\log 2, \tag{6.9}$$

which is the value of n at which $\mathbf{E}_n[\exp(uS_n)]$ starts getting large, one has for $p \ge 0$

$$Q_{n_1+p} \le c\mathrm{e}^{c2^p}\sqrt{\mathbf{E}_{n_1+p}^{\otimes 2}\big[\mathrm{e}^{2C\beta^2 D_{n_1+p}} - 1\big]}. \tag{6.10}$$

Thus it is left to estimate the last term, with Proposition A.5.

The case $B > B_c$. Thanks to Proposition A.5 there exists some $\beta_0 > 0$ such that for $\beta < \beta_0$ and for all $n \in \mathbb{N}$

$$\mathbf{E}_n^{\otimes 2}\big[\mathrm{e}^{2C\beta^2 D_n} - 1\big] \le c\beta_0^2\Phi^n \tag{6.11}$$

for some $\Phi < 1$. Choose $p_1 = p_1(n_1)$ such that $\mathrm{e}^{c2^{p_1}}\sqrt{\Phi^{n_1}} = 1$ (note that p_1 diverges with n_1) and then

$$Q_{n_1+p_1} \le c'\sqrt{\Phi^{p_1}} \stackrel{n_1\to\infty}{\longrightarrow} 0. \tag{6.12}$$

Then we use that

$$\mathbb{E}\big[\log\bar{Z}^\omega_{n,h}\big] \ge \log\bigg(\frac{\mathbb{E}[\bar{Z}^\omega_{n,h}]}{2}\bigg)\mathbb{P}\bigg(\bar{Z}^\omega_{n,h} \ge \frac{\mathbb{E}[\bar{Z}^\omega_{n,h}]}{2}\bigg) + \log\bigg(\frac{B-1}{B}\bigg), \tag{6.13}$$

where $\mathbb{P}(\bar{Z}^{\omega}_{n,h} \geq \mathbb{E}[\bar{Z}^{\omega}_{n,h}]/2) \geq 1 - 4Q_n$ from the Tchebyshev inequality. We apply this with $n = n_1 + p_1(n_1)$ to get (using also Theorem 2.1 and (5.9))

$$\begin{aligned}\mathtt{F}(\beta, h) &\geq \frac{1}{2^n}\mathbb{E}\big[\log \bar{Z}^{\omega}_{n,h}\big] - \frac{\log B}{2^n} \geq (1-4\eta)\frac{1}{2^n}\log\big(\mathbb{E}\big[\bar{Z}^{\omega}_{n,h}\big]\big) - \frac{c}{2^n} \\ &\geq (1-4\eta)\mathtt{F}^{\mathrm{a}}(\beta, h) - \frac{c'}{2^{p_1(n_1)}}2^{-n_1} \geq (1-5\eta)\mathtt{F}^{\mathrm{a}}(\beta, h),\end{aligned} \tag{6.14}$$

provided that n_1 is large enough to ensure both

$$Q_{n_1+p_1} \leq c'\Phi^{p_1(n_1)/2} \leq \eta \quad \text{and} \tag{6.15}$$

$$c' 2^{-p_1(n_1)} u^{\nu} \leq \eta \mathtt{F}^{\mathrm{a}}(\beta, h) \qquad \text{for all } u \in (0,1). \tag{6.16}$$

Note that the requirement on n_1 in (6.16) also depends only on η, cf. Theorem 3.1. Since n_1 is related to u via (6.9), one has actually to assume that $u \leq \epsilon(\eta)$ with ϵ sufficiently small, as required in Proposition 6.1.

The case $B < B_c$. Given $\eta > 0$ and $\beta \leq 1$, fix some $p_1 = p_1(\eta)$ such that (6.16) holds and assume that $c_1\beta^{2/(2-\nu)} \leq u \leq \epsilon(\eta)$ with $c_1 = c_1(\eta)$ to be chosen sufficiently large later (observe that if $\epsilon(\eta)$ is small one has that n_1 and p_1 are large, so the above requirement on p_1 is coherent). The definition of $n_1(u)$ (which gives $u = (B/2)^{n_1}$) and of ν (which gives $(2/B)^{\nu} = 2$) imply that

$$\beta^2 \leq c_1^{-1}\left(\frac{2}{B^2}\right)^{p_1(\eta)}\left(\frac{B^2}{2}\right)^{n_1+p_1(\eta)} \leq c_2\left(\frac{B^2}{2}\right)^{n_1+p_1(\eta)}, \tag{6.17}$$

where $c_2 = c_2(\eta)$ can be made arbitrarily small by choosing c_1 large. Then, again provided that c_2 is small enough (i.e. c_1 large enough), we can apply Proposition A.5 to get from (6.10)

$$Q_{n_1+p_1(\eta)} \leq c\mathrm{e}^{c2^{p_1(\eta)}}\sqrt{c\beta^2\left(\frac{2}{B^2}\right)^{n_1+p_1(\eta)}} \leq c'\mathrm{e}^{c2^{p_1(\eta)}}\sqrt{c_2(\eta)} \leq \eta. \tag{6.18}$$

From this point on, the proof proceeds like in the case $B > B_c$, starting from (6.13).

The case $B = B_c$. This is similar to the case $B < B_c$. The value of β_0 has to be chosen small enough to guarantee that Proposition A.5 is applicable. We skip details. □

7. Disorder relevance: critical point shift lower bounds

To prove disorder relevance, we give a finite size condition for delocalization, adapting the fractional moment method, first used in [8], and then in [15,16] for the pinning model with i.i.d. disorder.

7.1. *Fractional moment iteration*

For $\gamma < 1$ let x_γ to be the largest solution of

$$x = \frac{x^2 + (B-1)^\gamma}{B^\gamma}.$$

One can easily see that for γ sufficiently close to 1 (which we assume to be the case in the following) x_γ actually exists and is strictly less than 1. Moreover one has that x_γ increases to 1 as γ increases to 1. Then we have:

Proposition 7.1. *Take $\kappa < 1/2$. Then, setting $A_n := \mathbb{E}[(\bar{Z}^{\omega}_{n,h})^\gamma]$ with $\bar{Z}^{\omega}_{n,h}$ defined in* (5.2), *one has*

$$A_{n+1} \leq \frac{A_n^2 + (B-1)^\gamma}{B^\gamma}. \tag{7.1}$$

If there exists some n_0 such that $A_{n_0} \leq x_\gamma$, then $\mathtt{F}(\beta, h) = 0$.

Proof. If for some n_0 one has $A_{n_0} \le x_\gamma$, then iterating (7.1) one gets $A_n \le x_\gamma \le 1$ for all $n \ge n_0$. Using the Jensen's inequality one has

$$\frac{1}{n}\mathbb{E}\big[\log \bar{Z}^{\omega}_{n,h}\big] = \frac{1}{\gamma n}\mathbb{E}\big[\log\big(\bar{Z}^{\omega}_{n,h}\big)^{\gamma}\big] \le \frac{1}{\gamma n}\log A_n \tag{7.2}$$

which gives $\mathtt{F}(\beta,h) = \bar{\mathtt{F}}(\beta,h) = 0$ (equality of the two free energies was noted after (5.2)).

We now turn to the proof of (7.1). We define $Z^{\mu}_{n,h} = \mathbf{E}_n[\mathrm{e}^{H^{\omega}_{n,h}}\mathrm{e}^{\mu\kappa_n\beta^2(S_n)^2}]$ and use that $(S_{n+1})^2 \le 2(S_n)^2 + 2(S_n)^2$ to get the iteration

$$Z^{\mu}_{n+1,h} \le \frac{1}{B} Z^{2\kappa\mu,(1)}_{n,h} Z^{2\kappa\mu,(2)}_{n,h} + \frac{B-1}{B}, \tag{7.3}$$

where as usual the two partition functions in the r.h.s. refer to the first and second sub-system of size 2^n. From this, and using the inequality $(a+b)^\gamma \le a^\gamma + b^\gamma$ for any $a,b \ge 0$ and $\gamma \le 1$, one has

$$\mathbb{E}\big[\big(Z^{\mu}_{n+1,h}\big)^{\gamma}\big] \le \frac{1}{B^\gamma}\mathbb{E}\big[\big(Z^{2\kappa\mu,(1)}_{n,h} Z^{2\kappa\mu,(2)}_{n,h}\big)^{\gamma}\big] + \frac{(B-1)^\gamma}{B^\gamma}. \tag{7.4}$$

One then shows the following lemma.

Lemma 7.2. *If $\mu \ge \theta$ with $\theta = \frac{\kappa}{2(1-2\kappa)}$ as in (5.2),*

$$\mathbb{E}\big[\big(Z^{2\mu\kappa,(1)}_{n,h} Z^{2\mu\kappa,(2)}_{n,h}\big)^{\gamma}\big] \le \mathbb{E}\big[\big(Z^{\mu}_{n,h}\big)^{\gamma}\big]^2. \tag{7.5}$$

This gives directly (7.1), taking $\mu = \theta$ so that $Z^{\mu}_{n,h} = \bar{Z}^{\omega}_{n,h}$. □

Proof of Lemma 7.2. One sets

$$\Phi(t,\mu) := \log \mathbb{E}_t\big[\big(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\big)^{\gamma}\big], \tag{7.6}$$

where one defines $\mathbb{P}_t$ to be the law of a Gaussian vector $(\omega_1,\dots,\omega_{2^{n+1}})$ with correlations $\kappa_{ij}(t) = \kappa_p$ if $d(i,j) = p \le n$, and $\kappa_{ij}(t) = t\kappa_{n+1}$ if $d(i,j) = n+1$. Then one can compute the derivatives of Φ. Using the definition of $Z^{\mu}_{n,h}$ one has for $t \ge 0$, $\mu \in \mathbb{R}$

$$\begin{aligned}\frac{\partial\Phi}{\partial\mu}(t,\mu) &= \frac{\gamma\kappa_n\beta^2}{\mathbb{E}_t[(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h})^\gamma]} \\ &\quad\times \mathbb{E}_t\big[\mathbf{E}_n^{\otimes 2}\big[\big(\big(S_n^{(1)}\big)^2 + \big(S_n^{(2)}\big)^2\big)\mathrm{e}^{H^{\omega,(1)}_{n,h}+H^{\omega,(2)}_{n,h}}\mathrm{e}^{\mu\kappa_n((S_n^{(1)})^2+(S_n^{(2)})^2)}\big]\big(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\big)^{\gamma-1}\big]. \end{aligned} \tag{7.7}$$

