The Annals of Applied Probability
2013, Vol. 23, No. 3, 1254–1289
DOI: 10.1214/12-AAP874

DEGREE AND CLUSTERING COEFFICIENT IN SPARSE RANDOM INTERSECTION GRAPHS

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Dedicated to Professor Friedrich Götze on the occasion of his 60th birthday

We establish asymptotic vertex degree distribution and examine its relation to the clustering coefficient in two popular random intersection graph models of Godehardt and Jaworski [*Electron. Notes Discrete Math.* **10** (2001) 129–132]. For sparse graphs with a positive clustering coefficient, we examine statistical dependence between the (local) clustering coefficient and the degree. Our results are mathematically rigorous. They are consistent with the empirical observation of Foudalis et al. [In *Algorithms and Models for Web Graph* (2011) Springer] that, "clustering correlates negatively with degree." Moreover, they explain empirical results on k^{-1} scaling of the local clustering coefficient of a vertex of degree k reported in Ravasz and Barabási [*Phys. Rev. E* **67** (2003) 026112].

1. Introduction. In a recent paper [12], Foudalis et al. analyzed Facebook data and made the observation that "clustering correlates negatively with degree." Their empirical findings suggest that the chances of two neighbors of a given vertex to be adjacent is a decreasing function of the vertex degree. A reasonable question to ask is whether and how such a phenomenon can be explained with the aid of a known theoretical model.

This question is addressed in the present paper. We consider two simple random graph models admitting a power-law degree distribution and positive clustering coefficient.

Given a finite set W and a collection of its subsets $D_1, \dots, D_n$, the active intersection graph defines adjacency relation between the subsets by declaring two subsets adjacent whenever they share at least s common elements. The passive intersection graph defines adjacency relation between elements of W. A pair of elements is declared an edge if it is contained in s or more subsets. Here $s \geq 1$ is a model parameter. Both models have reasonable interpretations: in the active graph, two actors D_i and D_j establish a communication link whenever they have sufficiently many common interests; in the passive graph, students become acquaintances if they participate in sufficiently many joint projects. In order to model

Received October 2011; revised May 2012.

[1]Supported by Research Council of Lithuania Grant MIP-053/2011.

MSC2010 subject classifications. Primary 05C80, 91D30; secondary 05C07.

Key words and phrases. Clustering coefficient, power law, degree distribution, random intersection graph.

active and passive graphs with desired statistical properties, we choose subsets $D_1, \dots, D_n$ at random and obtain random intersection graphs. Such random graph models have been introduced in [13, 19]; see also [10].

We remark that the adjacency relations of random intersection graphs resemble that of some real networks, like, for example, the actor network, where two actors are linked by an edge whenever they have acted in the same film, or the collaboration network, where authors are declared adjacent whenever they have coauthored at least s papers. These networks exploit the underlying bipartite graph structure: actors are linked to films, and authors to papers. Newman et al. [22] pointed out that the clustering property of some social networks could be explained by the presence of such a bipartite graph structure; see also [1, 16], and references therein. In this respect, it is relevant to mention that the random intersection graphs of the present paper can be obtained from the random bipartite graph with bipartition $V \cup W$, where each vertex v_i of the set $V = \{v_1, \dots, v_n\}$ selects the set $D_i \subset W$ of its neighbors in the bipartite graph independently at random. In addition, we assume for simplicity that all elements of W have equal probabilities to be selected. Now, the active intersection graph defines the adjacency relation on the vertex set V: v_i and v_j are adjacent if they have at least s common neighbors in the bipartite graph. Similarly, vertices $w_i, w_j \in W$ are adjacent in the passive graph whenever they have at least s common neighbors in the bipartite graph. An attractive property of these models that motivated our study is that they capture some features of real networks and, therefore, they may be useful in better understanding the statistical properties of some social networks.

To summarize our paper, we note that both active and passive random intersection graphs admit a nonvanishing clustering coefficient although their "clustering mechanisms" are different. Both models admit a power-law (asymptotic) degree distribution but of a different structure. A common feature of these models is that the (asymptotic) degree distribution of graphs with a nontrivial clustering coefficient (i.e., one taking values other than 0 or 1) has a finite second moment. Another interesting fact is that, in many cases, the chance of two neighbors of a given vertex to be adjacent decays as ck^{-1}, where k is the vertex degree. For example, the k^{-1} *scaling* is shown for a power-law active random intersection graph with a positive clustering coefficient and integrable degree. This theoretical result agrees with the empirical findings reported in [24] showing the k^{-1} scaling in some real networks.

The paper is organized as follows. In Section 1, we introduce the active random intersection graph and collect results about the degree distribution, clustering coefficient and statistical dependence between the degree and the clustering coefficient of a given vertex. Section 2 is devoted to the passive random intersection graph. Proofs are given in Section 3.

In what follows, the expressions $o(\cdot)$, $O(\cdot)$ refer to the case where the size m of the auxiliary set W and the number n of subsets defining the intersection graph both tend to infinity (unless stated otherwise). Given a sequence $\{\xi_m\}$ of random variables, we write $\xi_m = o_P(1)$ if $\xi_m \to 0$ in probability, that is, $\forall \varepsilon > 0, \mathbf{P}(\xi_m >$

$\varepsilon) = o(1)$. We write $\xi_m = O_P(1)$ if the sequence is stochastically bounded, that is, $\forall\varepsilon > 0, \exists A > 0$: $\sup_m \mathbf{P}(|\xi_m| > A) < \varepsilon$.

2. Active intersection graph. Given a set of attributes $W = \{w_1, \dots, w_m\}$, an actor v is identified with the set $D(v)$ of attributes selected by v from W. We assume that the actors $v_1, \dots, v_n$ choose their attribute sets $D_i = D(v_i)$, $1 \le i \le n$, independently at random, and we declare v_i and v_j adjacent (denoted $v_i \sim v_j$) whenever they share at least s common attributes, that is, $|D_i \cap D_j| \ge s$. Here and below, $s \ge 1$ is the same for all pairs v_i, v_j. The graph on the vertex set $V = \{v_1, \dots, v_n\}$ defined by this adjacency relation is called the *active* random intersection graph; see [14]. Subsets of W of size s play a special role; we call them joints. They serve as witnesses of established links: $v_i \sim v_j$ whenever there exists a joint belonging to both D_i and D_j.

We first assume, for simplicity, that the random sets $D_1, \dots, D_n$ have the same probability distribution of the form

$$\mathbf{P}(|D_i| = A) = P(|A|)\binom{m}{|A|}^{-1} \qquad \text{for } A \subset W. \tag{2.1}$$

That is, given an integer k, all subsets $A \subset W$ of size $|A| = k$ receive equal chances, proportional to the weight $P(k)$, where P is a probability on $\{0, 1, \dots, m\}$. The random intersection graph defined in this way is denoted $G_s(n, m, P)$. One special case where all random sets are of the same (nonrandom) size has attracted particular attention in the literature (see, e.g., [3, 10, 14, 23, 25, 30]), as it provides a convenient model of a secure wireless network. We call it the *uniform* active random intersection graph and denote $G_s(n, m, \delta_x)$. Here δ_x is the probability distribution putting mass 1 on x, and x is the size of the random sets.

2.1. *Degree distribution.* We consider a sequence of random intersection graphs, where m and $n = n_m$ tend to infinity, and where $s = s_m$ and $P = P_m$ depend on m. We suppress the subscript m in what follows whenever this does not cause an ambiguity. By $X_i = |D_i|$ we denote the size of the attribute set D_i. Note that X_i is a random variable taking values in $\{0, 1, \dots, m\}$ and having the probability distribution P.

LEMMA 1. *Let $m \to +\infty$. Assume that $s = O(1)$ and $\mathbf{E}X_1^s \mathbb{I}_{\{X_1 > \varepsilon\sqrt{m}\}} = o(\mathbf{E}X_1^s)$ for any $\varepsilon > 0$. Then for any pair of vertices v_i, v_j, the edge probability in $G_s(n, m, P)$*

$$\mathbf{P}(v_i \sim v_j) = (1 + o(1))\binom{m}{s}^{-1}\left(\mathbf{E}\binom{X_1}{s}\right)^2. \tag{2.2}$$

In particular, we have $\mathbf{P}(v_i \sim v_j) = o(1)$.

We note that $\mathbf{E}\binom{X_i}{s}$ is the expected number of joints available to the vertex v_i. (2.2) may fail in the case where $s = s_m \to \infty$ as $m \to \infty$; see Example 2 below.

REMARK 1. In the particular case where s remains fixed as $n, m \to \infty$, Lemma 1 suggests that a sparse graph $G_s(n, m, P)$ is obtained when the random sets are of order

$$|D_i| = O_P\big(m^{1/2} n^{-1/(2s)}\big). \tag{2.3}$$

By sparse we mean that the degree of the typical vertex is stochastically bounded as $n \to \infty$.

Our next result shows that the degree of the typical vertex of a sparse random intersection graph is asymptotically Poissonian with (random) intensity parameter $Z_1 \mathbf{E} Z_1$. Here $Z_i = Z_{mi} = \binom{m}{s}^{-1/2} n^{1/2} \binom{X_i}{s}$ denotes the properly rescaled number of joints of the vertex v_i, $1 \le i \le n$. Let $d(v_i) = d_m(v_i)$ denote the degree of the vertex v_i in $G_s(n, m, P)$. Observe that, by symmetry, the random variables $d(v_1), \dots, d(v_n)$ have the same probability distribution.

THEOREM 2.1. *Let $m, n \to \infty$. Assume that $s = O(1)$ and*:

(i) *Z_{m1} converges in distribution to a random variable Z;*
(ii) *$\mathbf{E} Z < \infty$ and $\lim_{m\to\infty} \mathbf{E} Z_{m1} = \mathbf{E} Z$.*

Denote $\mu = \mathbf{E} Z$. We have, for $k = 0, 1, 2 \dots,$

$$\lim_{m\to\infty} \mathbf{P}\big(d_m(v_1) = k\big) = (k!)^{-1} \mathbf{E}\big((Z\mu)^k e^{-Z\mu}\big). \tag{2.4}$$

Formula (2.4) remains valid if the condition $s = O(1)$ is replaced by the following weaker condition: there exists $0 < a < 1$ such that $s \le am$ and $(s!/n)^{1/s} = o(m)$. In this case, we require, in addition to (i) *and* (ii), *that*:

(iii) $\frac{(s!)^{1/s}}{n^{(s+1)/s}} \sum_{k=2}^{n} (Z_{m1} Z_{mk})^{(s+1)/s} = o_P(1)$.

Formula (2.4) relates the vertex degree distribution to the distribution of sizes of the random sets. In particular, the more variable is the sequence of realized values $X_1, \dots, X_n$, the more irregular is the sequence of vertex degrees of a realized instance of the intersection graph. For example, we obtain a power-law distribution $p_k \sim c k^{-\gamma}$ as $k \to \infty$ whenever the limiting distribution of $\{Z_{m1}\}_m$ has a power law $\mathbf{P}(Z > t) \sim c' t^{1-\gamma}$ as $t \to \infty$. Here $\gamma > 2$, and c, c' denote some positive constants. Observe, that the asymptotic degree distribution defined by formula (2.4) has a first moment.

In the case where $s = 1$, the asymptotic degree distribution of an active random intersection graph was shown (in increasing generality) in [4, 9, 17, 27]; see also [6] and [26]. In particular, the result of Theorem 2.1, for $s = 1$, can be found in [4, 6] and [26]. For $s > 1$, the result of Theorem 2.1 is new. It also applies to the case where $s = s_m \to +\infty$ as $m \to \infty$. We note that limit (2.4) may fail when $s = s_m$ grows to infinity "sufficiently fast;" see Example 2 below. A general sufficient condition for the convergence to the Poisson mixture (2.4) is given in Theorem 2.2.

In Theorem 2.2, we assume that the random sets $D_1, \dots, D_n$ defining the intersection graph are independent but not necessarily identically distributed. We write, as before, $X_k = |D_k|$ and assume that for every $k = 1, \ldots, n$ and each $A \subset W$, we have $\mathbf{P}(D_k = A) = \binom{m}{|A|}^{-1}\mathbf{P}(X_k = |A|)$. Let $\overline{P}$ denote the distribution of the random vector $\overline{X} = (X_1, \ldots, X_n)$. By $\overline{G}_s(n, m, \overline{P})$ we denote the random intersection graph on the vertex set V, where $v_i, v_j \in V$ are adjacent whenever $|D_i \cap D_j| \geq s$. By $\overline{d}(v) = \overline{d}_m(v)$ we denote the degree of $v \in V$ in $\overline{G}_s(n, m, \overline{P})$. For a vector with nonnegative integer coordinates $\overline{x} = (x_1, \ldots, x_n)$, we write $u_k = \binom{x_1}{s}\binom{x_k}{s}$ and $x_k^+ = \max\{0, x_k - s\}$, and we denote

$$\lambda(\overline{x}) = \binom{m}{s}^{-1} \sum_{k=2}^{n} u_k, \qquad \kappa_1(\overline{x}) = \binom{m}{s}^{-2} \sum_{k=2}^{n} u_k^2,$$

$$\kappa_2(\overline{x}) = \frac{x_1^+}{m - x_1} \binom{m}{s}^{-1} \sum_{k=2}^{n} u_k x_k^+.$$

Our next result applies to a sequence of random intersection graphs $\overline{G}_s(n, m, \overline{P})$, where $m \to \infty$ and $n = n_m \to \infty$, and where $s = s_m$ and the distribution $\overline{P}$ of the random vector $\overline{X}$ depend on m.

THEOREM 2.2. *Assume that, as $m \to \infty$:*

(iv) $\lambda(\overline{X})$ *converges in distribution to a random variable* Λ;
(v) $\kappa_1(\overline{X}) = o_P(1)$;
(vi) $\kappa_2(\overline{X}) = o_P(1)$.
Then we have for $k = 0, 1, 2, \ldots,$

$$\lim_{m \to \infty} \mathbf{P}\big(\overline{d}_m(v_1) = k\big) = (k!)^{-1}\mathbf{E}\big(\Lambda^k e^{-\Lambda}\big). \tag{2.5}$$

We briefly remark that condition (v) is a kind of "asymptotic negligibility condition" imposed on the sequence of random variables $\binom{X_1}{s}\binom{X_k}{s}$, $2 \leq k \leq n$. Condition (vi) ensures that (the distributional limit of) $\lambda(\overline{X})$ alone determines the asymptotic degree distribution; see also Lemma 6 below.

In Examples 1 and 2 below, we consider *uniform* random intersection graphs. In Example 1, we formulate, in terms of $n = n_m$, $x = x_m$, and $s = s_m$, conditions that are sufficient for the convergence (2.5). Example 2 shows that conditions (iv) and (v) alone do not suffice to establish the convergence (2.5). Here we are in a situation where the edge probability formula (2.2) fails.

EXAMPLE 1. Let $\{x_m\}$ and $\{s_m\}$ be integer sequences such that $1 \leq x_m - s_m = o((m - x_m)^{1/2})$ and $\binom{x_m}{s_m}^2\binom{m}{s_m}^{-1} = o(1)$ as $m \to \infty$. Let n_m be an integer sequence such that the limit $\lim_m n_m\binom{x_m}{s_m}^2\binom{m}{s_m}^{-1}$ exists and is finite. We denote this limit by λ. The sequence of random intersection graphs $\{G_{s_m}(n_m, m, \delta_{x_m})\}$ satisfies the conditions of Theorem 2.2 with $\Lambda \equiv \lambda$. Therefore, the degree of the typical vertex has limiting Poisson distribution with mean λ.

EXAMPLE 2. Given m, let $s = 0.5m$, $x = (\varepsilon + 0.5)m$. Here $\varepsilon \in (0, 1)$ is a small absolute constant. We choose ε small enough so that $p^*(m) := \binom{x}{s}^2 \binom{m}{s}^{-1} = o(1)$ as $m \to \infty$. Next, we choose an integer sequence $n_m \uparrow +\infty$ such that $n_m p^*(m) \to 1$ as $m \to \infty$ and consider the sequence of random intersection graphs $\{G_s(n_m, m, \delta_x)\}_m$. It follows from the relation $n_m p^*(m) \to 1$ that conditions (iv)–(v) hold with $\Lambda \equiv 1$. Moreover, we show that $\overline{d}_m(v_1) = o_P(1)$ as $m \to \infty$. The latter relation contradicts to (2.5). Note that for this sequence of random intersection graphs, condition (vi) fails. Indeed, we have $\kappa_2(\overline{X}) = \frac{\varepsilon^2 m}{0.5-\varepsilon} n_m p^*(m) \to +\infty$ as $m \to \infty$. The proofs of the statements of Example 2 are given in Section 3.

2.2. *Clustering coefficient.* Typically, the adjacency relations between actors in real networks are not statistically independent events. Often, chances of a link $v' \sim v''$ increase as we learn that actors v' and v'' have a common neighbor, say, v. As a theoretical measure of such a statistical dependence, one can use the conditional probability (see, e.g., [9])

$$\alpha = \mathbf{P}\big(v' \sim v'' | v' \sim v, v'' \sim v\big).$$

In the literature (see [2, 20, 21, 29]), the empirical estimates of the conditional probability α,

$$\hat{\alpha} = n^{-1} \sum_{v \in V} \frac{N_3(v)}{N_2(v)} \quad \text{and} \quad \hat{\hat{\alpha}} = \frac{\sum_{v \in V} N_3(v)}{\sum_{v \in V} N_2(v)},$$

are called the clustering coefficient and the global clustering coefficient, respectively. Here n denotes the number of vertices of a graph, $N_3(v)$ is the number of unlabeled triangles having vertex v, $N_2(v)$ is the number of unlabeled 2-stars with the central vertex v. In this paper, the term clustering coefficient is used exclusively for the conditional probability α.

Note that, in the random graph $G_s(n, m, P)$, the conditional probability α does not depend on the choice of v, v', v''. It does not depend on n either. We write $\alpha = \alpha_s(m, P)$ in order to indicate the dependence on s, m and P.

We begin our analysis with the uniform random intersection graph $G_s(n, m, \delta_x)$ where, for large m, the asymptotics of the clustering coefficient is simple and transparent. Since we are interested in sparse random graphs, we may assume that $\mathbf{P}(v_i \sim v_j) = o(1)$ as $m \to \infty$. Observe that in the case of bounded s [i.e., $s = O(1)$ as $m \to \infty$], this assumption implies that $x^2 = o(m)$; see (2.3). In the following lemma, we consider a sequence of uniform random intersection graphs, where $x = x_m$ and $s = s_m$ depend on m, and $(x - s)^2 = o(m - x)$. Note that, for bounded s, the later condition is equivalent to $x^2 = o(m)$.

LEMMA 2. *Let $m \to +\infty$. Assume that $s < x$ and $(x - s)^2 = o(m - x)$. Then*

$$\alpha_s(m, \delta_x) = \binom{x}{s}^{-1} + o(1). \tag{2.6}$$

The assumption $s < x$ of the lemma excludes the trivial case $x = s$, where we have $\alpha_s(m, \delta_x) \equiv 1$. Lemma 2 brings some insight into the general model $G_s(n, m, P)$. Namely, in the case of a sparse random graph, it suggests that the clustering coefficient is nonvanishing whenever the sizes of random sets are stochastically bounded as $m \to \infty$. In fact, we need to impose an even stronger condition, which requires, in particular, that the moment of order $2s$ of $P = P_m$ is bounded as $m \to \infty$. For $k = 1, 2$, we denote $a_k = \int \binom{x}{s}^k P(dx) = \mathbf{E}\binom{X_1}{s}^k$.

LEMMA 3. *Assume that for $k = 1, 2$, we have, as $m \to \infty$,*

$$a_k > 0 \text{ and } \forall \varepsilon \in (0, 1) \qquad \frac{1}{a_k} \int_{x > \varepsilon\sqrt{m}} \binom{x}{s}^k P(dx) = o(1). \tag{2.7}$$

Then

$$\alpha_s(m, P) = a_1/a_2 + o(1). \tag{2.8}$$

Note that invoking in (2.8) the simple inequality $a_1^2 \le a_2$, we obtain $\alpha_s(m, P) \le a_2^{-1/2} + o(1)$. Hence, $a_2 \to \infty$ implies $\alpha_s(m, P) \to 0$.

Up to our best knowledge, the first result showing that a power-law (active) random intersection graph with $m \approx \beta n$, $\beta > 0$ admits a nonvanishing clustering coefficient is due to Deijfen and Kets [9]. They established a first-order asymptotics of $\alpha_1(m, P)$ as $m \to \infty$ in the particular case where P is a mixture of binomial distributions. Yagan and Makowski [30] evaluated the clustering coefficient $\alpha_1(m, \delta_x)$ of a uniform (active) random intersection graph and proved that it is always positive. Nonvanishing clustering coefficients of a random intersection digraph were studied in [5]. The effect of a positive clustering coefficient on the size of the largest component of $G_1(n, m, P)$ was studied in [6]. The effect on an epidemic spread was considered in [8].

2.3. *Clustering coefficient and degree.* Here we consider a sequence of sparse random intersection graphs $\{G_s(n, m, P)\}_m$ with nonvanishing clustering coefficient and nondegenerate asymptotic vertex degree distribution.

In our next theorem and its corollary, we express an approximate formula for the clustering coefficient (2.8) in terms of moments of the asymptotic vertex degree distribution. Recall the notation $Z_1 = \binom{X_1}{s}\binom{m}{s}^{-1/2} n^{1/2}$. Here X_1 is a random variable with the distribution P.

THEOREM 2.3. *Let $\beta > 0$. Let $m \to \infty$. Assume that $s = O(1)$ and*

$$\binom{m}{s} n^{-1} \to \beta. \tag{2.9}$$

Suppose that conditions (i) *and* (ii) *of Theorem 2.1 hold. In the case where*:

(ii′) $0 < \mathbf{E}Z^2 < \infty$ *and* $\lim_{m\to\infty} \mathbf{E}Z_1^2 = \mathbf{E}Z^2$, *we have*

$$\alpha_s(m, P) = \frac{1}{\sqrt{\beta}} \frac{\mathbf{E}Z}{\mathbf{E}Z^2} + o(1). \tag{2.10}$$

In the case where $\mathbf{E}Z^2 = \infty$, *we have* $\alpha_s(m, P) = o(1)$.

REMARK 2. Observe that under conditions (i) and (ii′), the asymptotic degree distribution exists and is defined by (2.4). Indeed, (i) and (ii′) imply (ii), and, therefore, one can apply Theorem 2.1. Furthermore, $\mathbf{E}Z^2 < \infty$ implies that the asymptotic degree distribution has a second moment. Assuming, in addition, that (2.9) holds, we conclude from Theorem 2.3 that the clustering coefficient does not vanish in the case where the (asymptotic) degree distribution has finite second moment. Moreover, if (2.9) fails and we have $n = o\big(\binom{m}{s}\big)$, then the clustering coefficient vanishes [i.e., $\alpha = o(1)$ as $n, m \to \infty$] in the case where (i) and (ii′) hold, as well as in the case where (i), (ii) hold and $\mathbf{E}Z^2 = \infty$. The proof of this statement is given in Section 3.

Next we express (2.10) in terms of moments of the asymptotic degree distribution. Let d_* be a random variable with the asymptotic degree distribution defined by (2.4), that is, we have

$$\mathbf{P}(d_* = k) = p_k, \qquad p_k = (k!)^{-1}\mathbf{E}\big((Z\mu)^k e^{-Z\mu}\big), \qquad k = 0, 1, 2, \dots. \tag{2.11}$$

Then $\mathbf{E}d_* = (\mathbf{E}Z)^2$ and $\mathbf{E}d_*^2 = (\mathbf{E}Z)^2\mathbf{E}Z^2 + (\mathbf{E}Z)^2$. Invoking these formulas in (2.10), we obtain the following:

COROLLARY 1. *Let* $m, n \to \infty$. *Assume that* $s = O(1)$ *and conditions* (i), (ii′) *and* (2.9) *hold. Then we have*

$$\alpha_s(m, P) = \frac{1}{\sqrt{\beta}} \frac{(\mathbf{E}d_*)^{3/2}}{\mathbf{E}d_*^2 - \mathbf{E}d_*} + o(1). \tag{2.12}$$

Foudalis et al. [12] made an interesting observation, based on an empirical study of a power-law network data, that "clustering correlates negatively with degree." More precisely, their numerical data suggest that the conditional probability

$$\alpha^{[k]} = \mathbf{P}\big(v' \sim v'' | v \sim v', v \sim v'', d(v) = k\big) \tag{2.13}$$

is a decreasing convex function of (the realized value k of) the degree $d(v)$ of vertex v. Theorem 2.4 establishes a first-order asymptotics of the conditional probability $\alpha^{[k]} = \alpha_s^{[k]}(m, P)$ as $m, n \to \infty$ for $G_s(n, m, P)$ model. The result of Theorem 2.4 provides a rigorous argument explaining empirical findings of [12] mentioned above; see also Example 3 below.

THEOREM 2.4. *Let $\beta > 0$. Let $m \to \infty$. Assume that $s = O(1)$ and that conditions* (i), (ii′) *and* (2.9) *hold. Then we have, for every $k = 2, 3, \dots,$*

$$\alpha_s^{[k]}(m, P) = \frac{1}{k} \frac{\mathbf{E}Z}{\sqrt{\beta}} \frac{p_{k-1}}{p_k} + o(1). \tag{2.14}$$

Note that, for a power-law asymptotic degree distribution $p_k \sim ck^{-\gamma}$, (2.14) implies

$$\alpha_s^{[k]}(m, P) \sim c'k^{-1} \qquad \text{as } k \to \infty \tag{2.15}$$

for any $\gamma > 3$. Here $c' = \beta^{-1/2}\mathbf{E}Z$. Hence, $\alpha^{[\cdot]}$ "correlates negatively with degree."

Next we illustrate (2.14) by two examples, where we assume, for simplicity, that $s \equiv 1$.

EXAMPLE 3. Fix $\gamma > 3$ and $\theta > 0$, and consider a sequence of random intersection graphs satisfying the conditions of Theorem 2.3 with Z having the power-law distribution $\mathbf{P}(Z > t) = (\theta t)^{1-\gamma}$ for $t \geq \theta^{-1}$. In this case, we have $p_k \sim (\gamma - 1)\theta^{1-\gamma}k^{-\gamma}$ as $k \to \infty$. Now (2.14) implies (2.15).

EXAMPLE 4. Fix $\mu > 0$ and consider a sequence $\{G_1(n, m, \delta_x)\}$ such that $x(n/m)^{1/2} \to \mu$ as $m \to \infty$. The sequence satisfies the conditions of Theorem 2.1 with Z having the degenerate distribution $\mathbf{P}(Z = \mu) = 1$. Hence, by Theorem 2.1, the asymptotic degree distribution is Poisson with mean μ^2. In this case, we have $p_{k-1}/p_k = k\mu^{-2}$, and (2.14) implies $\alpha^{[k]} \approx \mu^{-1}$. Hence, $\alpha^{[\cdot]}$ does not correlate with degree.

The rationale behind these examples is as follows. In a sparse active graph, an edge is typically realized by a single joint shared by the two vertices linked by this edge. Similarly, a triangle is realized by a single joint shared by the three vertices of the triangle. Therefore, two neighbors, say, v_i and v_j of a given vertex v_t establish a link whenever the joints responsible for the edges $v_i \sim v_t$ and $v_j \sim v_t$ match. In particular, given $X_t = |D_t|$, the probability that two neighbors of v_t are adjacent is approximately $\binom{X_t}{s}^{-1}$. Here $\binom{X_t}{s}$ is the number of joints available to v_t. Next, we remark that given X_t, the number of neighbors of v_t has binomial distribution $\mathrm{Bin}(n-1, p_{(t)})$ with $p_{(t)} \approx \binom{m}{s}^{-1}\binom{X_t}{s}\mathbf{E}\binom{X_1}{s}$; see Lemma 6 below. For large values of X_t, this number concentrates around its mean because of the concentration property of Binomial distribution. Hence, for large values of X_t, the degree $d(v_t)$ scales as $(n-1)p_{(t)} = \binom{X_t}{s}\tau$, where $\tau \approx \beta^{-1/2}\mathbf{E}Z$ as $n, m \to +\infty$. In particular, given a vertex v_t with a large degree, say, $d(v_t) = k$, it is reasonable to expect that $(n-1)p_{(t)} \approx k$, that is, $\binom{X_t}{s}^{-1} \approx k^{-1}\tau$. We summarize the argument as follows: The probability that two neighbors of a vertex of degree k are adjacent scales as $\binom{X_t}{s}^{-1} \approx k^{-1}\tau$, for $k \to +\infty$. This explains formula (2.14) in the case where the sequence $X_1, X_2, \dots, X_n$ exhibits a high variability, for example, it is a sample from a heavy-tailed distribution. The argument fails in the case of uniform random intersection graphs, since here all attribute sets are of the same size ($X_i =$ const) and, hence, the realized values of $d(v_t)$ and X_t do not correlate.

REMARK 3. Using the identity $\mathbf{E}d_* = (\mathbf{E}Z)^2$, we can express (2.14) solely in terms of the (asymptotic) degree distribution

$$\alpha_s^{[k]}(m, P) = \frac{1}{k} \frac{\sqrt{\mathbf{E}d_*}}{\sqrt{\beta}} \frac{\mathbf{P}(d_* = k-1)}{\mathbf{P}(d_* = k)} + o(1). \tag{2.16}$$

In the case where m is large and the distribution (2.11) is close to that of the degree sequence of the observed graph, we can replace the moments $\mathbf{E}d_*$, $\mathbf{E}d_*^2$ and probabilities $\mathbf{P}(d_* = k)$, $\mathbf{P}(d_* = k-1)$ in (2.12) and (2.16) by their estimates based on the observed degree sequence. In this way, we obtain estimates of α and $\alpha^{[k]}$ based on the degree sequence and involving the parameter β.

Finally, we note that the second moment $\mathbf{E}Z^2 < \infty$ required by Theorem 2.4 does not show up in (2.14). This observation suggests indirectly that the second moment condition could perhaps be replaced by the weaker first moment condition.

2.4. *Concluding remarks.* We remark that the asymptotic degree distribution of $G_s(n, m, P)$ as $n, m \to \infty$ is determined by the limiting distribution of the properly scaled number of joints of a typical vertex, that is, by the limiting distribution of $Z_1 = \binom{m}{s}^{-1} n^{1/2} \binom{X_1}{s}$, which we denote P_Z. Furthermore, the first-order asymptotics as $n, m \to +\infty$ of the clustering coefficients α and $\alpha^{[k]}$ is determined by P_Z and $\beta = \lim \binom{m}{s} n^{-1}$. One may observe that the parameter s does not show up in an explicit way neither in the description of the asymptotic degree distribution nor in the asymptotic formulas for α and $\alpha^{[k]}$. We expect that the role of the parameter s becomes more important in the case of denser graphs, for example, in the case where $\binom{m}{s} n^{-1}$ converges to zero sufficiently fast.

3. Passive intersection graph. In this section, we consider graphs on the vertex set $W = \{w_1, \dots, w_m\}$. Let $s \geq 1$ be an integer. Let $D_1, \dots, D_n$ be independent random subsets of W having the same probability distribution (2.1). We say that vertices $w, w' \in W$ are linked by D_j if $w, w' \in D_j$. For example, every $w' \in D_j \setminus \{w\}$ is linked to w by D_j. The links created by $D_1, \dots, D_n$ define a multigraph on the vertex set W. In the *passive* random intersection graph, two vertices $w, w' \in W$ are declared adjacent whenever there are at least s links between w and w'; that is, the pair $\{w, w'\}$ is contained in at least s subsets of the collection $\{D_1, \dots, D_n\}$; see [14]. We denote the passive random intersection graph $G_s^*(n, m, P)$. Here P is the common probability distribution of the random variables $X_1 = |D_1|, \dots, X_n = |D_n|$. We shall consider only the case where $s = 1$.

Before presenting our results, we introduce some notation. With a sequence of probabilities $Q = \{q_0, q_1, \dots\}$ such that $\sum_j q_j = 1$ and $\mu_Q = \sum_j j q_j < \infty$, we associate another sequence of probabilities $\tilde{Q} = \{\tilde{q}_0, \tilde{q}_1, \dots\}$ obtained as follows. In the case where $\mu_Q > 0$, we define $\tilde{q}_j = (j+1)q_{j+1}/\mu_Q$, $j = 0, 1, 2, \dots$. For $\mu_Q = 0$, we put $\tilde{q}_0 = 1$ and $\tilde{q}_j = 0$, $j = 1, 2, \dots$. In particular, we denote by $\tilde{P} = \tilde{P}_{X_1}$ the probability distribution on $\{0, 1, 2, \dots\}$ putting mass $(j+1)\mathbf{P}(X_1 = j+1)/\mathbf{E}X_1$ on an integer $j = 0, 1, 2, \dots$. By P_ξ we denote the probability distribution of a random variable ξ.

3.1. *Degree distribution.* Let $d = d(w_1)$ denote the degree of w_1 in $G_1^*(n, m, P)$, and let $L = L(w_1)$ denote the number of links incident to w_1. Our next theorem establishes the asymptotic degree distribution of a sequence of sparse passive random intersection graphs in the case where m and n are of the same order. By "sparse" we mean that the degree d remains stochastically bounded, that is, $d = O_P(1)$ as $m, n \to \infty$.

THEOREM 3.1. *Let $\beta > 0$. Let $m, n \to \infty$. Assume that $mn^{-1} \to \beta$ and*:

(vii) X_1 *converges in distribution to a random variable Z.*

(viii) $\mathbf{E}Z < \infty$ *and* $\lim_{m\to\infty} \mathbf{E}X_1 = \mathbf{E}Z$.

Then L converges in distribution to the compound Poisson random variable $d_* = \sum_{j=1}^{\Lambda} \tilde{Z}_j$. Here $\tilde{Z}_1, \tilde{Z}_2, \dots$ are independent random variables with common probability distribution $\tilde{P}_Z$, the random variable Λ is independent of the sequence $\tilde{Z}_1, \tilde{Z}_2, \dots$ and has Poisson distribution with mean $\mathbf{E}\Lambda = \beta^{-1}\mathbf{E}Z$.

If, in addition,

(ix) $\mathbf{E}Z^{4/3} < \infty$ *and* $\lim_{m\to\infty} \mathbf{E}X_1^{4/3} = \mathbf{E}Z^{4/3}$,

then d converges in distribution to d_*.

Theorem 3.1 shows how the (asymptotic) vertex degree distribution depends on the distribution of sizes of the random sets. In particular, we obtain a power-law degree distribution whenever the distribution of the sizes has a power-law. Moreover, since the distribution of $\tilde{Z}_1$ has a much heavier tail than that of Z, we can even obtain (asymptotic) degree distribution with infinite first moment. We refer to [11] for a survey of results on local and tail probabilities of sums of heavy-tailed random variables, including, in particular, random sums and compound Poisson random variables.

REMARK 4. Note that (vii) and (ix) imply (viii). Hence, condition (ix) is more restrictive than (viii). The fact that the asymptotic distribution of d only refers to the first moment of Z suggests indirectly that the $4/3$ moment condition (ix) could perhaps be waived.

In the next lemma, we collect several facts about the degree distribution in the case where either $n = o(m)$ or $m = o(n)$.