Thanks to Proposition B.3 one gets

$$\frac{\partial\Phi}{\partial t}(t,\mu) = \frac{\kappa_{n+1}}{\mathbb{E}_t[(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h})^\gamma]}\sum_{i=1}^{2^n}\sum_{j=2^n+1}^{2^{n+1}}\mathbb{E}_t\bigg[\frac{\partial^2}{\partial\omega_i\partial\omega_j}\big(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\big)^{\gamma}\bigg]. \tag{7.8}$$

For the values of i,j under consideration one has

$$\begin{aligned}&\frac{\partial}{\partial\omega_i\,\partial\omega_j}\big(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\big)^{\gamma} \\ &\quad= \gamma^2\beta^2\mathbf{E}_n^{\otimes 2}\big[\delta_i\delta_j\mathrm{e}^{H^{\omega,(1)}_{n,h}+H^{\omega,(2)}_{n,h}}\mathrm{e}^{\mu\kappa_n((S_n^{(1)})^2+(S_n^{(2)})^2)}\big]\big(Z^{\mu,(1)}_{n,h} Z^{\mu,(2)}_{n,h}\big)^{\gamma-1}. \end{aligned} \tag{7.9}$$

Therefore,

$$\begin{aligned}
&\sum_{i=1}^{2^n}\sum_{j=2^n+1}^{2^{n+1}}\frac{\partial^2}{\partial\omega_i\partial\omega_j}\big(Z_{n,h}^{\mu,(1)}Z_{n,h}^{\mu,(2)}\big)^\gamma\\
&\quad\le\frac{\gamma^2\beta^2}{2}\mathbf{E}_n^{\otimes 2}\big[\big(\big(S_n^{(1)}\big)^2+\big(S_n^{(2)}\big)^2\big)\mathrm{e}^{H_{n,h}^{\omega,(1)}+H_{n,h}^{\omega,(2)}}\mathrm{e}^{\mu\kappa_n((S_n^{(1)})^2+(S_n^{(2)})^2)}\big]\big(Z_{n,h}^{\mu,(1)}Z_{n,h}^{\mu,(2)}\big)^{\gamma-1},
\end{aligned}\tag{7.10}$$

and as a consequence, since we chose $\kappa_n=\kappa^n$

$$\frac{\partial\Phi}{\partial t}(t,\mu)\le\frac{\kappa}{2}\frac{\partial\Phi}{\partial\mu}(t,\mu).\tag{7.11}$$

Thus, the function $t\mapsto\Phi(t,\mu-\kappa t/2)$ is non-increasing and

$$\log\mathbb{E}\big[\big(Z_{n,h}^{\mu-\kappa/2,(1)}Z_{n,h}^{\mu-\kappa/2,(2)}\big)^\gamma\big]=\Phi(1,\mu-\kappa/2)\le\Phi(0,\mu)=2\log\mathbb{E}_t\big[\big(Z_{n,h}^{\mu}\big)^\gamma\big].\tag{7.12}$$

Then, one uses that for $\mu\ge\frac{\kappa}{2(1-2\kappa)}$ one has $2\mu\kappa\le\mu-\kappa/2$, which allows us to conclude. □

7.2. *Change of measure*

In this section we prove the lower bounds of Theorem 3.3 on the critical point shift for $B\le B_c$.

One fixes γ close to 1 such that x_γ is also close to 1, and proves that if $h=h_c^{\mathrm{a}}+u$ with $u>0$ small enough, one has $A_{n_0}:=\mathbb{E}[(\bar Z_{n_0,h}^\omega)]\le x_\gamma$ for some $n_0\in\mathbb{N}$. To this purpose, we introduce a change of measure in the spirit of [16]. Define

$$\begin{aligned}
g(\omega)&:=\mathbf{1}_{\{F(\omega)\le R\}}+\varepsilon_R\mathbf{1}_{\{F(\omega)>R\}},\\
F(\omega)&:=\langle V\omega,\omega\rangle-\mathbb{E}\big[\langle V\omega,\omega\rangle\big],
\end{aligned}\tag{7.13}$$

where the choices of the symmetric $2^n\times 2^n$ matrix V, of $R\in\mathbb{R}$ and $\varepsilon_R>0$ will be made later. Note that we have chosen F to be centered. Then using the Hölder inequality, one has

$$\mathbb{E}\big[\big(\bar Z_{n,h}^\omega\big)^\gamma\big]=\mathbb{E}\big[g(\omega)^{-\gamma}\big(g(\omega)\bar Z_{n,h}^\omega\big)^\gamma\big]\le\mathbb{E}\big[\big(g(\omega)\big)^{-\gamma/(1-\gamma)}\big]^{1-\gamma}\mathbb{E}\big[g(\omega)\bar Z_{n,h}^\omega\big]^\gamma.\tag{7.14}$$

Remark 7.3. *The original idea [15] is to take $g(\omega)=\frac{\mathrm{d}\check{\mathbb{P}}}{\mathrm{d}\mathbb{P}}$ where $\check{\mathbb{P}}$ is a new probability measure on $\{\omega_1,\dots,\omega_{2^n}\}$ such that $\check{\mathbb{P}}$ and $\mathbb{P}$ are mutually absolutely continuous. Then, to control both terms in (7.14), one has to choose $\check{\mathbb{P}}$ in a certain sense close enough to $\mathbb{P}$, such that the first term is close to 1, but also such that under the measure $\check{\mathbb{P}}$ the annealed partition function $\mathbb{E}[g(\omega)\bar Z_{n,h_c^{\mathrm{a}}}]=\check{\mathbb{E}}[\bar Z_{n,h_c^{\mathrm{a}}}]$ is small.*

The choice of g and F in (7.13) has the same effect of the change of measure in [15], that is inducing negative correlations between different ω_i, and the specific form (7.13) is chosen for technical reasons, to deal more easily with the case in which $\langle V\omega,\omega\rangle$ is large.

Let us first deal with the Radon–Nikodym part of (7.14): we make here the choice $\varepsilon_R:=\mathbb{P}(F(\omega)\ge R)^{1-\gamma}$. Then one has

$$\mathbb{E}\big[\big(g(\omega)\big)^{-\gamma/(1-\gamma)}\big]\le 1+(\varepsilon_R)^{-\gamma/(1-\gamma)}\mathbb{P}\big(F(\omega)\ge R\big)=1+\mathbb{P}\big(F(\omega)\ge R\big)^{1-\gamma}=1+\varepsilon_R.\tag{7.15}$$

We now use the following lemma to estimate ε_R in terms of R. We let $\|V\|^2=\sum_{i,j}V_{ij}^2$ and K denote the covariance matrix $(\kappa_{ij})_{1\le i,j\le 2^n}$.

Lemma 7.4. *If V is such that V_{ij} depends only on $d(i,j)$ and $\|V\|^2 = 1$, then one has* $\mathrm{Var}(F) < 2K_\infty^2$ *with K_∞ defined in* (4.2), *so that*

$$\mathbb{P}\big(F(\omega) \geq R\big) \leq \frac{2K_\infty}{R^2} \overset{R\to\infty}{\longrightarrow} 0. \tag{7.16}$$

Thus one gets that $\varepsilon_R \leq const \times R^{-2(1-\gamma)}$, which can be made arbitrarily small choosing R large.

Proof. We have that $\mathrm{Var}(F) = \mathbb{E}[\langle V\omega,\omega\rangle^2] - \mathbb{E}[\langle V\omega,\omega\rangle]^2$, and we can compute

$$\begin{aligned}\mathbb{E}\big[\langle V\omega,\omega\rangle^2\big] &= \sum_{i,j=1}^{2^n}\sum_{k,l=1}^{2^n} V_{ij}V_{kl}\mathbb{E}[\omega_i\omega_j\omega_k\omega_l] \\ &= \sum_{i,j=1}^{2^n}\sum_{k,l=1}^{2^n} V_{ij}V_{kl}(\kappa_{ij}\kappa_{kl}+\kappa_{ik}\kappa_{jl}+\kappa_{il}\kappa_{jk}) \\ &= \mathbb{E}\big[\langle V\omega,\omega\rangle\big]^2 + 2\,\mathrm{Tr}\big((VK)^2\big). \end{aligned}\tag{7.17}$$

We now use Lemma B.1, which says that V and K can be codiagonalized, and that the eigenvalues of K are bounded by K_∞, to get that $\mathrm{Tr}((VK)^2) \leq K_\infty^2\,\mathrm{Tr}(V^2) = K_\infty^2$ (recall that $\mathrm{Tr}(V^2) = \|V\|^2 = 1$, as V is symmetric). One finally gets that $\mathrm{Var}(F) \leq 2K_\infty^2$, and as F is centered, using Tchebyshev's inequality gives the result. □

Next, we study the second factor in the r.h.s. of (7.14):

$$\mathbb{E}\big[g(\omega)\bar{Z}^{\omega}_{n,h}\big] \leq \mathbb{E}\big[\mathbf{1}_{\{F(\omega)\leq R\}}\bar{Z}^{\omega}_{n,h}\big] + \varepsilon_R\mathbb{E}\big[\bar{Z}^{\omega}_{n,h}\big]. \tag{7.18}$$

To study the first term we define the measure $\widetilde{\mathbb{P}}$ on $\{\omega_1,\dots,\omega_{2^n}\}$ to be absolutely continuous with respect to $\mathbb{P}$, with Radon–Nikodym derivative given by $\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \frac{\bar{Z}^{\omega}_{n,h}}{\bar{Z}^{\mathrm{a}}_{n,h}}$. One then has

$$\mathbb{E}\big[\mathbf{1}_{\{F(\omega)\leq R\}}\bar{Z}^{\beta,\omega}_{n,h}\big] = \bar{Z}^{\mathrm{a}}_{n,h}\widetilde{\mathbb{P}}\big(F(\omega)\leq R\big). \tag{7.19}$$

We are now ready to choose $V = V_n$, and we do so as in [15]. We take V to be zero on the diagonal ($V_{ii} = 0$), and for $i,j \in \{1,\dots,2^n\}$

$$V_{ij} := \frac{\mathbf{E}_n[\delta_i\delta_j]}{Y_n}, \quad \text{if } i\neq j, \tag{7.20}$$

where

$$Y_n := \Bigg(\sum_{\substack{i,j=1\\ i\neq j}}^{2^n}\mathbf{E}_n[\delta_i\delta_j]^2\Bigg)^{1/2} \tag{7.21}$$

is used to normalize V. We stress that V satisfy the conditions of Lemma 7.4.