LEMMA 4. *Let $m, n \to \infty$. In the case where $n = o(m)$, we have*

$$\forall \varepsilon > 0 \qquad \mathbf{P}\bigl(d \geq \varepsilon m n^{-1} - 1\bigr) \geq \mathbf{P}(d \geq 1) - \varepsilon. \tag{3.1}$$

Now assume that $m = o(n)$. Denote $n_* = n\mathbf{P}(X_1 \geq 2)$. We distinguish three cases: (a) $n_* = o(m)$; (b) $m = o(n_*)$; (c) $mn_*^{-1} = \beta(1 + o(1))$ *for some $\beta > 0$. In the case* (a), *we have*

$$\forall \varepsilon > 0 \qquad \mathbf{P}\bigl(d \geq \varepsilon m \bigl(\max\{1, n_*\}\bigr)^{-1} - 1\bigr) \geq \mathbf{P}(d \geq 1) - \varepsilon. \tag{3.2}$$

In the case (b), *we have*

$$\forall C > 0 \qquad \mathbf{P}(d > C) = 1 - o(1). \tag{3.3}$$

In the case (c), *the conclusion of Theorem* 3.1 *holds if the conditions of the theorem are satisfied with the random variable X_1 replaced by the random variable X_{1*} having the distribution* $\mathbf{P}(X_{1*} = i) = \mathbf{P}(X_1 = i | X_1 \geq 2)$, $i \geq 2$.

REMARK 5. Observe that any of inequalities (3.1), (3.2) and (3.3) rules out the option of a nondegenerate and stochastically bounded d as $m, n \to \infty$. Therefore, a nontrivial asymptotic degree distribution is possible only in the cases where $c_1 < m^{-1}n < c_2$ or $c_1 < m^{-1}n\mathbf{P}(X_1 \geq 2) < c_2$ as $n, m \to \infty$ for some $c_1, c_2 > 0$. Furthermore, in the second case, the number $N = \sum_{i=1}^{n} \mathbb{I}_{\{X_i \geq 2\}}$ of random sets that define the edges of $G_1^*(n, m, P)$ concentrates around the expected value $n_* = \mathbf{E}N = n\mathbf{P}(X_1 \geq 2)$, and, for this reason, the asymptotic vertex degree distribution of $G_1^*(n, m, P)$ is the same as that of $G_1^*(\lfloor n_* \rfloor, m, P_*)$. Here $\lfloor n_* \rfloor$ denotes the largest integer not exceeding n_*, and P_* denotes the distribution of X_{1*}. Let us note that the asymptotic degree distribution of $G_1^*(\lfloor n_* \rfloor, m, P_*)$ can be obtained from Theorem 3.1.

The asymptotic degree distribution of the passive random intersection graph $G_1^*(n, m, P)$ has been studied by Jaworski and Stark [18]. They showed a necessary and sufficient condition for the convergence of the degree distribution and determined conditions for the convergence to a Poisson limit. They also asked what other possible limiting distributions are. Theorem 3.1 and Lemma 4 answer this question in the particular case of sparse graphs. Let us mention that the approach used in the proof of Theorem 3.1 is different from that of [18].

3.2. *Clustering coefficient and degree.* The clustering coefficient

$$\alpha^* = \alpha^*(n, m, P) = \mathbf{P}(w_2 \sim w_3 | w_1 \sim w_2, w_1 \sim w_3)$$

of a passive random intersection graph $G_1^*(n, m, P)$ has been studied in the recent paper by Godehardt et al. [15]. They showed, in particular, that

$$\alpha^*(n,m,P) = \frac{\beta_*^2 m^{-1}(\mathbf{E}(X_1)_2)^3 + \mathbf{E}(X_1)_3}{\beta_*(\mathbf{E}(X_1)_2)^2 + \mathbf{E}(X_1)_3} + o(1), \qquad \beta_* := nm^{-1}, \tag{3.4}$$

provided that $\mathbf{E}(X_1)_2 > 0$ and $\mathbf{E}(X_1)_2 = o(m^2 n^{-1})$ as $m, n \to \infty$; see Theorem 2 and Corollary 1 in [15]. Here and below we denote $(x)_k = x(x-1)\cdots(x-k+1)$.

We are interested in the relation between the clustering coefficient and the degree. Our first result expresses (3.4) in terms of moments of the (asymptotic) degree distribution.

THEOREM 3.2. *Let $\beta > 0$. Let $m, n \to \infty$ so that $mn^{-1} \to \beta$. Suppose that condition* (vii) *of Theorem* 3.1 *holds and that, in addition,* $0 < \mathbf{E}Z^2 < \infty$ *and* $\lim_{m\to\infty} \mathbf{E}X_1^2 = \mathbf{E}Z^2$. *Then the degree d converges in distribution to the compound Poisson random variable d_* defined in Theorem* 3.1.

In the case where $\mathbf{P}(Z \geq 2) > 0$, $\mathbf{E}Z^3 < \infty$ *and* $\lim_{m\to\infty} \mathbf{E}X_1^3 = \mathbf{E}Z^3$, *we have*

$$\alpha^*(n,m,P) = \frac{\mathbf{E}(d_*)_2 - (\mathbf{E}d_*)^2}{\mathbf{E}(d_*)_2} + o(1). \tag{3.5}$$

In the case $\mathbf{E}Z^3 = \infty$, *we have* $\alpha^*(n,m,P) = 1 + o(1)$.

Observe that $\mathbf{E}Z^3 < \infty$ if and only if $\mathbf{E}d_*^2 < \infty$. Hence, in order to obtain the clustering coefficient $\alpha^* < 1$ (as $n, m \to \infty$), we need to require $\mathbf{E}d_*^2 < \infty$.

REMARK 6. The result of Theorem 3.2 extends to the case where $m = o(n)$ and $mn_*^{-1} \to \beta$. Indeed, in this case, one can apply Theorem 3.2 to the random graph $G_1^*(\lfloor n_* \rfloor, m, P_*)$; see Remark 5.

Next, we examine the conditional probability

$$\alpha^{*[k]} = \alpha^{*[k]}(n,m,P) = \mathbf{P}\big(w_2 \sim w_3 | w_1 \sim w_2, w_1 \sim w_3, d(w_1) = k\big).$$

THEOREM 3.3. *Let $\beta > 0$. Let $m, n \to \infty$ so that $mn^{-1} \to \beta$. Assume that condition* (vii) *of Theorem* 3.1 *holds. Assume, in addition, that the random variable Z defined by* (vii) *has the third moment,* $\mathbf{P}(Z \geq 2) > 0$ *and* $\lim_{m\to\infty} \mathbf{E}X_1^3 = \mathbf{E}Z^3$. *Denote $d_{2*} = \sum_{j=1}^{\Lambda} (\tilde{Z}_j)_2$, where $\Lambda, \tilde{Z}_1, \tilde{Z}_2, \ldots$ are defined in Theorem* 3.1. *Then for every $k = 2, 3, \ldots$ satisfying $\mathbf{P}(d_* = k) > 0$, we have*

$$\alpha^{*[k]}(n,m,P) = \frac{1}{k(k-1)} \mathbf{E}(d_{2*} | d_* = k) + o(1). \tag{3.6}$$

EXAMPLE 5. Given $\beta > 0$ and an integer $x \geq 2$, consider the random graph $G_1^*(n, m, \delta_x)$, where $m = \lfloor \beta n \rfloor$ as $m, n \to \infty$. By Theorem 3.1, the degree d converges in distribution to the random variable $d_* = (x-1)\Lambda$. Furthermore, we have $d_{2*} = (x-1)_2\Lambda$. Here the random variable Λ has Poisson distribution with mean $\beta^{-1}(x-1)$. It follows from (3.6) that for every $k = t(x-1)$, $t = 1, 2, \ldots,$ we have

$$\alpha^{*[k]}(n, m, \delta_x) = \frac{x-2}{k-1} + o(1) \qquad \text{as } m, n \to \infty. \tag{3.7}$$

Hence, in this case, $\alpha^{*[k]}$ is of order k^{-1} as $k \to \infty$. Therefore, $\alpha^{*[\cdot]}$ "correlates negatively with degree."

In order to explain (3.7), we first note that a triangle in a sparse passive random intersection graph is typically realized by a single set D_j that covers all three vertices. Furthermore, with a high probability, any two sets D_i, D_j that cover a given vertex w_1 have no other element in common, that is, $D_i \cap D_j = w_1$. Therefore, almost all triangles incident to w_1 are realized by the sets $D_{i_1}, \ldots, D_{i_t}$ that cover w_1, and the number of such triangles N_Δ is approximately the sum of numbers of pairs $\{w', w''\}$ covered by D_{i_j}, $1 \leq j \leq t$. Hence, $N_\Delta \approx \binom{|D_{i_1}|-1}{2} + \cdots + \binom{|D_{i_t}|-1}{2}$. In addition, the degree $d(w_1) \approx |D_{i_1}| - 1 + \cdots + |D_{i_t}| - 1$. Hence, given $D_{i_1}, \ldots, D_{i_t}$, the probability that a pair of neighbors of w_1 makes a triangle with w_1 approximately equals $N_\Delta / \binom{d(w_1)}{2}$. In the case where the sizes of random sets do not deviate much from their average value $\mathbf{E}X_1$, we write $|D_{i_j}| \approx \mathbf{E}X_1$ and obtain $N_\Delta / \binom{d(w_1)}{2} \approx (\mathbf{E}X_1 - 2)/(d(w_1) - 1)$. This explains and generalizes (3.7).

In the case where sizes of random sets are heavy-tailed random variables, we expect a different pattern for large values of $d(w_1)$. Now the maximal size $\max_{1 \leq j \leq t} |D_{i_j}|$ is of the same order as the sum $\sum_{1 \leq j \leq t} |D_{i_j}|$. For this reason, the fraction $N_\Delta / \binom{d(w_1)}{2}$ stays bounded away from zero. Moreover, one may expect that, in this case, $\alpha^{*[k]} \to 1$ as $k \to \infty$.

4. Proofs. We begin with general lemmas that are used in the proofs below. Then we prove results for the active graph. Afterwards, we prove results for the passive graph. We note that the notation introduced in the proof of a particular lemma or theorem is only valid for that proof.

4.1. *General lemmas.* The following inequality is referred to as LeCam's lemma; see, for example, [28].

LEMMA 5. *Let $S = \mathbb{I}_1 + \mathbb{I}_2 + \cdots + \mathbb{I}_n$ be the sum of independent random indicators with probabilities $\mathbf{P}(\mathbb{I}_i = 1) = p_i$. Let Λ be Poisson random variable with mean $p_1 + \cdots + p_n$. The total variation distance between the distributions P_S and P_Λ of S and Λ satisfies the inequality*

$$d_{\mathrm{TV}}(P_S, P_\Lambda) \leq 2 \sum_i p_i^2. \tag{4.1}$$

LEMMA 6. *Given integers $1 \le s \le d_1 \le d_2 \le m$, let D_1, D_2 be independent random subsets of the set $W = \{1, \dots, m\}$ such that D_1 (resp., D_2) is uniformly distributed in the class of subsets of W of size d_1 (resp., d_2). The probabilities $\mathring{p} := \mathbf{P}(|D_1 \cap D_2| = s)$ and $\tilde{p} := \mathbf{P}(|D_1 \cap D_2| \ge s)$ satisfy*

$$\left(1 - \frac{(d_1 - s)(d_2 - s)}{m + 1 - d_1}\right) p^*_{d_1,d_2,s} \le \mathring{p} \le \tilde{p} \le p^*_{d_1,d_2,s}, \tag{4.2}$$

where we denote $p^*_{d_1,d_2,s} = \binom{d_1}{s}\binom{d_2}{s}\binom{m}{s}^{-1}$.

PROOF. It suffices to establish inequalities (4.2) for conditional probabilities given D_2. In order to prove the left inequality, we write $\mathring{p} = \binom{d_2}{s}\binom{m-d_2}{d_1-s}\binom{m}{d_1}^{-1} = y p^*_{d_1,d_2,s}$, where

$$y = \prod_{i=0}^{d_1-s-1}\left(1 - \frac{d_2 - s}{m - s - i}\right) \ge 1 - \sum_{i=0}^{d_1-s-1} \frac{d_2 - s}{m - s - i} \ge 1 - \frac{(d_2 - s)(d_1 - s)}{m + 1 - d_1}.$$

Let us show the right inequality of (4.2). Since, for every $D \subset D_2$ of size $|D| = s$, the number of subsets of W of size d_1 that contain D is at most $\binom{m-s}{d_1-s}$, we conclude that $\tilde{p} \le \binom{d_2}{s}\binom{m-s}{d_1-s}\binom{m}{d_1}^{-1}$. Note that the quantity in the right-hand side of the latter inequality equals $p^*_{d_1,d_2,s}$. $\square$

LEMMA 7. *Let $\{X_{n1}, X_{n2}, \dots, X_{nn}\}_{n\ge 1}$ be a collection of nonnegative random variables such that, for each n, the random variables $X_{n1}, \dots, X_{nn}$ are independent and identically distributed. Let $\alpha \ge 1$. Let $S_\alpha = n^{-\alpha}(X_{n1}^\alpha + \cdots + X_{nn}^\alpha)$. Assume that $\mathbf{E}X_{n1} < \infty$ for each n and that the sequence $\{X_{n1}\}_n$ converges in distribution to a random variable Z. Assume, in addition, that $\mathbf{E}Z < \infty$ and $\lim_n \mathbf{E}X_{n1} = \mathbf{E}Z$. Then*

$$\sup_n \mathbf{E}X_{n1}\mathbb{I}_{\{X_{n1} > x\}} \to 0 \qquad \text{as } x \to +\infty, \tag{4.3}$$

$$S_1 - \mathbf{E}Z = o_P(1) \qquad \text{as } n \to +\infty, \tag{4.4}$$

$$\forall \alpha > 1 \qquad S_\alpha = o_P(1) \qquad \text{as } n \to +\infty. \tag{4.5}$$

PROOF. Relations (4.3) and (4.4) are shown in the proof of Remark 1 in [4] and Corollary 1 in [5]. Let us prove (4.5). We shall show that $\mathbf{P}(S_\alpha > \varepsilon) \le \varepsilon\mathbf{E}Z + o(1)$ as $n \to \infty$ for each $\varepsilon \in (0, 1)$. Given ε, introduce the event $\mathcal{B} = \{\max_{1\le i\le n} X_{ni} < \varepsilon^{2/(\alpha-1)} n\}$ and note that $S_\alpha \le \varepsilon^2 S_1$ when the event $\mathcal{B}$ holds. Hence, we have

$$\mathbf{P}(S_\alpha > \varepsilon) \le \mathbf{P}\big(\{S_\alpha > \varepsilon\} \cap \mathcal{B}\big) + \mathbf{P}(\overline{\mathcal{B}}) \le \mathbf{P}\big(S_1 > \varepsilon^{-1}\big) + \mathbf{P}(\overline{\mathcal{B}}).$$

Here the complement event $\overline{\mathcal{B}}$ has the probability

$$\mathbf{P}(\overline{\mathcal{B}}) \le n\mathbf{P}\big(X_{n1} \ge \varepsilon^{2/(\alpha-1)} n\big) \le \varepsilon^{-2/(\alpha-1)}\mathbf{E}X_{n1}\mathbb{I}_{\{X_{n1} \ge \varepsilon^{2/(\alpha-1)} n\}} = o(1).$$

In the last step, we applied (4.3). Finally, the bound $\mathbf{P}(S_1 > \varepsilon^{-1}) \le \varepsilon\mathbf{E}S_1 = \varepsilon(\mathbf{E}Z + o(1))$ completes the proof. $\square$

4.2. *Active graph.* We first note that Lemma 1 is an immediate consequence of Lemma 6. Next, we prove results on the degree distribution: Theorems 2.2 and 2.1 and Example 2. Afterwards, we prove the statements related to clustering coefficient.

PROOF OF THEOREM 2.2. Let d_∞ be a random variable with the distribution defined by the right-hand side of (2.5). We write, for short, $d_m = \overline{d}_m(v_1)$. Let $f_m(t) = \mathbf{E}e^{itd_m}$ and $f(t) = \mathbf{E}e^{itd_\infty}$ denote the Fourier transforms of the probability distributions of d_m and d_∞. In order to prove the theorem, we show that $\lim_m f_m(t) = f(t)$ for every real t.

Given $0 < \delta < 0.01$ and an integer m, introduce the event $\mathcal{A} = \{\kappa_i(\overline{X}) < \delta, i = 1, 2\}$. Note that, by (v) and (vi), we have $\mathbf{P}(\mathcal{A}) = 1 - o(1)$ as $m \to \infty$. Therefore, we write

$$f_m(t) = \mathbf{E}\big(e^{itd_m}\mathbb{I}_{\mathcal{A}}\big) + o(1). \tag{4.6}$$

On the event $\mathcal{A}$, we approximate the conditional characteristic function

$$f_m(t; \overline{x}) := \mathbf{E}\big(e^{itd_m} | \overline{X} = \overline{x}\big)$$

by the Fourier transform of the Poisson distribution with mean $\lambda(\overline{x})$,

$$g_m(t; \overline{x}) = \exp\big\{\lambda(\overline{x})\big(e^{it} - 1\big)\big\}.$$

Since the conditional distribution of d_m, given the event $\{\overline{X} = \overline{x}\}$, is that of the sum of independent Bernoulli random variables with success probabilities

$$q_k = \mathbf{P}\big(|D_1 \cap D_k| \geq s | X_1 = x_1, X_k = x_k\big), \qquad 2 \leq k \leq n,$$

we write

$$f_m(t; \overline{x}) = \prod_{2\leq k\leq n} \big(1 + q_k(e^{it} - 1)\big) = \exp\bigg\{\sum_{2\leq k\leq n} \ln\big(1 + q_k(e^{it} - 1)\big)\bigg\}.$$

Note that (4.2) implies

$$\binom{m}{s}^{-1} u_k\bigg(1 - \frac{x_1^+ x_k^+}{m - x_1}\bigg) \leq q_k \leq \binom{m}{s}^{-1} u_k, \qquad 2 \leq k \leq n. \tag{4.7}$$

It follows from the right-hand side inequality of (4.7) and the inequality $\binom{m}{s}^{-2} u_k^2 \leq \delta \leq 0.01$, which holds on the event $\mathcal{A}$, that for each k, we have $|q_k(e^{it} - 1)| < 0.5$. Invoking the inequality $|\ln(1 + z) - z| \leq |z|^2$ for complex numbers z satisfying $|z| \leq 0.5$ (see, e.g., Proposition 8.46 of [7]), we obtain from (4.7) that

$$f_m(t; \overline{x}) = \exp\big\{\lambda(\overline{x})\big(e^{it} - 1\big) + r(t)\big\},$$

where $|r(t)| \leq 4\kappa_1(\overline{x}) + 2\kappa_2(\overline{x})$. Now, the inequalities $\kappa_i(\overline{x}) < \delta$, $i = 1, 2$, which hold on the event $\mathcal{A}$, imply that $|f_m(t; \overline{x}) - g_m(t; \overline{x})| \leq 7\delta$. Invoking this inequality in (4.6), we obtain

$$\big|f_m(t) - \mathbf{E}\exp\big\{\lambda(\overline{X})\big(e^{it} - 1\big)\big\}\big| \leq 7\delta + o(1) \qquad \text{as } n \to \infty.$$

Finally, the convergence in distribution of $\{\lambda(\overline{X})\}$ [i.e., condition (iv)] implies the convergence of the corresponding expectations of bounded continuous functions. Therefore, $\lim_m \mathbf{E}e^{\lambda(\overline{X})(e^{it}-1)} = f(t)$. We obtain the inequality $\limsup_m |f_m(t) - f(t)| \le 7\delta$, which holds for arbitrarily small $\delta > 0$. The proof of Theorem 2.2 is complete. □

PROOF OF THEOREM 2.1. Throughout the proof, we assume that the integers $n = n_m$, $s = s_m$, and the distributions of random variables $(X_1, \dots, X_n) = \overline{X}$ and $Z_i = \binom{X_i}{s} n^{1/2} \binom{m}{s}^{-1/2}$, $1 \le i \le n$, all depend on m.

We derive Theorem 2.1 from Theorem 2.2. To this aim, we verify conditions (v) and (vi) and show that condition (iv) holds with $\Lambda = \mu Z$. The fact that (i) and (ii) imply (iv) and (v) follows from Lemma 7. Here we show that (i), (ii) and (iii) imply (vi) in the case where $s \le am$ and $(s!/n)^{1/s} = o(m)$. In addition, we show that (i) and (ii) imply (iii) in the case where $s = O(1)$. Note that $s = O(1)$ means that the sequence $\{s_m\}$ is bounded, and this is a much more restrictive condition than $s \le am$ and $(s!/n)^{1/s} = o(m)$.

We start with the observation that the inequality $1 - y^{-1} \le \sqrt{1 - m^{-1}}$, which holds for $1 \le y \le m$, implies the inequality

$$\frac{\binom{x}{s}}{\sqrt{\binom{m}{s}}} = \frac{(x)_s}{\sqrt{(m)_s}} \frac{1}{\sqrt{s!}} \ge \left(\frac{x - s + 1}{\sqrt{m - s + 1}}\right)^s \frac{1}{\sqrt{s!}} \qquad \forall x \in [s, m]. \tag{4.8}$$

In order to show (vi), we prove that

$$\forall \varepsilon \in (0, 1) \qquad \limsup \mathbf{P}\bigl(\kappa_2(\overline{X}) > \varepsilon\bigr) \le \varepsilon. \tag{4.9}$$

Here and below, the limits are taken as $m \to \infty$. Given ε, we find a (sufficiently large) constant $A > 0$ such that $\mathbf{P}(Z_1 \ge A) \le A^{-1}\mathbf{E}Z_1 < \varepsilon$ uniformly in m. Here we applied Markov's inequality and used (ii). In view of the inequality $\mathbf{P}(Z_1 \ge A) < \varepsilon$, we can write

$$\mathbf{P}\bigl(\kappa_2(\overline{X}) > \varepsilon\bigr) \le \varepsilon + p_\varepsilon, \tag{4.10}$$

where $p_\varepsilon := \mathbf{P}(\{\kappa_2(\overline{X}) > \varepsilon\} \cap \{Z_1 < A\})$. Observing that on the event $\{Z_1 < A\}$ we have, by (4.8),

$$X_1 - s + 1 \le \sqrt{m - s + 1}\Bigl(A\sqrt{s!/n}\Bigr)^{1/s},$$

we obtain from the bound $(s!/n)^{1/s} = o(m)$ the inequality $X_1 \le am + o(m)$. It follows now that

$$Y_i := \frac{(X_1 - s + 1)(X_i - s + 1)}{m - X_1 + 1} \le \bigl((1 - a)^{-1} + o(1)\bigr)\frac{(X_1 - s + 1)(X_i - s + 1)}{m}.$$

Furthermore, invoking the inequality

$$\frac{(X_1 - s + 1)(X_i - s + 1)}{m} \le \left(s!\frac{\binom{X_1}{s}\binom{X_i}{s}}{\binom{m}{s}}\right)^{1/s}, \tag{4.11}$$

which follows from (4.8), we obtain the inequality

$$Y_i \le \big((1-a)^{-1} + o(1)\big)\big((Z_1 Z_i) s! n^{-1}\big)^{1/s}.$$

Hence, on the event $\{Z_1 \le A\}$, we have $\kappa_2(\overline{X}) \le ((1-a)^{-1} + o(1))\varkappa$. Here we denote $\varkappa = \frac{(s!)^{1/s}}{n^{(s+1)/s}} \sum_{k=2}^{n} t_k^{1+s}$ and $t_k = (Z_1 Z_k)^{1/s}$. Now we see that (iii) implies $p_\varepsilon = o(1)$, and, therefore, (4.9) follows from (4.10).

In the remaining part of the proof, we show that (i) and (ii), together with the condition $s = O(1)$, imply (iii). For $s \ge 2$, we write, by Hölder's inequality,

$$\frac{n^{(s+1)/s}}{(s!)^{1/s}} \varkappa = \sum_{k=2}^{n} t_k^{s-1} t_k^2 \le \left(\sum_{k=2}^{n} t_k^s\right)^{(s-1)/s} \left(\sum_{k=2}^{n} t_k^{2s}\right)^{1/s} \tag{4.12}$$

and observe that (4.12) implies

$$(s!)^{-1/s} \varkappa \le \big(\lambda(\overline{X})\big)^{(s-1)/s} \big(\kappa_1(\overline{X})\big)^{1/s}.$$

For $s = 1$, we have $\varkappa = \kappa_1(\overline{X})$. Now (iv) and (v), which follow from (i) and (ii), imply that $\varkappa = o_P(1)$ as $m \to \infty$. $\square$

PROOF OF EXAMPLE 2. Let D_1, D_2 be independent random subsets which are uniformly distributed in the class of subsets of $[m] = \{1, 2, \dots, m\}$ of size $x = (\varepsilon + 0.5)m$. Let $s = 0.5m$. Denote

$$p^*(m) = \binom{x}{s}^2 \binom{m}{s}^{-1}, \qquad p'(m) = \mathbf{P}\big(|D_1 \cap D_2| = s\big),$$
$$p''(m) = \mathbf{P}\big(|D_1 \cap D_2| \ge s\big).$$

We show that for a sufficiently small absolute constant $\varepsilon \in (0, 1)$, we have, as $m \to \infty$,

$$p^*(m) = o(1), \qquad p''(m) = o\big(p^*(m)\big). \tag{4.13}$$

Observe that, by the first relation of (4.13), we can construct an increasing integer sequence $\{n_m\}$ such that $n_m p^*(m) \to 1$ and $n_m (p^*(m))^2 \to 0$ as $m \to \infty$. Hence, conditions (iv) and (v) of Theorem 2.2 are fulfilled with $\Lambda \equiv 1$. In addition, by the second relation of (4.13), the average degree $\mathbf{E} d_m(v_1) = (n_m - 1) p''(m)$ satisfies $\mathbf{E} d_m(v_1) = o(1)$. Hence, $d_m(v_1) = o_P(1)$, and this means that (2.5) fails.

Let us prove (4.13). By Stirling's formula, as $m \to \infty$, we have $\binom{m}{s} \sim 2^m (0.5\pi m)^{-1/2}$ and

$$\binom{x}{s}^2 \sim \left(\frac{x}{x-s}\right)^{2(x-s)} \left(\frac{x}{s}\right)^{2s} \frac{x}{2\pi s(x-s)} = \frac{0.5+\varepsilon}{\pi \varepsilon m} A_\varepsilon^m,$$

where $A_\varepsilon = (1 + (2\varepsilon)^{-1})^{2\varepsilon}(1 + 2\varepsilon) \to 1$ as $\varepsilon \to 0$. In particular, for a small $\varepsilon \in (0, 1)$, we have $A_\varepsilon < 2$. The latter inequality implies $p^*(m) = o(1)$.

Let us prove the second relation of (4.13). We denote $p'_r = \mathbf{P}(|D_1 \cap D_2| = r)$ and write

$$p''(m) = \sum_{r=s}^{x} p'_r = p'_s \left(1 + \sum_{k=1}^{x-s} (p'_{s+k}/p'_s)\right). \tag{4.14}$$

It follows from the identities $p'_r = \binom{x}{r}\binom{m-x}{x-r}\binom{m}{x}^{-1}$ and $\frac{p'_{s+k}}{p'_s} = \frac{(x-s)_k (x-s)_k}{(s+k)_k (m-2x+s+k)_k}$ that

$$\frac{p'_{s+k}}{p'_s} \le \frac{(x-s)^{2k}}{s^k (m-2x+s)^k} = B_\varepsilon^k,$$

where $B_\varepsilon = 4\varepsilon^2/(1-4\varepsilon) \to 0$ as $\varepsilon \to 0$. In particular, we have $B_\varepsilon < 0.07$ for $\varepsilon < 0.1$. Invoking these inequalities in (4.14) and observing that $p'_s = p'(m)$, we obtain, for $\varepsilon < 0.1$,

$$p'(m) < p''(m) < 1.1 p'(m).$$

We complete the proof of (4.13) by showing that $p'(m) = o(p^*(m))$. We have

$$\frac{p'(m)}{p^*(m)} = \frac{(m-x)_{x-s}}{(m-s)_{x-s}} \le \left(\frac{m-x}{m-s}\right)^{x-s} = (1-2\varepsilon)^{\varepsilon m} = o(1) \qquad \text{as } m \to \infty. \qquad \square$$

Before proving the statements related to clustering coefficient, we introduce some notation. For a real number a, we denote $a_+ = \max\{0, a\}$. Recall that $V = \{v_1, \dots, v_n\}$ is the vertex set of $G_s(n, m, P)$, and $X_i = |D_i|$ denotes the size of the attribute set D_i of v_i, $1 \le i \le n$. By $\tilde{\mathbf{P}}(\cdot) = \mathbf{P}(\cdot | D_1, X_2, \dots, X_n)$ we denote the conditional probability given D_1 and $X_2, \dots, X_n$. Write, for short, $M = \binom{m}{s}$ and denote $Y_i = \binom{X_i}{s}$, $Z_i = Y_i n^{1/2} M^{-1/2}$, and $a_k = \mathbf{E} Y_1^k$. We write $D_{ij} = D_i \cap D_j$ and introduce the events $\mathcal{E}_{ij} = \{v_i \sim v_j\}$,

$$\mathcal{A}_1 = \{v_1 \sim v_2, v_1 \sim v_3, v_2 \sim v_3\}, \qquad \mathcal{A}_2 = \{|D_{12}| = |D_{13}| = |D_{23}| = s\},$$
$$\mathcal{A}_3 = \{D_{12} = D_{13} = D_{23}\} \cap \mathcal{A}_2, \qquad \mathcal{B} = \{v_1 \sim v_2, v_1 \sim v_3\},$$
$$\mathcal{C} = \{|D_{23} \setminus D_1| \ge 1, |D_{12}| = |D_{13}| = s\}, \qquad \mathcal{D}_{ij} = \{|D_{ij}| \ge s+1\}.$$

We have

$$\alpha = \alpha_s(m, P) = \mathbf{P}(\mathcal{A}_1)/\mathbf{P}(\mathcal{B}). \tag{4.15}$$

PROOF OF LEMMA 2. Here we consider the graph $G_s(n, m, \delta_x)$, where attribute sets of vertices are of size x, that is, $X_i = x$, $1 \le i \le n$. We write $\mathring{p} = \mathbf{P}(|D_{ij}| = s)$.

We note that (2.6) follows from (4.15) and the identities, which we prove below:

$$\mathbf{P}(\mathcal{B}) = \mathbf{P}(v_1 \sim v_2)\mathbf{P}(v_1 \sim v_3) = \mathring{p}^2(1 + o(1)), \tag{4.16}$$

$$\mathbf{P}(\mathcal{A}_3) \le \mathbf{P}(\mathcal{A}_1) \le \mathbf{P}(\mathcal{A}_3) + o(\mathring{p}^2), \tag{4.17}$$

$$\mathbf{P}(\mathcal{A}_3) = \binom{x}{s}^{-1} \mathring{p}^2(1 + o(1)). \tag{4.18}$$

We obtain the first identity of (4.16) from the corresponding identity of conditional probabilities given D_1 that holds almost surely,

$$\mathbf{P}(\mathcal{B}) = \mathbf{P}(\mathcal{B}|D_1) = \mathbf{P}(v_1 \sim v_2|D_1)\mathbf{P}(v_1 \sim v_3|D_1) = \mathbf{P}(v_1 \sim v_2)\mathbf{P}(v_1 \sim v_3).$$

The second identity of (4.16) follows from Lemma 6. In order to show (4.18), we write $\mathbf{P}(\mathcal{A}_3) = p_* \mathring{p}$, where $p_* = \mathbf{P}(D_3 \cap (D_1 \cup D_2) = D_{12} || D_{12}| = s)$, and evaluate

$$p_* = \frac{\binom{m-(2x-s)}{x-s}}{\binom{m}{x}} = \frac{(m-2x+s)_{x-s}}{(m-x)_{x-s}} \frac{\mathring{p}}{\binom{x}{s}} = (1+o(1)) \frac{\mathring{p}}{\binom{x}{s}}.$$

Let us prove (4.17). Since the event $\mathcal{A}_3$ implies $\mathcal{A}_2$, and $\mathcal{A}_2$ implies $\mathcal{A}_1$, we have

$$\mathbf{P}(\mathcal{A}_1) = \mathbf{P}(\mathcal{A}_3) + \mathbf{P}(\mathcal{A}_2 \setminus \mathcal{A}_3) + \mathbf{P}(\mathcal{A}_1 \setminus \mathcal{A}_2). \tag{4.19}$$

It remains to show that $\mathbf{P}(\mathcal{A}_2 \setminus \mathcal{A}_3), \mathbf{P}(\mathcal{A}_1 \setminus \mathcal{A}_2) = o(\mathring{p}^2)$. Noting that the event $\mathcal{A}_1 \setminus \mathcal{A}_2$ implies at least one of events $\mathcal{D}_{ij}$, $1 \le i < j \le 3$, we write, by the union bound and symmetry,

$$\begin{aligned} \mathbf{P}(\mathcal{A}_1 \setminus \mathcal{A}_2) &\le \sum_{1 \le i < j \le 3} \mathbf{P}(\mathcal{D}_{ij} \cap \mathcal{A}_1) = 3\mathbf{P}(\mathcal{D}_{12} \cap \mathcal{A}_1) \\ &\le 3\mathbf{P}(\mathcal{D}_{12} \cap \{v_1 \sim v_3\}). \end{aligned} \tag{4.20}$$

Now, conditioning on D_1, we obtain from inequalities of Lemma 6 that

$$\begin{aligned} \mathbf{P}(\mathcal{D}_{12} \cap \{v_1 \sim v_3\}|D_1) &= \mathbf{P}(\mathcal{D}_{12}|D_1) \times \mathbf{P}(v_1 \sim v_3|D_1) \\ &\le \binom{x}{s+1}^2 \binom{m}{s+1}^{-1} \times \binom{x}{s}^2 \binom{m}{s}^{-1} \\ &= \frac{(x-s)^2}{(m-s)(s+1)} \binom{x}{s}^4 \binom{m}{s}^{-2} \\ &= o(\mathring{p}^2). \end{aligned}$$

We conclude that $\mathbf{P}(\mathcal{D}_{12} \cap \{v_1 \sim v_3\}) = o(\mathring{p}^2)$. Hence, $\mathbf{P}(\mathcal{A}_1 \setminus \mathcal{A}_2) = o(\mathring{p}^2)$. Next, observing that the event $\mathcal{A}_2 \setminus \mathcal{A}_3$ implies $\mathcal{C}$, we write

$$\mathbf{P}(\mathcal{A}_2 \setminus \mathcal{A}_3) \le \mathbf{P}(\mathcal{C}) \tag{4.21}$$

and evaluate $\mathbf{P}(\mathcal{C}) = \tilde{p}_1 \tilde{p}_2$, where

$$\tilde{p}_1 = \mathbf{P}(|D_{23} \setminus D_1| \ge 1 || D_{12}| = |D_{13}| = s), \qquad \tilde{p}_2 = \mathbf{P}(|D_{12}| = |D_{13}| = s).$$

Here $\tilde{p}_1$ is the probability that two independent subsets $D_2 \setminus D_1$ and $D_3 \setminus D_1$ of $W \setminus D_1$ of size $|D_2 \setminus D_1| = |D_3 \setminus D_1| = x - s$ do intersect. By Lemma 6, $\tilde{p}_1 \le (x-s)^2(m-x)^{-1}$. Hence, $\tilde{p}_1 = o(1)$. Finally, the simple inequality $\tilde{p}_2 \le \mathbf{P}(\mathcal{B}) = (1+o(1))\mathring{p}^2$ implies $\mathbf{P}(\mathcal{C}) = o(\mathring{p}^2)$. □

PROOF OF LEMMA 3. The proof goes along the lines of the proof of Lemma 2. In view of (4.15), relation (2.8) would follow if we show that for any $\varepsilon \in (0,1)$, we have, as $m \to \infty$,

$$(1-3\varepsilon^2+o(1))a_2a_1^2M^{-2} \le \mathbf{P}(\mathcal{B}) \le a_2a_1^2M^{-2}, \tag{4.22}$$

$$\mathbf{P}(\mathcal{A}_3) \le \mathbf{P}(\mathcal{A}_1) \le \mathbf{P}(\mathcal{A}_3) + \big(\varepsilon^2+o(1)\big)a_2a_1^2M^{-2}, \tag{4.23}$$

$$(1-2\varepsilon^2+o(1))a_1^3M^{-2} \le \mathbf{P}(\mathcal{A}_3) \le a_1^3M^{-2}. \tag{4.24}$$

We fix ε and introduce the indicator functions $\mathbb{I}_i = \mathbb{I}_{\{X_i<\varepsilon\sqrt{m}\}}$ and write $\overline{\mathbb{I}}_i = 1-\mathbb{I}_i$ and $\mathbb{I}_* = \mathbb{I}_1\mathbb{I}_2\mathbb{I}_3$.