One can compute easily Y_n, since from Proposition 2.4 we have $\mathbf{E}_n[\delta_i\delta_j] = B^{-n-d(i,j)+1}$, and one finds (cf. [15], Eq. (8.23))

$$Y_n = \begin{cases} \sqrt{n} & \text{if } B = B_c := \sqrt{2}, \\ \Theta\Big(\Big(\frac{2}{B^2}\Big)^n\Big) & \text{if } B < B_c\,, \end{cases} \tag{7.22}$$

where $X = \Theta(Y)$ means that $X \geq cY$ for some positive constant c.

Proposition 7.5. *We choose $V = V_n$ as in (7.20) and (7.21), and $R = R_n := \frac{1}{2}\widetilde{\mathbb{E}}[F(\omega)]$. Then there exists some $\delta > 0$ small such that, if $u(2/B)^n \le \delta$, one has*

$$R := \frac{1}{2}\widetilde{\mathbb{E}}\big[F(\omega)\big] \ge c\beta^2 Y_n. \tag{7.23}$$

Therefore, from (7.22), R can be made arbitrarily large with n. Moreover there exists a constant $\zeta > 0$ which does not depend on n, such that

$$\widetilde{\mathbb{P}}\big(F(\omega) \ge R\big) = \widetilde{\mathbb{P}}\bigg(F(\omega) \ge \frac{1}{2}\widetilde{\mathbb{E}}\big[F(\omega)\big]\bigg) \ge \zeta. \tag{7.24}$$

Combining this proposition to (7.18) and (7.19), one gets that

$$\mathbb{E}\big[g(\omega)\bar{Z}^{\omega}_{n,h}\big] \le \bar{Z}^{\mathrm{a}}_{n,h}(1 - \zeta + \varepsilon_R). \tag{7.25}$$

Recalling the equality (5.29) (which is the analog of Theorem 3.1 for the alternative partition function $\bar{Z}^{\mathrm{a}}_{n,h}$), one has for $\kappa < B^2/4 \wedge 1/2$

$$\bar{Z}^{\mathrm{a}}_{n,h} \le \mathbf{E}_n\big[\mathrm{e}^{c_1' u S_n}\big] + c_2'\beta^2\bigg(\frac{4\kappa}{B^2}\bigg)^n \le \mathrm{e}^{c\delta} + \delta, \tag{7.26}$$

provided that $u \le \delta(B/2)^n$ with δ small (to be able to apply Lemma A.1 to $\mathbf{E}_n[\mathrm{e}^{c_1' u S_n}]$), and that $n \ge n_\delta$ to deal with the term $(4\kappa/B^2)^n$. Therefore, if δ and ε_R was chosen small enough (that is smaller than some constant $c = c(\zeta)$), one has for $n \ge n_\delta$ that $\mathbb{E}[g(\omega)\bar{Z}^{\beta,\omega}_{n,h}] \le 1 - \zeta/2$ for all $u \le \delta(B/2)^n$. This and (7.15) bound the two terms in (7.14), so that one has

$$A_n := \mathbb{E}\big[\big(\bar{Z}^{\omega}_{n,h}\big)^\gamma\big] \le (1 + \varepsilon_R)(1 - \zeta/2)^\gamma \le 1 - \zeta/3 \le x_\gamma, \tag{7.27}$$

where the two last inequalities hold if ε_R is small and γ close to 1. To sum up, for δ, β small and R large enough, one has that $A_n \le x_\gamma$ for all $u \le \delta(B/2)^n$, and so $\mathtt{F}(\beta, h_c^{\mathrm{a}} + u) = 0$.

Then, let us check how large has to be n so that our choice of $R := \frac{1}{2}\widetilde{\mathbb{E}}[F(\omega)]$ becomes large. From Proposition 7.5 one has that $R \ge c\beta^2 Y_n$ so that one has to take $\beta^2 Y_n \ge C$ for some constant C large enough. From (7.22), in order to have $\beta^2 Y_n \ge C$,

- if $B < B_c$, it is enough to take n larger than $n_0 := \log(C'\beta^{-2})/\log(2/B^2)$;
- if $B = B_c$, one has to take n larger than $n_0 := c'\beta^{-4}$.

Then for $n = n_0$ one gets that R is large, but one also needs to take $u \le \delta(2/B)^{n_0}$ to ensure that $A_{n_0 \vee n_\delta} \le x_\gamma$. Notice that from the choice of n_0 above, the condition on u translates into

$$u \le \begin{cases} c'\beta^{2\log(2/B)/\log(2/B^2)} = c'\beta^{2/(2-\nu)} & \text{if } B < B_c, \\ \mathrm{e}^{-c\beta^{-4}} & \text{if } B = B_c, \end{cases} \tag{7.28}$$

where we also used that $\nu = \log 2/\log(2/B)$. One then gets the desired bounds (3.10) and (3.11) on the difference between quenched and annealed critical points.

7.3. *Proof of Proposition 7.5*

To compute $\widetilde{\mathbb{E}}[F(\omega)]$, we define for any $1 \le i, j \le 2^n$

$$U_{ij} := \widetilde{\mathbb{E}}[\omega_i\omega_j] = \frac{1}{\bar{Z}^{\mathrm{a}}_{n,h}}\mathbf{E}_n\mathbb{E}\big[\omega_i\omega_j\mathrm{e}^{\bar{H}^{\omega}_{n,h}}\big]. \tag{7.29}$$

A Gaussian integration by parts gives easily

$$U_{ij} = \kappa_{ik} + u_{ij} := \kappa_{ij} + \beta^2 \sum_{k,l=1}^{2^n} \kappa_{ik}\kappa_{jl} \bar{\mathbf{E}}^{\mathrm{a}}_{n,h}[\delta_k \delta_l], \tag{7.30}$$

where $\bar{\mathbf{E}}^{\mathrm{a}}_{n,h}$ denotes expectation w.r.t. the measure whose density with respect to $\mathbf{P}_n$ is $\exp(\bar{H}^{\mathrm{a}}_{n,h})/\bar{Z}^{\mathrm{a}}_{n,h}$. We then compare $\bar{\mathbf{E}}^{\mathrm{a}}_{n,h}[\delta_k\delta_l]$ with $\mathbf{E}_n[\delta_k\delta_l]$, using that $h = h_c^{\mathrm{a}} + u$, $0 \le u \le \delta(B/2)^n$:

$$\bar{\mathbf{E}}^{\mathrm{a}}_{n,h}[\delta_k\delta_l] = \frac{1}{\bar{Z}^{\mathrm{a}}_{n,h}} \mathbf{E}_n\big[\delta_k\delta_l e^{\bar{H}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}}} e^{uS_n}\big] \le e^{2c_1\beta^2} \mathbf{E}_n\big[\delta_k\delta_l e^{e^{c_1\beta^2} uS_n}\big] \le c' \mathbf{E}_n[\delta_k\delta_l], \tag{7.31}$$

where in the first inequality we used Remark 5.4 and also the fact that $\bar{Z}^{\mathrm{a}}_{n,h} \ge \bar{Z}^{\mathrm{a}}_{n,h_c^{\mathrm{a}}} \ge 1$, and in the second inequality we used that $u(2/B)^n \le \delta$ to apply Corollary A.2. The same argument easily gives $\bar{\mathbf{E}}^{\mathrm{a}}_{n,h}[\delta_k\delta_l] \ge c\mathbf{E}_n[\delta_k\delta_l]$ in the range of u considered, so that $c\beta^2 a_{ij} \le u_{ij} \le c'\beta^2 a_{ij}$, where

$$a_{ij} := \sum_{k,l=1}^{2^n} \kappa_{ik}\kappa_{jl} \mathbf{E}_n[\delta_k\delta_l] \ge Y_n (KVK)_{ij} \tag{7.32}$$

(the inequality is due to the fact that V is zero on the diagonal). We finally get

$$\widetilde{\mathbb{E}}\big[F(\omega)\big] = \widetilde{\mathbb{E}}\big[\langle V\omega, \omega\rangle\big] - \mathbb{E}\big[\langle V\omega,\omega\rangle\big] = \sum_{i,j=1}^{2^n} V_{ij}(\kappa_{ij} + u_{ij}) - \mathbb{E}\big[\langle V\omega,\omega\rangle\big] = \sum_{i,j=1}^{2^n} V_{ij}u_{ij}, \tag{7.33}$$

so that we only have to compute $\sum_{i,j=1}^{2^n} V_{ij}a_{ij} \ge Y_n \operatorname{Tr}(VKVK)$. Since $\|V\|^2 = 1$ and all eigenvalues of K are between 1 and K_∞, one has $\operatorname{Tr}((VK)^2) = \Theta(1)$. Altogether, we get (7.23).