Let us show (4.22). We write $\tilde{\mathbf{P}}(\mathcal{B}) = \tilde{\mathbf{P}}(v_1\sim v_2)\tilde{\mathbf{P}}(v_1\sim v_3)$ and apply (4.2) to each probability $\tilde{\mathbf{P}}(v_1\sim v_i)$, $i=1,2$. We have

$$Y_1^2Y_2Y_3M^{-2}(1-r)\mathbb{I}_* \le \tilde{\mathbf{P}}(\mathcal{B})\mathbb{I}_* \le \tilde{\mathbf{P}}(\mathcal{B}) \le Y_1^2Y_2Y_3M^{-2}. \tag{4.25}$$

Here

$$r = 1-\left(1-\frac{(X_1-s)_+(X_2-s)_+}{m-X_1+1}\right)\left(1-\frac{(X_1-s)_+(X_3-s)_+}{m-X_1+1}\right).$$

Observing that $\mathbb{I}_*=1$ implies $(X_i-s)_+ \le \varepsilon\sqrt{m}$ for each i, we write $\mathbb{I}_*r \le 2\varepsilon^2(1-m^{-1/2})^{-1}$. Now (4.25) implies the inequalities

$$\begin{aligned} Y_1^2Y_2Y_3M^{-2}\big(1-2\varepsilon^2\big(1-m^{-1/2}\big)^{-1}-(1-\mathbb{I}_*)\big) &\le \tilde{\mathbf{P}}(\mathcal{B}) \\ &\le Y_1^2Y_2Y_3M^{-2}. \end{aligned} \tag{4.26}$$

Finally, taking the expected values and using the simple bounds

$$\begin{aligned} \mathbf{E}Y_1^2Y_2Y_3M^{-2}(1-\mathbb{I}_*) &\le \mathbf{E}Y_1^2Y_2Y_3M^{-2}(\overline{\mathbb{I}}_1+\overline{\mathbb{I}}_2+\overline{\mathbb{I}}_3) \\ &= o(1)a_2a_1^2M^{-2}, \end{aligned} \tag{4.27}$$

we obtain (4.22). In the very last last step of (4.27) we used (2.7) in the form $\mathbf{E}Y_i^k\overline{\mathbb{I}}_i = o(a_k)$.

In order to show (4.23), we combine (4.19), (4.20) and (4.21) with the following inequalities:

$$\begin{aligned} \mathbf{P}\big(\mathcal{D}_{12}\cap\{v_1\sim v_3\}\big) &\le \big(\varepsilon^2+o(1)\big)a_2a_1^2M^{-2}, \\ \mathbf{P}(\mathcal{C}) &\le \big(\varepsilon^2+o(1)\big)a_2a_1^2M^{-2}. \end{aligned} \tag{4.28}$$

Let us consider the first probability. Since the event $\mathcal{D}_{12}$ implies $\{v_1\sim v_2\}$, we can write

$$\tilde{\mathbf{P}}\big(\mathcal{D}_{12}\cap\{v_1\sim v_3\}\big) \le \tilde{\mathbf{P}}\big(\mathcal{D}_{12}\cap\{v_1\sim v_3\}\big)\mathbb{I}_* + \tilde{\mathbf{P}}(\mathcal{B})(1-\mathbb{I}_*). \tag{4.29}$$

Next, we apply (4.2) to each probability of the right-hand side,

$$\tilde{\mathbf{P}}\big(\mathcal{D}_{12}\cap\{v_1\sim v_3\}\big) \le \frac{\binom{X_1}{s+1}\binom{X_2}{s+1}}{\binom{m}{s+1}}\frac{Y_1Y_3}{M}, \qquad \tilde{\mathbf{P}}(\mathcal{B}) \le \frac{Y_1^2Y_2Y_3}{M^2}, \tag{4.30}$$

and write $\frac{\binom{X_1}{s+1}\binom{X_2}{s+1}}{\binom{m}{s+1}} = \frac{Y_1Y_2}{M} r'$, where $r' = \frac{(X_1-s)_+(X_2-s)_+}{(m-s)(s+1)}$. Finally, collecting inequalities (4.30) and $r'\mathbb{I}_* \le \varepsilon^2$ in (4.29) and then taking the expected value in (4.29), we obtain the first bound of (4.28). In order to prove the second bound of (4.28), we write

$$\tilde{\mathbf{P}}(\mathcal{C}) = \tilde{p}_1\tilde{p}_2 \le \tilde{p}_1\tilde{p}_2\mathbb{I}_* + \tilde{p}_2(1-\mathbb{I}_*), \tag{4.31}$$

where $\tilde{p}_1 = \tilde{\mathbf{P}}(|D_{23}\setminus D_1| \ge 1 \,|\, |D_{12}| = |D_{13}| = s)$ and $\tilde{p}_2 = \tilde{\mathbf{P}}(|D_{12}| = |D_{13}| = s)$, and apply (4.2) to $\tilde{p}_1$ and $\tilde{p}_2$,

$$\tilde{p}_1 \le \frac{(X_2-s)_+(X_3-s)_+}{m-X_1}, \qquad \tilde{p}_2 \le \frac{Y_1^2Y_2Y_3}{M^2}.$$

We collect these inequalities in (4.31) and observe that $\tilde{p}_1\mathbb{I}_* \le \varepsilon^2/(1-m^{-1/2})$. Now taking the expected value in (4.31), we obtain the second bound of (4.28).

Let us show (4.24). We write $\mathcal{A}_3$ in the form $\mathcal{L}_1 \cap \mathcal{L}_2$, where $\mathcal{L}_1 = \{D_3 \cap (D_1 \cup D_2) = D_{12}\}$, $\mathcal{L}_2 = \{|D_{12}| = s\}$, and decompose $\tilde{\mathbf{P}}(\mathcal{A}_3) = \tilde{p}_3\tilde{p}_4$. Here $\tilde{p}_3 = \tilde{\mathbf{P}}(\mathcal{L}_1|\mathcal{L}_2)$ and $\tilde{p}_4 = \tilde{\mathbf{P}}(\mathcal{L}_2)$ satisfy

$$(1-\tilde{r}_3)Y_3M^{-1} \le \tilde{p}_3 \le Y_3M^{-1}, \qquad \tilde{r}_3 := \frac{(X_1+X_2-2s)(X_3-s)}{m-X_3}, \tag{4.32}$$

$$(1-\tilde{r}_4)Y_1Y_2M^{-1} \le \tilde{p}_4 \le Y_1Y_2M^{-1}, \qquad \tilde{r}_4 := \frac{(X_1-s)_+(X_2-s)_+}{m-X_1+1}. \tag{4.33}$$

Note that (4.33) is an immediate consequence of (4.2). In order to get (4.32), we write

$$\tilde{p}_3 = \frac{\binom{m-X_1-X_2+s}{X_3-s}}{\binom{m}{X_3}} = \theta\frac{Y_3}{M}, \qquad \theta := \frac{(m-X_1-X_2+s)_{X_3-s}}{(m-s)_{X_3-s}}, \tag{4.34}$$

and apply to θ the chain of inequalities

$$\frac{(a)_r}{(a+b)_r} > \frac{(a-r)^r}{(a+b-r)^r} = \left(1-\frac{b}{a+b-r}\right)^r \ge 1-\frac{br}{a+b-r}.$$

Combining (4.34), (4.32), (4.33) and $\tilde{\mathbf{P}}(\mathcal{A}_3) = \tilde{p}_3\tilde{p}_4$, we write

$$(1-\tilde{r}_3)(1-\tilde{r}_4)Y_1Y_2Y_3M^{-2}\mathbb{I}_* \le \tilde{\mathbf{P}}(\mathcal{A}_3)\mathbb{I}_* \le \tilde{\mathbf{P}}(\mathcal{A}_3) \le Y_1Y_2Y_3M^{-2}. \tag{4.35}$$

Finally, invoking in (4.35) the inequalities $\tilde{r}_i\mathbb{I}_* \le \varepsilon^2(1-m^{-1/2})^{-1}$ and then taking the expected values, we obtain (4.24). $\square$

PROOF OF THEOREM 2.3. In the case where (ii′) holds, we apply Lemma 3. Note that conditions (i) and (ii′) imply the uniform integrability of the sequences of random variables $\{Z_{m1}^2\}_m$ and $\{Z_{m1}\}_m$; that is, for $k = 1, 2$, we have

$$\forall\varepsilon > 0, \exists\Delta > 0 \text{ such that } \forall m \qquad \mathbf{E}Z_{m1}^k\mathbb{I}_{\{Z_{m1}>\Delta\}} < \varepsilon; \tag{4.36}$$

see Lemma 7. Hence, (2.7) holds, and Theorem 2.3 follows from Lemma 3.

Now assume that $\mathbf{E}Z^2 = +\infty$. In order to prove that $\alpha = o(1)$, we show [see (4.15)] that

$$\mathbf{P}(\mathcal{A}_1) = O(n^{-2}) \quad \text{and} \quad \liminf_m n^2 \mathbf{P}(\mathcal{B}) = +\infty. \tag{4.37}$$

Before the proof of (4.37), we introduce some notation. Denote

$$h \to \varphi(h) = \sup_m \mathbf{E}X_{m1}^s \mathbb{I}_{\{X_{m1} > h\}}, \qquad \varepsilon_h = \varphi(h^{1/2}), \qquad \delta_h^{2s} = \max\{h^{-1}, \varepsilon_h\},$$

and observe, that (i), (ii) and (2.9) imply $\varphi(h) = o(1)$ as $h \to \infty$. Hence, $\delta_h = o(1)$ as $h \to +\infty$. Furthermore, by Chebyshev's inequality,

$$\begin{aligned} \mathbf{P}(X_{m1} > m\delta_m) &\le (m\delta_m)^{-s}\varphi(m\delta_m) \le (m\delta_m)^{-s}\varepsilon_m \le m^{-s}\varepsilon_m^{1/2} \\ &= o(m^{-s}). \end{aligned} \tag{4.38}$$

For $k = 2, 3$ and $t = 1, 3$, denote $D_k^* := D_k \setminus D_1$ and introduce the events $\mathcal{H}_t = \{|D_t| \le m\delta_m\}$,

$$\mathcal{Q}_i = \{|D_1 \cap D_2 \cap D_3| = i\}, \qquad \mathcal{Q}_i^* = \{|D_2^* \cap D_3^*| \ge s - i\}, \qquad 0 \le i \le s-1,$$
$$\mathcal{Q}_s = \{|D_1 \cap D_2 \cap D_3| \ge s\}, \qquad \mathcal{Q}_s^* = \{|D_2^* \cap D_3^*| \ge 0\}, \qquad \mathcal{H} = \mathcal{H}_1 \cap \mathcal{H}_3.$$

Let us prove the first bound of (4.37). We denote $R = \mathbf{P}(\mathcal{A}_1 \cap \overline{\mathcal{H}})$ and write

$$\mathbf{P}(\mathcal{A}_1) = \mathbf{P}(\mathcal{A}_1 \cap \mathcal{H}) + R = \tilde{p}_0 + \cdots + \tilde{p}_s + R, \tag{4.39}$$

where $\tilde{p}_i = \mathbf{P}(\mathcal{B} \cap \mathcal{Q}_i \cap \mathcal{Q}_i^* \cap \mathcal{H})$. Note that the remainder R is negligibly small,

$$R \le \mathbf{P}(v_2 \sim v_3)\mathbf{P}(\overline{\mathcal{H}}_1) + \mathbf{P}(v_1 \sim v_2)\mathbf{P}(\overline{\mathcal{H}}_3) = o(m^{-2s}).$$

In the last step, we used (4.38) and the inequality $\mathbf{P}(v_2 \sim v_3) \le a_1^2 M^{-1}$, which follows from (4.2). Let us estimate $\tilde{p}_i = \mathbf{P}(\mathcal{Q}_i^* | \mathcal{B} \cap \mathcal{Q}_i \cap \mathcal{H})\mathbf{P}(\mathcal{B} \cap \mathcal{Q}_i \cap \mathcal{H}) =: \tilde{p}_{i1}\tilde{p}_{i2}$. Note that, on the event $\mathcal{B} \cap \mathcal{H}_1$, the sets D_2^* and D_3^* are random subsets of $W \setminus D_1$ of sizes $|D_k^*| \le X_k - s$, where $|W \setminus D_1| \ge m(1 - \delta_m) =: m'$ is of order $(1 - o(1))m$. By (4.2), the probability that their intersection has at least $s - i$ elements is bounded from above by $\mathbf{E}\binom{X_2}{s-i}\binom{X_3}{s-i}\binom{m'}{s-i}^{-1}$. Hence, $\tilde{p}_{i1} = O(m^{i-s})$. Next, we estimate

$$\begin{aligned} \tilde{p}_{i2} &\le \mathbf{P}(\{v_1 \sim v_2\} \cap \mathcal{Q}_i \cap \mathcal{H}_3) = \mathbf{P}(|D_2^{**} \cap D_1^{**}| \ge s - i | \mathcal{Q}_i \cap \mathcal{H}_3)\mathbf{P}(\mathcal{Q}_i \cap \mathcal{H}_3) \\ &=: \tilde{p}_{i2}'\tilde{p}_{i2}''. \end{aligned}$$

Here the random subsets $D_k^{**} := D_k \setminus D_3$ of $W \setminus D_3$ are of sizes at most $X_k - i \le X_k$, $k = 1, 2$. Since on the event $\mathcal{H}_3$ we have $|W \setminus D_3| \ge m'$, inequality (4.2) implies $\tilde{p}_{i2}' \le \mathbf{E}\binom{X_1}{s-i}\binom{X_2}{s-i}\binom{m'}{s-i}^{-1} = O(m^{i-s})$. Finally, we estimate $\tilde{p}_{i2}''$. Let S

count the number of subsets A_i of W of size i covered by the intersection $D_1 \cap D_2 \cap D_3$, that is, $S = \sum_{A_i \subset W} \mathbb{I}_{\{A_i \subset D_1 \cap D_2 \cap D_3\}}$. We have

$$\tilde{p}''_{i2} \le \mathbf{P}(S \ge 1) \le \mathbf{E}S = \binom{m}{i} \left(\mathbf{P}(A_i \in D_1)\right)^3 = \binom{m}{i}^{-2} \left(\mathbf{E}\binom{X_1}{i}\right)^3.$$

Hence, $\tilde{p}''_{i2} = O(m^{-2i})$. Collecting these bounds, we conclude that

$$\tilde{p}_i = \tilde{p}_{i1}\tilde{p}_{i2} \le \tilde{p}_{i1}\tilde{p}'_{i2}\tilde{p}''_{i2} = O(m^{-2s}).$$

Now the first bound of (4.37) follows from (4.39).

Let us prove the second identity of (4.37). Given $t < m$, denote $\mathbb{I}_{it} = \mathbb{I}_{\{X_i \le t\}}$ and write [see (4.2)]

$$\begin{aligned} \mathbf{P}(\mathcal{B}) &= \mathbf{E}\tilde{\mathbf{P}}(\mathcal{B}) \ge \mathbf{E}\mathbb{I}_{1t}\mathbb{I}_{2t}\mathbb{I}_{3t}\tilde{\mathbf{P}}(\mathcal{B}) \\ &\ge M^{-2}\mathbf{E}\mathbb{I}_{1t}\mathbb{I}_{2t}\mathbb{I}_{3t}Y_1^2Y_2Y_3\left(1 - t^2/(m-t)\right)^2. \end{aligned} \tag{4.40}$$

We have, by (i) and (ii), that $\mathbf{E}\mathbb{I}_{it}Y_i \to a_1$ as $m, t \to +\infty$. Furthermore, (i) and $\mathbf{E}Z^2 = \infty$ implies $\mathbf{E}Y_1^2\mathbb{I}_{1t} \to +\infty$ as $m, t \to \infty$. Therefore, letting $t = t_m \to +\infty$ and $t_m = o(m^{1/2})$, we obtain (4.37). □

PROOF OF REMARK 2. Here we assume that $\beta_m := \binom{m}{s}n^{-1}$ tend to $+\infty$ as $m, n \to \infty$. In the case where (i), (ii) and (ii′) hold, the relation $\alpha_s(m, P) = o(1)$ is an immediate consequence of Lemma 3. In the case where (i) and (ii) hold and $\mathbf{E}Z^2 = \infty$, we show (4.37) by the same argument as that used in the proof of Theorem 2.3 above. In particular, in the proof of the first bound of (4.37), we choose $\delta_m = \beta_m^{-1/(4s)}$, and, for $\tau_m = \binom{m\delta_m}{s}\beta_m^{-1/2}$, we write [cf. (4.38)]

$$\mathbf{P}(X_{m1} > m\delta_m) = \mathbf{P}(Z_{m1} > \tau_m) \le \tau_m^{-1}\mathbf{E}Z_{m1}\mathbb{I}_{\{Z_{m1} > \tau_m\}} = o(\tau_m^{-1}).$$

Here $\tau_m^{-1} \le c(s)m^{-s/2}n^{-1/2}\delta_m^{-s} = o(n^{-1})$. In the proof of the second identity of (4.37), we choose $t = t_m = m^{1/2}n^{-1/(4s)}$ in (4.40). Finally, from (4.37) and (4.15) we deduce the relation $\alpha_s(m, P) = o(1)$. □

PROOF OF THEOREM 2.4. We start with some notation. By $f_i(\lambda) = e^{-\lambda}\lambda^i/i!$ we denote Poisson's probability. Observing that the absolute value of the derivative of the function $\lambda \to f_i(\lambda)$ is bounded by 1, we write $|f_i(\lambda_1) - f_i(\lambda_2)| \le |\lambda_1 - \lambda_2|$, by the mean value theorem. We denote

$$H_1 = M^{-2}Y_1Y_2Y_3 = n^{-2}\beta_m^{-1/2}Z_1Z_2Z_3, \qquad H_2 = n^{-2}Z_1^2Z_2Z_3,$$

$\mu_1 = \mathbf{E}Z_1$, and $\beta_m = M/n$, and introduce the event $\mathcal{K} = \{\sum_{j=4}^n \mathbb{I}_{\{v_1 \sim v_j\}} = k - 2\}$. We also note that, for $i = 1, 2$, the moments $\mathbf{E}Z_1^i$ and $a_i = \mathbf{E}Y_1^i = \beta_m^{-i/2}\mathbf{E}Z_1^i$ are bounded from above and that they are bounded away from zero as $m, n \to \infty$.