We now prove (7.24). Using the Paley–Zygmund inequality, we get that

$$\widetilde{\mathbb{P}}\big(F(\omega) \ge R\big) = \widetilde{\mathbb{P}}\left(F(\omega) \ge \frac{1}{2}\widetilde{\mathbb{E}}\big[F(\omega)\big]\right) \ge \frac{\widetilde{\mathbb{E}}[F(\omega)]^2}{4\widetilde{\mathbb{E}}[F(\omega)^2]}, \tag{7.34}$$

so that we only have to prove the following:

$$\widetilde{\operatorname{Var}}\big(F(\omega)\big) = \widetilde{\mathbb{E}}\big[\langle V\omega,\omega\rangle^2\big] - \widetilde{\mathbb{E}}\big[\langle V\omega,\omega\rangle\big]^2 = O\big(\widetilde{\mathbb{E}}\big[F(\omega)\big]^2\big). \tag{7.35}$$

Indeed from this it follows immediately that there exists some constant $\zeta > 0$ such that $\widetilde{\mathbb{E}}[F(\omega)]^2/\widetilde{\mathbb{E}}[F(\omega)^2] \ge \zeta$.

We now prove (7.35), studying $\widetilde{\mathbb{E}}[\langle V\omega,\omega\rangle^2] = \sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}\widetilde{\mathbb{E}}[\omega_i\omega_j\omega_k\omega_l]$, starting with the computation, for any $1 \le i,j,k,l \le 2^n$, of

$$\widetilde{\mathbb{E}}[\omega_i\omega_j\omega_k\omega_l] = \frac{1}{\bar{Z}^{\mathrm{a}}_{n,h}} \mathbf{E}_n \mathbb{E}\big[\omega_i\omega_j\omega_k\omega_l e^{\bar{H}^{\omega}_{n,h}}\big]. \tag{7.36}$$

Again, a Gaussian integration by parts gives, after elementary computations,

$$\begin{aligned}\widetilde{\mathbb{E}}[\omega_i\omega_j\omega_k\omega_l] &= A_{ijkl} + B_{ijkl} \\ &:= [\kappa_{ij}U_{kl} + \kappa_{ik}U_{jl} + \kappa_{il}U_{jk} + \kappa_{jk}u_{il} + \kappa_{jl}u_{ik} + \kappa_{kl}u_{ij}] \\ &\quad + \beta^4 \sum_{r,s,t,v=1}^{2^n} \kappa_{ir}\kappa_{js}\kappa_{kt}\kappa_{lv} \bar{\mathbf{E}}^{\mathrm{a}}_{n,h}[\delta_r\delta_s\delta_t\delta_v].\end{aligned} \tag{7.37}$$

We estimate $\widetilde{\mathbb{E}}[\langle V\omega,\omega\rangle^2]$ by analyzing separately A_{ijkl} and B_{ijkl}.

Contribution from B_{ijkl}: we have

$$B_{ijkl} \le c\beta^4 \sum_{r,s,t,v=1}^{2^n} \kappa_{ir}\kappa_{js}\kappa_{kt}\kappa_{lv}\mathbf{E}_n[\delta_r\delta_s\delta_t\delta_v], \tag{7.38}$$

where we used again Proposition 3.2 and Corollary A.2 as in (7.31) (recall that we consider $u \le \delta(B/2)^n$). Then defining

$$W_{ij} := \frac{\mathbf{E}_n[\delta_i\delta_j]}{Y_n} = V_{ij} + \frac{\mathbf{1}_{\{i=j\}}}{Y_n B^n} \tag{7.39}$$

we get

$$\begin{aligned}\sum_{i,j,k,l=1}^{2^n} V_{ij}V_{kl}B_{ijkl} &\le c\beta^4 \sum_{r,s,t,v=1}^{2^n} (KWK)_{rs}(KWK)_{tv}\mathbf{E}_n[\delta_r\delta_s\delta_t\delta_v] \\ &\le c'\beta^4 \sum_{\substack{r,s,t,v=1\\ r\neq s,t\neq v}}^{2^n} W_{rs}W_{tv}\mathbf{E}_n[\delta_r\delta_s\delta_t\delta_v] + c''\beta^4 \sum_{r,t,v=1}^{2^n} W_{rr}W_{tv}\mathbf{E}_n[\delta_r\delta_t\delta_v],\end{aligned} \tag{7.40}$$

where we used the following claim:

Claim 7.6. *There exists a constant $c' > 0$ such that for every $1 \le i, j \le 2^n$, $(WK)_{ij} \le c'W_{ij}$ and $(KW)_{ij} \le c'W_{ij}$.*

Proof. We write $q = d(i,j)$, so $W_{ij} =: W_q$, and

$$(WK)_{ij} = \sum_{l=1}^{2^n} W_{il}\kappa_{lj} = \sum_{p=0}^{q-1} 2^{p-1}W_p\kappa_q + \sum_{p=0}^{q-1} 2^{p-1}W_q\kappa_p + \sum_{p=q+1}^{n} 2^{p-1}W_p\kappa_p, \tag{7.41}$$

where we decomposed the sum according to the positions of l ($d(i,l) = p < q$, $d(i,l) = q$ or $d(i,l) > q$). Using that W_p is decreasing with p, we get that the second and the third term are both smaller than $(\sum 2^p\kappa_p)W_q$. We only have to deal with the first term, using the explicit expression of W_p, together with Proposition 2.4:

$$\sum_{p=0}^{q-1} 2^{p-1}W_p = \frac{1}{Y_n}B^{-n}\sum_{p=0}^{q-1}\left(\frac{2}{B}\right)^{p-1} \le c\frac{1}{Y_n}B^{-n}\left(\frac{2}{B}\right)^q = c2^q W_q, \tag{7.42}$$

so that the first term in (7.41) is smaller than $c2^q\kappa_q W_q$. One then has that $(WK)_{ij} \le c'W_{ij}$, and the same computations also gives that $(KW)_{ij} \le c'W_{ij}$. □

The main term in the r.h.s. of (7.40) is the first one, for which we have

Lemma 7.7. *Let $B \le B_c$. There exists a constant $c > 0$ such that*

$$\sum_{\substack{r,s,t,v=1\\ r\neq s,t\neq v}}^{2^n} V_{rs}V_{tv}\mathbf{E}_n[\delta_r\delta_s\delta_t\delta_v] = \frac{1}{Y_n^2}\sum_{\substack{r,s,t,v=1\\ r\neq s,t\neq v}}^{2^n} \mathbf{E}_n[\delta_r\delta_s]\mathbf{E}_n[\delta_t\delta_v]\mathbf{E}_n[\delta_r\delta_s\delta_t\delta_v] \le cY_n^2. \tag{7.43}$$

This can be found in the proof of Lemma 4.4 of [15] for $B = B_c$; the proof is easily extended to the case $B < B_c$.

As for the remaining terms in (7.40), it is not hard to see, using repeatedly Proposition 2.4, that they give a contribution of order $o(Y_n^2)$. For instance, one has

$$\beta^4 \sum_{\substack{r,t,v=1\\ t\neq v}}^{2^n} W_{rr} W_{tv} \mathbf{E}_n[\delta_r \delta_t \delta_v] \le c\beta^4 \frac{2^n}{B^n Y_n^2} \sum_{p=0}^{n} 2^p B^{-n-p} \sum_{q=0}^{n} 2^q B^{-n-p-q} = \beta^4 o\bigl(Y_n^2\bigr). \tag{7.44}$$

Altogether one has

$$\sum_{i,j,k,l=1}^{2^n} V_{ij} V_{kl} B_{ijkl} = \beta^4 O\bigl(Y_n^2\bigr) = O\bigl(\widetilde{\mathbb{E}}\bigl[F(\omega)\bigr]^2\bigr), \tag{7.45}$$

cf. (7.23).

Contribution of A_{ijkl}: recalling that $U_{ij} = \kappa_{ij} + u_{ij}$, we have $\kappa_{ij} U_{kl} + \kappa_{kl} u_{ij} \le U_{ij} U_{kl}$. Thus, we get

$$\sum_{i,j,k,l=1}^{2^n} V_{ij} V_{kl} (\kappa_{ij} U_{kl} + \kappa_{kl} u_{ij}) \le \left(\sum_{i,j=1}^{2^n} V_{ij} U_{ij} \right)^2 = \widetilde{\mathbb{E}}\bigl[\langle V\omega, \omega\rangle\bigr]^2, \tag{7.46}$$

that we recall is not $O(\widetilde{\mathbb{E}}[F(\omega)]^2)$, but will be canceled in the variance. The other contributions are, thanks to symmetry of V, all equal to (or smaller than)

$$\sum_{i,j,k,l=1}^{2^n} V_{ij} V_{kl} \kappa_{ik} U_{jl} = \sum_{i,j,k,l=1}^{2^n} V_{ij} V_{kl} \kappa_{ik} \kappa_{jl} + \sum_{i,j,k,l=1}^{2^n} V_{ij} V_{kl} \kappa_{ik} u_{jl}, \tag{7.47}$$

where the first term is $\mathrm{Tr}((VK)^2)$ which is bounded as remarked before. Thanks to the estimate $u_{jl} \le c'\beta^2 a_{jl} = c'\beta^2 Y_n (KWK)_{jl}$, the second term is bounded above by a constant times

$$\begin{aligned}\beta^2 Y_n \sum_{i,j,k,l=1}^{2^n} V_{ij} V_{kl} \kappa_{ik} (KWK)_{jl} &\le \beta^2 Y_n \,\mathrm{Tr}\bigl((WK)^3\bigr) \\ &\le c\beta^2 Y_n \,\mathrm{Tr}\bigl(W^2\bigr) \le 2c\beta^2 Y_n = O\bigl(\widetilde{\mathbb{E}}\bigl[F(\omega)\bigr]\bigr).\end{aligned} \tag{7.48}$$

We used Lemma B.1 to codiagonalize W and K and to bound the eigenvalues of K by a constant, and then the fact that the eigenvalues λ_i of W are also bounded, so that $\sum |\lambda_i|^3 \le c \sum |\lambda_i|^2 = c\,\mathrm{Tr}(W^2) = O(1)$. Indeed, $\mathrm{Tr}(W^2) = \mathrm{Tr}(V^2) + \sum_i W_{ii}^2 = 1 + (2/B^2)^n Y_n^{-2} = 1 + o(1)$. Putting together (7.37) with the estimates (7.45), (7.46) and (7.48) we have

$$\begin{aligned}\widetilde{\mathrm{Var}}\bigl(F(\omega)\bigr) &= \widetilde{\mathbb{E}}\bigl(\langle V\omega, \omega\rangle^2\bigr) - \bigl(\widetilde{\mathbb{E}}\langle V\omega, \omega\rangle\bigr)^2 \\ &= \sum_{ijkl} (A_{ijkl} + B_{ijkl}) V_{ij} V_{kl} - \bigl(\widetilde{\mathbb{E}}\langle V\omega, \omega\rangle\bigr)^2 \\ &= O\bigl(\widetilde{\mathbb{E}}\bigl[F(\omega)\bigr]^2\bigr)\end{aligned} \tag{7.49}$$

and (7.35) is proven.