Let us prove (2.14). Given integer $k \geq 2$, we write

$$\alpha^{[k]} = \frac{\mathbf{P}(\mathcal{A}_1 \cap \{d(v_1) = k\})}{\mathbf{P}(\mathcal{B} \cap \{d(v_1) = k\})} = \frac{p^*}{p^{**}}, \tag{4.41}$$

where $p^* = \mathbf{P}(\mathcal{A}_1 \cap \mathcal{K})$ and $p^{**} = \mathbf{P}(\mathcal{B} \cap \mathcal{K})$. We shall show that, for any $\varepsilon \in (0, 1)$,

$$p^* = n^{-2} \beta_m^{-1/2} \mu_1 (k-1) \mathbf{E} f_{k-1}(\mu_1 Z_1) + R_1, \tag{4.42}$$

$$p^{**} = n^{-2} (k-1) k \mathbf{E} f_k(\mu_1 Z_1) + R_2, \tag{4.43}$$

where the remainder terms are negligibly small,

$$|R_1| \leq n^{-2} c(\beta) \big(a_2 a_1^2 + a_1^2 + a_1^3\big)\big(\varepsilon + o(1)\big),$$
$$|R_2| \leq n^{-2} c(\beta) \big(a_2 a_1^2 + a_1^2\big)\big(\varepsilon + o(1)\big).$$

Here the number $c(\beta)$ depends only on β. Note that (i) and (ii) imply $\mu_1 \to \mu$ and the convergence in distribution $\mu_1 Z_1 \to \mu Z$. Hence, we have the convergence of the expectations $\mathbf{E} f_i(\mu_1 Z_1) \to \mathbf{E} f_i(\mu Z) = p_i$. Now (4.42), (4.43) and (2.9) imply $p^* = n^{-2} \beta^{-1/2} \mu (k-1) p_{k-1} + o(n^{-2})$ and $p^{**} = n^{-2}(k-1) k p_k + o(n^{-2})$. Substitution of these identities into (4.41) shows (2.14).

In the remaining part of the proof, we show (4.42) and (4.43) for a given $\varepsilon \in (0, 1)$. For this purpose, we write

$$p^* = \mathbf{E}\tilde{\mathbf{P}}(\mathcal{A}_1)\tilde{\mathbf{P}}(\mathcal{K}), \qquad p^{**} = \mathbf{E}\tilde{\mathbf{P}}(\mathcal{B})\tilde{\mathbf{P}}(\mathcal{K}) \tag{4.44}$$

and show that the conditional probabilities $\tilde{\mathbf{P}}(\mathcal{A}_1)$, $\tilde{\mathbf{P}}(\mathcal{B})$ and $\tilde{\mathbf{P}}(\mathcal{K})$ can be approximated by

$$\tilde{\mathbf{P}}(\mathcal{K}) \approx f_{k-2}(\mu_1 Z_1), \qquad \tilde{\mathbf{P}}(\mathcal{A}_1) \approx H_1, \qquad \tilde{\mathbf{P}}(\mathcal{B}) \approx H_2. \tag{4.45}$$

Note that substitution of (4.45) into (4.44) gives the leading terms of (4.42) and (4.43). In the remaining part of the proof, we show the validity of such an approximation.

Note that since conditions of Theorem 2.4 are more restrictive than those of Lemma 3 (see the proof of Theorem 2.3), we can rightfully use the inequalities established in the proof of Lemma 3.

Approximation of $\tilde{\mathbf{P}}(\mathcal{K})$. We first establish an upper bound for $\Delta = \tilde{\mathbf{P}}(\mathcal{K}) - f_{k-2}(\mu_1 Z_1)$,

$$|\Delta| \leq 2\tilde{\kappa}_1 + \tilde{\kappa}_2 + |\tilde{\mu} - \mu_1| Z_1. \tag{4.46}$$

Here

$$\tilde{\mu} = n^{-1} \sum_{i=4}^{n} Z_i, \qquad \tilde{\kappa}_1 = n^{-2} \sum_{i=4}^{n} Z_1^2 Z_i^2,$$
$$\tilde{\kappa}_2 = n^{-1} \sum_{i=4}^{n} Z_1 Z_i \frac{(X_1 - s)(X_i - s)}{m - X_1 + 1}.$$

We show (4.46) in two steps. In the first step, we apply Le Cam's inequality (4.1) to the sum $\sum_{i=4}^{n} \mathbb{I}_{\{v_1 \sim v_i\}}$ of conditionally independent random indicators,

$$\left| \tilde{\mathbf{P}}(\mathcal{K}) - f_{k-2}\left(\sum_{i=4}^{n} \tilde{\mathbf{P}}(\mathcal{E}_{1i}) \right) \right| \le 2 \sum_{i=4}^{n} \bigl(\tilde{\mathbf{P}}(\mathcal{E}_{1i}) \bigr)^2. \tag{4.47}$$

In the second step, using the mean value theorem, we replace the argument of f_{k-2} by $\mu_1 Z_1$ in (4.47). The approximation error estimate (4.46) now follows from the inequalities

$$\sum_{i=4}^{n} \bigl(\tilde{\mathbf{P}}(\mathcal{E}_{1i}) \bigr)^2 \le \tilde{\kappa}_1, \qquad \tilde{\mu} Z_1 - \tilde{\kappa}_2 \le \sum_{i=4}^{n} \tilde{\mathbf{P}}(\mathcal{E}_{1i}) \le \tilde{\mu} Z_1, \tag{4.48}$$

which are simple consequences of the inequalities of Lemma 6.

Next, using (4.46), we replace $\tilde{P}(\mathcal{K})$ by $f_{k-2}(\mu_1 Z_1)$ in (4.44) and show that the error of the replacement is negligibly small. We denote $\mathbb{I}_A = \mathbb{I}_{\{Z_1 \le A\}}$ and $\overline{\mathbb{I}}_A = 1 - \mathbb{I}_A$ and observe, that in view of (4.36), we can find $A > 1$ such that, uniformly in m,

$$\mathbf{E} Z_1^k \overline{\mathbb{I}}_A < \varepsilon \mathbf{E} Z_1^k, \qquad k = 1, 2. \tag{4.49}$$

We write

$$\tilde{\mathbf{P}}(\mathcal{A}_1) \tilde{\mathbf{P}}(\mathcal{K}) = \tilde{\mathbf{P}}(\mathcal{A}_1) f_{k-2}(\mu_1 Z_1) \mathbb{I}_A + r_1 + r_2, \tag{4.50}$$

$$\tilde{\mathbf{P}}(\mathcal{B}) \tilde{\mathbf{P}}(\mathcal{K}) = \tilde{\mathbf{P}}(\mathcal{B}) f_{k-2}(\mu_1 Z_1) \mathbb{I}_A + r_3 + r_4, \tag{4.51}$$

where

$$r_1 = \tilde{\mathbf{P}}(\mathcal{A}_1) \tilde{\mathbf{P}}(\mathcal{K}) \overline{\mathbb{I}}_A, \qquad r_2 = \tilde{\mathbf{P}}(\mathcal{A}_1) \Delta \mathbb{I}_A,$$
$$r_3 = \tilde{\mathbf{P}}(\mathcal{B}) \tilde{\mathbf{P}}(\mathcal{K}) \overline{\mathbb{I}}_A, \qquad r_4 = \tilde{\mathbf{P}}(\mathcal{B}) \Delta \mathbb{I}_A,$$

and show that

$$\begin{aligned} \mathbf{E} r_i &\le \varepsilon M^{-2} a_2 a_1^2 \qquad \text{for } i = 1, 3, \\ \mathbf{E} |r_j| &= o\bigl(M^{-1} n^{-1}\bigr) a_1^2 \qquad \text{for } j = 2, 4. \end{aligned} \tag{4.52}$$

To prove (4.52), we write, using the inequalities $\tilde{\mathbf{P}}(\mathcal{A}_1) \le \tilde{\mathbf{P}}(\mathcal{B})$ and $\tilde{\mathbf{P}}(\mathcal{B}) \le H_2$ [see (4.2)],

$$r_i \le H_2 \overline{\mathbb{I}}_A \qquad \text{for } i = 1, 3 \quad \text{and} \quad |r_j| \le H_2 |\Delta| \mathbb{I}_A \qquad \text{for } j = 2, 4 \tag{4.53}$$

and notice that, for $i = 1, 3$, inequalities (4.52) follow from (4.49) and, for $j = 2, 4$, inequalities (4.52) follow from the bound $\mathbf{E} Z_1^2 |\Delta| \mathbb{I}_A = o(1)$. Let us prove this bound. We note that (4.46), combined with the simple inequality $|\Delta| \le 1$, implies

$$|\Delta| \le 1 \wedge (2\tilde{\kappa}_1) + 1 \wedge \bigl(Z_1 |\tilde{\mu} - \mu_1| \bigr) + 1 \wedge \tilde{\kappa}_2.$$

Here we use the notation $a \wedge b = \min\{a, b\}$. Hence, for $A > 1$, we can write

$$Z_1^2|\Delta|\mathbb{I}_A \le c_A\big(1 \wedge \kappa_1^* + 1 \wedge |\tilde{\mu} - \mu_1| + 1 \wedge \kappa_2^*\big), \tag{4.54}$$

where the number $c_A > 0$ does not depend on m, and where

$$\kappa_1^* := n^{-2}\sum_{i=4}^{n} Z_i^2, \qquad \kappa_2^* := n^{-1}m^{-1}\sum_{i=4}^{n} Z_i(X_i - s).$$

The bound $\mathbf{E}Z_1^2|\Delta|\mathbb{I}_A = o(1)$ follows from (4.54) and from the bounds

$$\kappa_1^* = o_P(1), \qquad \tilde{\mu} - \mu_1 = o_P(1), \qquad \kappa_2^* = o_P(1). \tag{4.55}$$

The first and second bounds of (4.55) are shown in Lemma 7. To obtain the third bound, we write $\mathbf{E}\kappa_2^* \le m^{-1}\mathbf{E}Z_4X_4 = o(1)$; see (i), (ii′).

Approximation of $\tilde{\mathbf{P}}(\mathcal{A}_1)$. Now we replace $\tilde{\mathbf{P}}(\mathcal{A}_1)$ by H_1 in the right-hand side of (4.50) and show that the error of the replacement is negligibly small. We proceed in two steps. First, we approximate $\tilde{\mathbf{P}}(\mathcal{A}_1) \approx \tilde{\mathbf{P}}(\mathcal{A}_3)$, and, second, we approximate $\tilde{\mathbf{P}}(\mathcal{A}_3) \approx H_1$. In the first step, we combine the inequalities [see (4.20), (4.21) and (4.28)]

$$\begin{aligned}
\tilde{\mathbf{P}}(\mathcal{A}_3) \le \tilde{\mathbf{P}}(\mathcal{A}_1) &= \tilde{\mathbf{P}}(\mathcal{A}_3) + \tilde{\mathbf{P}}(\mathcal{A}_1 \setminus \mathcal{A}_2) + \tilde{\mathbf{P}}(\mathcal{A}_2 \setminus \mathcal{A}_3),\\
\mathbf{E}\tilde{\mathbf{P}}(\mathcal{A}_1 \setminus \mathcal{A}_2) &\le 3\mathbf{P}\big(\mathcal{D}_{12} \cap \{v_1 \sim v_3\}\big) \le 3\big(\varepsilon^2 + o(1)\big)M^{-2}a_2a_1^2,\\
\mathbf{E}\tilde{\mathbf{P}}(\mathcal{A}_2 \setminus \mathcal{A}_3) &\le \mathbf{P}(\mathcal{C}) \le \big(\varepsilon^2 + o(1)\big)M^{-2}a_2a_1^2
\end{aligned}$$

and show that

$$\begin{aligned}
\tilde{\mathbf{P}}(\mathcal{A}_3)f_{k-2}(\mu_1Z_1)\mathbb{I}_A &\le \tilde{\mathbf{P}}(\mathcal{A}_1)f_{k-2}(\mu_1Z_1)\mathbb{I}_A\\
&\le \tilde{\mathbf{P}}(\mathcal{A}_3)f_{k-2}(\mu_1Z_1)\mathbb{I}_A + r_5,
\end{aligned} \tag{4.56}$$

where $\mathbf{E}r_5 \le 4(\varepsilon^2 + o(1))M^{-2}a_2a_1^2$. In the second step, we apply (4.35) and write

$$H_1f_{k-2}(\mu_1Z_1)\mathbb{I}_A - r_6 \le \tilde{\mathbf{P}}(\mathcal{A}_3)f_{k-2}(\mu_1Z_1)\mathbb{I}_A \le H_1f_{k-2}(\mu_1Z_1)\mathbb{I}_A, \tag{4.57}$$

where $\mathbf{E}r_6 \le (2\varepsilon^2 + o(1))M^{-2}a_1^3$. Finally, using the inequality $1 - \bar{\mathbb{I}}_A = \mathbb{I}_A \le 1$, we write

$$\mathbf{E}H_1f_{k-2}(\mu_1Z_1) - r_7 = \mathbf{E}H_1f_{k-2}(\mu_1Z_1)\mathbb{I}_A \le \mathbf{E}H_1f_{k-2}(\mu_1Z_1), \tag{4.58}$$

where $r_7 = \mathbf{E}H_1f_{k-2}(\mu_1Z_1)\bar{\mathbb{I}}_A \le \mathbf{E}H_1\bar{\mathbb{I}}_A \le \varepsilon M^{-2}a_1^3$. In the last step, we used (4.49). Finally, from (4.44), (4.50), (4.56), (4.57) and (4.58) we obtain (4.42).

Approximation of $\tilde{\mathbf{P}}(\mathcal{B})$. Now we replace $\tilde{\mathbf{P}}(\mathcal{B})$ by H_2 in the right-hand side of (4.51). For this purpose, we combine (4.25) and (4.26) with the inequality $f_{k-2}(\mu Z_1) \le 1$ and obtain

$$H_2f_{k-2}(\mu_1Z_1)\mathbb{I}_A - r_8 \le \tilde{\mathbf{P}}(\mathcal{B})f_{k-2}(\mu_1Z_1)\mathbb{I}_A \le H_2f_{k-2}(\mu_1Z_1)\mathbb{I}_A, \tag{4.59}$$

where $r_8 \le (3\varepsilon + (1 - \mathbb{I}_*))H_2$ is negligibly small, because $\mathbf{E}r_8 \le (3\varepsilon + o(1)) \times M^{-2}a_2a_1^2$. Next, we proceed as in (4.58) above and write

$$\mathbf{E}H_2 f_{k-2}(\mu_1 Z_1) - r_9 = \mathbf{E}H_2 f_{k-2}(\mu_1 Z_1)\mathbb{I}_A \le \mathbf{E}H_2 f_{k-2}(\mu_1 Z_1), \tag{4.60}$$

where $r_9 = \mathbf{E}H_2 f_{k-2}(\mu_1 Z_1)\overline{\mathbb{I}}_A \le \mathbf{E}H_2\overline{\mathbb{I}}_A \le \varepsilon M^{-2}a_2a_1^2$. Finally, from (4.44), (4.51), (4.59) and (4.60) we obtain (4.43). □

4.3. *Passive graph.*

PROOF OF THEOREM 3.1. The proof consists of two parts. In the first part, we establish the convergence of the Fourier transforms

$$\lim_{m\to\infty} \mathbf{E}e^{itL} = \mathbf{E}e^{itd_*}. \tag{4.61}$$

In the second part, we show that, as $m \to \infty$,

$$\mathbf{P}(d \ne L) = o(1). \tag{4.62}$$

Part 1. Here we prove (4.61). Before the proof, we introduce some notation. We denote $\mathbb{I}_j(w) = \mathbb{I}_{\{w \in D_j\}}$ and write

$$L = (X_1 - 1)_+\mathbb{I}_1(w_1) + \cdots + (X_n - 1)_+\mathbb{I}_n(w_1). \tag{4.63}$$

Here we use the notation $a_+ = \max\{0, a\}$. Given $\mathbb{X} = (X_1, \ldots, X_n)$, we generate independent Poisson random variables $\eta_1(X_1), \ldots, \eta_n(X_n)$ with mean values $\mathbf{E}(\eta_j(X_j)|X_j) = m^{-1}X_j$. We also generate independent random variables $Y_1(\mathbb{X}), Y_2(\mathbb{X}), \ldots$ with the common probability distribution defined as follows. In the case where $S(X) = X_1 + \cdots + X_n$ is positive, we put

$$\mathbf{P}\big(Y_1(\mathbb{X}) = j|\mathbb{X}\big) = (j+1)\frac{n_{j+1}}{S(X)}, \qquad n_{j+1} = \sum_{k=1}^{n} \mathbb{I}_{\{X_k = j+1\}}, \qquad j = 0, 1, \ldots.$$

For $S(X) = 0$, we put $Y_1(\mathbb{X}) \equiv 0$. By $\eta_1, \ldots, \eta_n$ (resp., $Y_1, Y_2, \ldots$) we denote the corresponding unconditional random variables, that is, η_j is the outcome of a two-step procedure: first, we generate $\mathbb{X}$ and then generate $\eta_j(X_j)$. Let $\eta(\mathbb{X}) = \eta_1(X_1) + \cdots + \eta_n(X_n)$ and $\eta = \eta_1 + \cdots + \eta_n$. Introduce the random variables

$$\tau = (X_1 - 1)_+\eta_1 + \cdots + (X_n - 1)_+\eta_n, \qquad \xi = \sum_{j=1}^{\eta} Y_j \tag{4.64}$$

and observe that, given $\mathbb{X}$, their conditional distributions $P_\tau(\mathbb{X})$ and $P_\xi(\mathbb{X})$ are compound Poisson and that $P_\tau(\mathbb{X}) \equiv P_\xi(\mathbb{X})$. Here and below in this proof, by $P_\zeta(\mathbb{X})$ (and $\mathbf{E}_{\mathbb{X}}\zeta$) we denote the conditional distribution (and conditional expectation) of a random variable ζ given $\mathbb{X}$.