Appendix A: Pure model estimates

We first give some estimates on the partition function of a system of size n.

Lemma A.1.

(1) *There exist constants* $a_0 > 0$ *and* $c_0 > 0$ *such that for any* $n \geq 0$, *if* $u \leq a_0(B/2)^n$ *one has*

$$\mathbf{E}_n\big[\exp(uS_n)\big] \leq \exp\big(c_0 u(2/B)^n\big). \tag{A.1}$$

(2) *There exists a constant* $c > 0$ *such that for any* $n \geq 0$ *and* $u \geq 0$ *one has*

$$\mathbf{E}_n\big[\exp(uS_n)\big] \leq c \exp\big(cu^{\nu} 2^n\big), \tag{A.2}$$

where ν *is as in* (2.21).

Proof. For the first inequality, the same type of computation was already done in [14], and we give here only an outline of the proof. The partition function R_k of the pure model satisfies the iteration

$$\begin{cases} R_0 = \mathrm{e}^u, \\ R_{k+1} = \dfrac{R_k^2 + B - 1}{B}. \end{cases} \tag{A.3}$$

Defining $P_k := R_k - 1$ it is easy to show by recurrence that $P_k \leq c_0 u(\frac{2}{B})^k$ for every $k \leq n$ (because we stay in the linear regime for the chosen value of u), so that for $k = n$ we get the result.

For the second inequality, we use that for any $n \geq 0$ and $u \geq 0$,

$$\frac{1}{2^n} \log \mathbf{E}_n\big[\exp(uS_n)\big] \leq \mathtt{F}(u) + \frac{c(B)}{2^n}, \tag{A.4}$$

from [14], Theorem 1.1, and this gives immediately the result, using (2.20). $\square$

Defining for any subset $I \subset \{1, \ldots, 2^n\}$ $\delta_I := \prod_{i \in I} \delta_i$, and $\delta_I = 1$ if $I = \varnothing$, one wants to compare $\mathbf{E}_n[\delta_I \mathrm{e}^{uS_n}]$ and $\mathbf{E}_n[\delta_I]$ when the partition function $Z_{n,h}^{\mathrm{pure}}$ is still in the linear regime $0 \leq u \leq a_0(B/2)^n$, the bound $\mathbf{E}_n[\delta_I \mathrm{e}^{uS_n}] \geq \mathbf{E}_n[\delta_I]$ being trivial.

Corollary A.2. *There exist constants* $a_0 > 0$ *and* $c' > 0$ *such that for any* $n \geq 0$ *and any non-empty subset* $I \subset \{1, \ldots, 2^n\}$, *if* $0 \leq u \leq a_0(B/2)^n$ *one has*

$$\mathbf{E}_n\big[\delta_I \exp(uS_n)\big] \leq \exp\left(c' u \left(\frac{2}{B}\right)^n\right)^{|I|} \mathbf{E}_n[\delta_I]. \tag{A.5}$$

Proof. We prove by iteration on n that for all non-empty subsets $I \subset \{1, \ldots, 2^n\}$, if $u \leq a_0(B/2)^n$ one has

$$\mathbf{E}_n\big[\delta_I \exp(uS_n)\big] \leq \exp\left(c_0 u \sum_{k=0}^{n} \left(\frac{2}{B}\right)^k\right)^{|I|} \mathbf{E}_n[\delta_I], \tag{A.6}$$

where c_0 is the constant obtained in Lemma A.1.

The case $n = 0$ is trivial. Let us assume that we have the assumption for $n \geq 0$ and prove it for $n + 1$. Take I a non-empty subset of $\{1, \ldots, 2^{n+1}\}$. As in the proof of Lemma 5.3, one decomposes I into its "left" and "right" part and writes $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]$ and $|I| = |I_1| + |\widetilde{I}_2|$.

If $I_1, \widetilde{I}_2 \neq \varnothing$, using the induction hypothesis, one easily has

$$\begin{aligned} \mathbf{E}_{n+1}\big[\delta_I \exp(uS_{n+1})\big] &= \frac{1}{B}\mathbf{E}_n\big[\delta_{I_1} \exp(uS_n)\big]\mathbf{E}_n\big[\delta_{\widetilde{I}_2} \exp(uS_n)\big] \\ &\leq \exp\left(c_0 u \sum_{k=0}^{n} \left(\frac{2}{B}\right)^k\right)^{|I_1| + |\widetilde{I}_2|} \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}], \end{aligned} \tag{A.7}$$

which gives the right bound.

If $I_1 = \varnothing$ (or analogously if $\widetilde{I}_2 = \varnothing$), one has $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{\widetilde{I}_2}]$ and

$$\begin{aligned}\mathbf{E}_{n+1}\big[\delta_I \exp(uS_{n+1})\big] &= \frac{1}{B}\mathbf{E}_n\big[\exp(uS_n)\big]\mathbf{E}_n\big[\delta_{\widetilde{I}_2}\exp(uS_n)\big] \\ &\le \mathrm{e}^{c_0 u(2/B)^{n+1}} \exp\left(c_0 u \sum_{k=0}^{n}\left(\frac{2}{B}\right)^k\right)^{|\widetilde{I}_2|} \frac{1}{B}\mathbf{E}_n[\delta_{\widetilde{I}_2}],\end{aligned} \tag{A.8}$$

where the first part is dealt with Lemma A.1, and the second one with the induction hypothesis. □

Theorem A.3. *Let $B \in (1,2)$. Let $(b_n)_{n\ge 0}$ be a sequence that goes to 0 as n goes to infinity. There exists a constant $c_b > 0$ such that for all $n \ge 0$ and every $0 \le u \le b_n(\frac{B^2}{4} \wedge \frac{1}{2})^n$ one has*

$$\mathbf{E}_n\big[\exp\big(u(S_n)^2\big)\big] \le \exp\left(c_b u\left(\frac{4}{B^2}\right)^n\right). \tag{A.9}$$

Corollary A.4. *Let $B \in (1,2)$, $\kappa < \frac{B^2}{4} \wedge \frac{1}{2}$ and note $\varphi := (2\kappa) \vee \frac{4\kappa}{B^2} < 1$. Then for every $A > 0$ there exists a constant $c_A > 0$ such that for any $n \ge 0$, any $u \in [0, A]$ and any subset I of $\{1, \dots, 2^n\}$, one has*

$$\mathbf{E}_n\big[\delta_I \exp\big(u\kappa^n (S_n)^2\big)\big] \le \big(\mathrm{e}^{c_A u \varphi^n}\big)^{n|I|+1}\mathbf{E}_n[\delta_I]. \tag{A.10}$$

Note that if $I = \varnothing$, the statement is implied by Theorem A.3.

Proof of Theorem A.3. The proof relies on Lemma A.1. Consider $u \le b_n(\frac{B^2}{4} \wedge \frac{1}{2})^n$. One writes

$$J := \mathbf{E}_n\left[\exp\left(\frac{1}{2}u(S_n)^2\right)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-z^2/2}\mathbf{E}_n\big[\exp(z\sqrt{u}S_n)\big]\,\mathrm{d}z. \tag{A.11}$$

One sets $\Delta := \frac{a}{\sqrt{u}}(\frac{B}{2})^n$, where a is a constant that will be chosen small. Note that thanks to our choice of u, one has $\Delta \ge a b_n^{-1/2}$ that goes to infinity as n grows to infinity. Then one decomposes the integral J according to the values of z, and writes $J = J_1 + J_2$, where

$$\begin{aligned} J_1 &:= \frac{1}{\sqrt{2\pi}}\int_{z\le\Delta} \mathrm{e}^{-z^2/2}\mathbf{E}_n\big[\exp(z\sqrt{u}S_n)\big]\,\mathrm{d}z, \\ J_2 &:= \frac{1}{\sqrt{2\pi}}\int_{z\ge\Delta} \mathrm{e}^{-z^2/2}\mathbf{E}_n\big[\exp(z\sqrt{u}S_n)\big]\,\mathrm{d}z. \end{aligned} \tag{A.12}$$

To bound J_1, one chooses $a \le a_0$ with a_0 as in Lemma A.1, such that for the values of z considered one has $z\sqrt{u} \le a_0(B/2)^n$ and then one applies Lemma A.1(1) to get

$$J_1 \le \frac{1}{\sqrt{2\pi}}\int_{z\le\Delta} \mathrm{e}^{-z^2/2}\exp\big(cz\sqrt{u}(2/B)^n\big)\,\mathrm{d}z \le \exp\left(\frac{c^2}{2}u\big(4/B^2\big)^n\right). \tag{A.13}$$

We deal with the term J_2, decomposing again according to the values of z. Let us first introduce some notations: we define the sequence $(\Delta_k)_{k\ge 0}$ by the iteration

$$\begin{cases} \Delta_0 = \Delta, \\ \Delta_{k+1} = \Delta(\Delta_k)^{2/\nu}(> \Delta_k > 1) \end{cases} \tag{A.14}$$

and define also $m = \inf\{k, \Delta_k \ge A\sqrt{u}2^n\}$, for some A chosen large enough later. We point out that m is finite. Indeed for a fixed large n, if $\nu \le 2$, then $\Delta_k \ge \Delta^{k+1}$ and goes to infinity as k goes to infinity. Otherwise, if $\nu > 2$, Δ_k goes to $\Delta^{\nu/(\nu-2)}$ as k goes to infinity. Then, we just need to check that $\Delta^{\nu/(\nu-2)} \geq A\sqrt{u}2^n$ if n is large. Using the value of $\nu = \log 2/\log(2/B)$ one has $2^{1/\nu} = 2/B$, so that $\Delta^\nu = a^\nu u^{-\nu/2}2^{-n}$. Then

$$\frac{\Delta^\nu}{(\sqrt{u}2^n)^{\nu-2}} = a^\nu \frac{u^{-\nu/2}2^{-n}}{u^{\nu/2-1}2^{n(\nu-2)}} = a^\nu \bigl(u2^n\bigr)^{-(\nu-1)} \geq a^\nu b_n^{1-\nu}, \tag{A.15}$$

where we used that $u2^n \leq b_n$. As $\nu > 1$, it remains only to take n large.