Let us prove (4.61). For this purpose, we write $\mathbf{E}e^{itL} - \mathbf{E}e^{itd_*} = I_1 + I_2$, where $I_1 = \mathbf{E}e^{itL} - \mathbf{E}e^{it\xi}$ and $I_2 = \mathbf{E}e^{it\xi} - \mathbf{E}e^{itd_*}$, and show that $I_1, I_2 = o(1)$. In order to show the first bound, we write

$$I_1 = \mathbf{E}(\mathbf{E}_{\mathbb{X}}e^{itL} - \mathbf{E}_{\mathbb{X}}e^{it\xi}) = \mathbf{E}\tilde{\Delta}, \qquad \tilde{\Delta} = (\mathbf{E}_{\mathbb{X}}e^{itL} - \mathbf{E}_{\mathbb{X}}e^{it\tau})$$

and invoke the inequalities

$$2^{-1}|\mathbf{E}_{\mathbb{X}}e^{itL} - \mathbf{E}_{\mathbb{X}}e^{it\tau}| \leq d_{\mathrm{TV}}(P_L(\mathbb{X}), P_\tau(\mathbb{X})) \leq 2m^{-2}\sum_{j=1}^{n} X_j^2. \tag{4.65}$$

Here d_{TV} denotes the total variation distance. Indeed, we obtain from (4.65) that, for any $\varepsilon > 0$,

$$|I_1| \leq \varepsilon + 2\mathbf{P}(|\tilde{\Delta}| > \varepsilon) \leq \varepsilon + 2\mathbf{P}\left(4m^{-2}\sum_{j=1}^{n} X_j^2 > \varepsilon\right) = \varepsilon + o(1).$$

In the last step, we applied (4.5). Hence, $I_1 = o(1)$. Let us prove (4.65). The first inequality of (4.65) follows immediately from the definition of the total variation distance. The second one is a simple consequence of LeCam's inequality (4.1). Indeed, we have, by the triangle inequality,

$$d_{\mathrm{TV}}(P_L(\mathbb{X}), P_\tau(\mathbb{X})) \leq \sum_{j=0}^{n-1} d_{\mathrm{TV}}(P_{\tau_j}(\mathbb{X}), P_{\tau_{j+1}}(\mathbb{X})),$$

where $\tau_0 = L$, $\tau_n = \tau$, and, for $1 \leq j \leq n-1$,

$$\tau_j = \sum_{t=1}^{j}(X_t - 1)_+\eta_t + \sum_{t=j+1}^{n}(X_t - 1)_+\mathbb{I}_t(w_1).$$

Furthermore, noting that the sums τ_j and τ_{j+1} differ only by one term, we obtain

$$d_{\mathrm{TV}}(P_{\tau_j}(\mathbb{X}), P_{\tau_{j+1}}(\mathbb{X})) \leq d_{\mathrm{TV}}(P_{\mathbb{I}_{j+1}}(\mathbb{X}), P_{\eta_{j+1}}(\mathbb{X})).$$

Finally, invoking the inequality $d_{\mathrm{TV}}(P_{\mathbb{I}_{j+1}}(\mathbb{X}), P_{\eta_{j+1}}(\mathbb{X})) \leq 2m^{-2}X_{j+1}^2$ [see (4.1)], we arrive at (4.65).

The proof of the second bound $I_2 = o(1)$ is based on the fact that $\eta \to \Lambda$ and $Y_1 \to \tilde{Z}_1$ in probability as $m \to \infty$. Details of the proof are given in the separate Lemma 8.

Part 2. We write, by inclusion–exclusion, $L \geq d \geq L - T$, where the number

$$T = \sum_{1 \leq i < j \leq n} \sum_{w \in W \setminus \{w_1\}} \mathbb{I}_i(w_1)\mathbb{I}_j(w_1)\mathbb{I}_i(w)\mathbb{I}_j(w)$$

is at least as large as the number of vertices w having two or more links to w_1. Hence, we have $\mathbf{P}(d \neq L) \leq \mathbf{P}(T \geq 1)$. In order to prove (4.62), we show that $\mathbf{P}(T \geq 1) \leq \varepsilon + o(1)$ for every $\varepsilon \in (0, 1)$. For this purpose, we write

$$\mathbf{P}(T \geq 1) = \mathbf{E}\mathbf{P}(T \geq 1|\mathbb{X}) \leq \varepsilon + \mathbf{P}(\mathbf{P}(T \geq 1|\mathbb{X}) \geq \varepsilon) \leq \varepsilon + \mathbf{P}(\mathbf{E}_{\mathbb{X}}T \geq \varepsilon)$$

and invoke the bound $\mathbf{E}_{\mathbb{X}}T = o_P(1)$. Let us prove this bound. We write

$$\mathbf{E}_{\mathbb{X}}T = \sum_{1\le i<j\le n}(n-1)\frac{(X_i)_2}{(m)_2}\frac{(X_j)_2}{(m)_2} \le (n-1)m^{-4}\big(X_1^2+\cdots+X_n^2\big)^2$$

and use the bound $n^{-3/2}(X_1^2+\cdots+X_n^2) = o_P(1)$, which follows from (4.5) (applied to random variables $X_i^{4/3}$, $1\le i\le n$, and $\alpha = 3/2$). $\square$

LEMMA 8. *Assume that the conditions of Theorem 3.1 hold. For ξ defined in (4.64) and every real t, we have* $\lim_{m\to+\infty}\mathbf{E}e^{it\xi} = \mathbf{E}e^{itd_*}$.

PROOF. We write $I_2 = \mathbf{E}e^{it\xi} - \mathbf{E}e^{itd_*}$.

The cases $\mathbf{E}Z = 0$ and $\mathbf{E}Z > 0$ are considered separately. In the case where $\mathbf{E}Z = 0$, the random variable d_* is degenerate, $\mathbf{P}(d_* = 0) = 1$ and the lemma follows from the relation

$$\mathbf{P}(\xi = 0) \ge \mathbf{P}(\eta = 0) = \mathbf{E}\mathbf{P}(\eta = 0|\mathbb{X}) = \mathbf{E}e^{-S(X)/m} = 1 - o(1).$$

In the last step, we combined the bound $n^{-1}S(X) - \mathbf{E}Z = o_P(1)$ [see (4.4)] with $\mathbf{E}Z = 0$.

Next, we consider the case where $\mathbf{E}Z > 0$. We need some more notation. Denote

$$f(t) = \mathbf{E}e^{itd_*}, \qquad f_{\mathbb{X}}(t) = \mathbf{E}_{\mathbb{X}}e^{itY_1}, \qquad f_*(t) = \mathbf{E}e^{it\tilde{Z}_1},$$

$$\lambda_{\mathbb{X}} = \mathbf{E}_{\mathbb{X}}\eta,\ \lambda_* = \mathbf{E}\Lambda,$$

$$\Delta = \mathbf{E}_{\mathbb{X}}e^{it\xi} - f(t), \qquad \delta = \big(f_{\mathbb{X}}(t) - 1\big)\lambda_{\mathbb{X}} - \big(f_*(t) - 1\big)\lambda_*$$

and write the Fourier transforms of the compound Poisson distributions P_{d_*} and $P_\xi(\mathbb{X})$ in the form $\mathbf{E}e^{itd_*} = e^{(f_*(t)-1)\lambda_*}$ and $\mathbf{E}_{\mathbb{X}}e^{it\xi} = e^{(f_{\mathbb{X}}(t)-1)\lambda_{\mathbb{X}}}$, respectively. Introduce the events

$$\mathcal{A}(\varepsilon) = \big\{\big|n^{-1}S(X) - \mathbf{E}Z\big| < 2^{-1}\varepsilon\min\{1,\beta,\mathbf{E}Z\}\big\}, \qquad \varepsilon > 0,$$

and the function $t\to a(t) = \sup_m\{(\mathbf{E}X_1)^{-1}\mathbf{E}X_1\mathbb{I}_{\{X_1\ge t\}}\}$. It follows from Lemma 7 that conditions (vii) and (viii) imply that $n^{-1}S(X) - \mathbf{E}Z = o_P(1)$ as $m\to\infty$ and $a(t) = o(1)$ as $t\to+\infty$. Hence, we have

$$\forall\varepsilon > 0 \qquad \mathbf{P}\big(\mathcal{A}(\varepsilon)\big) = 1 - o(1) \qquad \text{as } m\to\infty. \tag{4.66}$$

In addition, for every $\varepsilon > 0$, we can choose a positive integer t_ε such that

$$a(t_\varepsilon) < \varepsilon \quad \text{and} \quad (\mathbf{E}Z)^{-1}\mathbf{E}Z\mathbb{I}_{\{Z\ge t_\varepsilon\}} < \varepsilon. \tag{4.67}$$

In order to prove $I_2 = o(1)$, we show that there exists a number $c > 0$, depending only on $\mathbf{E}Z$ and β, such that, for any $\varepsilon\in(0,1)$,

$$\limsup_n |I_2| \le c\varepsilon. \tag{4.68}$$

In the proof of (4.68), we assume that m, n are so large that $\beta \le 2m/n \le 4\beta$ and $\mathbf{E}X_1 \le 2\mathbf{E}Z \le 4\mathbf{E}X_1$. In particular, on the event $\mathcal{A}(\varepsilon)$, we have

$$m^{-1}S(X) \le 3\beta^{-1}\mathbf{E}Z \quad \text{and} \quad S(X) > 2^{-1}n\mathbf{E}Z \ge 4^{-1}n\mathbf{E}X_1. \tag{4.69}$$

We fix ε and write

$$I_2 = \mathbf{E}\Delta = I_{21} + I_{22}, \qquad I_{21} = \mathbf{E}\Delta\mathbb{I}_{\mathcal{A}(\varepsilon)}, \qquad I_{22} = \mathbf{E}\Delta(1 - \mathbb{I}_{\mathcal{A}(\varepsilon)}). \tag{4.70}$$

Note that $|\Delta| \le 2$ (as the absolute value of the Fourier transform of a probability distribution is at most 1). This inequality, together with (4.66), implies $I_{22} = o(1)$ as $m \to \infty$. Next, we estimate I_{21}. Combining the identity $\Delta = f(t)(e^{\delta} - 1)$ with the inequalities $|f(t)| \le 1$ and $|e^{\delta} - 1| \le \sum_{k\ge 1} |\delta|^k/k! \le |\delta|e^{|\delta|}$, we obtain

$$|I_{21}| \le \mathbf{E}|\delta|e^{|\delta|}\mathbb{I}_{\mathcal{A}(\varepsilon)} \le e^{8\lambda_*}\mathbf{E}|\delta|\mathbb{I}_{\mathcal{A}(\varepsilon)}.$$

In the last step, we used the bound $|\delta| \le 8\lambda_*$, which follows from the inequality $|\delta| \le 2\lambda_{\mathbb{X}} + 2\lambda_*$ and the inequality $\lambda_{\mathbb{X}} \le 3\lambda_*$; see (4.69).

Next, we show that $\mathbf{E}|\delta|\mathbb{I}_{\mathcal{A}(\varepsilon)} \le (2 + 7\lambda_*)\varepsilon + o(1)$. We write

$$\delta = \big(f_{\mathbb{X}}(t) - 1\big)(\lambda_{\mathbb{X}} - \lambda_*) + \big(f_{\mathbb{X}}(t) - f_*(t)\big)\lambda_*,$$

estimate $|\delta| \le 2|\lambda_{\mathbb{X}} - \lambda_*| + \lambda_*|f_{\mathbb{X}}(t) - f_*(t)|$ and substitute

$$|\lambda_{\mathbb{X}} - \lambda_*| \le \big|n^{-1}S(X) - \mathbf{E}Z\big|nm^{-1} + \big|nm^{-1} - \beta^{-1}\big|\mathbf{E}Z.$$

In this way, we obtain $\mathbf{E}|\delta|\mathbb{I}_{\mathcal{A}(\varepsilon)} \le 2I_{31} + 2I_{32} + \lambda_* I_{33}$, where

$$\begin{aligned} I_{31} &= nm^{-1}\mathbf{E}\big|n^{-1}S(X) - \mathbf{E}Z\big|\mathbb{I}_{\mathcal{A}(\varepsilon)} \le \varepsilon, \\ I_{32} &= \big|nm^{-1} - \beta^{-1}\big|\mathbf{E}Z = o(1), \\ I_{33} &= \mathbf{E}\big|f_{\mathbb{X}}(t) - f_*(t)\big|\mathbb{I}_{\mathcal{A}(\varepsilon)}. \end{aligned} \tag{4.71}$$

It remains to estimate I_{33}. We expand $f_{\mathbb{X}}(t) - f_*(t) = R_1 + R_2 + R_3$, where

$$\begin{aligned} R_1 &= \sum_{0\le k\le t_\varepsilon} e^{itk}\delta_P(k), \qquad \delta_P(k) = \mathbf{P}\big(Y_1(\mathbb{X}) = k|\mathbb{X}\big) - \mathbf{P}(\tilde{Z}_1 = k), \\ |R_2| &\le \mathbf{P}(\tilde{Z}_1 \ge t_\varepsilon + 1) = (\mathbf{E}Z)^{-1}\mathbf{E}Z\mathbb{I}_{\{Z\ge t_\varepsilon+2\}} < \varepsilon, \\ |R_3| &\le \mathbf{P}\big(Y_1(\mathbb{X}) \ge t_\varepsilon + 1|\mathbb{X}\big) = \big(S(X)\big)\sum_{j=1}^{n} X_j\mathbb{I}_{\{X_j>t_\varepsilon+2\}} \end{aligned} \tag{4.72}$$

and observe that the last inequality of (4.69) implies

$$\mathbf{E}|R_3|\mathbb{I}_{\mathcal{A}(\varepsilon)} \le 4(\mathbf{E}X_1)^{-1}\mathbf{E}\left(n^{-1}\sum_{j=1}^{n} X_j\mathbb{I}_{\{X_j>t_\varepsilon+2\}}\right) \le 4a(t_\varepsilon + 2) < 4\varepsilon. \tag{4.73}$$

Finally, we write $|R_1| \le \sum_{0\le k\le t_\varepsilon} |\delta_P(k)|$ and expand $\delta_P(k) = \sum_{j=1}^{3} r_j(k+1)$, where

$$r_1(k) = \frac{k}{S(X)}\big(n_k - np_x(k)\big), \qquad r_2(k) = kp_x(k)\left(\frac{n}{S(X)} - \frac{1}{\mathbf{E}Z}\right),$$

and $r_3(k) = \frac{k}{\mathbf{E}Z}(p_x(k) - p_z(k))$. Here we denote $p_x(k) = \mathbf{P}(X_1 = k)$ and $p_z(k) = \mathbf{P}(Z = k)$. Since the number t_ε is fixed, it follows from (vii) that

$$\sum_{0\le k\le t_\varepsilon} \big|r_3(k+1)\big| = o(1) \qquad \text{as } m \to \infty. \tag{4.74}$$

Furthermore, on the event $\mathcal{A}(\varepsilon)$, we obtain from (4.69) that

$$\big|r_1(k)\big| \le 4(\mathbf{E}X_1)^{-1}k\left|\frac{n_k}{n} - p_k(x)\right|, \qquad \big|r_2(k)\big| \le 2\varepsilon(\mathbf{E}X_1)^{-1}kp_k(x).$$

Hence, we have $\sum_{0\le k\le t_\varepsilon} |r_2(k+1)|\mathbb{I}_{\mathcal{A}(\varepsilon)} \le 2\varepsilon$ and

$$\begin{aligned}
\mathbf{E}\sum_{0\le k\le t_\varepsilon} \big|r_1(k+1)\big|\mathbb{I}_{\mathcal{A}(\varepsilon)} &\le \frac{4}{\mathbf{E}X_1}\sum_{1\le k\le t_\varepsilon+1} k\left(\mathbf{E}\left(\frac{n_k}{n} - p_k(x)\right)^2\right)^{1/2} \\
&\le \frac{4}{\mathbf{E}X_1} n^{-1/2} \sum_{1\le k\le t_\varepsilon+1} k \\
&= o(1)
\end{aligned}$$

as $m \to \infty$. We conclude that $\mathbf{E}|R_1|\mathbb{I}_{\mathcal{A}(\varepsilon)} \le 2\varepsilon + o(1)$. This bound, together with (4.72) and (4.73), shows the bound $I_{33} \le 7\varepsilon + o(1)$. The proof is complete. □

PROOF OF LEMMA 4. Let us prove (3.2). We only consider the case where $\varepsilon m(\max\{1, n_*\})^{-1} \ge 2$, since otherwise (3.2) is obvious. Given $0 < k \le 2^{-1}\varepsilon m$, introduce the random variables

$$d_A(k) = \sum_{i=1}^{n} \mathbb{I}_{\{w_1\in D_i\}}\mathbb{I}_{\{2\le X_i<\varepsilon m/k\}}, \qquad d_B(k) = \sum_{i=1}^{n} \mathbb{I}_{\{w_1\in D_i\}}\mathbb{I}_{\{X_i\ge\varepsilon m/k\}}$$

and note that the equality of events $\{d \ge 1\} = \{d_A(k) \ge 1\} \cup \{d_B(k) \ge 1\}$ implies

$$\mathbf{P}\big(d_B(k) \ge 1\big) \ge \mathbf{P}(d \ge 1) - \mathbf{P}\big(d_A(k) \ge 1\big). \tag{4.75}$$

Combining (4.75) with the inequalities $\mathbf{P}(d \ge \varepsilon mk^{-1} - 1) \ge \mathbf{P}(d_B(k) \ge 1)$ and

$$\mathbf{P}\big(d_A(k) \ge 1\big) \le \mathbf{E}\big(\mathbf{E}\big(d_A(k)|X_1, \dots, X_n\big)\big) = \mathbf{E}\sum_{i=1}^{n} X_i m^{-1}\mathbb{I}_{\{2\le X_i<\varepsilon m/k\}} < \varepsilon n_* k^{-1},$$

we obtain $\mathbf{P}(d \ge \varepsilon mk^{-1} - 1) \ge \mathbf{P}(d \ge 1) - \varepsilon n_* k^{-1}$. Finally, we choose $k = \max\{1, n_*\}$ and obtain (3.2).