One decomposes J_2 as follows:

$$J_2 = \sum_{k=0}^{m-1} \frac{1}{\sqrt{2\pi}} \int_{\Delta_k}^{\Delta_{k+1}} \mathrm{e}^{-z^2/2} \mathbf{E}_n\bigl[\exp(z\sqrt{u}S_n)\bigr]\,\mathrm{d}z \\ + \frac{1}{\sqrt{2\pi}} \int_{\Delta_m}^{+\infty} \mathrm{e}^{-z^2/2} \mathbf{E}_n\bigl[\exp(z\sqrt{u}S_n)\bigr]\,\mathrm{d}z. \tag{A.16}$$

Each term of the sum in (A.16) can be dealt with Lemma A.1(2). One gets

$$\frac{1}{\sqrt{2\pi}} \int_{\Delta_k}^{\Delta_{k+1}} \mathrm{e}^{-z^2/2} \mathbf{E}_n\bigl[\exp(z\sqrt{u}S_n)\bigr]\,\mathrm{d}z \\ \leq \mathbf{E}_n\bigl[\exp(\Delta_{k+1}\sqrt{u}S_n)\bigr] P(\mathcal{N} \geq \Delta_k) \\ \leq c_1 \exp\bigl(c_2 2^n u^{\nu/2} (\Delta_{k+1})^\nu\bigr) \exp\bigl(-c(\Delta_k)^2\bigr), \tag{A.17}$$

where $\mathcal{N}$ stands for a standard centered Gaussian. Now recall the definition of Δ_k and Δ, that gives $(\Delta_{k+1})^\nu = \Delta^\nu(\Delta_k)^2 = a^\nu u^{-\nu/2}2^{-n}(\Delta_k)^2$, so that one can bound the term in (A.17) by

$$c_1 \exp\bigl((c_2 a^\nu - c)(\Delta_k)^2\bigr) \leq c_1 \exp\bigl(-c(\Delta_k)^2/2\bigr), \tag{A.18}$$

where the inequality is valid provided one has chosen a sufficiently small.

Let us now deal with the last term in (A.16), trivially bounding $S_n \leq 2^n$:

$$\frac{1}{\sqrt{2\pi}} \int_{\Delta_m}^{\infty} \mathrm{e}^{-z^2/2} \mathbf{E}_n\bigl[\exp(z\sqrt{u}S_n)\bigr]\,\mathrm{d}z \\ \leq \frac{1}{\sqrt{2\pi}} \int_{\Delta_m}^{\infty} \mathrm{e}^{-z^2/2} \mathrm{e}^{z\sqrt{u}2^n}\,\mathrm{d}z \\ = \mathrm{e}^{u4^n/2} P\bigl(\mathcal{N} \geq \Delta_m - \sqrt{u}2^n\bigr) \\ \leq \mathrm{e}^{A^{-2}(\Delta_m)^2} \mathrm{e}^{-c(1-A^{-1})^2(\Delta_m)^2} \leq \mathrm{e}^{-c(\Delta_m)^2/2}, \tag{A.19}$$

where we used that $\sqrt{u}2^n \leq A^{-1}\Delta_m$, and supposed that A was chosen large enough for the last inequality.

We finally get that for n large one has

$$J_2 \leq c_1 \sum_{k=0}^{m} \mathrm{e}^{-c(\Delta_k)^2/2} \leq \begin{cases} C\mathrm{e}^{-c\Delta^2/2} & \text{if } \nu \leq 2, \\ Cm\mathrm{e}^{-c\Delta^2/2} & \text{if } \nu > 2, \end{cases} \tag{A.20}$$

where in the case $\nu \leq 2$ we used that $\Delta_k \geq \Delta^{k+1}$. Note that for $\nu > 2$, using (A.15), one also can bound m from above as follows: since $\Delta_k = \Delta^{(1-(2/\nu)^{k+1})/(1-2/\nu)}$,

$$\frac{\Delta_k}{\sqrt{u}2^n} = \frac{\Delta^{\nu/(\nu-2)}}{\sqrt{u}2^n} \Delta^{-\nu/(\nu-2)(2/\nu)^{k+1}} \geq a^{\nu/(\nu-2)} b_n^{(1-\nu)/(\nu-2)} \Delta^{-c'(2/\nu)^k}. \tag{A.21}$$

So if one takes $k \geq -\log\log\Delta/\log(2/\nu)$ one gets that $\Delta_k \geq a^{\nu/(\nu-2)} b_n^{(1-\nu)/(\nu-2)} \mathrm{e}^{-c'} \sqrt{u}2^n$. If n is large enough this implies that $m \leq \mathit{const} \times \log\log\Delta$.

Then one easily gets that $J_2 = o(\Delta^{-2})$, with $\Delta^{-2} = O(u(4/B^2)^n)$, so that combining with the bound on J_1 one has

$$J \le \exp\left(\frac{c_0^2}{2}u\left(4/B^2\right)^n\right) + o\left(u\left(4/B^2\right)^n\right). \tag{A.22}$$

□

Proof of Corollary A.4. We proceed by induction. Fix $A > 0$ and $u \le A$, and take the constant c_A obtained in Theorem A.3 for the sequence $b_n = A(\frac{4\kappa}{B^2} \wedge 2\kappa)^n$. The case $n = 0$ is trivial. Suppose now that the assumption is true for some n, and take I a subset of $\{1, \dots, 2^{n+1}\}$.

Suppose $I \neq \varnothing$ (otherwise one already has the result from Theorem A.3). As in the proof of Lemma 5.3, one decomposes I into its "left" and "right" part and $\mathbf{E}_{n+1}[\delta_I] = \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]$. Using that $(S_{n+1})^2 \le 2(S_n^{(1)})^2 + 2(S_n^{(2)})^2$ one gets

$$\begin{aligned}
&\mathbf{E}_{n+1}\left[\delta_I \exp\left(u\kappa^{n+1}(S_{n+1})^2\right)\right] \\
&\quad \le \frac{1}{B}\mathbf{E}_n\left[\delta_{I_1}\exp\left((2\kappa)u\kappa^n(S_n)^2\right)\right]\mathbf{E}_n\left[\delta_{\widetilde{I}_2}\exp\left((2\kappa)u\kappa^n(S_n)^2\right)\right] \\
&\quad \le \frac{1}{B}\mathbf{E}_n[\delta_{I_1}]\mathbf{E}_n[\delta_{\widetilde{I}_2}]\left(e^{c_A u(2\kappa)\varphi^n}\right)^{n|I_1|+n|\widetilde{I}_2|+2} \le \mathbf{E}_{n+1}[\delta_I]\left(e^{c_A u 2\kappa\varphi^n}\right)^{(n+1)|I|+1},
\end{aligned} \tag{A.23}$$

where for the second inequality we used the recursion assumption and for the last one the assumption $|I| \ge 1$. Now one just uses that $2\kappa \le \varphi$ to conclude. □

From Corollary A.4 one can deduce the following proposition, useful to control the variance of the partition function (see Section 6). Define as in (6.5) $D_n := \sum_{i,j=1}^{2^n} \kappa_{ij}\delta_i\delta'_j$, where δ and δ' are the populations at generation n of two independent GW trees.