The proof of (3.1) is much the same, but now we choose $k = n$.

Let us prove (3.3). Given $N = \sum_{i=1}^{n} \mathbb{I}_{\{X_i \geq 2\}}$, let Y_N be a random variable with binomial distribution $\mathrm{Bin}(N, 2m^{-1})$. We have

$$\mathbf{P}(d \geq C) \geq \mathbf{P}(Y_N \geq C) \geq \mathbf{P}\big(Y_N \geq C | N \geq 2^{-1}n_*\big)\mathbf{P}\big(N \geq 2^{-1}n_*\big) = 1 - o(1).$$

In the last step, we used the simple facts about the binomial distribution that $\mathbf{E}N = n_* \to \infty$ implies $\mathbf{P}(N \geq 2^{-1}n_*) = 1 - o(1)$ and that $\mathbf{E}(Y_N | N = 2^{-1}n_*) = n_* m^{-1} \to +\infty$ implies

$$\mathbf{P}\big(Y_N \geq C | N \geq 2^{-1}n_*\big) \geq \mathbf{P}\big(Y_N \geq C | N = 2^{-1}n_*\big) = 1 - o(1). \qquad \square$$

PROOF OF THEOREM 3.2. The convergence of the degree distribution follows from Theorem 3.1. Assuming, in addition, that $\lim_{m\to\infty} \mathbf{E}X_1^3 = \mathbf{E}Z^3 < \infty$, we obtain the convergence of moments $\mathbf{E}(X_1)_k \to \mathbf{E}(Z)_k$, $k = 2, 3$, as $m, n \to \infty$. Hence, we can write (3.4) in the form

$$\alpha^*(n, m, P) = \frac{\mathbf{E}(Z)_3}{\beta^{-1}(\mathbf{E}(Z)_2)^2 + \mathbf{E}(Z)_3} + o(1).$$

Invoking the identities $\mathbf{E}(Z)_2 = \beta \mathbf{E}d_*$ and $\mathbf{E}(Z)_3 = \beta(\mathbf{E}(d_*)_2 - (\mathbf{E}d_*)^2)$, we obtain (3.5).

In the case where $\mathbf{E}Z^2 < \infty$ and $\mathbf{E}Z^3 = \infty$, we have $\mathbf{E}(X_1)_2 \to \mathbf{E}(Z)_2 < \infty$ and $\mathbf{E}(X_1)_3 \to \infty$. Therefore, (3.4) implies $\alpha^* \to 1$. $\square$

PROOF OF THEOREM 3.3. In the proof, we use some new notation. Let $\{w_2^*, w_3^*\}$ be a random pair of vertices uniformly distributed in the set of all pairs from $W \setminus \{w_1\}$. Let $\overline{\delta} = (\delta_1, \delta_2, \dots, \delta_n)$ denote an ordered collection of subsets of W. We call $\overline{\delta}$ a set-valued vector. Introduce the random set-valued vector $\overline{D} = (D_1^*, D_2^*, \dots, D_n^*)$, where $D_i^* = D_i$ for $w_1 \in D_i$ and $D_i^* = \varnothing$ otherwise. Denote $\overline{X} = (X_1^*, \dots, X_n^*)$, where $X_i^* = |D_i^*|$. Introduce the function $\overline{\delta} \to h(\overline{\delta}) = 2\sum_{i=1}^{n} \binom{|\delta_i|-1}{2} \mathbb{I}_{\{\delta_i \neq \varnothing\}}$ and the random variable $H = h(\overline{D}) = \sum_{i=1}^{n} (X_i^* - 1)_2 \mathbb{I}_{\{X_i^* \geq 1\}}$. A collection $\mathcal{I}$ of set-valued vectors is identified with the event $\overline{D} \in \mathcal{I}$. We denote $\mathcal{S}_{\mathcal{I}} = \mathbf{E}h(\overline{D})\mathbb{I}_{\mathcal{I}} = \sum_{\overline{\delta} \in \mathcal{I}} h(\overline{\delta})\mathbf{P}(\overline{D} = \overline{\delta})$. By $\mathbf{P}_{\overline{\delta}}$ we denote the conditional probability given the event $\{\overline{D} = \overline{\delta}\}$. Denote $n_{\overline{\delta}} = \sum_{i=1}^{n} \mathbb{I}_{\{\delta_i = \varnothing\}}$.

Now we fix an integer $k \geq 2$ such that $\mathbf{P}(d_* = k) > 0$ and assume that m, n are so large that $\mathbf{P}(d = k) \geq 2^{-1}\mathbf{P}(d_* = k) > 0$. Introduce the events

$$\Delta = \{w_1 \sim w_2^*, w_1 \sim w_3^*, w_2^* \sim w_3^*\}, \qquad \mathcal{A} = \{d = k\}, \qquad \mathcal{A}_k = \{d = L, L = k\}$$

and the probabilities

$$p_\star = \mathbf{P}\big(w_1 \sim w_2^*, w_1 \sim w_3^*, d = k\big),$$
$$\tilde{p}_\star = \mathbf{P}\big(w_1 \sim w_2^*, w_1 \sim w_3^*, d = k, d \neq L\big).$$

Here $\{w_2^*, w_3^*\}$ is a random subset of $W \setminus \{w_1\}$ uniformly distributed in the class of subsets of size 2. We assume that $\{w_2^*, w_3^*\}$ is independent of the random sets $D_1, \dots, D_n$ defining the intersection graph.

In order to prove (3.6), we write $\alpha^{*[k]}$ in the form

$$\alpha^{*[k]} = \frac{\mathbf{P}(\Delta \cap \mathcal{A})}{p_\star}$$

and invoke the identities

$$\mathbf{P}(\Delta \cap \mathcal{A}) = \frac{1}{(m-1)_2}\mathbf{E}d_{2*}\mathbb{I}_{\{d_*=k\}} + o(m^{-2}), \tag{4.76}$$

$$p_\star = \binom{k}{2}\binom{m-1}{2}^{-1}\mathbf{P}(d_* = k) + o(m^{-2}). \tag{4.77}$$

In order to show (4.77), we note that every pair $\{w', w''\} \subset W \setminus \{w_1\}$ has the same probability to be covered by the neighborhood of w_1. Hence, by symmetry we have

$$\begin{aligned} p_\star &= \binom{k}{2}\binom{m-1}{2}^{-1}\mathbf{P}(d = k), \\ \tilde{p}_\star &= \binom{k}{2}\binom{m-1}{2}^{-1}\mathbf{P}(d = k, d \neq L). \end{aligned} \tag{4.78}$$

Now (4.77) follows from the first identity and the convergence $\mathbf{P}(d = k) \to \mathbf{P}(d_* = k)$.

Let us show (4.76). Observe that $\mathbf{P}(\Delta \cap \mathcal{A} \cap \{d \neq L\}) \leq \tilde{p}_\star$, and by (4.78) and (4.62) we have $\tilde{p}_\star = o(m^{-2})$. Hence,

$$\mathbf{P}(\Delta \cap \mathcal{A}) = \mathbf{P}(\Delta \cap \mathcal{A}_k) + o(m^{-2}). \tag{4.79}$$

Next, we expand, by the total probability formula,

$$\mathbf{P}(\Delta \cap \mathcal{A}_k) = \sum_{\overline{\delta} \in \mathcal{A}_k} \mathbf{P}_{\overline{\delta}}(\Delta)\mathbf{P}(\overline{D} = \overline{\delta}) \tag{4.80}$$

and note that $\mathbf{P}_{\overline{\delta}}(\mathcal{D}_{\overline{\delta}}) \leq \mathbf{P}_{\overline{\delta}}(\Delta) \leq \mathbf{P}_{\overline{\delta}}(\mathcal{C}_{\overline{\delta}}) + \mathbf{P}_{\overline{\delta}}(\mathcal{D}_{\overline{\delta}})$, where

$$\mathcal{C}_{\overline{\delta}} = \{\exists D_j : w_2^*, w_3^* \in D_j \text{ and } w_1 \notin D_j\}, \qquad \mathcal{D}_{\overline{\delta}} = \{\exists D_j : w_1, w_2^*, w_3^* \in D_j\}.$$

Observe that

$$\mathbf{P}_{\overline{\delta}}(\mathcal{D}_{\overline{\delta}}) = \frac{h(\overline{\delta})}{(m-1)_2} \tag{4.81}$$

and, since $n_{\overline{\delta}} < n$,

$$\mathbf{P}_{\overline{\delta}}(\mathcal{C}_{\overline{\delta}}) = 1 - \left(1 - \frac{\mathbf{E}(X')_2}{(m-1)_2}\right)^{n_{\overline{\delta}}} < n\frac{\mathbf{E}(X')_2}{(m-1)_2} = O(m^{-1}). \tag{4.82}$$

Here X' denotes the random variable $|D_i|$ conditioned on the event $w_1 \notin D_i$. Collecting (4.81) and (4.82) in (4.80), we obtain

$$\mathbf{P}(\Delta \cap \mathcal{A}_k) = \frac{1}{(m-1)_2} \mathcal{S}_{\mathcal{A}_k} + O(m^{-3}). \tag{4.83}$$

Now we replace the event $\mathcal{A}_k$ by $\mathcal{A}$ in (4.83). For the left-hand side, we apply (4.79). For the right-hand side, we apply the inequalities

$$\mathcal{S}_{\mathcal{A}} \geq \mathcal{S}_{\mathcal{A}_k} \geq \mathcal{S}_{\mathcal{A}} - \binom{k}{2} \mathbf{P}(d = k, d \neq L) = \mathcal{S}_{\mathcal{A}} - o(1) \tag{4.84}$$

[here we used $h(\bar{\delta}) \leq \binom{k}{2}$ and (4.62)]. We obtain

$$\mathbf{P}(\Delta \cap \mathcal{A}) = \frac{1}{(m-1)_2} \mathcal{S}_{\mathcal{A}} + o(m^{-2}).$$

Finally, (4.76) follows by the convergence of $\mathcal{S}_{\mathcal{A}} = \mathbf{E} H \mathbb{I}_{\{d=k\}}$ to $\mathbf{E} d_{2*} \mathbb{I}_{\{d_*=k\}}$ as $m, n \to \infty$. We derive this convergence from the weak convergence of bivariate random vectors $(H, L) \to (d_{2*}, d_*)$, which is obtained using the same argument as that of the proof of Theorem 3.1. $\square$

Acknowledgments. I would like to thank anonymous referees for their valuable comments and suggestions.

REFERENCES

[1] BARBOUR, A. D. and REINERT, G. (2011). The shortest distance in random multi-type intersection graphs. *Random Structures Algorithms* **39** 179–209. MR2850268

[2] BARRAT, A. and WEIGT, M. (2000). On the properties of small-world networks. *Eur. Phys. J. B* **13** 547–560.

[3] BLACKBURN, S. R. and GERKE, S. (2009). Connectivity of the uniform random intersection graph. *Discrete Math.* **309** 5130–5140. MR2548914

[4] BLOZNELIS, M. (2008). Degree distribution of a typical vertex in a general random intersection graph. *Lith. Math. J.* **48** 38–45. MR2398169

[5] BLOZNELIS, M. (2010). A random intersection digraph: Indegree and outdegree distributions. *Discrete Math.* **310** 2560–2566. MR2669380

[6] BLOZNELIS, M. (2010). The largest component in an inhomogeneous random intersection graph with clustering. *Electron. J. Combin.* **17** Research Paper 110, 17. MR2679564

[7] BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, MA. MR0229267

[8] BRITTON, T., DEIJFEN, M., LAGERÅS, A. N. and LINDHOLM, M. (2008). Epidemics on random graphs with tunable clustering. *J. Appl. Probab.* **45** 743–756. MR2455182

[9] DEIJFEN, M. and KETS, W. (2009). Random intersection graphs with tunable degree distribution and clustering. *Probab. Engrg. Inform. Sci.* **23** 661–674. MR2535025

[10] ESCHENAUER, L. and GLIGOR, V. D. (2002). A key-management scheme for distributed sensor networks. In *Proceedings of the 9th ACM Conference on Computer and Communications Security, Washington, DC* 41–47.

[11] FOSS, S., KORSHUNOV, D. and ZACHARY, S. (2011). *An Introduction to Heavy-Tailed and Subexponential Distributions*. ACM, New York. MR2810144

[12] FOUDALIS, I., JAIN, K., PAPADIMITRIOU, C. and SIDERI, M. (2011). Modeling social networks through user background and behavior. In *Algorithms and Models for the Web Graph. Lecture Notes in Computer Science* **6732** 85–102. Springer, Heidelberg. MR2842315

[13] GODEHARDT, E. and JAWORSKI, J. (2001). Two models of random intersection graphs and their applications. *Electron. Notes Discrete Math.* **10** 129–132.

[14] GODEHARDT, E. and JAWORSKI, J. (2003). Two models of random intersection graphs for classification. In *Exploratory Data Analysis in Empirical Research* 67–81. Springer, Berlin. MR2074223

[15] GODEHARDT, E., JAWORSKI, J. and RYBARCZYK, K. (2012). Clustering coefficients of random intersection graphs. In *Studies in Classification, Data Analysis and Knowledge Organization* 243–253. Springer, Berlin.

[16] GUILLAUME, J.-L. and LATAPY, M. (2004). Bipartite structure of all complex networks. *Inform. Process. Lett.* **90** 215–221. MR2054656

[17] JAWORSKI, J., KAROŃSKI, M. and STARK, D. (2006). The degree of a typical vertex in generalized random intersection graph models. *Discrete Math.* **306** 2152–2165. MR2255609

[18] JAWORSKI, J. and STARK, D. (2008). The vertex degree distribution of passive random intersection graph models. *Combin. Probab. Comput.* **17** 549–558. MR2433940

[19] KAROŃSKI, M., SCHEINERMAN, E. R. and SINGER-COHEN, K. B. (1999). On random intersection graphs: The subgraph problem. *Combin. Probab. Comput.* **8** 131–159. MR1684626

[20] NEWMAN, M. E. J. (2003). Properties of highly clustered networks. *Phys. Rev. E* **68** 026121.

[21] NEWMAN, M. E. J., STROGATZ, S. H. and WATTS, D. J. (2001). Random graphs with arbitrary degree distributions and their applications. *Phys. Rev. E* **64** 026118.

[22] NEWMAN, M. E. J., WATTS, D. J. and STROGATZ, S. H. (2002). Random graph models of social networks. *Proc. Natl. Acad. Sci. USA* **99** (Suppl. 1) 2566–2572.

[23] NIKOLETSEAS, S., RAPTOPOULOS, C. and SPIRAKIS, P. G. (2011). On the independence number and Hamiltonicity of uniform random intersection graphs. *Theoret. Comput. Sci.* **412** 6750–6760. MR2885093

[24] RAVASZ, L. and BARABÁSI, A. L. (2003). Hierarchical organization in complex networks. *Phys. Rev. E* **67** 026112.

[25] RYBARCZYK, K. (2011). Diameter, connectivity, and phase transition of the uniform random intersection graph. *Discrete Math.* **311** 1998–2019. MR2818878

[26] RYBARCZYK, K. (2012). The degree distribution in random intersection graphs. In *Challenges at the Interface of Data Analysis, Computer Science, and Optimization* 291–299. Springer, Berlin.

[27] STARK, D. (2004). The vertex degree distribution of random intersection graphs. *Random Structures Algorithms* **24** 249–258. MR2068868

[28] STEELE, J. M. (1994). Le Cam's inequality and Poisson approximations. *Amer. Math. Monthly* **101** 48–54. MR1252705

[29] WATTS, D. J. and STROGATZ, S. H. (1998). Collective dynamics of "small-world" networks. *Nature* **393** 440–442.

[30] YAGAN, O. and MAKOWSKI, A. M. (2009). Random key graphs—Can they be small worlds? In *NETCOM'09: Proceedings of the* 2009 *First International Conference on Networks & Communications* 313–318. IEEE Computer Society, Washington, DC.

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