Proposition A.5. *Let $B \in (1,2)$, $\kappa < \frac{1}{2} \wedge \frac{B^2}{4}$ and set $\varphi = (2\kappa) \wedge (4\kappa/B^2) < 1$.*

- *If $B > B_c$, then for every $\Phi \in (\frac{2}{B^2} \vee \varphi, 1)$ there exist some $u_0 > 0$ and some constant $c > 0$, such that for every $n \in \mathbb{N}$, $u \in [0, u_0]$ one has*

$$\mathbf{E}_n^{\otimes 2}\left[\exp(uD_n)\right] \le 1 + cu\Phi^n. \tag{A.24}$$

- *If $B < B_c$ there exist some $a_1 > 0$ and some constant $c > 0$, such that for every $n \in \mathbb{N}$, if $u \le a_1(\frac{B^2}{2})^n$ one has*

$$\mathbf{E}_n^{\otimes 2}\left[\exp(uD_n)\right] \le 1 + cu\left(\frac{2}{B^2}\right)^n. \tag{A.25}$$

- *If $B = B_c$, there exists some u_0 such that if $u \le u_0$ then for all $n \le \frac{1}{2}u^{-1/3}$ one has*

$$\mathbf{E}_n^{\otimes 2}\left[\exp(uD_n)\right] \le 1 + 2u^{1/3}. \tag{A.26}$$

Proof. One has

$$\begin{aligned}
D_{n+1} &= D_n^{(1)} + D_n^{(2)} + \kappa_{n+1}\left(S_n^{(1)}S'^{(2)}_n + S'^{(1)}_n S'^{(2)}_n\right) \\
&\le D_n^{(1)} + D_n^{(2)} + \frac{\kappa_n}{2}\left(\left(S_n^{(1)}\right)^2 + \left(S'^{(2)}_n\right)^2 + \left(S_n^{(2)}\right)^2 + \left(S'^{(1)}_n\right)^2\right).
\end{aligned} \tag{A.27}$$

Since clearly D_{n+1} vanishes when either of the two GW trees is empty, one has for every $v \in [0,1]$

$$\begin{aligned}
\mathbf{E}_{n+1}^{\otimes 2}\left[e^{vD_{n+1}}\right] &\le \frac{1}{B^2}\mathbf{E}_n^{\otimes 2}\left[e^{vD_n}\exp\left(\frac{v}{2}\kappa_n\left((S_n)^2 + \left(S'_n\right)^2\right)\right)\right]^2 + \frac{B^2-1}{B^2} \\
&\le \frac{1}{B^2}e^{c_0 v\varphi^n}\mathbf{E}_n^{\otimes 2}\left[\exp\left(ve^{c_0 v(\varphi')^n}D_n\right)\right]^2 + \frac{B^2-1}{B^2},
\end{aligned} \tag{A.28}$$

where in the second inequality we expanded e^{vD_n} as in Remark 5.2 and used Corollary A.4 to get the constant $c_0 > 0$ for $\varphi := (2\kappa) \vee \frac{4\kappa}{B^2}$ and some $\varphi' \in (\varphi, 1)$. Then we set $v_0 \le 1$ and for $n \ge 0$ define $v_{n+1} := v_n e^{-c_0 v_n (\varphi')^n} \le v_0$. Define $X_n := \mathbf{E}_n^{\otimes 2}[\exp(v_n D_n)] - 1$, so that using the previous inequality one has

$$X_{n+1} \le \frac{1}{B^2} e^{c_0 v_n \varphi^n} (X_n + 1)^2 - \frac{1}{B^2} \le \frac{2e^{c_0 v_0 \varphi^n}}{B^2} X_n \left(1 + \frac{X_n}{2}\right) + c v_0 \varphi^n. \tag{A.29}$$

We consider the different cases $B < B_c$, $B = B_c$ and $B > B_c$ separately, but each time we estimate from above $\mathbf{E}_n^{\otimes 2}[e^{v_n D_n}]$. One then easily deduces Proposition A.5 using that there exists a constant c_1 such that $v_n \ge c_1 v_0$, and then $\mathbf{E}_n^{\otimes 2}[e^{c_1 v_0 D_n}] \le 1 + X_n$. One concludes taking $u := c_1 v_0$.

In the sequel we actually study the iteration

$$\widehat{X}_{n+1} = \frac{2e^{w_n}}{B^2} \widehat{X}_n \left(1 + \frac{\widehat{X}_n}{2}\right) + (c/c_0) w_n, \quad \widehat{X}_0 = X_0, \tag{A.30}$$

where we defined $w_n := c_0 v_0 \varphi^n$. Clearly, $X_n \le \widehat{X}_n$ for every n.

– Take $B > B_c := \sqrt{2}$. Let us fix some $\Phi \in (\frac{2}{B^2} \vee \varphi, 1)$. One has that $X_0 \le C_0 v_0$ and one shows easily by iteration, using (A.30) and the definition of w_n, that $\widehat{X}_n \le C_n \Phi^n v_0$, with $(C_n)_{n\in\mathbb{N}}$ an increasing sequence satisfying

$$C_{n+1} = C_n e^{w_n} \left(1 + \frac{1}{2} C_n v_0 \Phi^n\right) + c' \varphi^n \Phi^{-(n+1)} \tag{A.31}$$

(use that $\Phi > (2/B^2)$). Then we show that provided that v_0 has been chosen small enough, $(C_n)_{n\in\mathbb{N}}$ is a bounded sequence. Indeed, using that $C_n \ge C_0$ one has

$$\begin{aligned} C_{n+1} &\le C_n e^{w_n} \left(1 + \frac{1}{2} C_n v_0 \Phi^n + c' \Phi^{-1} C_n^{-1} (\varphi/\Phi)^n\right) \\ &\le C_n e^{w_n} \exp\left(\frac{1}{2} C_n v_0 \Phi^n\right) \exp\bigl(c''(\varphi/\Phi)^n\bigr) \le A \exp\left(\frac{1}{2} v_0 \sum_{k=0}^{n} C_k \Phi^k\right), \end{aligned} \tag{A.32}$$

where we noted $A := \prod_{n=0}^{\infty} e^{w_n} e^{c''(\varphi/\Phi)^n}$, with $A < +\infty$ thanks to the definition of w_n and using that $\Phi > \varphi$. It is then not difficult to see that if v_0 is chosen small enough, more precisely such that $A \exp(v_0 C_0 \sum_{k=0}^{n} \Phi^k) \le 2C_0$, then C_n remains smaller than $2C_0$ for every $n \in \mathbb{N}$. From this, one gets that $X_n \le 2C_0 \Phi^n v_0$ for every n.

– Take $B < B_c$. The idea is that if X_0 is small enough, (A.30) can be approximated by the iteration $X_{n+1} \le \frac{2}{B^2} X_n$ while X_n remains small. For any fixed $n \ge 0$, one chooses $v_0 = a(B^2/2)^n$ with a small (chosen in a moment), and one has $X_0 \le C_0 a (\frac{B^2}{2})^n$. Then one shows by iteration that

$$\widehat{X}_k \le C_k a \bigl(B^2/2\bigr)^{n-k} \tag{A.33}$$

for some increasing sequence $(C_k)_{k\in\mathbb{N}}$ verifying

$$C_{k+1} = e^{w_k} C_k \left(1 + \frac{C_k}{2} a \left(\frac{B^2}{2}\right)^{n-k}\right) + a^{-1} \left(\frac{B^2}{2}\right)^{k+1-n} w_k. \tag{A.34}$$

One then shows with the same method as in the case $B > B_c$ that C_n is bounded by some constant C uniformly in n, provided that a had been chosen small enough. Thus taking $k = n$ one has $X_n \le ca = cv_0 (2/B^2)^n$.

– Take $B = B_c = \sqrt{2}$. The iteration (A.30) gives

$$X_{n+1} \le e^{w_n} X_n \left(1 + \frac{X_n}{2}\right) + (c/c_0) w_n \tag{A.35}$$

and we recall that $w_n = c_0 v_0 \varphi^n$. Take $v_0 = \varepsilon^3$, so that $X_0 \le \varepsilon$ for ε small. We now show that if $\varepsilon \le \varepsilon_0$ with ε_0 chosen small enough, one has for all $n \le \frac{1}{2}\varepsilon^{-1}$ that $X_n \le \varepsilon(1+n\varepsilon)$. We prove this by induction. For $n=0$ this is just because one chose $X_0 \le \varepsilon$. If $X_n \le \varepsilon(1+n\varepsilon)$ and $n\varepsilon \le 1/2$, one has (note that $w_n \le c_0\varepsilon^3$ for all n)

$$\begin{aligned} X_{n+1} &\le e^{c_0\varepsilon^3}\varepsilon(1+n\varepsilon)\left(1+\frac{1}{2}\varepsilon(1+n\varepsilon)\right)+c\varepsilon^3, \\ &\le \varepsilon\big[\big(1+c_0'\varepsilon^3\big)(1+n\varepsilon)(1+3\varepsilon/4)+c\varepsilon^2\big] \\ &\le \varepsilon\big[1+\varepsilon\big(n+3/4+c_0'\varepsilon^2+c\varepsilon\big)\big] \le \varepsilon\big(1+(n+1)\varepsilon\big), \end{aligned} \tag{A.36}$$

provided that $\varepsilon \le \varepsilon_0$ with ε_0 small enough. This concludes the induction step. Thus one has that for all $n \le \frac{1}{2}\varepsilon^{-1}$, $X_n \le 2\varepsilon$, with $\varepsilon = v_0^{1/3}$. $\square$

Appendix B: Hierachically correlated Gaussian vectors

Lemma B.1. *Let $m(\cdot)$ be a function from $\mathbb{N}$ to $\mathbb{R}$ and for $n \in \mathbb{N}$ let Let $M := M^{(n)} = (M_{ij})_{1\le i,j\le 2^n}$ be the $2^n \times 2^n$ matrix with entries $M_{ij} := m(d(i,j))$. Then, the eigenvectors of such a matrix do not depend on the function $m(\cdot)$, and the eigenvalues are*

$$\begin{aligned} \lambda_0 &= m(0) + \sum_{k=1}^{n} 2^{k-1} m(k), \quad \textit{with multiplicity } 1, \\ \lambda_p &= m(0) + \sum_{k=1}^{n-p} 2^{k-1} m(k) - 2^{n-p} m(n+1-p), \quad \textit{with multiplicity } 2^{p-1}, \textit{ for } 1 \le p \le n. \end{aligned} \tag{B.1}$$

This comes directly from the fact that

$$M^{(n)} = \begin{pmatrix} & M^{(n-1)} & & \begin{matrix} m(n) & \cdots & m(n) \\ \vdots & & \vdots \\ m(n) & \cdots & m(n) \end{matrix} \\ \begin{matrix} m(n) & \cdots & m(n) \\ \vdots & & \vdots \\ m(n) & \cdots & m(n) \end{matrix} & & & M^{(n-1)} \end{pmatrix}, \tag{B.2}$$

where each block is of size 2^{n-1}. One computes the eigenvalues: the eigenvector $(1,\dots,1)$ gives λ_0, the eigenvector $(1,\dots,1,-1,\dots,-1)$ gives λ_1. Then the eigenvectors $(X,0)$ and $(0,X)$ with $X \ne (1,\dots,1)$ being an eigenvector of $M^{(n-1)}$ give all the others eigenvalues, which are the eigenvalue associated to X with $M^{(n-1)}$, but with multiplicity multiplied by 2.

Remark B.2. *Lemma B.1 shows that the spectral radius of $M^{(n)}$ is upper bounded by $\sum_{p=0}^{\infty} 2^p|m(p)|$. Also, two matrices with entries depending only on the distances $d(i,j)$ can be codiagonalized, as the eigenvectors do not depend on the values of the entries, and one can describe the diagonalizing orthogonal matrix Ω*

$$\Omega = \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & 1 & \sqrt{2} & 0 & \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & 1 & -\sqrt{2} & 0 & \\ 1 & -1 & 0 & \sqrt{2} & \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 1 & -1 & 0 & -\sqrt{2} & \end{pmatrix} \tag{B.3}$$

such that $\Omega^t K \Omega = \mathrm{Diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_2, \dots)$ with λ_i given in Lemma B.1.

Let $\omega = \{\omega_i\}_{i\in\mathbb{N}}$ be the centered Gaussian family with correlation structure $\mathbb{E}[\omega_i\omega_j] = \kappa_{d(i,j)}$. The following Proposition gives the dependence on κ_n of a smooth function of $\omega_1, \dots, \omega_{2^n}$:

Proposition B.3. *If $f : \mathbb{R}^{2^n} \mapsto \mathbb{R}$ is twice differentiable and grows at most polynomially at infinity, one has*

$$\frac{\partial}{\partial\kappa_n}\mathbb{E}\big[f(\omega_1, \dots, \omega_{2^n})\big] = \sum_{i=1}^{2^{n-1}} \sum_{j=2^{n-1}+1}^{2^n} \mathbb{E}\bigg[\frac{\partial^2 f}{\partial\omega_i\partial\omega_j}(\omega)\bigg]. \tag{B.4}$$

Proof. Thanks to Remark B.3, one has

$$\mathbb{E}\big[f(\omega_1, \dots, \omega_{2^n})\big] = \widetilde{\mathbb{E}}\big[f(\Omega\omega)\big], \tag{B.5}$$

with Ω defined in (B.3), and where $\widetilde{\mathbb{P}}$ stands for the law of a centered Gaussian vector of covariance matrix $\Delta := \mathrm{Diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_2, \dots)$. The eigenvalues λ_i and their multiplicity are given in Lemma B.1. Then, as only $\lambda_0 = \kappa_0 + \sum_{k=1}^{n} 2^{k-1}\kappa_k$ and $\lambda_1 = \kappa_0 + \sum_{k=1}^{n-1} 2^{k-1}\kappa_k - 2^{n-1}\kappa_n$ depend on κ_n one gets

$$\frac{\partial}{\partial\kappa_n}\mathbb{E}\big[f(\omega)\big] = 2^{n-1}\frac{\partial}{\partial\lambda_0}\widetilde{\mathbb{E}}\big[f(\Omega\omega)\big] - 2^{n-1}\frac{\partial}{\partial\lambda_1}\widetilde{\mathbb{E}}\big[f(\Omega\omega)\big]. \tag{B.6}$$

Then one uses the classical Gaussian fact that if ω is a centered Gaussian variable of variance σ^2 and g is a differentiable function which grows at most polynomially at infinity,

$$\frac{\partial}{\partial\sigma^2}\mathbb{E}\big[g(\omega)\big] = \frac{1}{2}\mathbb{E}\bigg[\frac{\partial^2 g}{\partial\omega^2}(\omega)\bigg]. \tag{B.7}$$

Plugging this result in (B.6) one gets

$$\begin{aligned}
&\frac{1}{2^{n-1}}\frac{\partial}{\partial\kappa_n}\mathbb{E}\big[f(\omega_1, \dots, \omega_{2^n})\big] \\
&\quad= \frac{1}{2}\sum_{i,j=1}^{2^n}\Omega_{i1}\Omega_{j1}\widetilde{\mathbb{E}}\bigg[\frac{\partial^2 f}{\partial x_i\,\partial x_j}\bigg|_{x=\Omega\omega}\bigg] - \frac{1}{2}\sum_{i,j=1}^{2^n}\Omega_{i2}\Omega_{j2}\widetilde{\mathbb{E}}\bigg[\frac{\partial^2 f}{\partial x_i\,\partial x_j}\bigg|_{x=\Omega\omega}\bigg] \\
&\quad= \frac{1}{2^n}\sum_{\substack{i,j=1\\ d(i,j)=n}}^{2^n}\mathbb{E}\bigg[\frac{\partial^2 f}{\partial\omega_i\,\partial\omega_j}(\omega)\bigg],
\end{aligned} \tag{B.8}$$

where in the second equality we used the values of Ω_{k1} and Ω_{k2}. □

Remark B.4. *With the same type of computations, since Ω is explicit, one can also compute the derivative with respect to κ_p for $p \le n$, and after some computations, one gets*

$$\frac{\partial}{\partial\kappa_p}\mathbb{E}\big[f(\omega_1, \dots, \omega_{2^n})\big] = \frac{1}{2}\sum_{\substack{i,j=1\\ d(i,j)=p}}^{2^n}\mathbb{E}\bigg[\frac{\partial^2 f}{\partial\omega_i\,\partial\omega_j}(\omega)\bigg]. \tag{B.9}$$

Acknowledgments

We wish to thank Bernard Derrida for a very interesting discussion and Giambattista Giacomin for comments on the manuscript. This work was partly done while the authors were visiting the Department of Mathematics of the University of Roma Tre, with financial support by ERC "Advanced Grant" PTRELSS 228032.

References

[1] K. S. Alexander. The effect of disorder on polymer depinning transitions. *Commun. Math. Phys.* **279** (2008) 117–146. MR2377630

[2] K. S. Alexander and N. Zygouras. Quenched and annealed critical points in polymer pinning models. *Commun. Math. Phys.* **291** (2009) 659–689. MR2534789

[3] Q. Berger and H. Lacoin. Sharp critical behavior for random pinning model with correlated environment. *Stochastic Process. Appl.* **122** (2012) 1397–1436.

[4] P. M. Bleher. The renormalization group on hierarchical lattices. In *Stochastic Methods in Mathematics and Physics (Karpacz, 1988)* 171–201. World Sci. Publ., Teaneck, NJ, 1989. MR1124692

[5] D. Cheliotis and F. den Hollander. Variational characterization of the critical curve for pinning of random polymers. *Ann. Probab.* **41** (2013) 1767–1805.

[6] P. Collet, J.-P. Eckmann, V. Glaser and A. Martin. Study of the iterations of a mapping associated to a spin glass model. *Commun. Math. Phys.* **94** (1984) 353–370. MR0763384

[7] B. Derrida and E. Gardner. Renormalization group study of a disordered model. *J. Phys. A* **17** (1984) 3223–3236. MR0772152

[8] B. Derrida, G. Giacomin, H. Lacoin and F. L. Toninelli. Fractional moment bounds and disorder relevance for pinning models. *Commun. Math. Phys.* **287** (2009) 867–887. MR2486665

[9] B. Derrida, V. Hakim and J. Vannimenus. Effect of disorder on two-dimensional wetting. *J. Statist. Phys.* **66** (1992) 1189–1213. MR1156401

[10] F. J. Dyson. Existence of a phase-transition in a one-dimensional Ising ferromagnet. *Commun. Math. Phys.* **12** (1969) 91–107. MR0436850

[11] G. Giacomin. *Random Polymer Models*. Imperial College Press, London, 2007. MR2380992

[12] G. Giacomin. Renewal convergence rates and correlation decay for homogeneous pinning models. *Electron. J. Probab.* **13** (2008) 513–529. MR2386741

[13] G. Giacomin. *Disorder and Critical Phenomena Through Basic Probability Models*. *Lecture Notes in Mathematics* **2025**. Springer, Berlin, 2011. MR2816225

[14] G. Giacomin, H. Lacoin and F. Toninelli. Hierarchical pinning model, quadratic maps and quenched disorder. *Probab. Theory Related Fields* **148** (2010) 159–175. MR2653225

[15] G. Giacomin, H. Lacoin and F. L. Toninelli. Marginal relevance of disorder for pinning models. *Commun. Pure Appl. Math.* **63** (2010) 233–265. MR2588461

[16] G. Giacomin, H. Lacoin and F. L. Toninelli. Disorder relevance at marginality and critical point shift. *Ann. Inst. Henri Poincaré Probab. Stat.* **47** (2011) 148–175. MR2779401

[17] G. Giacomin and F. L. Toninelli. Smoothing effect of quenched disorder on polymer depinning transitions. *Commun. Math. Phys.* **266** (2006) 1–16. MR2231963

[18] A. B. Harris. Effect of random defects on the critical behaviour of ising models. *J. Phys. C* **7** (1974) 1671–1692.

[19] H. Lacoin. The martingale approach to disorder irrelevance for pinning models. *Electron. Commun. Probab.* **15** (2010) 418–427. MR2726088

[20] H. Lacoin and F. L. Toninelli. A smoothing inequality for hierarchical pinning models. In *Spin Glasses: Statics and Dynamics* 271–278. A. Boutet de Monvel and A. Bovier (eds.). *Progress in Probability* **62**. Birkhäuser Verlag, Basel, 2009. MR2761990

[21] J. Poisat. On quenched and annealed critical curves of random pinning model with finite range correlations. *Ann. Inst. Henri Poincaré Probab. Stat.* **49** (2013) 456–482.

[22] F. L. Toninelli. Critical properties and finite-size estimates for the depinning transition of directed random polymers. *J. Statist. Phys.* **126** (2007) 1025–1044. MR2311896

[23] F. L. Toninelli. A replica-coupling approach to disordered pinning models. *Commun. Math. Phys.* **280** (2008) 389–401. MR2395475

[24] A. Weinrib and B. I. Halperin. Critical phenomena in systems with long-range-correlated quenched disorder. *Phys. Rev. B* **27** (1983) 413–427.